

# What is categorification?

Greg Stevenson

BIREP  
Universität Bielefeld

20/03/12

## What is decategorification?

C - an essentially small category

## What is decategorification?

$C$  - an essentially small category

### Definition

The **decategorification** of  $C$ ,  $K(C)$ , is the set of isomorphism classes of objects of  $C$ .

## What is decategorification?

$C$  - an essentially small category

### Definition

The **decategorification** of  $C$ ,  $K(C)$ , is the set of isomorphism classes of objects of  $C$ .

Defines a decategorification functor

$$K: \text{Cat} \longrightarrow \text{Set}$$

## What is decategorification?

$C$  - an essentially small category

### Definition

The **decategorification** of  $C$ ,  $K(C)$ , is the set of isomorphism classes of objects of  $C$ .

which it is perhaps wiser to write as

$$K: 1\text{-Cat} \longrightarrow 0\text{-Cat}$$

## What is decategorification?

$C$  - an essentially small category

### Definition

The **decategorification** of  $C$ ,  $K(C)$ , is the set of isomorphism classes of objects of  $C$ .

which it is perhaps wiser to write as

$$K: 1\text{-Cat} \longrightarrow 0\text{-Cat}$$

More generally we have decategorifications  $n\text{-Cat} \longrightarrow m\text{-Cat}$  for  $n \geq m$ .

## Extra structure

$K(\mathcal{C})$  often inherits structure from  $\mathcal{C}$ .

## Extra structure

$K(\mathcal{C})$  often inherits structure from  $\mathcal{C}$ .

### Example

If  $\mathcal{C}$  is monoidal then  $K(\mathcal{C})$  is a monoid.

## Extra structure

$K(\mathcal{C})$  often inherits structure from  $\mathcal{C}$ .

### Example

If  $\mathcal{C}$  is monoidal then  $K(\mathcal{C})$  is a monoid.

E.g.  $K(\text{FinSet}) \cong \mathbb{N}$  as semirings.

## Extra structure

$K(\mathcal{C})$  often inherits structure from  $\mathcal{C}$ .

### Example

If  $\mathcal{C}$  is monoidal then  $K(\mathcal{C})$  is a monoid.

E.g.  $K(\text{FinSet}) \cong \mathbb{N}$  as semirings.

We should consider variants. A construction factoring through  $K$  is a “decategorification”.

## Extra structure

$K(\mathcal{C})$  often inherits structure from  $\mathcal{C}$ .

### Example

If  $\mathcal{C}$  is monoidal then  $K(\mathcal{C})$  is a monoid.  
E.g.  $K(\text{FinSet}) \cong \mathbb{N}$  as semirings.

We should consider variants. A construction factoring through  $K$  is a “decategorification”.

### Definition

$\mathcal{A}$  - essentially small abelian category,  $R$  - unital commutative ring

## Extra structure

$K(C)$  often inherits structure from  $C$ .

### Example

If  $C$  is monoidal then  $K(C)$  is a monoid.  
E.g.  $K(\text{FinSet}) \cong \mathbb{N}$  as semirings.

We should consider variants. A construction factoring through  $K$  is a “decategorification”.

### Definition

$A$  - essentially small abelian category,  $R$  - unital commutative ring  
the  **$R$ -decategorification** of  $A$  is

$$R \otimes_{\mathbb{Z}} K_0(A).$$

# Naive categorification

# Naive categorification

## (Naive) Definition

Categorification is the reverse process of decategorification.

# Naive categorification

## (Naive) Definition

Categorification is the reverse process of decategorification.

Explicitly we take some set-theoretic structure and try to make the following replacements:

# Naive categorification

## (Naive) Definition

Categorification is the reverse process of decategorification.

Explicitly we take some set-theoretic structure and try to make the following replacements:

sets  $\mapsto$  categories

# Naive categorification

## (Naive) Definition

Categorification is the reverse process of decategorification.

Explicitly we take some set-theoretic structure and try to make the following replacements:

sets  $\mapsto$  categories  
functions  $\mapsto$  functors

# Naive categorification

## (Naive) Definition

Categorification is the reverse process of decategorification.

