1. Macdonald Polynomials

Λ symmetric functions over $\mathbb{Q}$

- $\{s_\mu(z)\}_{\mu \text{ partition}}$ basis of Schur polynomials

$$\bigoplus_{n\geq 0} \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathrm{Rep}(\mathfrak{S}_n) \xrightarrow{\sim} \Lambda$$

$$[M] \longmapsto F_M(z) \qquad \text{Frobenius characteristic}$$

$$[S(\mu)] \longmapsto s_\mu(z)$$

irreducible rep^n attached to μ

Macdonald '88:

$$\Lambda_{q,t} = \Lambda \otimes_{\mathbb{Q}} \mathbb{Q}(q,t)$$

coefficients are now rational functions in 2 variables

$\{ M_\mu(z;q,t) \}_\mu$ basis of Macdonald polynomials.

They have interesting specializations in q & t.

$$M_\mu(z;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)\, s_\lambda(z)$$

Integrality Conjecture

(Garsia-Remmel; Tessler; Knop; Sahi; Kirrilov; Noumi '96)

$$K_{\lambda\mu}(q,t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$$

Positivity Conjecture

(Haiman '01; Grojnowski-Haiman; Assaf)

$$K_{\lambda\mu}(q,t) \in \mathbb{N}[q^{\pm 1}, t^{\pm 1}]$$

i.e. under relation with $\mathrm{Rep}(\mathfrak{S}_n)$ Macdonald polynomials count dimensions of components of <u>bigraded</u> representations

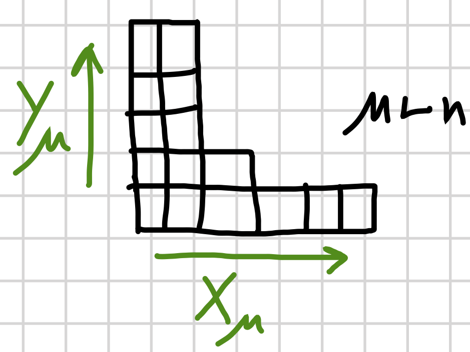
2. HILBERT SCHEMES

$V=\mathbb{C}^n$; $\mathrm{Hilb}^n(\mathbb{C}^2)$

$$\begin{aligned} \mathrm{Hilb}^n(\mathbb{C}^2) &= \{(X,Y,v)\in \mathrm{End}(V)^2\times V : [X,Y]=0,\ \mathbb{C}\langle X,Y\rangle\cdot v=V\}/GL(V) \\ &= \left\{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{I} = n \right\} \end{aligned}$$

$(\mathbb{C}^*)^2 \curvearrowright \mathrm{Hilb}^n(\mathbb{C}^2)$ fixed points $\longleftrightarrow$ monomial ideals, I_μ

$\updownarrow$



$$\begin{tikzcd}
\text{(reduced) } Z \arrow[r, dashed] \arrow[d, dashed, "\pi"'] \arrow[dr, phantom, "\square"] & (\mathbb{C}^2)^n \arrow[d, "\text{orbit}"] \\
\mathrm{Hilb}^n(\mathbb{C}^2) \arrow[r, "\text{supp}"'] & (\mathbb{C}^2)^n/\mathfrak{S}_n = \mathrm{Sym}^n(\mathbb{C}^2)
\end{tikzcd}$$

Z is called the isospectral Hilbert scheme.

THEOREM (HAIMAN) Z is Cohen-Macaulay.

$\rightsquigarrow$ $\pi_* \mathcal{O}_{Z_n} \cong \mathcal{P}$ a vector bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$

$$F_{\mathcal{P}(I_\mu)}(z; q, t) = M_\mu(z; q, t).$$

3. Grothendieck-Springer Resolution

$G = GL(V)$; $\mathfrak{g} = \mathrm{End}(V)$; $\mathfrak{h} = \mathbb{C}^n$

The Grothendieck-Springer resolution is a smooth variety

$$\tilde{\mathfrak{g}} = \{ (X; 0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset V) : \dim F_i = i, \ X(F_i) \subseteq F_i \}$$
$$\subseteq \mathfrak{g} \times \mathcal{B}$$

It admits a pair of morphisms

$\mu(X, F_\cdot) = X$

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr] & & \mathfrak{h} \arrow[dl] \\ & \mathfrak{g}/\!/G = \mathfrak{h}/\mathfrak{S}_n & \end{tikzcd}$$

$\nu(X, F_\cdot) = (X|_{F_1/F_0}, X|_{F_2/F_1}, \ldots)$

Over the set of endomorphisms whose eigenvalues are distinct the morphism μ is an $\mathfrak{S}_n$-covering.

Over the set of nilpotent endomorphisms the map μ is a resolution of singularities

$$\mu : T^*\mathcal{B} = \mu^{-1}(\mathcal{N}) \longrightarrow \mathcal{N}$$

the Springer resolution.

Hotta–Kashiwara '84 $\quad HC = \int_{\mu\times\nu} \mathcal{O}_{\tilde{\mathfrak{g}}}$ a holonomic $D_{\mathfrak{g}\times\mathfrak{h}}$-Module

the Harish-Chandra D-module.

