THICK SUBCATEGORIES OF THE BOUNDED DERIVED CATEGORY

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GROUND RULES.

$k =$ a field of characteristic p, algebraically closed.

$S := k[x_1, \ldots, k_n]$, a polynomial ring in n variables.

$G =$ a finite group.

$k =$ a field of characteristic p, algebraically closed.

Note that kG is a Hopf algebra, and hence there is a tensor product operation on kG-modules. That is, if M and N are kG-modules, then the module $M \otimes N = M \otimes_k N$ is a kG-module by the rule $g(m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$.

Recall that the group algebra kG is self-injective, meaning that projective modules are injective and *vice versa*.

THE CATEGORIES.

Let R be a ring. The derived categrory $\mathbf{D}(R)$ is the category consisting of objects which are complexes of R-modules, with morphisms being chain maps of complexes - except that we invert any quasi-isomorphism (a chain maps that induces an isomorphism on homology).

$\mathbf{D}^{\mathrm{b}}(R)$ is the subcategory of bounded, finitely generated complexes.

The stable category, $\mathbf{stmod}(kG)$, has objects consisting of all finitely generated kG-modules. If M and N are kG-modules then the set of morphisms from M to N in the stable category is the group

$$\mathrm{Hom}_{\mathbf{stmod}(kG)}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N).$$

Here $\mathrm{PHom}_{kG}(M,N)$ is the set of all homomorphisms from M to N that factor through a projective module. The stable category is triangulated with the triangles corresponding roughly to exact sequences in the module category.

THE CATEGORIES

There are functors

$$\mathbf{stmod}(kG) \leftrightarrows^{f}_{g} \mathbf{D}^{\mathrm{b}}(kG)$$

Here, f takes a module to a complex consisting of the module in degree zero. The reverse functor collapses a complex down to a single module.

TRIANGULATED CATEGORIES.

A triangulated category has a translation functor τ. The category has triangles which are sequences of modules and maps of the form

$$\cdots \longrightarrow \tau^{-1}(N) \xrightarrow{\tau^{-1}(\gamma)} L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \tau(L) \xrightarrow{\tau(\alpha)} \tau(M) \longrightarrow \cdots$$

The triangles and the translation functor must satisfy several axioms. (*e.g.* Any morphism $\alpha : L \rightarrow M$ must fit into a triangle.)

In the case of $\mathbf{D}^{\mathrm{b}}(R)$, the translation functor τ is the shift $\tau(F_*) = F[1]$, where $F[1]_r = F_{r-1}$ and the boundary map on F is multiplied by -1.

TRIANGULATED CATEGORIES.

In **stmod**(kG), the translation functor is Ω^{-1}. If M is a kG-module, $\Omega^{-1}(M)$ is the cokernel of the inclusion of M in its injective hull: $0 \longrightarrow M \longrightarrow \mathrm{InjHull} \longrightarrow \Omega^{-1}(M) \longrightarrow 0$

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Exercise: Given an exact sequence

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0,$$

add a projective module P to N, so that the map $\beta' : M \to N \oplus P$ is injective. Show that the cokernel of β' is $\Omega^{-1}(L)$, so that we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\beta'} N \oplus P \xrightarrow{\gamma} \Omega^{-1}(L) \longrightarrow 0.$$

THICK SUBCATEGORIES.

If $\mathcal{S}$ is a subcategory of a triangulated category $\mathcal{C}$, then $\mathcal{S}$ is thick if it is triangulated, closed under finite direct sums (coproducts) and direct summands.

In particular if $\mathcal{S}$ is thick and if $A \to B \to C \to \tau(A)$ is a triangle with two of the three objects A, B and C in $\mathcal{S}$, the third object is also in $\mathcal{S}$.

Example: Take the set of all bounded complexes in $\mathbf{D}^{\mathrm{b}}(R)$ whose terms are all projective.

What is this?

THE THEOREMS.

THEOREM (HOPKINS)

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THEOREM (BENSON-C.-RICKARD)

The (tensor ideal) thick subcategories of $\mathbf{stmod}(kG)$ *are determined by the support varieties of the objects.*

"Tensor ideal" means that if M is in the subcategory, the so is $M \otimes_k N$ for any module N.

SUPPORT VARIETIES FOR S.

Let Spec(S) denote the spectrum of S, the space of all prime ideals in S with the Zariski topology ($V \subset \text{Spec}(S)$ is closed if there is an ideal $I \subset S$ such that V is the set of all prime ideals that contain I.)

Then the support of an object $F_* \in \mathbf{D}^{\mathrm{b}}(S)$ is the variety of the annihilator of the homology of F_*.

