On Andersen and Jantzen filtrations

Johannes Kübel
University of Erlangen-Nürnberg

DFG Schwerpunkttagung, Schloss Thurnau

March 20, 2012
1. The category $\mathcal{O}_A$

2. Sheaves on moment graphs

3. Andersen filtration

4. Jantzen filtration
Basic setting
Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
Basic setting

- \( \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h} \) semisimple complex Lie algebra with Borel and Cartan subalgebra
- \( S = S(\mathfrak{h}) = U(\mathfrak{h}) \)
Basic setting

- \( g \supset b \supset h \) semisimple complex Lie algebra with Borel and Cartan subalgebra
- \( S = S(h) = U(h) \)
- \( S(0) \) the localization of \( S \) at the maximal ideal generated by \( h \subset S \).
Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of $S$ at the maximal ideal generated by $\mathfrak{h} \subset S$
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresponding to $\mathfrak{b}$ and $\rho$ the half sum of positive roots
Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S_{(0)}$ the localization of $S$ at the maximal ideal generated by $\mathfrak{h} \subset S$
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and $\rho$ the half sum of positive roots
- For $\alpha \in R^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresp. reflection by $s_\alpha$
Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S(0)$ the localization of $S$ at the maximal ideal generated by $\mathfrak{h} \subset S$
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots corresp. to $\mathfrak{b}$ and $\rho$ the half sum of positive roots
- For $\alpha \in R^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresp. reflection by $s_\alpha$
- $W$ Weyl group with $\rho$-shifted dot-action on $\mathfrak{h}^*$
Basic setting

- $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ semisimple complex Lie algebra with Borel and Cartan subalgebra
- $S = S(\mathfrak{h}) = U(\mathfrak{h})$
- $S(0)$ the localization of $S$ at the maximal ideal generated by $\mathfrak{h} \subset S$
- $R^+ \subset R \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with positive roots correspond to $\mathfrak{b}$ and $\rho$ the half sum of positive roots
- For $\alpha \in R^+$ we denote its coroot by $\alpha^\vee \in \mathfrak{h}$ and the corresponding reflection by $s_\alpha$
- $W$ Weyl group with $\rho$-shifted dot-action on $\mathfrak{h}^*$
- $A$ a local, commutative $S$-algebra with structure map $\tau : S \to A$
- (in this talk $A = S(0)$ or $A = S(0)/S(0)\mathfrak{h} = \mathbb{C}$)
The category $\mathcal{O}_A$ is the full subcategory of $\mathcal{U}(g)$-$A$-bimodules whose objects $M$ satisfy:

$M$ is finitely generated over $g \otimes \mathbb{C}A$ and

$M \cong \bigoplus_{\lambda \in h^*} M_{\lambda}$ where $M_{\lambda} = \{ m \in M | hm = (\lambda + \tau)(h)m \forall h \in h \}$

$M$ is locally $b$-$A$-finite

$\mathcal{O}_A$ is abelian

Simple modules: highest weight modules $L_A(\lambda)$ ($\lambda \in h^*$) exist in $\mathcal{O}_A$

$\mathcal{O}_A \cong \bigoplus_{\lambda} \mathcal{O}_A,\lambda$ where $\lambda$ runs over anti-dominant weights (block decomposition).
Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(g)$-$A$-bimodules whose objects $M$ satisfy

$$M \text{ is finitely generated over } g \otimes \mathbb{C} A$$

$$M \sim = \bigoplus_{\lambda} M_{\lambda}$$

where $M_{\lambda} = \{ m \in M | hm = (\lambda + \tau)(h)m \ \forall h \in h^* \}$

$M$ is locally $A$-$A$-finite

$\mathcal{O}_A$ is abelian

simple modules: highest weight modules $L_A(\lambda)$ ($\lambda \in h^*$)

projective covers $P_A(\lambda)$ of $L_A(\lambda)$ exist in $\mathcal{O}_A$

$\mathcal{O}_A \sim = \bigoplus_{\lambda} \mathcal{O}_A,\lambda$, where $\lambda$ runs over anti-dominant weights (block decomposition).
Definition
The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy
- $M$ is finitely generated over $\mathfrak{g} \otimes_{\mathbb{C}} A$.
The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $\mathfrak{g} \otimes \mathbb{C} A$

- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
The category $\mathcal{O}_A$ is the full subcategory of $U(g)$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $g \otimes \mathbb{C} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
- $M$ is locally $\mathfrak{b}$-$A$-finite.
Definition

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $\mathfrak{g} \otimes \mathbb{C} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_{\lambda}$ where
  $M_{\lambda} = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
- $M$ is locally $\mathfrak{b}$-$A$-finite

- $\mathcal{O}_A$ is abelian
The category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $\mathfrak{g} \otimes \mathbb{C} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
- $M$ is locally $\mathfrak{b}$-$A$-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda) \ (\lambda \in \mathfrak{h}^*)$
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

**Definition**

The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $\mathfrak{g} \otimes_\mathbb{C} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where
  $M_\lambda = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
- $M$ is locally $\mathfrak{b}$-$A$-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda)$ ($\lambda \in \mathfrak{h}^*$)
- projective covers $P_A(\lambda)$ of $L_A(\lambda)$ exist in $\mathcal{O}_A$
The deformed category $\mathcal{O}_A$ is the full subcategory of $U(\mathfrak{g})$-$A$-bimodules whose objects $M$ satisfy

- $M$ is finitely generated over $\mathfrak{g} \otimes \mathbb{C} A$
- $M \cong \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{ m \in M \mid hm = (\lambda + \tau)(h)m \ \forall h \in \mathfrak{h} \}$
- $M$ is locally $\mathfrak{b}$-$A$-finite

