Positivity in cluster algebras

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A type-A cluster structure on polynomial rings

$$X = \left\{ \begin{bmatrix} t_1 & 1 & & 0 \\ 1 & t_2 & 1 & \\ & 1 & \ddots & 1 \\ 0 & & 1 & t_m \end{bmatrix} : t_1, \ldots, t_m \in \mathbb{R} \right\}$$

$\mathbb{R}[X]$ = $\mathbb{R}$-algebra generated by

non-zero principal minors

$$\Delta_{[i,j]} = \det \begin{bmatrix} t_i & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & t_j \end{bmatrix} \qquad (i \leq j),$$

$\simeq \mathbb{R}[t_1, \ldots t_m]$ being $t_i = \Delta_{[i,i]}$

Relations: $1 \leq i < j < k < \ell \leq m+2$

$$\boxed{\Delta_{[i,k-2]}\Delta_{[j,\ell-2]} = \Delta_{[i,j-2]}\Delta_{[k,\ell-2]} + \Delta_{[i,\ell-2]}\Delta_{[j,k-2]}}$$

convention:

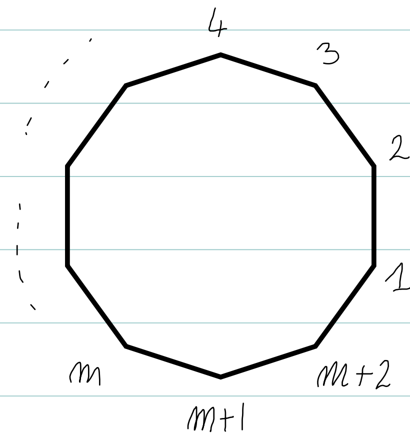
$$\Delta_{[i,j]} = 1 \quad \text{if} \quad j < i$$

A model for this algebra is the

Ptolemy algebra:

defined by generators and relations as follows

<u>GENERATORS</u>: $P_{m-1} :=$



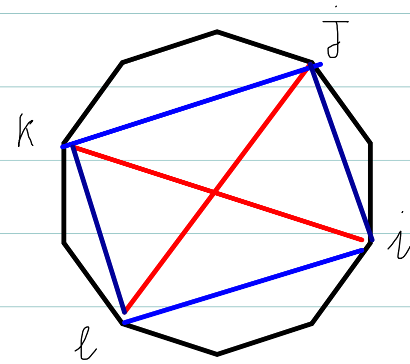
$D_{m-1} := \{ \text{edges and diagonals of } P_{m-1} \}$

$= \{ [i,j] \mid 1 \leq i < j \leq m+2 \}$

<u>Define</u> a generator $X([i,j])$ $\forall [i,j] \in D$

with $X([i,i+1]) = 1$ for $1 \leq i \leq m+1$

PTOLEMY RELATIONS:



$$X([ik])\,X([j\ell]) = X([i\ell])\,X([j,k]) + X([ij])\,X([k\ell])$$

the isomorphism is given by

$$\Delta_{[i,j-2]} \xrightarrow{\sim} X([i,j])$$

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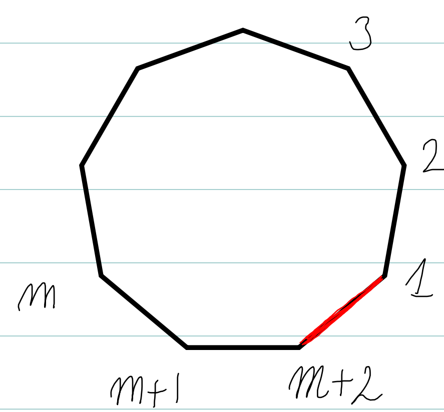
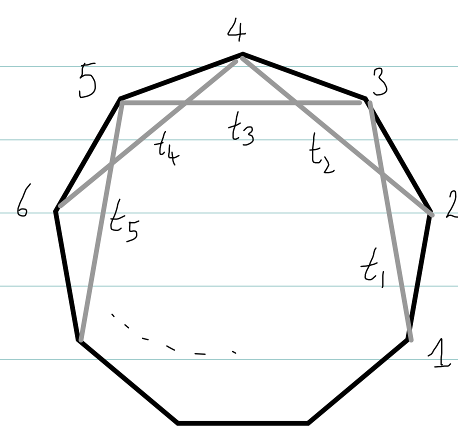
Special elements:

$X([1,m+2]) = \Delta_{[1,m]} = \det$

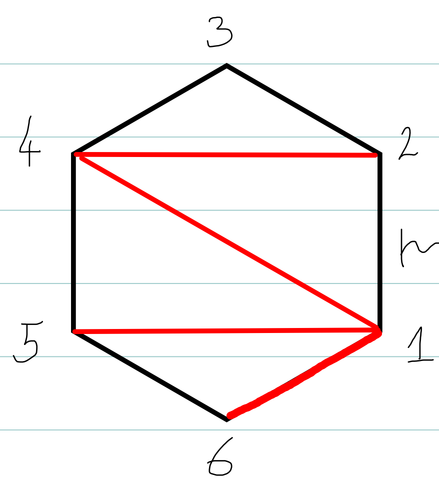


$t_i = X[i,i+2]$



From the relations it follows that there is a bijection:

$$\{\text{triangulations of } P_{m-1}\} \xrightarrow{1-1} \{\text{max. algebraically indipendent set of generators}\}$$



$$\longmapsto \{X([1,4]),\ X([24]),\ X([15]),\ \det\}$$

Remark: det appears in every such alg. indipendent set.

Definition

A set $\{X(d_1), \dots, X(d_{m-1})\}$, d_i: diagonal of P_{m-1}, $\{d_1, \dots, d_{m-1}, [1, m+2]\}$ is a triangulation, is called a cluster

$X(d)$, d a diagonal, is called a cluster variable.

$X([1, m+2]) = \det$ is called the frozen cluster variable or coefficient.

Remark:

the monomials $X(d_1)^{a_1}\cdots X(d_{m-1})^{a_{m-1}}\det^{a_m}$

$a_1,\dots,a_m\geq 0$, for a given cluster

$\{X(d_1),\dots,X(d_{m-1}),\det\}$ are

linearly independent over $\mathbb{Z}$.

$\Rightarrow$ the monomials $\{X(d_1)^{a_1}\cdots X(d_{m-1})^{a_{m-1}}\}_{a_i\geq 0}$

are linearly independent over $\mathbb{Z}[\det]$.

Remark:

We look at $\mathbb{R}[X]$ as $\mathbb{R}[\det]$- alg.

generated by $\Delta_{[i,j]} \neq \det$.

We could have considered $X \cap GL(m)$

whose coordinate ring is the

$\mathbb{R}[\det^{\pm 1}]$- alg. gen. by $\Delta_{[i,j]} \neq \det$.

$\rightsquigarrow$ In the definition of a

cluster algebra there is always the choice

if the multiplicative inverses of

the coefficients are in the ground

ring or not.

Definition

A cluster monomial is a monomial $X(d_1)^{a_1} \cdots X(d_{m-1})^{a_{m-1}}$ where $\{X(d_1), \ldots, X(d_{m-1})\}$ is a cluster.

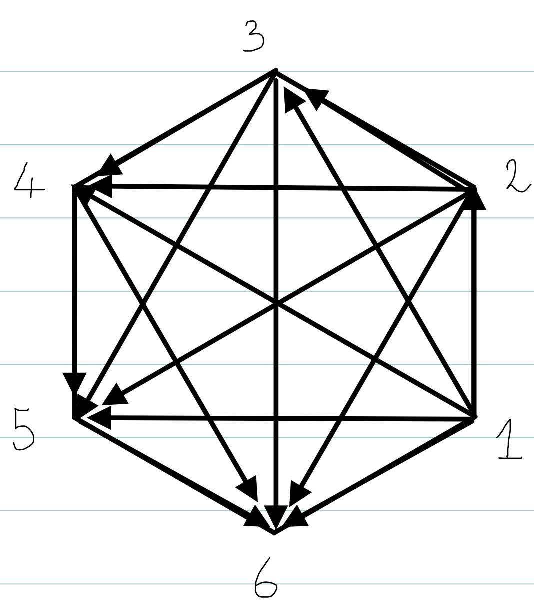
Question

Are cluster monomials linearly independent over $\mathbb{R}[\det]$, or $\mathbb{R}[\det^{\pm 1}]$?

