On Andersen and Jantzen filtrations

Johannes Kübel

University of Erlangen-Nürnberg

DFG Schwerpunkttagung, Schloss Thurnau

March 20, 2012

Basic setting

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of S at the maximal ideal generated by $\mathfrak{h} \subset S$.

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of S at the maximal ideal generated by $\mathfrak{h} \subset S$.
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and ρ the half sum of positive roots

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of S at the maximal ideal generated by $\mathfrak{h} \subset S$.
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and ρ the half sum of positive roots
- For $\alpha \in \mathrm{R}^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresp. reflection by s_α

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of S at the maximal ideal generated by $\mathfrak{h} \subset S$.
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and ρ the half sum of positive roots
- For $\alpha \in \mathrm{R}^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresp. reflection by s_α
- $\mathcal{W}$ Weyl group with ρ-shifted dot-action on $\mathfrak{h}^*$

Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of S at the maximal ideal generated by $\mathfrak{h} \subset S$.
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and ρ the half sum of positive roots
- For $\alpha \in \mathrm{R}^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresp. reflection by s_α
- $\mathcal{W}$ Weyl group with ρ-shifted dot-action on $\mathfrak{h}^*$
- A a local, commutative S-algebra with structure map $\tau : S \to A$
  (in this talk $A = S_{(0)}$ or $A = S_{(0)}/S_{(0)}\mathfrak{h} = \mathbb{C}$)

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \; \forall h \in \mathfrak{h}\}$

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \; \forall h \in \mathfrak{h}\}$
- M is locally $\mathfrak{b}$-A-finite

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h}\}$
- M is locally $\mathfrak{b}$-A-finite

- $\mathcal{O}_A$ is abelian

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \; \forall h \in \mathfrak{h}\}$
- M is locally $\mathfrak{b}$-A-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda)$ $(\lambda \in \mathfrak{h}^*)$

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h}\}$
- M is locally $\mathfrak{b}$-A-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda)$ $(\lambda \in \mathfrak{h}^*)$
- projective covers $P_A(\lambda)$ of $L_A(\lambda)$ exist in $\mathcal{O}_A$

Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-A-bimodules whose objects M satisfy

- M is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h}\}$
- M is locally $\mathfrak{b}$-A-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda)$ $(\lambda \in \mathfrak{h}^*)$
- projective covers $P_A(\lambda)$ of $L_A(\lambda)$ exist in $\mathcal{O}_A$
- $\mathcal{O}_A \cong \bigoplus_\lambda \mathcal{O}_{A,\lambda}$ where λ runs over anti-dominant weights (block decomposition).

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A
- $\mathcal{O}_A$ contains deformed Verma modules:
  $\Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$ where A_λ is the $\mathfrak{b}$-A-bimodule A on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[\mathfrak{b}, \mathfrak{b}]$ by 0.

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A
- $\mathcal{O}_A$ contains deformed Verma modules:
  $\Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$ where A_λ is the $\mathfrak{b}$-A-bimodule A on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[\mathfrak{b}, \mathfrak{b}]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A
- $\mathcal{O}_A$ contains deformed Verma modules: $\Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$ where A_λ is the $\mathfrak{b}$-A-bimodule A on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[\mathfrak{b}, \mathfrak{b}]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

Definition

A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a Δ_A- and a ∇_A-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A
- $\mathcal{O}_A$ contains deformed Verma modules: $\Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$ where A_λ is the $\mathfrak{b}$-A-bimodule A on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[\mathfrak{b}, \mathfrak{b}]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

Definition

A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a Δ_A- and a ∇_A-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

- indecomposable deformed tilting modules are parametrized by their highest weight

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over A
- $\mathcal{O}_A$ contains deformed Verma modules: $\Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda$ where A_λ is the $\mathfrak{b}$-A-bimodule A on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[\mathfrak{b}, \mathfrak{b}]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

Definition

A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a Δ_A- and a ∇_A-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

- indecomposable deformed tilting modules are parametrized by their highest weight
- $K_A(\lambda) :=$ indecomposable deformed tilting module with highest weight $\lambda \in \mathfrak{h}^*$

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_{\mathbb{C}}(\lambda) = \Delta(\lambda)$,

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_{\mathbb{C}}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_{\mathbb{C}}(\lambda) = P(\lambda)$,

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_{\mathbb{C}}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_{\mathbb{C}}(\lambda) = P(\lambda)$,
- $K_A(\lambda) \otimes_A \mathbb{C} \cong K_{\mathbb{C}}(\lambda) = K(\lambda)$,