- $\mathcal{O}_A$ is abelian
- simple modules: highest weight modules $L_A(\lambda)$ ($\lambda \in \mathfrak{h}^*$)
- projective covers $P_A(\lambda)$ of $L_A(\lambda)$ exist in $\mathcal{O}_A$
- $\mathcal{O}_A \cong \bigoplus_{\lambda} \mathcal{O}_{A,\lambda}$ where $\lambda$ runs over anti-dominant weights (block decomposition).
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration
d
\[ \mathcal{O}_A \longrightarrow \mathcal{O}_A^{\text{op}} \]
duality on modules which are free over $A$
contains deformed Verma modules:

\[ \Delta_A(\lambda) := U(g) \otimes U(b) \cdot \lambda \]
where $A_\lambda$ is the $b$-bimodule $A$ on which $h$ acts by $\lambda + \tau$ and $[b, b]$ by 0.

\[ \nabla_A(\lambda) := d(\Delta_A(\lambda)) \]

Definition
A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a $\Delta_A$- and a $\nabla_A$-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

indecomposable deformed tilting modules are parametrized by their highest weight

\[ K_A(\lambda) := \text{indecomposable deformed tilting module with highest weight } \lambda \in h^* \]
The category $\mathcal{O}_A$

Sheaves on moment graphs

Andersen filtration

Jantzen filtration

- $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{\text{op}}$ duality on modules which are free over $A$
• $d : \mathcal{O}_A \longrightarrow \mathcal{O}^{op}_A$ duality on modules which are free over $A$

• $\mathcal{O}_A$ contains deformed Verma modules:
  $\Delta_A(\lambda) := U(g) \otimes_{U(b)} A_{\lambda}$ where $A_{\lambda}$ is the $b$-$A$-bimodule $A$ on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[b, b]$ by 0.
\begin{itemize}
  \item $d : \mathcal{O}_A \longrightarrow \mathcal{O}_A^{op}$ duality on modules which are free over $A$
  \item $\mathcal{O}_A$ contains deformed Verma modules:
    \[ \Delta_A(\lambda) := U(g) \otimes_{U(b)} A_{\lambda} \] where $A_{\lambda}$ is the $b$-$A$-bimodule $A$ on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[b, b]$ by $0$.
  \item $\nabla_A(\lambda) := d(\Delta_A(\lambda))$
\end{itemize}
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

- $d : \mathcal{O}_A \to \mathcal{O}_A^{op}$ duality on modules which are free over $A$
- $\mathcal{O}_A$ contains deformed Verma modules:
  $\Delta_A(\lambda) := U(g) \otimes_{U(b)} A_\lambda$ where $A_\lambda$ is the $b$-$A$-bimodule $A$ on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[b, b]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

**Definition**

A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a $\Delta_A$- and a $\nabla_A$-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.
- \( d : \mathcal{O}_A \rightarrow \mathcal{O}_A^{op} \) duality on modules which are free over \( A \)
- \( \mathcal{O}_A \) contains deformed Verma modules:
  \( \Delta_A(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} A_\lambda \) where \( A_\lambda \) is the \( \mathfrak{b}-A \)-bimodule \( A \) on which \( \mathfrak{h} \) acts by \( \lambda + \tau \) and \([\mathfrak{b}, \mathfrak{b}]\) by 0.
- \( \nabla_A(\lambda) := d(\Delta_A(\lambda)) \)

**Definition**

A deformed tilting module is an object \( K \in \mathcal{O}_A \) which has a \( \Delta_A \)- and a \( \nabla_A \)-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

- indecomposable deformed tilting modules are parametrized by their highest weight
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

- $d : \mathcal{O}_A \to \mathcal{O}_A^{op}$ duality on modules which are free over $A$
- $\mathcal{O}_A$ contains deformed Verma modules:
  \[ \Delta_A(\lambda) := U(g) \otimes_{U(b)} A_\lambda \]
  where $A_\lambda$ is the $b$-$A$-bimodule $A$ on which $\mathfrak{h}$ acts by $\lambda + \tau$ and $[b, b]$ by 0.
- $\nabla_A(\lambda) := d(\Delta_A(\lambda))$

**Definition**

A deformed tilting module is an object $K \in \mathcal{O}_A$ which has a $\Delta_A$- and a $\nabla_A$-flag, i.e. a finite filtration with subquotients isomorphic to Verma (resp. Nabla) modules.

- Indecomposable deformed tilting modules are parametrized by their highest weight
- $K_A(\lambda) :=$ indecomposable deformed tilting module with highest weight $\lambda \in \mathfrak{h}^*$
The category $O_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

For $A = S(0)$ and $\delta: A \to A/\mathfrak{a}$ we get a base change functor
$\cdot \otimes A C: O_A \to O_C$ and the category $O_C$ coincides with the usual
BGG-category $O_g$.

$\Delta_A(\lambda) \otimes A C \sim = \Delta_C(\lambda) = \Delta(\lambda),$

$P_A(\lambda) \otimes A C \sim = P_C(\lambda) = P(\lambda),$

$K_A(\lambda) \otimes A C \sim = K_C(\lambda) = K(\lambda),$

$\text{Hom}_{O_A}(P, \Delta_A(\lambda)) \otimes A C \sim \to \text{Hom}_{O_A}(P \otimes A C, \Delta(\lambda))$