Explicitly we take some set-theoretic structure and try to make the following replacements:

sets	$\mapsto$	categories
functions	$\mapsto$	functors
equations	$\mapsto$	natural isomorphisms

# Examples

- FinSet and  $k$ -vect categorify the semiring  $\mathbb{N}$ .

# Examples

- FinSet and  $k$ -vect categorify the semiring  $\mathbb{N}$ .
- $\mathbb{N}$ -graded vector spaces over  $k$ , which are of finite dimension over  $k$ , categorify the semiring  $\mathbb{N}[t]$  via

$$\bigoplus_{i \geq 0} V_i \mapsto \sum_{i \geq 0} (\dim_k V_i) t^i$$

# Examples

- FinSet and  $k$ -vect categorify the semiring  $\mathbb{N}$ .
- $\mathbb{N}$ -graded vector spaces over  $k$ , which are of finite dimension over  $k$ , categorify the semiring  $\mathbb{N}[t]$  via

$$\bigoplus_{i \geq 0} V_i \mapsto \sum_{i \geq 0} (\dim_k V_i) t^i$$

- the categorification of a sheaf is a stack

Our naive definition covers examples which are “meaningless”.

Our naive definition covers examples which are “meaningless”.

$A$  - finite free  $\mathbb{Z}$ -algebra with basis  $\{a_r \mid r = 1, \dots, n\}$ .

Our naive definition covers examples which are “meaningless”.

$A$  - finite free  $\mathbb{Z}$ -algebra with basis  $\{a_r \mid r = 1, \dots, n\}$ .

$$a_r a_s = \sum_{t=1}^n b_{rs}^t a_t \quad \text{where } b_{rs}^t \in \mathbb{N} \forall r, s, t.$$

Our naive definition covers examples which are “meaningless”.

$A$  - finite free  $\mathbb{Z}$ -algebra with basis  $\{a_r \mid r = 1, \dots, n\}$ .

$$a_r a_s = \sum_{t=1}^n b_{rs}^t a_t \quad \text{where } b_{rs}^t \in \mathbb{N} \forall r, s, t.$$

A “trivial” categorification of the regular representation of  $A$  is given by  $(A, F_1, \dots, F_n)$ :

$$A = \bigoplus_{t=1}^n k\text{-vect} \quad \text{and} \quad F_r(x_s) = \bigoplus_{t=1}^n x_t^{\oplus b_{rs}^t}.$$

Here  $x_s$  is the simple object which is a copy of  $k$  in the  $s$ th position.

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ .

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ .  
The  $[F_r]: K_0(A) \rightarrow K_0(A)$  define an action of  $A$  on  $K_0(A)$  via

$$\left(\sum_{r=1}^n m_r a_r\right) \cdot [x_s] = \sum_{r=1}^n m_r [F_r(x_s)].$$

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ . The  $[F_r]: K_0(A) \rightarrow K_0(A)$  define an action of  $A$  on  $K_0(A)$  via

$$\left(\sum_{r=1}^n m_r a_r\right) \cdot [x_s] = \sum_{r=1}^n m_r [F_r(x_s)].$$

$K_0(A) \rightarrow A$  defined by  $[x_s] \mapsto a_s$  is an isomorphism of left  $A$ -modules.

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ . The  $[F_r]: K_0(A) \rightarrow K_0(A)$  define an action of  $A$  on  $K_0(A)$  via

$$\left(\sum_{r=1}^n m_r a_r\right) \cdot [x_s] = \sum_{r=1}^n m_r [F_r(x_s)].$$

$K_0(A) \rightarrow A$  defined by  $[x_s] \mapsto a_s$  is an isomorphism of left  $A$ -modules.

## The point

There is no “interesting higher structure” in this categorification.

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ . The  $[F_r]: K_0(A) \rightarrow K_0(A)$  define an action of  $A$  on  $K_0(A)$  via

$$\left(\sum_{r=1}^n m_r a_r\right) \cdot [x_s] = \sum_{r=1}^n m_r [F_r(x_s)].$$

$K_0(A) \rightarrow A$  defined by  $[x_s] \mapsto a_s$  is an isomorphism of left  $A$ -modules.

### The point

There is no “interesting higher structure” in this categorification.

### The philosophy (sort of)

We **should not** use our set-theoretic object to build a category.