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_{\mathbb{C}}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_{\mathbb{C}}(\lambda) = P(\lambda)$,
- $K_A(\lambda) \otimes_A \mathbb{C} \cong K_{\mathbb{C}}(\lambda) = K(\lambda)$,
- $$\mathrm{Hom}_{\mathcal{O}_A}(P, \Delta_A(\lambda)) \otimes_A \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(P \otimes_A \mathbb{C}, \Delta(\lambda))$$

For $A = S_{(0)}$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_\mathbb{C}$ and the category $\mathcal{O}_\mathbb{C}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_\mathbb{C}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_\mathbb{C}(\lambda) = P(\lambda)$,
- $K_A(\lambda) \otimes_A \mathbb{C} \cong K_\mathbb{C}(\lambda) = K(\lambda)$,
- $$\mathrm{Hom}_{\mathcal{O}_A}(P, \Delta_A(\lambda)) \otimes_A \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(P \otimes_A \mathbb{C}, \Delta(\lambda))$$
- $$\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K) \otimes_A \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), K \otimes_A \mathbb{C})$$

The associated moment graph

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

- $\mathcal{V} := \mathcal{W}$ vertices

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

- $\mathcal{V} := \mathcal{W}$ vertices
- $\mathcal{E} := \{\{x, y\} \in \mathcal{P}(\mathcal{V}) \,|\, \exists \beta \in \mathrm{R}^+ : x = s_\beta y\}$ edges

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

- $\mathcal{V} := \mathcal{W}$ vertices
- $\mathcal{E} := \{\{x, y\} \in \mathcal{P}(\mathcal{V}) \,|\, \exists \beta \in \mathrm{R}^+ : x = s_\beta y\}$ edges
- $\alpha(\{x, s_\beta x\}) = \beta^\vee$ labeling

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

- $\mathcal{V} := \mathcal{W}$ vertices
- $\mathcal{E} := \{\{x, y\} \in \mathcal{P}(\mathcal{V}) \mid \exists \beta \in \mathrm{R}^+ : x = s_\beta y\}$ edges
- $\alpha(\{x, s_\beta x\}) = \beta^\vee$ labeling
- Define the order on $\mathcal{V}$ by

$$w \leq w' \Leftrightarrow w \cdot \lambda \leq w' \cdot \lambda$$

for all $w, w' \in \mathcal{W}$.

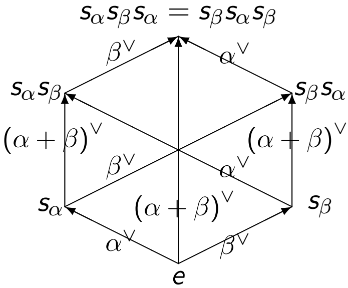
Example:

Example:

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\lambda = -2\rho$

Example:

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\lambda = -2\rho$



Sheaves on $\mathcal{G}$

Sheaves on $\mathcal{G}$

Definition

An A-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties

Sheaves on $\mathcal{G}$

Definition

An A-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties

- $\mathcal{M}_x$ is an A-module for any $x \in \mathcal{V}$

Sheaves on $\mathcal{G}$

Definition

An A-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties

- $\mathcal{M}_x$ is an A-module for any $x \in \mathcal{V}$
- $\mathcal{M}_E$ is an A-module for all $E \in \mathcal{E}$ with $\alpha(E)\mathcal{M}_E = 0$

Sheaves on $\mathcal{G}$

Definition

An A-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties

- $\mathcal{M}_x$ is an A-module for any $x \in \mathcal{V}$
- $\mathcal{M}_E$ is an A-module for all $E \in \mathcal{E}$ with $\alpha(E)\mathcal{M}_E = 0$
- $\rho_{x,E} : \mathcal{M}_x \to \mathcal{M}_E$ is a homomorphism of A-modules for $x \in \mathcal{V}$, $E \in \mathcal{E}$ with $x \in E$.

Sheaves on $\mathcal{G}$

Definition

An A-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties

- $\mathcal{M}_x$ is an A-module for any $x \in \mathcal{V}$
- $\mathcal{M}_E$ is an A-module for all $E \in \mathcal{E}$ with $\alpha(E)\mathcal{M}_E = 0$
- $\rho_{x,E} : \mathcal{M}_x \to \mathcal{M}_E$ is a homomorphism of A-modules for $x \in \mathcal{V}$, $E \in \mathcal{E}$ with $x \in E$.

For $A = S$ consider all S-modules and S-linear maps as graded modules and graded morphisms in degree zero.