They presented it; they related it to Springer theory.

$$\rightsquigarrow \quad CH(HC) = \mathfrak{X} \subset T^*(\mathfrak{g}\times\mathfrak{h}) \cong \mathfrak{g}\times\mathfrak{g}\times\mathfrak{h}\times\mathfrak{h}$$

where

$$\begin{tikzcd}
\mathfrak{X}^{\text{(reduced)}} \arrow[rr, dashed] \arrow[d, dashed, "p"'] \arrow[drr, phantom, "\square"] & & \mathfrak{h}\times\mathfrak{h} \arrow[d] \\
\mathfrak{C}=\{(X,Y)\in\mathfrak{g}\times\mathfrak{g} : [X,Y]=0\} \arrow[r] & \mathfrak{C}/\!/G = & \mathfrak{h}\times\mathfrak{h}/\mathfrak{S}_n
\end{tikzcd}$$

THEOREM (GINZBURG '10)

a) $\exists$ filtration on HC s.t. $\mathrm{gr}\,HC = \psi_* \mathcal{O}_{\mathfrak{X}_{norm}}$ where $\psi : \mathfrak{X}_{norm} \longrightarrow \mathfrak{X}$ is the normalization morphism.

b) $\mathfrak{X}_{norm}$ is Cohen-Macaulay and Gorenstein

($\Rightarrow \mathfrak{X}_{norm}/\mathfrak{S}_n = \mathfrak{C}_{norm}$ is Cohen-Macaulay)

4. Back to Hilbert Schemes

$$\mathcal{C}^\circ = \{ (X,Y) \in \mathcal{C} : \exists v \text{ s.t. } \mathbb{C}\langle X,Y\rangle v = V \} \subseteq \mathcal{C}_{sm}$$

Then $\mathcal{R} := p_* \mathcal{O}_{\mathcal{X}_{norm}} \big|_{\mathcal{C}^\circ}$ is a vector bundle

$\circlearrowleft$

$(\mathbb{C}^*)^2 \times G \times \mathfrak{S}_n$: the fibres carry the regular representation of $\mathfrak{S}_n$.

$$\begin{tikzcd} & \{(X,Y,v) : [X,Y]=0,\ \mathbb{C}\langle X,Y\rangle v = V\} \arrow[dl, "\delta"', "\text{forget } v"] \arrow[dr, "\rho", "\text{orbit}"'] & \\ \mathcal{C}^\circ & & \mathrm{Hilb}^n(\mathbb{C}^2) \end{tikzcd}$$

$\Longrightarrow$ $\widetilde{\mathcal{P}} := (\rho_* \delta^* \mathcal{R})^G$ is a $(\mathbb{C}^*)^2 \times \mathfrak{S}_n$-equivariant vector bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$

THEOREM (G.) $F_{\widetilde{\mathcal{P}}(I_\mu)}(z; q,t) = M_\mu(z; q,t)$

$\Longrightarrow$ Macdonald positivity (again)

Proof: Macdonald polynomials are characterized by 3 properties

(A) $M_\mu[(1-q)z; q,t] \in \mathbb{Q}(q,t)\{s_\lambda(z) : \lambda \geq \mu\}$

(B) $M_\mu[(1-t)z; q,t] \in \mathbb{Q}(q,t)\{s_\lambda(z) : \lambda \geq \mu'\}$

(C) $M_\mu[1; q,t] = 1.$

These have clear representation theoretic meaning:

(A) $\sum_k (-1)^k \quad [\widetilde{P}(I_\mu) \otimes \Lambda^k \mathfrak{h}_1] \in \mathbb{Q}(q,t)\{[S_\lambda] : \lambda \geq \mu\}$

(B) $\sum_k (-1)^k \quad [\widetilde{P}(I_\mu) \otimes \Lambda^k \mathfrak{h}_2] \in \mathbb{Q}(q,t)\{[S_\lambda] : \lambda \geq \mu'\}$

(C) $[\widetilde{P}(I_\mu) : \mathrm{triv}] = 1.$

Obviously (A) will be the crucial point, by a Fourier dual argument.

$$\begin{array}{ccc} I_\mu & \longleftrightarrow & (X_\mu, Y_\mu) \in \mathcal{C}^\circ \quad \text{(principal nilpotent pair)} \\ & & \downarrow \\ & & (X_\mu, Y_\mu, 0, 0) \in \mathcal{X} \end{array}$$

$$\sum (-1)^k [\widehat{\mathcal{P}}(I_\mu) \otimes \Lambda^k \mathfrak{z}_1] \longleftrightarrow \sum (-1)^k \left[(\mathrm{gr}\, HC)_{(X_\mu, Y_\mu, 0, 0)} \otimes \Lambda^k \mathfrak{z}_1 \right]$$

$$\longleftrightarrow (\mathrm{gr} HC)_{(X_\mu, Y_\mu, 0,0)} / \mathfrak{z}_1$$

$$\longleftrightarrow \mathrm{gr}\, (HC|_{\mathfrak{g} \times \{0\}}) \quad \text{at } (X_\mu, Y_\mu)$$

$$CH(HC|_{\mathfrak{g}\times\{0\}}) = \{ (X,Y) : [X,Y]=0, X \text{ nilpotent} \}$$

$$= \bigcup_{\tau} \overline{T^*_{G_\tau} \mathfrak{g}}$$

But $(HC|_{\mathfrak{g}\times\{0\}})_\lambda$ is D-module on $\mathcal{N}$ corresponding to the Springer representation

$$\Longrightarrow CH((HC|_{\mathfrak{g}\times\{0\}})_\lambda) \subseteq \bigcup_{\tau\leq\lambda} \overline{T^*_{G_\tau}\mathfrak{g}}$$

$\therefore (X_\mu, Y_\mu) \in CH((HC|_{\mathfrak{g}\times\{0\}})_\lambda)$ only if $\mu \leq \lambda$. $\square$