Hopkins says: If $\mathcal{S}$ is a thick subcategory of $\mathbf{D}^{\mathrm{b}}(S)$, then there exists a subset $\mathcal{W} \subseteq \text{Spec}(S)$, which is closed under specialization (if $U \subset V$ and $V \in \mathcal{W}$, then $U \in \mathcal{W}$) such that $\mathcal{S} = \mathbf{D}^{\mathrm{b}}(S)_{\mathcal{W}}$, the full subcategory of objects whose supports are in $\mathcal{W}$.

SUPPORT VARIETIES IN kG.

Suppose that M is a finitely generated kG-module. Then the ring $\mathrm{Ext}^*_{kG}(M, M)$ is a finitely generated module over the cohomology ring $\mathrm{H}^*(G, k) \cong \mathrm{Ext}^*_{kG}(k, k)$. (Take an extension of k by k, tensor it with M, and get an extension of M by M.)

Let $J(M)$ be the annihilator of $\mathrm{Ext}^*_{kG}(M, M)$ in $\mathrm{H}^*(G, k)$.

Let $V_G(M)$ be the variety of $J(M)$, the set of all ideals that contain $J(M)$.

The variety of the module M has the following properties

PROPERTIES OF SUPPORT VARIETIES.

THEOREM

Suppose that L, M and N are finitely generated kG-modules. Then

1. *The module M is projective if and only if $V_G(M) = \{0\}$.*
2. $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.
3. $V_G(M) = \cup_{E \in \mathcal{E}\mathcal{A}} \operatorname{res}^*_{G,E}(V_E(M))$.
4. *if* $\to L \to M \to N \to \Omega^{-1}(L)$ *is a triangle, then* $V_G(M) \subseteq V_G(L) \cup V_G(N)$.
5. $V_G(M \otimes N) \cong V_G(M) \cap V_G(N)$.

BCR says: If $\mathcal{S}$ is a thick subcategory of $\mathbf{stmod}(kG)$, then there exists a subset $\mathcal{W} \subseteq \operatorname{Spec}^*(\mathrm{H}^*(G,k))$, which is closed under specialization such that $\mathcal{S} = \mathbf{stmod}(kG)_{\mathcal{W}}$, the full subcategory of objects whose supports are in $\mathcal{W}$.

GENERATION.

The thick subcategory generated by an object M, $\mathrm{Thick}(M)$, is the smallest thick subcategory that contains M.

In $\mathbf{D}(S)$, $\mathrm{Thick}(S) = \mathbf{D}^{\mathrm{b}}(S)$.

If G is a p-group, then in $\mathbf{D}(kG)$, $\mathrm{Thick}(k) = \mathbf{D}^{\mathrm{b}}(kG)$.

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From now on we assume that $G = \langle g_1, \dots, g_n \rangle \cong (\mathbb{Z}/2\mathbb{Z})^n$ is an elementary abelian group of order 2^n. ($g_i g_j = g_j g_i$, $g_i^2 = 1$.)

Note that if $z_i = g_i - 1$, then $z_i^2 = g_i^2 - 1^2 = 0$. So $kG = k[z_1, \dots, z_n]/(z_1^2, \dots, z_n^2)$.

COMPUTING GROUP COHOMOLOGY.

By definition, $\mathrm{H}^n(G,k) \cong \mathrm{Ext}^n_{kG}(k,k) = \mathrm{H}^n(\mathrm{Hom}_{kG}(P_*,k))$, where $P_* \to k$ is a projective resolution of k:

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

Let $\Lambda = \Lambda(z_1,\dots,z_n)$ denote the exterior algebra generated by $z_1 \dots, z_n$. If $p = 2$, then $\Lambda = kG$.

We define a complex $J = \Lambda \otimes \mathrm{Hom}_k(S,k)$ with differential $\delta = \sum_{i=1}^n z_i \otimes x_i$.

Let S^n denote the space of homogeneous polynomials of degree n in S. Let $T_n = \mathrm{Hom}_k(S^n,k)$. Then J has the form.

$$\cdots \longrightarrow \Lambda \otimes T_2 \longrightarrow \Lambda \otimes T_1 \longrightarrow \Lambda \otimes T_0 (\xrightarrow{\varepsilon} k) \longrightarrow 0$$

EXAMPLE: $n = 1$.

Let $f_i \in T_i$ be the function such that $f_i(x^i) = 1$ and $f_i(x^j) = 0$ for $j \neq i$.

Note that $xf_i = f_{i-1}$ since $(xf_i)(x^j) = f_i(xx^j) = f_i(x^{j+1})$.