$\text{Hom}_{O_A}(\Delta_A(\lambda), K) \otimes A C \sim \to \text{Hom}_{O_A}(\Delta(\lambda), K \otimes A C)$
For $A = S_{(0)}$ and $\delta : A \to A/\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_\mathbb{C}$ and the category $\mathcal{O}_\mathbb{C}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$. 
For $A = S(0)$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor
\[
\mathcal{O}_{\delta}^{} : \mathcal{O}_A \to \mathcal{O}_C
\]
and the category $\mathcal{O}_C$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_C(\lambda) = \Delta(\lambda)$,
For $A = S(0)$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_\mathbb{C}$ and the category $\mathcal{O}_\mathbb{C}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_\mathbb{C}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_\mathbb{C}(\lambda) = P(\lambda)$,
For $A = S_0$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_C$ and the category $\mathcal{O}_C$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_C(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_C(\lambda) = P(\lambda)$,
- $K_A(\lambda) \otimes_A \mathbb{C} \cong K_C(\lambda) = K(\lambda)$,
For $A = S(0)$ and $\delta : A \to A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor
\[
\cdot \otimes_A \mathbb{C} : \mathcal{O}_A \to \mathcal{O}_\mathbb{C}
\]
and the category $\mathcal{O}_\mathbb{C}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_A(\lambda) \otimes_A \mathbb{C} \cong \Delta_\mathbb{C}(\lambda) = \Delta(\lambda)$,
- $P_A(\lambda) \otimes_A \mathbb{C} \cong P_\mathbb{C}(\lambda) = P(\lambda)$,
- $K_A(\lambda) \otimes_A \mathbb{C} \cong K_\mathbb{C}(\lambda) = K(\lambda)$,
- $\text{Hom}_{\mathcal{O}_A}(P, \Delta_A(\lambda)) \otimes_A \mathbb{C} \sim \text{Hom}_{\mathcal{O}}(P \otimes_A \mathbb{C}, \Delta(\lambda))$
For $A = S_{(0)}$ and $\delta : A \rightarrow A/A\mathfrak{h} \cong \mathbb{C}$ we get a base change functor $\cdot \otimes_{A} \mathbb{C} : \mathcal{O}_{A} \rightarrow \mathcal{O}_{\mathbb{C}}$ and the category $\mathcal{O}_{\mathbb{C}}$ coincides with the usual BGG-category $\mathcal{O}$ of $\mathfrak{g}$.

- $\Delta_{A}(\lambda) \otimes_{A} \mathbb{C} \cong \Delta_{\mathbb{C}}(\lambda) = \Delta(\lambda)$,
- $P_{A}(\lambda) \otimes_{A} \mathbb{C} \cong P_{\mathbb{C}}(\lambda) = P(\lambda)$,
- $K_{A}(\lambda) \otimes_{A} \mathbb{C} \cong K_{\mathbb{C}}(\lambda) = K(\lambda)$,

\[
\text{Hom}_{\mathcal{O}_{A}}(P, \Delta_{A}(\lambda)) \otimes_{A} \mathbb{C} \xrightarrow{\sim} \text{Hom}_{\mathcal{O}}(P \otimes_{A} \mathbb{C}, \Delta(\lambda))
\]

and

\[
\text{Hom}_{\mathcal{O}_{A}}(\Delta_{A}(\lambda), K) \otimes_{A} \mathbb{C} \xrightarrow{\sim} \text{Hom}_{\mathcal{O}}(\Delta(\lambda), K \otimes_{A} \mathbb{C})
\]
The associated moment graph

The category $O_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

For simplicity: $\lambda \in h^*$ integral, regular and anti-dominant.

$\Rightarrow$ categorical structure of $O_A, \lambda$ governed by $W$.

Associate to $O_A, \lambda$ an ordered, labeled graph $G = (V, E, \alpha, \leq)$

$V$: vertices

$E$: $\{\{x, s^\beta y\} \in P(V) | \exists \beta \in \mathbb{R}^+: x = s^\beta y\}$ edges

$\alpha(\{x, s^\beta x\}) = \beta$ labeling

Define the order on $V$ by $w \leq w' \iff w \cdot \lambda \leq w' \cdot \lambda$ for all $w, w' \in W$. 
The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$. 
The category $O_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.
$\Rightarrow$ categorical structure of $O_{A,\lambda}$ governed by $\mathcal{W}$.
Associate to $O_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$
The associated moment graph

For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.

$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.

Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$

- $\mathcal{V} := \mathcal{W}$ vertices
For simplicity: $\lambda \in \mathfrak{h}^*$ integral, regular and anti-dominant.

$\Rightarrow$ categorical structure of $\mathcal{O}_{A,\lambda}$ governed by $\mathcal{W}$.

Associate to $\mathcal{O}_{A,\lambda}$ an ordered, labeled graph $\mathcal{G} = (V, E, \alpha, \leq)$

- $V := \mathcal{W}$ vertices
- $E := \{ \{x, y\} \in \mathcal{P}(V) \mid \exists \beta \in \mathbb{R}^+ : x = s_\beta y \}$ edges
The associated moment graph

For simplicity: \( \lambda \in \mathfrak{h}^* \) integral, regular and anti-dominant.
\( \Rightarrow \) categorical structure of \( \mathcal{O}_A, \lambda \) governed by \( \mathcal{W} \).
Associate to \( \mathcal{O}_A, \lambda \) an ordered, labeled graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq) \)

- \( \mathcal{V} := \mathcal{W} \) vertices
- \( \mathcal{E} := \{\{x, y\} \in \mathcal{P}(\mathcal{V}) | \exists \beta \in \mathbb{R}^+ : x = s_\beta y\} \) edges
- \( \alpha(\{x, s_\beta x\}) = \beta^\vee \) labeling
The associated moment graph

For simplicity: \( \lambda \in \mathfrak{h}^* \) integral, regular and anti-dominant.
\[ \Rightarrow \] categorical structure of \( \mathcal{O}_{A,\lambda} \) governed by \( \mathcal{W} \).