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag".

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag".
There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag".
There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.
$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag". There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$. $\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

Theorem ((Braden-MacPherson), (Fiebig))

For every $w \in \mathcal{V}$ there exists an up to isomorphism unique graded S-sheaf $\mathcal{B}^{\uparrow}(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag". There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.
$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

Theorem ((Braden-MacPherson), (Fiebig))

For every $w \in \mathcal{V}$ there exists an up to isomorphism unique graded S-sheaf $\mathcal{B}^{\uparrow}(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties

- $\mathcal{B}^{\uparrow}(w)_w \cong S$ *and* $\mathcal{B}^{\uparrow}(w)_x$ *graded free with* $\mathcal{B}^{\uparrow}(w)_x = 0$ *unless* $w \leq x$

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag".
There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.
$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

Theorem ((Braden-MacPherson), (Fiebig))

For every $w \in \mathcal{V}$ there exists an up to isomorphism unique graded S-sheaf $\mathcal{B}^{\uparrow}(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties

- $\mathcal{B}^{\uparrow}(w)_w \cong S$ *and* $\mathcal{B}^{\uparrow}(w)_x$ *graded free with* $\mathcal{B}^{\uparrow}(w)_x = 0$ *unless* $w \leq x$
- $\mathcal{B}^{\uparrow}(w)$ *is indecomposable and projective in* $\mathcal{C}_S(\mathcal{G})$

$\mathcal{C}_A(\mathcal{G}) :=$ category of A-sheaves on $\mathcal{G}$ that admit a "Verma flag".
There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.
$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

Theorem ((Braden-MacPherson), (Fiebig))

For every $w \in \mathcal{V}$ there exists an up to isomorphism unique graded S-sheaf $\mathcal{B}^{\uparrow}(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties

- $\mathcal{B}^{\uparrow}(w)_w \cong S$ *and* $\mathcal{B}^{\uparrow}(w)_x$ *graded free with* $\mathcal{B}^{\uparrow}(w)_x = 0$ *unless* $w \leq x$
- $\mathcal{B}^{\uparrow}(w)$ *is indecomposable and projective in* $\mathcal{C}_S(\mathcal{G})$

Definition

Denote by $\mathcal{B}^{\downarrow}(w)$ the Braden-MacPherson sheaf on the same moment graph $\mathcal{G}$ with reversed order.

An equivalence of categories

An equivalence of categories

$\mathcal{O}_{A,\lambda}^{VF} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

An equivalence of categories

$\mathcal{O}_{A,\lambda}^{VF} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

Theorem (Fiebig-localisation)

There is an equivalence of exact categories

$$\mathbb{V} : \mathcal{O}_{A,\lambda}^{VF} \to \mathcal{C}_A(\mathcal{G})$$

An equivalence of categories

$\mathcal{O}_{A,\lambda}^{VF} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

Theorem (Fiebig-localisation)

There is an equivalence of exact categories

$$\mathbb{V} : \mathcal{O}_{A,\lambda}^{VF} \to \mathcal{C}_A(\mathcal{G})$$

- $\mathbb{V}(P_A(x \cdot \lambda)) = \mathcal{B}^{\uparrow}(x) \otimes_S A =: \mathcal{B}_A^{\uparrow}(x)$

An equivalence of categories

$\mathcal{O}^{VF}_{A,\lambda} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

Theorem (Fiebig-localisation)

There is an equivalence of exact categories

$$\mathbb{V} : \mathcal{O}^{VF}_{A,\lambda} \to \mathcal{C}_A(\mathcal{G})$$

- $\mathbb{V}(P_A(x \cdot \lambda)) = \mathcal{B}^{\uparrow}(x) \otimes_S A =: \mathcal{B}^{\uparrow}_A(x)$
- $\mathbb{V}(K_A(x \cdot \lambda)) = \mathcal{B}^{\downarrow}(x) \otimes_S A =: \mathcal{B}^{\downarrow}_A(x)$

An equivalence of categories

$\mathcal{O}_{A,\lambda}^{VF} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

Theorem (Fiebig-localisation)

There is an equivalence of exact categories

$$\mathbb{V} : \mathcal{O}_{A,\lambda}^{VF} \to \mathcal{C}_A(\mathcal{G})$$

- $\mathbb{V}(P_A(x \cdot \lambda)) = \mathcal{B}^{\uparrow}(x) \otimes_S A =: \mathcal{B}_A^{\uparrow}(x)$
- $\mathbb{V}(K_A(x \cdot \lambda)) = \mathcal{B}^{\downarrow}(x) \otimes_S A =: \mathcal{B}_A^{\downarrow}(x)$
- $\mathbb{V}(\Delta_A(x \cdot \lambda)) = \mathcal{V}_A(x)$ *(sky scraper sheaf)*

Let $w_\circ \in \mathcal{W}$ be the longest element.