Let us orient every edge and diagonal $[i,j]$ of P_{m-1} as $i \longrightarrow j$

Let $\overrightarrow{D}_{m-1}$ be the set of all such elements

Example: $m=4$. $\overrightarrow{D}_3$ consists of

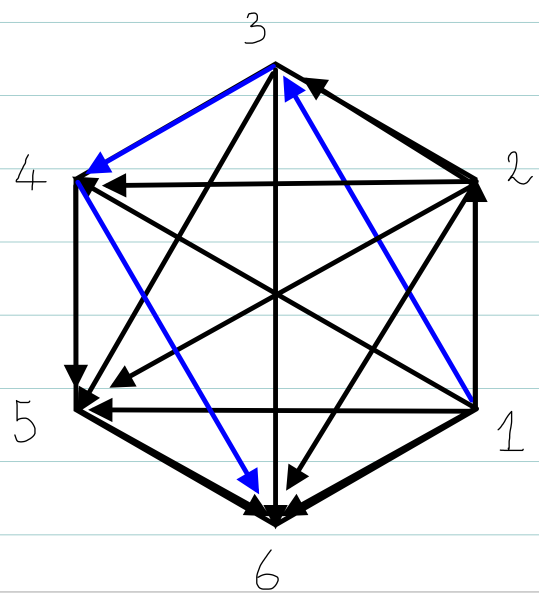


Definition

A cluster walk w from a vertex i_0 to a vertex i_1 of P_{m-1} is a path $w = d_1 \cdots d_\ell$ where

1) $d_i \in \overrightarrow{D}$ and $i_0 \xrightarrow{d_1} \xrightarrow{d_2} \cdots \xrightarrow{d_\ell} i_1$
2) ℓ is odd
3) d_{2k} is an edge of P_{m-1}

Example: cluster walks from 1 to 6



[13][34][46]

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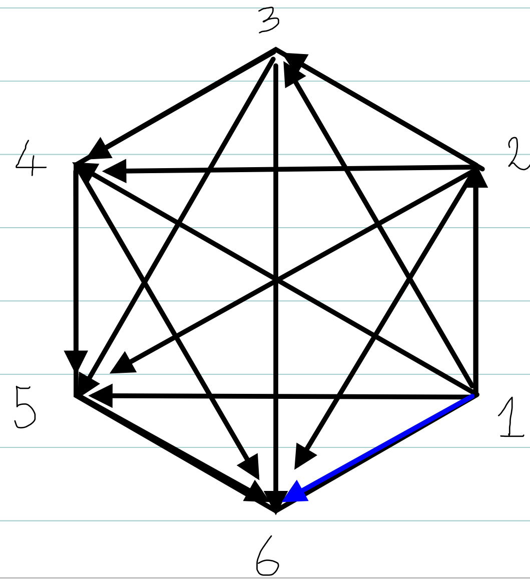

$[13][34][46]$, $[16]$

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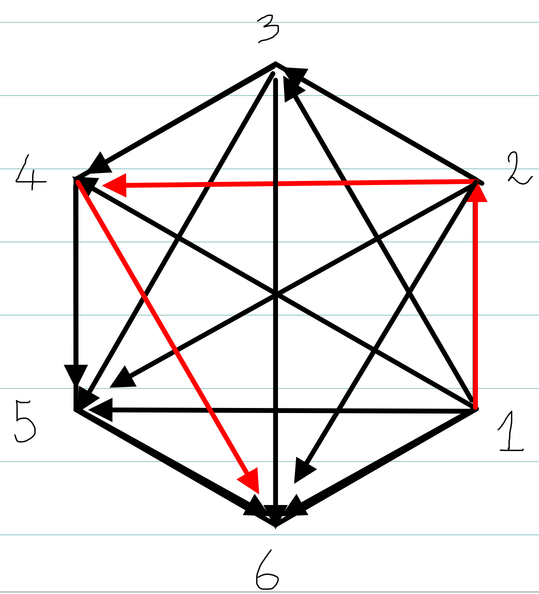

[13][34][46], [16]

no cluster walk:

[12][24][46]

the weight of a cluster walk

$w = d_1 \cdots d_\ell$ is the monomial

$$X(w) = X(d_1)X(d_3)\cdots X(d_\ell)$$

Remark

the weight of a cluster walk

is a cluster monomial

Let us denote by $\mathcal{W}(i_0, i_1)$

the set of cluster walks from

i_0 to i_1.