Associate to \( \mathcal{O}_{A,\lambda} \) an ordered, labeled graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq) \)

- \( \mathcal{V} := \mathcal{W} \) vertices
- \( \mathcal{E} := \{\{x, y\} \in \mathcal{P}(\mathcal{V}) \mid \exists \beta \in \mathbb{R}^+ : x = s_\beta y\} \) edges
- \( \alpha(\{x, s_\beta x\}) = \beta^\vee \) labeling
- Define the order on \( \mathcal{V} \) by

\[ w \leq w' \iff w \cdot \lambda \leq w' \cdot \lambda \]

for all \( w, w' \in \mathcal{W} \).
Example:

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\lambda = -2 \rho e_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$.
Example:

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\lambda = -2\rho$
Example:

For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\lambda = -2\rho$
The category $\mathcal{O}_A$ on moment graphs
Andersen filtration
Jantzen filtration

Sheaves on $\mathcal{G}$

Definition
An $A$-sheaf on the moment graph $\mathcal{G}$ is a tuple $\mathcal{M} := (\{ M_x \}, \{ M_E \}, \{ \rho_x, E \})$ with the properties

- $M_x$ is an $A$-module for any $x \in V$.
- $M_E$ is an $A$-module for all $E \in E$ with $\alpha(E) M_E = 0$.
- $\rho_x, E : M_x \to M_E$ is a homomorphism of $A$-modules for $x \in E$, $E \in E$ with $x \in E$.

For $A = S$ consider all $S$-modules and $S$-linear maps as graded modules and graded morphisms in degree zero.
Sheaves on $\mathcal{G}$

**Definition**

An $A$-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{\mathcal{M}_x\}, \{\mathcal{M}_E\}, \{\rho_{x,E}\})$ with the properties
Sheaves on $\mathcal{G}$

**Definition**

An $A$-sheaf on the moment graph $\mathcal{G}$ is a tupel

$\mathcal{M} := (\{M_x\}, \{M_E\}, \{\rho_{x,E}\})$ with the properties

- $M_x$ is an $A$-module for any $x \in \mathcal{V}$
An $A$-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{M_x\}, \{M_E\}, \{\rho_{x,E}\})$ with the properties

- $M_x$ is an $A$-module for any $x \in \mathcal{V}$
- $M_E$ is an $A$-module for all $E \in \mathcal{E}$ with $\alpha(E)M_E = 0$
Sheaves on $\mathcal{G}$

**Definition**

An $A$-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{M_x\}, \{M_E\}, \{\rho_{x,E}\})$ with the properties

- $M_x$ is an $A$-module for any $x \in \mathcal{V}$
- $M_E$ is an $A$-module for all $E \in \mathcal{E}$ with $\alpha(E)M_E = 0$
- $\rho_{x,E} : M_x \rightarrow M_E$ is a homomorphism of $A$-modules for $x \in \mathcal{V}$, $E \in \mathcal{E}$ with $x \in E$.
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

Sheaves on $\mathcal{G}$

Definition

An $A$-sheaf on the moment graph $\mathcal{G}$ is a tupel $\mathcal{M} := (\{M_x\}, \{M_E\}, \{\rho_{x,E}\})$ with the properties

- $M_x$ is an $A$-module for any $x \in V$
- $M_E$ is an $A$-module for all $E \in \mathcal{E}$ with $\alpha(E)M_E = 0$
- $\rho_{x,E} : M_x \to M_E$ is a homomorphism of $A$-modules for $x \in V$, $E \in \mathcal{E}$ with $x \in E$.

For $A = S$ consider all $S$-modules and $S$-linear maps as graded modules and graded morphisms in degree zero.
The category $\mathcal{O}_A$

Sheaves on moment graphs

Andersen filtration

Jantzen filtration

$\mathcal{A}(G) := \text{category of } A\text{-sheaves on } G \text{ that admit a "Verma flag".}$

There is an exact structure on $\mathcal{A}(G)$ depending on the order on $G$.

$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

Theorem ((Braden-MacPherson), (Fiebig))

For every $w \in V$ there exists an up to isomorphism unique graded $S$-sheaf $B^\uparrow(w) \in \mathcal{A}(G)$ with the properties $B^\uparrow(w) w \sim S$ and $B^\uparrow(w) x$ graded free with $B^\uparrow(w) x = 0$ unless $w \leq x$ $B^\uparrow(w)$ is indecomposable and projective in $\mathcal{A}(G)$.

Definition

Denote by $B^\downarrow(w)$ the Braden-MacPherson sheaf on the same moment graph $G$ with reversed order.
$\mathcal{C}_A(\mathcal{G}) := \text{category of } A\text{-sheaves on } \mathcal{G} \text{ that admit a "Verma flag".}$
\[ C_A(\mathcal{G}) := \text{category of } A\text{-sheaves on } \mathcal{G} \text{ that admit a "Verma flag".} \]

There is an exact structure on \( C_A(\mathcal{G}) \) depending on the order on \( \mathcal{G} \).
$\mathcal{C}_A(\mathcal{G}) :=$ category of $A$-sheaves on $\mathcal{G}$ that admit a ”Verma flag”. There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$. ⇒ Notion of projective objects makes sense. Different order leads to different projectives.
The category $\mathcal{O}_A$

Sheaves on moment graphs
Andersen filtration
Jantzen filtration

$\mathcal{C}_A(\mathcal{G}) :=$ category of $A$-sheaves on $\mathcal{G}$ that admit a ”Verma flag”.
There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.
$\Rightarrow$ Notion of projective objects makes sense. Different order leads to different projectives.

**Theorem ((Braden-MacPherson), (Fiebig))**

For every $w \in \mathcal{V}$ there exists an up to isomorphism unique graded $S$-sheaf $\mathcal{B}^\uparrow(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties
\( \mathcal{C}_A(\mathcal{G}) := \text{category of } A\text{-sheaves on } \mathcal{G} \text{ that admit a } "\text{Verma flag}". \)
There is an exact structure on \( \mathcal{C}_A(\mathcal{G}) \) depending on the order on \( \mathcal{G} \).
⇒ Notion of projective objects makes sense. Different order leads to different projectives.