Taking  $K_0(A)$  yields  $\mathbb{Z}^n$  with a canonical choice of basis  $[x_r]$ . The  $[F_r]: K_0(A) \rightarrow K_0(A)$  define an action of  $A$  on  $K_0(A)$  via

$$\left(\sum_{r=1}^n m_r a_r\right) \cdot [x_s] = \sum_{r=1}^n m_r [F_r(x_s)].$$

$K_0(A) \rightarrow A$  defined by  $[x_s] \mapsto a_s$  is an isomorphism of left  $A$ -modules.

### The point

There is no “interesting higher structure” in this categorification.

### The philosophy (sort of)

We **should not** use our set-theoretic object to build a category.  
We **should** find a category which builds our set-theoretic object.

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

- $A$  - noetherian, artinian  $k$ -linear abelian category

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

- $A$  - noetherian, artinian  $k$ -linear abelian category
- $(E, F)$  - adjoint pair of endofunctors  $A \rightarrow A$

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

- $A$  - noetherian, artinian  $k$ -linear abelian category
- $(E, F)$  - adjoint pair of endofunctors  $A \rightarrow A$

such that

- $[E]$  and  $[F]$  give a representation of  $\mathfrak{sl}_2$  on  $\mathbb{Q} \otimes K_0(A)$

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

- $A$  - noetherian, artinian  $k$ -linear abelian category
- $(E, F)$  - adjoint pair of endofunctors  $A \rightarrow A$

such that

- $[E]$  and  $[F]$  give a representation of  $\mathfrak{sl}_2$  on  $\mathbb{Q} \otimes K_0(A)$
- simple objects of  $A$  give weight vectors

## An example: weak $\mathfrak{sl}_2(\mathbb{Q})$ categorification

We describe a concrete, and interesting, definition of a categorification due to Chuang and Rouquier.

### Definition

A **weak  $\mathfrak{sl}_2$  categorification** is a triple  $(A, E, F)$  where:

- $A$  - noetherian, artinian  $k$ -linear abelian category
- $(E, F)$  - adjoint pair of endofunctors  $A \rightarrow A$

such that

- $[E]$  and  $[F]$  give a representation of  $\mathfrak{sl}_2$  on  $\mathbb{Q} \otimes K_0(A)$
- simple objects of  $A$  give weight vectors
- $F$  is isomorphic to a left adjoint of  $E$

# Example

$$A = k\text{-vect} \oplus k[x]/(x^2)\text{-mod} \oplus k\text{-vect}.$$

# Example

$$A = k\text{-vect} \oplus k[x]/(x^2)\text{-mod} \oplus k\text{-vect}.$$

We have functors

$$k\text{-vect} \begin{array}{c} \xrightarrow{\text{ind}} \\ \xleftarrow{\text{res}} \end{array} k[x]/(x^2)\text{-mod}$$

# Example

$$A = k\text{-vect} \oplus k[x]/(x^2)\text{-mod} \oplus k\text{-vect}.$$

We have functors

$$k\text{-vect} \begin{array}{c} \xrightarrow{\text{ind}} \\ \xleftarrow{\text{res}} \end{array} k[x]/(x^2)\text{-mod}$$

Set

$$E = \begin{pmatrix} 0 & 0 & 0 \\ \text{ind} & 0 & 0 \\ 0 & \text{res} & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & \text{res} & 0 \\ 0 & 0 & \text{ind} \\ 0 & 0 & 0 \end{pmatrix}$$

This gives a weak categorification  $(A, E, F)$  of the 3-dimensional irreducible  $\mathfrak{sl}_2$  representation.

# Example

$$A = k\text{-vect} \oplus k[x]/(x^2)\text{-mod} \oplus k\text{-vect}.$$

We have functors

$$k\text{-vect} \begin{array}{c} \xrightarrow{\text{ind}} \\ \xleftarrow{\text{res}} \end{array} k[x]/(x^2)\text{-mod}$$

Set

$$E = \begin{pmatrix} 0 & 0 & 0 \\ \text{ind} & 0 & 0 \\ 0 & \text{res} & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & \text{res} & 0 \\ 0 & 0 & \text{ind} \\ 0 & 0 & 0 \end{pmatrix}$$

This gives a weak categorification  $(A, E, F)$  of the 3-dimensional irreducible  $\mathfrak{sl}_2$  representation.