Proposition [C.I., 2011]

Given $i \in [1, m+2]$, and a nonneg. integer k so that $i+k \leq m+2$ we have

$$\boxed{t_i t_{i+1} \cdots t_{i+k} = \sum_{w \in \mathcal{W}(i, i+k+2)} X(w)}$$

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Remark:

LHS = square-free standard mon. in t_i's

RHS = sum of cluster monomials

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Define the degree

$$\deg X([i,j]) = j - i - 1$$

e.g.

$$\deg X[i, i+2] = \deg t_i = 1$$

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Define the degree

$$\deg X([i,j]) = j - i - 1$$

e.g. $w = d_1 \cdots d_\ell \in \mathcal{W}(i_0, i_1)$

$$\deg w = \sum_i \deg(d_i) = i_1 - i_0 - \ell$$

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$\deg \text{LHS} = k+1$

For $w \in \mathcal{W}(i, i+k+2)$ $w = d_1 \ldots d_\ell$

$\deg X(w) = k+2-\ell \leq k+1$

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For $w \in \mathcal{W}(i, i+k+2)$ $w = d_1 \cdots d_\ell$

$\deg X(w) = k+2-\ell \le k+1$

$\exists!\, w \in \mathcal{W}(i, i+k+2)$ s.t. $\deg X(w) = k+1$

Namely $w = [i, i+k+2]$

Proposition [C.I., 2011]

Given $i \in [1, m+2]$, and a nonneg. integer k so that $i+k \leq m+2$ we have

$$\boxed{t_i t_{i+1} \cdots t_{i+k} = \sum_{w \in \mathcal{W}(i, i+k+2)} X(w)}$$

<u>COR</u>: there is a unitriangular transformation wrt. deg between standard monomials in the t_i's and cluster monomials (possibly multiplied by det)

Since standard monomials in the t_i's form a linear basis of $\mathbb{R}[X] \simeq \mathbb{R}[t_1, \dots, t_m]$ we get:

Corollary:

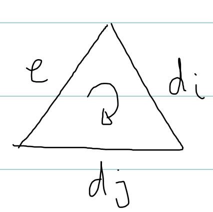
the set of cluster monomials form a $\mathbb{R}[\det]$-linear basis of $\mathbb{R}[X]$.

Question:

In which sense this construction provides a type A cluster structure on $\mathbb{R}[t_1, \dots, t_m]$?

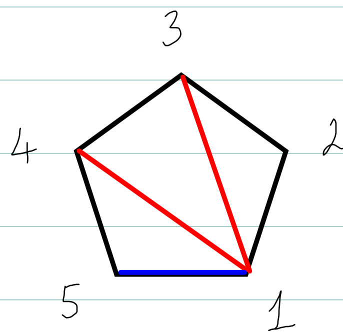
For a triangulation $\{d_1, \dots, d_{m-1}, [1, m+2]\}$ we associate a quiver whose vertices are $d_1, \dots, d_{m-1}, [1, m+2]$

arrows: there is an arrow $d_i \longrightarrow d_j$ if $\exists$ an edge e and a triangle

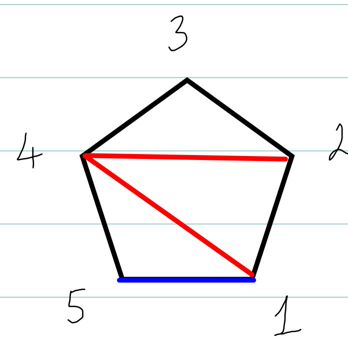


Example:

$[13] \longrightarrow [14] \longrightarrow [15]$



$[24] \longleftarrow [14] \longrightarrow [15]$



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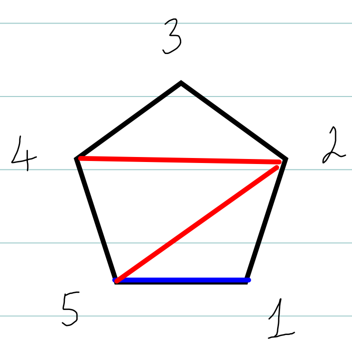
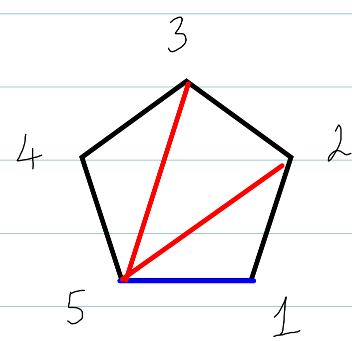
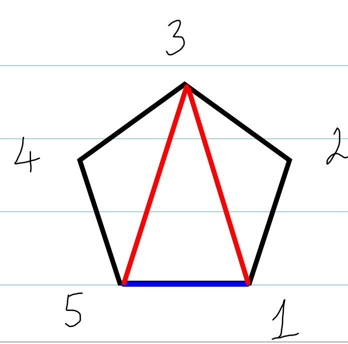


$[35] \longleftarrow [25] \longleftarrow [15]$



$$\begin{tikzcd} & {[13]} \arrow[dr] & \\ {[35]} \arrow[ur] & & {[15]} \arrow[ll] \end{tikzcd}$$



One can show easily that in P_{m-1} there is a triangulation whose quiver is of type A_m and by general results on cluster algebras one knows that every other quiver appearing this way is either of type A_m or wild.