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})$$

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})$$

- $F(\mathcal{B}^{\uparrow}(x)) = \mathcal{B}^{\downarrow}(w_\circ x)$

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})$$

- $F(\mathcal{B}^\uparrow(x)) = \mathcal{B}^\downarrow(w_\circ x)$
- $F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x)$

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})$$

- $F(\mathcal{B}^{\uparrow}(x)) = \mathcal{B}^{\downarrow}(w_\circ x)$
- $F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x)$

Lift F via Fiebig-localisation to an equivalence

Let $w_\circ \in \mathcal{W}$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})$$

- $F(\mathcal{B}^{\uparrow}(x)) = \mathcal{B}^{\downarrow}(w_\circ x)$
- $F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x)$

Lift F via Fiebig-localisation to an equivalence

$$T : \mathcal{O}_{A,\lambda}^{VF} \to (\mathcal{O}_{A,\lambda}^{VF})^{op}$$

with $T(P_A(\mu)) = K_A(w_\circ \cdot \mu)$ and $T(\Delta_A(\mu)) = \Delta_A(w_\circ \cdot \mu)$ $(\mu \in \mathcal{W} \cdot \lambda)$

Andersen filtration

Andersen filtration

$A = \mathbb{C}[[t]]$ with structure morphism $\tau : S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[t]]$ induced by $\mathbb{C}\rho \subset \mathfrak{h}^*$.

Andersen filtration

$A = \mathbb{C}[[t]]$ with structure morphism $\tau : S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[t]]$ induced by $\mathbb{C}\rho \subset \mathfrak{h}^*$.
$K \in \mathcal{O}_A$ a tilting object.

Andersen filtration

$A = \mathbb{C}[[t]]$ with structure morphism $\tau : S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[t]]$ induced by $\mathbb{C}\rho \subset \mathfrak{h}^*$.

$K \in \mathcal{O}_A$ a tilting object.

Composition gives a non-degenerate bilinear form of free A-modules:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K) \times \mathrm{Hom}_{\mathcal{O}_A}(K, \nabla_A(\lambda)) \to \\ \to \mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), \nabla_A(\lambda)) \cong A\end{aligned}$$

Define filtration on $\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by

Define filtration on $\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by

$$F_A^i(K,\lambda) := \{\varphi \in \mathrm{Hom}(\Delta_A(\lambda), K) \mid \psi \circ \varphi \in t^i A \ \forall \psi \in \mathrm{Hom}(K, \nabla_A(\lambda))\}$$

Define filtration on $\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by

$$F_A^i(K,\lambda) := \{\varphi \in \mathrm{Hom}(\Delta_A(\lambda), K) \,|\, \psi\circ\varphi \in t^i A \ \forall \psi \in \mathrm{Hom}(K, \nabla_A(\lambda))\}$$

Definition

The image $F^i(K \otimes_A \mathbb{C}, \lambda)$ of the filtration $F_A^i(K,\lambda)$ under the surjection $\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), K \otimes_A \mathbb{C})$ is called **Andersen filtration** on $\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), K \otimes_A \mathbb{C})$

Jantzen filtration

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. *Then* $\Delta(\mu)$ *has a filtration:*

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. *Then* $\Delta(\mu)$ *has a filtration:*

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. *Then* $\Delta(\mu)$ *has a filtration:*

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

- $\Delta(\mu)^1$ *is the maximal submodule of* $\Delta(\mu)$

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

- *$\Delta(\mu)^1$ is the maximal submodule of $\Delta(\mu)$*
- *Sum formula:*

$$\sum_{i>0} \mathrm{ch}\Delta(\mu)^i = \sum_{\alpha \in R^+, s_\alpha \cdot \mu < \mu} \mathrm{ch}\Delta(s_\alpha \cdot \mu)$$

Jantzen filtration

Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

- *$\Delta(\mu)^1$ is the maximal submodule of $\Delta(\mu)$*
- *Sum formula:*

$$\sum_{i>0} \mathrm{ch}\Delta(\mu)^i = \sum_{\alpha\in R^+, s_\alpha\cdot\mu<\mu} \mathrm{ch}\Delta(s_\alpha \cdot \mu)$$

Let $P \in \mathcal{O}$ be projective. Then the Jantzen filtration on $\Delta(\mu)$ induces a filtration on $\mathrm{Hom}_{\mathfrak{g}}(P, \Delta(\mu))$.