**Theorem ((Braden-MacPherson), (Fiebig))**

*For every \( w \in \mathcal{V} \) there exists an up to isomorphism unique graded \( S \)-sheaf \( \mathcal{B}^{\uparrow}(w) \in \mathcal{C}_S(\mathcal{G}) \) with the properties*

- \( \mathcal{B}^{\uparrow}(w)_w \cong S \) and \( \mathcal{B}^{\uparrow}(w)_x \) graded free with \( \mathcal{B}^{\uparrow}(w)_x = 0 \) unless \( w \leq x \)
$\mathcal{C}_A(\mathcal{G}) :=$ category of $A$-sheaves on $\mathcal{G}$ that admit a "Verma flag". There is an exact structure on $\mathcal{C}_A(\mathcal{G})$ depending on the order on $\mathcal{G}$.  
⇒ Notion of projective objects makes sense. Different order leads to different projectives.

**Theorem (Braden-MacPherson), (Fiebig))**

For every $w \in \mathcal{W}$ there exists an up to isomorphism unique graded $S$-sheaf $B^\uparrow(w) \in \mathcal{C}_S(\mathcal{G})$ with the properties

- $B^\uparrow(w)_w \cong S$ and $B^\uparrow(w)_x$ graded free with $B^\uparrow(w)_x = 0$ unless $w \leq x$
- $B^\uparrow(w)$ is indecomposable and projective in $\mathcal{C}_S(\mathcal{G})$
\( \mathcal{C}_A(\mathcal{G}) := \text{category of } A\text{-sheaves on } \mathcal{G} \text{ that admit a "Verma flag".} \)

There is an exact structure on \( \mathcal{C}_A(\mathcal{G}) \) depending on the order on \( \mathcal{G} \).

⇒ Notion of projective objects makes sense. Different order leads to different projectives.

**Theorem ([(Braden-MacPherson), (Fiebig)])**

For every \( w \in \mathcal{V} \) there exists an up to isomorphism unique graded \( S \)-sheaf \( \mathcal{B}^\uparrow(w) \in \mathcal{C}_S(\mathcal{G}) \) with the properties

- \( \mathcal{B}^\uparrow(w)_w \cong S \) and \( \mathcal{B}^\uparrow(w)_x \) graded free with \( \mathcal{B}^\uparrow(w)_x = 0 \) unless \( w \leq x \)
- \( \mathcal{B}^\uparrow(w) \) is indecomposable and projective in \( \mathcal{C}_S(\mathcal{G}) \)

**Definition**

Denote by \( \mathcal{B}^\downarrow(w) \) the Braden-MacPherson sheaf on the same moment graph \( \mathcal{G} \) with reversed order.
An equivalence of categories
An equivalence of categories

\[ \mathcal{O}^{VF}_{A,\lambda} \subset \mathcal{O}_{A,\lambda} \]  
full subcategory of modules with a deformed Verma flag,  
\( A = S_{(0)} \).
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

An equivalence of categories

$\mathcal{O}^{VF}_{A,\lambda} \subset \mathcal{O}_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

Theorem (Fiebig-localisation)

There is an equivalence of exact categories

$$\nabla : \mathcal{O}^{VF}_{A,\lambda} \rightarrow \mathcal{C}_A(\mathcal{G})$$
An equivalence of categories

\[ \mathcal{O}_{A,\lambda}^{VF} \subset \mathcal{O}_{A,\lambda} \] full subcategory of modules with a deformed Verma flag, \( A = S_{(0)} \).

**Theorem (Fiebig-localisation)**

*There is an equivalence of exact categories*

\[ \mathbb{V} : \mathcal{O}_{A,\lambda}^{VF} \to \mathcal{C}_A(\mathcal{G}) \]

\[ \mathbb{V}(P_A(x \cdot \lambda)) = \mathcal{B}^\uparrow(x) \otimes_S A =: \mathcal{B}^\uparrow_A(x) \]
An equivalence of categories

\( \mathcal{O}^{VF}_{A,\lambda} \subset \mathcal{O}_{A,\lambda} \) full subcategory of modules with a deformed Verma flag, \( A = S_{(0)} \).

**Theorem (Fiebig-localisation)**

*There is an equivalence of exact categories*

\[ \nabla : \mathcal{O}^{VF}_{A,\lambda} \rightarrow \mathcal{C}_A(\mathfrak{g}) \]

- \( \nabla(P_A(x \cdot \lambda)) = \mathcal{B}^\uparrow(x) \otimes_S A =: \mathcal{B}^\uparrow_A(x) \)
- \( \nabla(K_A(x \cdot \lambda)) = \mathcal{B}^\downarrow(x) \otimes_S A =: \mathcal{B}^\downarrow_A(x) \)
The category $O_A$

Sheaves on moment graphs

Andersen filtration

Jantzen filtration

An equivalence of categories

$O_{A,\lambda}^{VF} \subset O_{A,\lambda}$ full subcategory of modules with a deformed Verma flag, $A = S_{(0)}$.

**Theorem (Fiebig-localisation)**

There is an equivalence of exact categories

$$\nabla : O_{A,\lambda}^{VF} \to C_A(\mathcal{G})$$

- $\nabla(P_A(x \cdot \lambda)) = B^\uparrow(x) \otimes_S A =: B^\uparrow_A(x)$
- $\nabla(K_A(x \cdot \lambda)) = B^\downarrow(x) \otimes_S A =: B^\downarrow_A(x)$
- $\nabla(\Delta_A(x \cdot \lambda)) = \mathcal{V}_A(x)$ (sky scraper sheaf)
The category $O_A$

Sheaves on moment graphs
Andersen filtration
Jantzen filtration

Let $w^\circ \in W$ be the longest element.