Total positivity in X

$$X_{\geq 0} := \{ x \in X \mid \underbrace{\Delta_{[1,j]}(x) > 0, \quad \forall\, 1 \leq j}_{\text{system of } 2^m - 1 \text{ inequalities}} \}$$

Remark:

this system is redundant

Prop. [Fomin-Zelevinsky]

It's enough to check the inequalities in a cluster of $\mathbb{R}[X]$.

Total positivity in X

$$X_{\geq 0} := \{ x \in X \mid \Delta_{[1,j]}(x) > 0, \ \forall i \leq j \}$$

Prop. [Fomin-Zelevinsky]

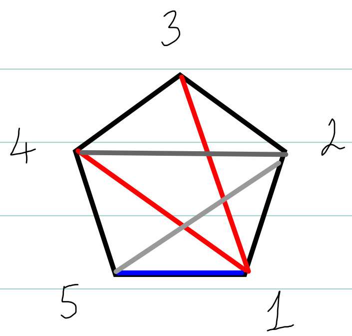
It's enough to check the inequalities in a cluster of $\mathbb{R}[X]$.

proof: Every cluster variable Δ is obtained from a cluster $\mathcal{C} = \{ \Delta_1, \ldots, \Delta_{m-1}, \det \}$ by iterated applications of the Ptolemy rels.:

$$\Rightarrow \Delta = \frac{\dfrac{f_1(\Delta_1, \ldots, \Delta_{m-1}, \det)}{f_2(\Delta_1, \ldots \ldots, \det)}}{\dfrac{f_3(\Delta_1, \ldots, \det)}{f_4(\Delta_1, \ldots, \det)}} \qquad f_i \in \mathbb{Z}_{\geq 0}[\Delta_1, \ldots, \det]$$

$\blacksquare$

Example: $X = \left\{ \begin{bmatrix} t_1 & 1 & 0 \\ 1 & t_2 & 1 \\ 0 & 1 & t_3 \end{bmatrix} \right\}$



$\longleftrightarrow \{ \Delta_{[11]}, \Delta_{[12]}, \det \}$

$$\Delta_{[22]} = (\Delta_{[12]} + 1)\,\Delta_{[11]}^{-1} = X([2,4])$$

$$\Delta_{[33]} = (\Delta_{[11]} + \det)\,\Delta_{[12]}^{-1}$$

$$\Delta_{[23]} = X([25]) = \frac{X([24])\det + 1}{X([14])}$$

$$= \frac{\dfrac{\det(\Delta_{[12]} + 1) + \Delta_{[11]}}{\Delta_{[11]}}}{\Delta_{[12]}}$$

Some references on this section

[Geiss, Leclerc, Schröer: "Factorial cluster algebras", 2011]

[Fomin, Zelevinsky: "Cluster algebras II"]

[J. Scott: "Cluster structure on the coordinate ring of Grassmannians"]

[Yang, Zelevinsky: "Cluster algebras of finite type via Coxeter elements and principal minors"]

Some references on this section

[G. Cerulli Irelli, A. D'Andrea, "Positivstellensatz for cluster algebras of geometric type", in preparation]

[T. Holms, P. Jørgensen, M. Rubey: "Ptolemy diagrams and torsion pairs in the Dynkin type A_n"]

2. Fomin-Zelevinsky cluster algebras

$m \geq 1$

$\mathcal{F}_m = \mathbb{Q}(x_1, \dots, x_m)$

Q: quiver with no loops

no oriented 2-cycles, m vertices

$\rightsquigarrow \mathcal{A}(Q) \subset \mathcal{F}_m$: cluster algebra associated with Q.

GENERATORS AND RELATIONS of $\mathcal{A}(Q)$ are encoded into the seeds mutation-equivalent to the initial seed $(Q, \underline{x})$

A seed is a pair $(H, \underline{u})$ where

-) H is a good quiver on m vertices
-) $\underline{u} = (u_1, \dots, u_m)$ is a free-generating set of $\mathcal{F}_m$.