Example

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.
Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.
Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

- $L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.
Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

- $L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.
- sum formula:

$$\sum_{i>0} \mathrm{ch}\Delta(w \cdot \lambda)^i = \mathrm{ch}\Delta(s_\alpha \cdot \lambda) + \mathrm{ch}\Delta(s_\beta \cdot \lambda)$$

$$= \mathrm{ch}L(s_\alpha \cdot \lambda) + \mathrm{ch}L(s_\beta \cdot \lambda) + 2\mathrm{ch}L(\lambda)$$

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.

Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

- $L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.
- sum formula:

$$\sum_{i>0} \mathrm{ch}\Delta(w \cdot \lambda)^i = \mathrm{ch}\Delta(s_\alpha \cdot \lambda) + \mathrm{ch}\Delta(s_\beta \cdot \lambda)$$

$$= \mathrm{ch}L(s_\alpha \cdot \lambda) + \mathrm{ch}L(s_\beta \cdot \lambda) + 2\mathrm{ch}L(\lambda)$$

$\Rightarrow \Delta(w \cdot \lambda)^2 = L(\lambda)$

Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.
Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

- $L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.
- sum formula:

$$\sum_{i>0} \mathrm{ch}\Delta(w \cdot \lambda)^i = \mathrm{ch}\Delta(s_\alpha \cdot \lambda) + \mathrm{ch}\Delta(s_\beta \cdot \lambda)$$

$$= \mathrm{ch}L(s_\alpha \cdot \lambda) + \mathrm{ch}L(s_\beta \cdot \lambda) + 2\mathrm{ch}L(\lambda)$$

$\Rightarrow \Delta(w \cdot \lambda)^2 = L(\lambda)$
$\Rightarrow \Delta(w \cdot \lambda)$ has composition factors $L(w \cdot \lambda)$, $L(s_\beta \cdot \lambda)$, $L(s_\alpha \cdot \lambda)$ and $L(\lambda)$ each occuring with multiplicity one.

$T : \mathcal{O}_{A,\lambda}^{VF} \to (\mathcal{O}_{A,\lambda}^{VF})^{op}$ induces an isomorphism:

$T : \mathcal{O}_{A,\lambda}^{VF} \to (\mathcal{O}_{A,\lambda}^{VF})^{op}$ induces an isomorphism:

$$T : \mathrm{Hom}_{\mathcal{O}_A}(P_A(x\cdot\lambda), \Delta_A(y\cdot\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(w_\circ y\cdot\lambda), K_A(w_\circ x\cdot\lambda))$$

$T : \mathcal{O}_{A,\lambda}^{VF} \to (\mathcal{O}_{A,\lambda}^{VF})^{op}$ induces an isomorphism:

$$T : \mathrm{Hom}_{\mathcal{O}_A}(P_A(x\cdot\lambda), \Delta_A(y\cdot\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(w_\circ y\cdot\lambda), K_A(w_\circ x\cdot\lambda))$$

Theorem

The isomorphism $\varphi := T \otimes_A \mathrm{id}_{\mathbb{C}}$ we get after base change:

$$\varphi : \mathrm{Hom}_{\mathfrak{g}}(P(x\cdot\lambda), \Delta(y\cdot\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\Delta(w_\circ y\cdot\lambda), K(w_\circ x\cdot\lambda))$$

identifies the Jantzen filtration on $\mathrm{Hom}_{\mathfrak{g}}(P(x\cdot\lambda), \Delta(y\cdot\lambda))$ with the Andersen filtration on $\mathrm{Hom}_{\mathfrak{g}}(\Delta(w_\circ y\cdot\lambda), K(w_\circ x\cdot\lambda))$.

References

- A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, Adv. Soviet Math., 1993.
- T. Braden and R. MacPherson, *From moment graphs to intersection cohomology*, Math. Ann., 2001
- P. Fiebig, *Sheaves on moment graphs and a localization of Verma flags*, Adv. Math., 2008.
- J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, Springer, 1979.
- J. Kübel, *From Jantzen to Andersen filtration via tilting equivalence*, preprint 2010, *to appear in Math. Scand.*
- J. Kübel, *Tilting modules in category $\mathcal{O}$ and sheaves on moment graphs*, preprint 2012
- W. Soergel, *Andersen Filtration and hard Lefschetz*, Geom. Funct. Anal., 2008