Proposition ((Fiebig), (Lanini))

There is an equivalence of categories $F: CS(G) \to C^{op}S(G)$

$F(B^{\uparrow}(x)) = B^{\downarrow}(w^\circ x)$

$Lift F via Fiebig-localisation to an equivalence $T: O_{VF_A,\lambda} \to (O_{VF_A,\lambda})^{op}$ with $T(P_{A}(\mu)) = K_{A}(w^\circ \cdot \mu)$ and $T(\Delta_{A}(\mu)) = \Delta_{A}(w^\circ \cdot \mu)$ ($\mu \in W \cdot \lambda$)
Let $w_0 \in \mathcal{W}$ be the longest element.
Let \( w_\circ \in \mathcal{W} \) be the longest element.

**Proposition ((Fiebig), (Lanini))**

*There is an equivalence of categories*

\[
F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}_S^{op}(\mathcal{G})
\]
Let $w_\circ \in W$ be the longest element.

**Proposition** ((Fiebig), (Lanini))

*There is an equivalence of categories*

$$F : \mathcal{C}_S(G) \rightarrow \mathcal{C}_S^{\text{op}}(G)$$

- $F(\mathcal{B}^\uparrow(x)) = \mathcal{B}^\downarrow(w_\circ x)$
Let \( w_\circ \in \mathcal{W} \) be the longest element.

**Proposition ((Fiebig), (Lanini))**

*There is an equivalence of categories*

\[
F : \mathcal{C}_S(\mathcal{G}) \rightarrow \mathcal{C}_S^{\text{op}}(\mathcal{G})
\]

- \( F(\mathcal{B}^\uparrow(x)) = \mathcal{B}^\downarrow(w_\circ x) \)
- \( F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x) \)
Let $w_\circ \in \mathcal{W}$ be the longest element.

**Proposition ((Fiebig), (Lanini))**

*There is an equivalence of categories*

\[
F : \mathcal{C}_S(\mathcal{G}) \rightarrow \mathcal{C}_S(\mathcal{G})^{\text{op}}
\]

- $F(\mathcal{B}^\uparrow(x)) = \mathcal{B}^\downarrow(w_\circ x)$
- $F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x)$

Lift $F$ via Fiebig-localisation to an equivalence
Let $w_\circ \in \mathcal{W}$ be the longest element.

**Proposition (Fiebig, Lanini)**

*There is an equivalence of categories*

$$F : \mathcal{C}_S(\mathcal{G}) \to \mathcal{C}^{op}_S(\mathcal{G})$$

- $F(\mathcal{B}^\uparrow(x)) = \mathcal{B}^\downarrow(w_\circ x)$
- $F(\mathcal{V}_S(x)) = \mathcal{V}_S(w_\circ x)$

Lift $F$ via Fiebig-localisation to an equivalence

$$T : \mathcal{O}^{VF}_{A,\lambda} \to (\mathcal{O}^{VF}_{A,\lambda})^{op}$$

with $T(P_A(\mu)) = K_A(w_\circ \cdot \mu)$ and $T(\Delta_A(\mu)) = \Delta_A(w_\circ \cdot \mu)$

$(\mu \in \mathcal{W} \cdot \lambda)$
Andersen filtration

The category $O_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

$A = C[[t]]$ with structure morphism $\tau: S = C[h^*] \to C[[t]]$
induced by $C_\rho \subset h^*$.

$K \in O_A$ a tilting object.
Composition gives a non-degenerate bilinear form of free $A$-modules:

$$\text{Hom}_{O_A}(\Delta_A(\lambda), K) \times \text{Hom}_{O_A}(K, \nabla_A(\lambda)) \to \text{Hom}_{O_A}(\Delta_A(\lambda), \nabla_A(\lambda)) \cong A.$$
Andersen filtration

\[ A = \mathbb{C}[[t]] \] with structure morphism \( \tau : S = \mathbb{C}[\mathfrak{h}^*] \rightarrow \mathbb{C}[[t]] \)
induced by \( \mathbb{C}\rho \subset \mathfrak{h}^* \).
$A = \mathbb{C}[[t]]$ with structure morphism $\tau : S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[t]]$

induced by $\mathbb{C}\rho \subset \mathfrak{h}^*$.

$K \in \mathcal{O}_A$ a tilting object.
Andersen filtration

$A = \mathbb{C}[[t]]$ with structure morphism $\tau : S = \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[[t]]$
induced by $\mathbb{C}\rho \subset \mathfrak{h}^*$.
$K \in \mathcal{O}_A$ a tilting object.
Composition gives a non-degenerate bilinear form of free $A$-modules:

$$
\text{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K) \times \text{Hom}_{\mathcal{O}_A}(K, \nabla_A(\lambda)) \rightarrow
\rightarrow \text{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), \nabla_A(\lambda)) \cong A
$$
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

Define filtration on $\text{Hom}(\Delta A(\lambda), K)$ by

$$F_i A(K, \lambda) := \{ \phi \in \text{Hom}(\Delta A(\lambda), K) | \psi \circ \phi \in \text{t}_A \forall \psi \in \text{Hom}(K, \nabla A(\lambda)) \}$$

Definition
The image $F_i A(K \otimes A C, \lambda)$ of the filtration $F_i A(K, \lambda)$ under the surjection $\text{Hom}(\Delta A(\lambda), K) \twoheadrightarrow \text{Hom}(\Delta(\lambda), K \otimes A C)$ is called Andersen filtration on $\text{Hom}(\Delta(\lambda), K \otimes A C)$.
Define filtration on $\text{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by
Define filtration on $\text{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by

$$F^i_A(K, \lambda) := \{ \varphi \in \text{Hom}(\Delta_A(\lambda), K) \mid \psi \circ \varphi \in t^i A \ \forall \psi \in \text{Hom}(K, \nabla_A(\lambda)) \}$$
Define filtration on $\operatorname{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K)$ by

$$F^i_A(K, \lambda) := \{ \varphi \in \operatorname{Hom}(\Delta_A(\lambda), K) | \psi \circ \varphi \in t^i A \ \forall \psi \in \operatorname{Hom}(K, \nabla_A(\lambda)) \}$$