Given $k \in H_0$ the mutation of $(H, \underline{u})$ at k is the pair $(\mu_k(H), \mu_k(\underline{u}))$ where:

-) $\mu_k(\underline{u}) = \underline{u} \setminus \{u_k\} \cup \{u'_k\}$

$$u'_k = \frac{\prod_{k \to j \in H_1} u_j + \prod_{i \to k} u_i}{u_k}$$

•) $\mu_K(H)$ is the good quiver H' obtained from H in 3 steps

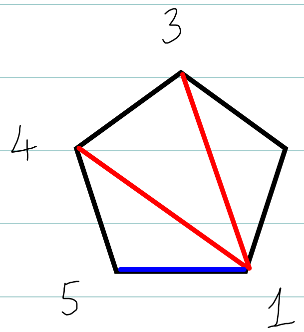
1. For every hook $i \to K \to j$ add an arrow $i \to j$

2. Reverse all arrows incident to K.

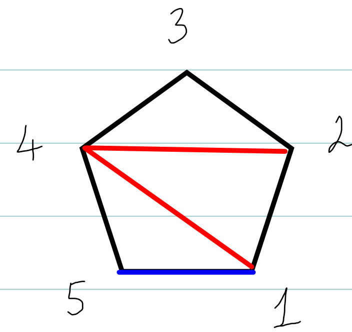
3. Remove a maximal set of oriented 2-cycles.

Example

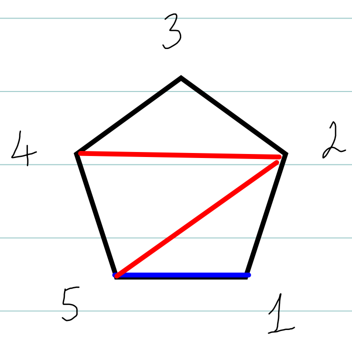
$$[13] \to [14] \to [15] \quad \xrightarrow{\mu_{[13]}} \quad [24] \leftarrow [14] \to [15]$$



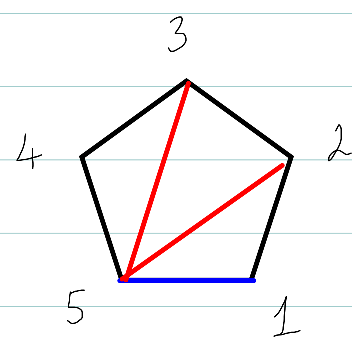
$\mu_{[14]}$



$$[24] \to [25] \leftarrow [15] \quad \xrightarrow{\mu_{[24]}} \quad [35] \leftarrow [25] \leftarrow [15]$$



$\mu_{[25]}$



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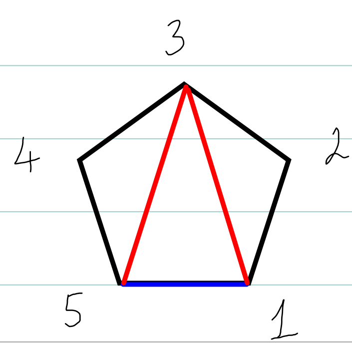
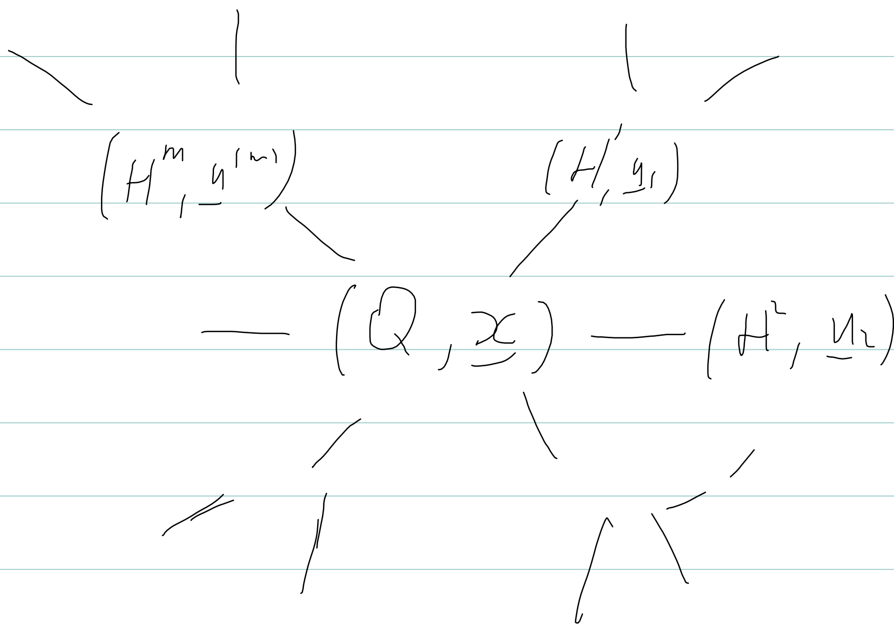