It admits a structure of  **$\mathfrak{sl}_2$ -categorification** in a stronger sense (coherence data).

## The point

Chuang and Rouquier have used these categorifications of  $\mathfrak{sl}_2$  actions to prove

### Broué's abelian defect group conjecture for symmetric groups

Let  $A$  be a block of a symmetric group  $G$  over  $\mathbb{Z}_{(p)}$ ,  $D$  a defect group and  $B$  the corresponding block of  $N_G(D)$ . If  $D$  is abelian then  $A$  and  $B$  are splendidly Rickard equivalent.

## Cluster categories

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$ .

# Cluster categories

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$ .

## Definition

The **cluster category**  $C_Q$  is the orbit category

$$D^b(Q\text{-rep}) / (S^{-1} \circ \Sigma^2)^{\mathbb{Z}},$$

$S$  - a Serre functor,  $\Sigma$  - the suspension.

# Cluster categories

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$ .

## Definition

The **cluster category**  $C_Q$  is the orbit category

$$D^b(Q\text{-rep}) / (S^{-1} \circ \Sigma^2)^{\mathbb{Z}},$$

$S$  - a Serre functor,  $\Sigma$  - the suspension.

The cluster category is triangulated by a theorem of Keller.

# Cluster categories

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$ .

## Definition

The **cluster category**  $C_Q$  is the orbit category

$$D^b(Q\text{-rep}) / (S^{-1} \circ \Sigma^2)^{\mathbb{Z}},$$

$S$  - a Serre functor,  $\Sigma$  - the suspension.

The cluster category is triangulated by a theorem of Keller.

## Definition

$T$  in  $C_Q$  is **rigid** if  $\text{Hom}_{C_Q}(T, \Sigma T) = 0$ .

## Another categorification

The cluster category categorifies the combinatorial data giving a cluster algebra in the following sense:

## Another categorification

The cluster category categorifies the combinatorial data giving a cluster algebra in the following sense:

### Theorem (Caldero-Keller)

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$

$A_Q$  - associated cluster algebra

## Another categorification

The cluster category categorifies the combinatorial data giving a cluster algebra in the following sense:

### Theorem (Caldero-Keller)

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$

$A_Q$  - associated cluster algebra

- There is an explicit bijection

$$K(\{\text{rigid objects in } \mathcal{C}_Q\}) \xrightarrow{\sim} \{\text{cluster variables in } A_Q\}$$

## Another categorification

The cluster category categorifies the combinatorial data giving a cluster algebra in the following sense:

### Theorem (Caldero-Keller)

$Q$  - finite quiver without oriented cycles on  $\{1, \dots, n\}$

$A_Q$  - associated cluster algebra

- There is an explicit bijection

$$K(\{\text{rigid objects in } \mathcal{C}_Q\}) \xrightarrow{\sim} \{\text{cluster variables in } A_Q\}$$

- the clusters correspond to the **cluster-tilting subsets** i.e., non-isomorphic rigid indecomposables  $\{T_1, \dots, T_n\}$  such that

$$\text{Hom}_{\mathcal{C}_Q}(T_i, \Sigma T_j) = 0 \quad \forall i, j.$$

## A different categorification

This example was “additive” - direct sum gives multiplication of cluster variables

## A different categorification

This example was “additive” - direct sum gives multiplication of cluster variables

Hernandez, Leclerc - “multiplicative” categorification

## A different categorification

This example was “additive” - direct sum gives multiplication of cluster variables

Hernandez, Leclerc - “multiplicative” categorification

M - monoidal abelian category

## A different categorification

This example was “additive” - direct sum gives multiplication of cluster variables

Hernandez, Leclerc - “multiplicative” categorification

M - monoidal abelian category

$$K_0(M) \cong A_Q$$

## A different categorification

This example was “additive” - direct sum gives multiplication of cluster variables

Hernandez, Leclerc - “multiplicative” categorification

$\mathcal{M}$  - monoidal abelian category

$$K_0(\mathcal{M}) \cong A_Q$$

such that cluster monomials and variables come from certain simples