**Definition**

The image $F^i(K \otimes_A \mathbb{C}, \lambda)$ of the filtration $F^i_A(K, \lambda)$ under the surjection $\operatorname{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), K) \to \operatorname{Hom}_\mathcal{O}(\Delta(\lambda), K \otimes_A \mathbb{C})$ is called **Andersen filtration** on $\operatorname{Hom}_\mathcal{O}(\Delta(\lambda), K \otimes_A \mathbb{C})$
The category $\mathcal{O}_A$ 
Sheaves on moment graphs 
Andersen filtration 
Jantzen filtration

**Theorem (Jantzen)**

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$
\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \cdots \supset 0
$$

with the properties

$\Delta(\mu)^1$ is the maximal submodule of $\Delta(\mu)$

*Sum formula:*

$$
\sum_{i > 0} \text{ch} \Delta(\mu)^i = \sum_{\alpha \in R^+, s \alpha \cdot \mu < \mu} \text{ch} \Delta(s \alpha \cdot \mu)
$$

Let $P \in \mathcal{O}$ be projective. Then the Jantzen filtration on $\Delta(\mu)$ induces a filtration on $\text{Hom}_g(P, \Delta(\mu))$. 

**Jantzen filtration**
Jantzen filtration

**Theorem (Jantzen)**

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:
Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$
\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0
$$
Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties
Theorem (Jantzen)

Let \( \mu \in \mathfrak{h}^* \). Then \( \Delta(\mu) \) has a filtration:

\[
\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0
\]

with the properties

- \( \Delta(\mu)^1 \) is the maximal submodule of \( \Delta(\mu) \)
Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

- $\Delta(\mu)^1$ is the maximal submodule of $\Delta(\mu)$
- Sum formula:

$$\sum_{i > 0} \text{ch}\Delta(\mu)^i = \sum_{\alpha \in R^+, s_\alpha \cdot \mu < \mu} \text{ch}\Delta(s_\alpha \cdot \mu)$$
Theorem (Jantzen)

Let $\mu \in \mathfrak{h}^*$. Then $\Delta(\mu)$ has a filtration:

$$\Delta(\mu) = \Delta(\mu)^0 \supset \Delta(\mu)^1 \supset \Delta(\mu)^2 \supset \ldots \supset 0$$

with the properties

- $\Delta(\mu)^1$ is the maximal submodule of $\Delta(\mu)$
- Sum formula:

$$\sum_{i>0} \text{ch} \Delta(\mu)^i = \sum_{\alpha \in R^+, s_\alpha \cdot \mu < \mu} \text{ch} \Delta(s_\alpha \cdot \mu)$$

Let $P \in \mathcal{O}$ be projective. Then the Jantzen filtration on $\Delta(\mu)$ induces a filtration on $\text{Hom}_g(P, \Delta(\mu))$. 
Example

The category $O_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

Example

$g = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^* \text{ regular, integral and anti-dominant.}$
$s_\alpha, s_\beta \in W$ simple reflections,
$w = s_\alpha s_\beta$.

Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

$L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.

Sum formula:

$\sum_{i > 0} \text{ch} \Delta(w \cdot \lambda) = \text{ch} \Delta(s_\alpha \cdot \lambda) + \text{ch} \Delta(s_\beta \cdot \lambda) = \text{ch} L(s_\alpha \cdot \lambda) + \text{ch} L(s_\beta \cdot \lambda) + 2 \text{ch} L(\lambda)$

$\Rightarrow \Delta(w \cdot \lambda)$ has composition factors $L(w \cdot \lambda), L(s_\beta \cdot \lambda), L(s_\alpha \cdot \lambda)$ and $L(\lambda)$ each occurring with multiplicity one.
Example

g = \mathfrak{sl}_3(\mathbb{C}), \lambda \in h^* \text{ regular, integral and anti-dominant. } s_\alpha, s_\beta \in \mathcal{W} \text{ simple reflections, } w = s_\alpha s_\beta.
Example

\[ g = \mathfrak{sl}_3(\mathbb{C}), \ \lambda \in \mathfrak{h}^* \text{ regular, integral and anti-dominant.} \ s_\alpha, s_\beta \in \mathcal{W} \]

simple reflections, \( w = s_\alpha s_\beta \).

Q: Simple subquotients of \( \Delta(w \cdot \lambda) \) and its multiplicities?
Example

\[ g = \mathfrak{sl}_3(\mathbb{C}), \lambda \in \mathfrak{h}^* \text{ regular, integral and anti-dominant.} \quad s_\alpha, s_\beta \in \mathcal{W} \]

simple reflections, \( w = s_\alpha s_\beta \).

Q: Simple subquotients of \( \Delta(w \cdot \lambda) \) and its multiplicities?

- \( L(\lambda) \) and \( L(w \cdot \lambda) \) occur exactly once.
Example

$\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\lambda \in \mathfrak{h}^*$ regular, integral and anti-dominant. $s_\alpha, s_\beta \in \mathcal{W}$ simple reflections, $w = s_\alpha s_\beta$.

Q: Simple subquotients of $\Delta(w \cdot \lambda)$ and its multiplicities?