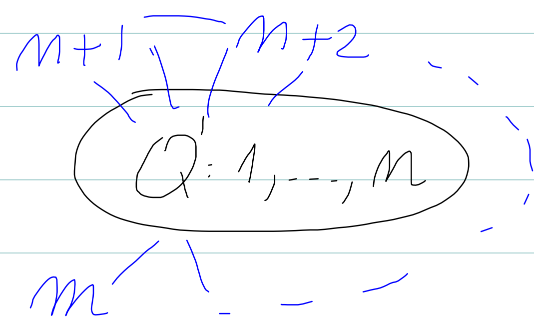
Definition

In a seed $(H,\underline{u})$ of $\mathcal{F}_m$, $\underline{u}=(u_1,\dots,u_m)$ is called its **cluster**, $u_1,\dots,u_m$ are its **cluster variables**.



$A(Q) := \mathbb{Z}$-subalgebra of $\mathcal{F}_m$ generated by the cluster variables of the seeds mut.-eq. to $(Q,\underline{x})$.

Sometimes we need to freeze some vertices of Q:



i.e. do not mutate at vertices $n+1, \dots, m$.

$\rightsquigarrow \mathcal{A}(Q', \{n+1, \dots, m\}) :=$ $\mathbb{Z}[x_{n+1}, \dots, x_m]$-subalg. gen. by the cl. variables obtained by successive mutations at vert. $1, \dots, n$.

Laurent phenomenon

Given a cluster variable x of $\mathcal{A}(Q)$

and a cluster $\mathcal{C}=\{u_1,\dots,u_m\}$

$\exists\ f,g\in\mathbb{Z}[u_1,\dots,u_m]$ s.t.

$$x=\frac{f(u_1,\dots,u_m)}{g(u_1,\dots,u_m)}$$

Thm [Fomin–Zelevinsky, '01]

$g(u_1,\dots,u_m)$ is a monomial.

Remark

The proof of Laurent phenomenon in

$\mathbb{R}[X]$ is as complicated as the general case

Positivity

An element of $A(Q)$ is called positive if its Laurent expansion in every cluster has non-negative coefficients.

Problem

In the cluster algebra $\mathbb{R}[X]$ above we have hence 2 notions of positivity:

- the FZ-positivity $\mathbb{R}[X]^+$
- the dual Total positivity:

$$\mathbb{R}[X]^+_{tot} = \{ f \in \mathbb{R}[X] \mid f(x) \geq 0 \ \forall x \in X_{>0} \}$$

Comparison between $\mathbb{R}[X]^+$ and $\mathbb{R}[X]^+_{tot}$

A cone of a real field F is

a subset $T \subset F$ s.t.

$$T \cdot T \subset T, \quad T + T \subset T, \quad F^2 \subset T$$

We embed $\mathbb{R}[X] \hookrightarrow \mathbb{R}(X)$ and

consider the cones of $\mathbb{R}(X)$

$$\mathbb{R}(X)^+_{tot} = \{ f \in \mathbb{R}(X) \mid f(x) \geq 0 \ \forall x \in X_{\geq 0} \}$$

$$\mathbb{R}(X)^+ = \bigcap \{ T \mid T \subset \mathbb{R}(X) \text{ cone}, T \supseteq \mathbb{R}[X]^+ \}$$

Thm [C.I.-D'Andrea]:

$$\mathbb{R}(X)^+ = \mathbb{R}(X)^+_{tot}$$

Remark

this theorem holds in
every coordinate ring with
cluster structure
under the hypothesis that
its cluster variables are positive!

A cluster monomial is a monomial in cluster variables all belonging to the same cluster.

Conjecture [Fomin-Zelevinsky, '01]

-) the cluster monomials are linearly independent over the ground ring
-) the cluster monomials are positive.