Remember that $\Lambda = k[z]/z^2$. So J looks like

$$\begin{tikzcd} \cdots \arrow[r] & \Lambda \otimes f_3 \arrow[r, "z"] & \Lambda \otimes f_2 \arrow[r, "z"] & \Lambda \otimes f_1 \arrow[r, "z"] & \Lambda \otimes f_0 \arrow[r] & 0 \end{tikzcd}$$

or

$$\begin{tikzcd} \cdots \arrow[r] & \Lambda \arrow[r, "z"] & \Lambda \arrow[r, "z"] & \Lambda \arrow[r, "z"] & \Lambda \arrow[r] & 0 \end{tikzcd}$$

Example: $n = 2$.

Let $f_{i,j} \in T_{i+j}$ be the function such that $f_{i,j}(x_1^r x_2^s) = 1$ if $r = i$ and $s = j$, and 0 otherwise.

Note that $x_1 f_{i,j} = f_{i-1,j}$ since $(x_1 f_{i,j})(x_1^r x_2^s) = f_{i,j}(x_1 x_1^r x_2^s)$ $= f_{i,j}(x_1^{r+1} x_2^s)$. Likewise $x_2 f_{i,j} = f_{i,j-1}$.
So J looks like

$$\begin{tikzcd}
 & \Lambda \otimes f_{3,0} \arrow[dr, "z_1"] & & & & \\
 & \oplus & \Lambda \otimes f_{2,0} \arrow[dr, "z_1"] & & & \\
 & \Lambda \otimes f_{2,1} \arrow[ur, "z_2"'] \arrow[dr, "z_1"] & \oplus & \Lambda \otimes f_{1,0} \arrow[dr, "z_1"] & & \\
\cdots \arrow[r] & \oplus & \Lambda \otimes f_{1,1} \arrow[ur, "z_2"'] \arrow[dr, "z_1"] & \oplus & \Lambda \otimes f_{0,0} \arrow[r] & 0 \\
 & \Lambda \otimes f_{1,2} \arrow[ur, "z_2"'] \arrow[dr, "z_1"] & \oplus & \Lambda \otimes f_{0,1} \arrow[ur, "z_2"'] & & \\
 & \oplus & \Lambda \otimes f_{0,2} \arrow[ur, "z_2"'] & & & \\
 & \Lambda \otimes f_{0,3} \arrow[ur, "z_2"'] & & & &
\end{tikzcd}$$

EXAMPLE: $n = 2$.

Or, cleaned up:

$$\begin{tikzcd}
 & \Lambda \arrow[dr, "z_1"] & & & & \\
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\end{tikzcd}$$

This is a projective resolution of k as a Λ-module.

WHAT DO WE DO WITH THIS?

Following work of Avramov, Buchweitz, Iyengar and Miller.

We regard S and Λ as differential Graded (DG) algebras with zero differential. Then J is a DG $(\Lambda \otimes S)$-module. Consider $\mathbf{D}(\Lambda)$ and $\mathbf{D}(S)$ as the categories of DG modules over Λ and S. Define a functor

$$h : \ \mathbf{D}(\Lambda) \ \longrightarrow \ \mathbf{D}(S)$$

$$M \ \mapsto \ \mathrm{Hom}_{\Lambda}(J, M) = \mathrm{Hom}_{\Lambda}(\Lambda \otimes \mathrm{Hom}_k(S, k), M)$$

Note that $h(k) = \mathrm{Hom}_{\Lambda}(\Lambda \otimes \mathrm{Hom}_k(S, k), k) \cong S$. So $h(\mathbf{D}^{\mathrm{b}}(\Lambda)) = h(\mathrm{Thick}(k)) = \mathrm{Thick}(S) = \mathbf{D}^f(S)$.

THE EQUIVALENCE.

THEOREM

The functor h induces and equivalence of categories $\mathbf{D}^{\mathrm{b}}(\Lambda) \to \mathbf{D}^{\mathrm{f}}(S)$.

So define the support of a Λ-module M to be the support of the S-module $h(M) = \mathrm{Hom}_\Lambda(J, M)$, which is the annihilator in S of the homology of $h(M)$.

Now because J is a Λ-projective resolution of k, $S = h(k) = \mathrm{Ext}^*_\Lambda(k, k)$ which (when $p = 2$) is $\mathrm{H}^*(G, k)$. Moreover, $h(M) = \mathrm{Ext}^*_{kG}(k, M)$.

So ($p = 2$), the support of $h(M)$ is the subvariety of $\mathrm{Spec}^*(\mathrm{H}^*(G, k))$ of the annihilator of $\mathrm{Ext}_{kG}(k, M)$. This is equivalent to the usual definition of $V_G(M)$.

CONCLUSIONS

Hence, if $p = 2$ and G is an elementary abelian 2-group, we have

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If $p > 2$, then we can prove the same, but we need an intermediate step through a Koszul algebra whose homology is the exterior algebra Λ. The functor we get is not an equivalence - but it is good enough.

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If G is not elementary abelian - Well, that's another story.

Come to my seminar lecture in Bielefeld on Friday.

THANKS.

Thank you.