- $L(\lambda)$ and $L(w \cdot \lambda)$ occur exactly once.
- Sum formula:

$$\sum_{i > 0} \text{ch} \Delta(w \cdot \lambda)^i = \text{ch} \Delta(s_\alpha \cdot \lambda) + \text{ch} \Delta(s_\beta \cdot \lambda)$$

$$= \text{ch} L(s_\alpha \cdot \lambda) + \text{ch} L(s_\beta \cdot \lambda) + 2 \text{ch} L(\lambda)$$
Example

\( g = \mathfrak{sl}_3(\mathbb{C}) \), \( \lambda \in \mathfrak{h}^* \) regular, integral and anti-dominant. \( s_\alpha, s_\beta \in \mathcal{W} \) simple reflections, \( w = s_\alpha s_\beta \).

Q: Simple subquotients of \( \Delta(w \cdot \lambda) \) and its multiplicities?

- \( L(\lambda) \) and \( L(w \cdot \lambda) \) occur exactly once.
- sum formula:

\[
\sum_{i > 0} \text{ch} \Delta(w \cdot \lambda)^i = \text{ch} \Delta(s_\alpha \cdot \lambda) + \text{ch} \Delta(s_\beta \cdot \lambda)
\]

\[
= \text{ch} L(s_\alpha \cdot \lambda) + \text{ch} L(s_\beta \cdot \lambda) + 2 \text{ch} L(\lambda)
\]

\( \Rightarrow \Delta(w \cdot \lambda)^2 = L(\lambda) \)
Example

\[ \mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}), \ \lambda \in \mathfrak{h}^\ast \ regular, \ integral \ and \ anti-dominant. \ s_\alpha, s_\beta \in \mathcal{W} \]

simple reflections, \( w = s_\alpha s_\beta \).

Q: Simple subquotients of \( \Delta(w \cdot \lambda) \) and its multiplicities?

- \( L(\lambda) \) and \( L(w \cdot \lambda) \) occur exactly once.

- sum formula:

\[
\sum_{i>0} \text{ch} \Delta(w \cdot \lambda)^i = \text{ch} \Delta(s_\alpha \cdot \lambda) + \text{ch} \Delta(s_\beta \cdot \lambda) \\
= \text{ch} L(s_\alpha \cdot \lambda) + \text{ch} L(s_\beta \cdot \lambda) + 2 \text{ch} L(\lambda)
\]

⇒ \( \Delta(w \cdot \lambda)^2 = L(\lambda) \)

⇒ \( \Delta(w \cdot \lambda) \) has composition factors \( L(w \cdot \lambda), L(s_\beta \cdot \lambda), L(s_\alpha \cdot \lambda) \) and \( L(\lambda) \) each occurring with multiplicity one.
The category $\mathcal{O}_A$ of sheaves on moment graphs induces an isomorphism:

$$T: \text{Hom}_{\mathcal{O}_A}(\mathcal{P}(x \cdot \lambda), \Delta(y \cdot \lambda)) \sim \rightarrow \text{Hom}_{\mathcal{O}_A}(\Delta(\omega \circ y \cdot \lambda), K(\omega \circ x \cdot \lambda)).$$

Theorem

The isomorphism $\phi := T \otimes A \text{id}$ we get after base change:

$$\phi: \text{Hom}_g(\mathcal{P}(x \cdot \lambda), \Delta(y \cdot \lambda)) \sim \rightarrow \text{Hom}_g(\Delta(\omega \circ y \cdot \lambda), K(\omega \circ x \cdot \lambda)).$$

Identifies the Jantzen filtration on $\text{Hom}_g(\mathcal{P}(x \cdot \lambda), \Delta(y \cdot \lambda))$ with the Andersen filtration on $\text{Hom}_g(\Delta(\omega \circ y \cdot \lambda), K(\omega \circ x \cdot \lambda)).$
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

$T : \mathcal{O}_A^{VF} \rightarrow (\mathcal{O}_A^{VF})^{op}$ induces an isomorphism:
The category $\mathcal{O}_A$
Sheaves on moment graphs
Andersen filtration
Jantzen filtration

$T : \mathcal{O}_{A,\lambda}^{VF} \rightarrow (\mathcal{O}_{A,\lambda}^{VF})^{op}$ induces an isomorphism:

$T : \text{Hom}_{\mathcal{O}_A}(P_A(x\cdot \lambda), \Delta_A(y\cdot \lambda)) \sim \text{Hom}_{\mathcal{O}_A}(\Delta_A(w\circ y\cdot \lambda), K_A(w\circ x\cdot \lambda))$
The category $\mathcal{O}_A$

Sheaves on moment graphs

Andersen filtration

Jantzen filtration

Let $T : \mathcal{O}^{VF}_{A, \lambda} \to (\mathcal{O}^{VF}_{A, \lambda})^{op}$ induces an isomorphism:

$$T : \text{Hom}_{\mathcal{O}_A}(P_A(x \cdot \lambda), \Delta_A(y \cdot \lambda)) \sim \text{Hom}_{\mathcal{O}_A}(\Delta_A(w \circ y \cdot \lambda), K_A(w \circ x \cdot \lambda))$$

**Theorem**

The isomorphism $\varphi := T \otimes_A \text{id}_\mathbb{C}$ we get after base change:

$$\varphi : \text{Hom}_g(P(x \cdot \lambda), \Delta(y \cdot \lambda)) \sim \text{Hom}_g(\Delta(w \circ y \cdot \lambda), K(w \circ x \cdot \lambda))$$

identifies the Jantzen filtration on $\text{Hom}_g(P(x \cdot \lambda), \Delta(y \cdot \lambda))$ with the Andersen filtration on $\text{Hom}_g(\Delta(w \circ y \cdot \lambda), K(w \circ x \cdot \lambda))$. 
References

- J. Kübel, *Tilting modules in category $\mathcal{O}$ and sheaves on moment graphs*, preprint 2012