Theorem [C.I., B. Keller, D. Labardini-Fragoso, P.G.-Plamondon, 2012]

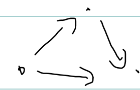
The cluster monomials of $\mathcal{A}(Q; \{x_{n+1}, \dots, x_m\})$ are lin. independent over $\mathbb{Z}[x_{n+1}, \dots, x_m]$.

Atomic Basis

A positive element of $\mathcal{A}(Q)$ is called positive indecomposable if it cannot be written as a sum of two (distinct) positive elements.

Conjecture [Sherman-Zelevinsky]
Cluster monomials are positive indecomposable.

The conjecture was proved:

•) Q: Dynkin, $Q =$  [C.I.]

•) $Q: \cdot \rightrightarrows \cdot$ [Sherman-Zelevinsky]

•) Q of Type $\tilde{A}_n$ [Dupont-Thomas]

•) Musiker-Schiffler-Williams have a candidate atomic basis in cluster algebras from Riemann surfaces.

All this proof are based on the existence of an atomic basis:

A linear basis $\mathcal{B}$ of $A(Q)$ is called atomic if

$$A(Q)^+ = \{\text{non-neg. lin. comb. of els. of } \mathcal{B}\}$$

If an atomic basis exists then its elements are positive indecomp.

In the cluster algebras above it is shown an explicit atomic basis which contains the cluster monomials.

Thm [C.I. - D. Labardini-Fragoso]

Q: Dynkin. The cluster monomials form an atomic basis of $\mathcal{A}(Q, \{x_{n+1}, \dots, x_m\})$ (over $\mathbb{Z}[x_{n+1}, \dots, x_m]$)

COROLLARY

An element $f \in \mathbb{R}[X]$ is positive iff f is $\mathbb{Z}_{\geq 0}[\det]$-linear combination of cluster monomials.

Geometric Realization of Atomic Basis.

For every cluster variable x of $\mathcal{A}(Q)$ there exists a representation M of Q such that the expansion of x is

$$C_Q(x) = \sum_{\underline{e} \in \mathbb{Z}_{\geq 0}^m} \chi(Gr_{\underline{e}}(M))\, \underline{x}^{B\underline{e} + \underline{g}_M}$$

where M is fin. dim. and nilpotent

$Gr_{\underline{e}}(M) = \{ N \subseteq M \mid \underline{\dim} N = \underline{e} \}$

quiver Grassmannian

χ: its Euler characteristic

Geometric Realization of Atomic Basis.

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where

$B \in \mathcal{M}_{m \times m}(\mathbb{Z})$

$b_{ij} = \#\{j \to i \in Q_1\} - \#\{i \to j \in Q_1\}$

Geometric Realization of Atomic Basis.

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where

$0 \to M \to I_0 \to I_1$ inj. res.

$\underline{g}_M := [\mathrm{soc}\, I_1] - [\mathrm{soc}\, I_0] \in K_0(\mathrm{mod}\, kQ)$

$\rightsquigarrow$ the g-vector of M.

this formula is due
to many authors
Caldero–Chapoton
Caldero–Keller
Palu
Derksen–Weyman–Zelevinsky
Plamondon.

Kronecker cluster algebra

$Q: 1 \rightrightarrows 2$, $\mathcal{F} = \mathbb{Q}(x_1, x_2)$

cluster variables: $\{x_m\}_{m \in \mathbb{Z}}$

$$x_m x_{m+2} = x_{m+1}^2 + 1$$

cluster monomials

$\{x_m^a x_{m+1}^b\}_{m \in \mathbb{Z},\ a, b \in \mathbb{Z}_{\geq 0}}$

Atomic basis [S. Z.]

$\{\text{cl. mon.}\} \sqcup \{z_n : n \geq 1\}$

Here $\{Z_n : n \geq 1\}$ are defined by:

$$Z_1 := x_0 x_3 - x_1 x_2$$

$$Z_2 := Z_1^2 - 2$$

$$Z_{n+1} = Z_1 Z_n - Z_{n-1}$$

<u>Thm</u> [C.I.-Esposito]

$$Z_n = \sum_{\underline{e}} \chi_c\left(Gr_{\underline{e}}(R_n^{sm})\right) \underline{x}^{B\underline{e} + \underline{g}_{R_n}}$$

$$R_n := \mathbb{C}^n \underset{=}{\overset{J_n(0)}{\rightrightarrows}} \mathbb{C}^n$$

$Gr_e(R_n)^{sm} :=$ smooth part of $Gr_e(R_n)$

Thank You!