

t-structures, recollements and silting objects

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Thurnau, March 21, 2012

Overview

- ① t-structures and co-t-structures;
- ② Recollements and glueing;
- ③ The piecewise hereditary case;
- ④ Silting objects;
- ⑤ Glueing of silting objects;
- ⑥ Glueing of tilting objects.

t-structures and co-t-structures

$\mathcal{D}$ triangulated.

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$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 0}[-1]$.

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Definition (Bondarko, Pauksztello'10)

A **co-t-structure** in $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ s.t.

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- **Co-heart:** $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$. It is additive but, in general, not abelian!
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satisfying:

- ❶ $(i^*, i_*, i^!)$, $(j_!, j^*, j_*)$ are adjoint triples;
- ❷ i_* , j_* , $j_!$ are fully faithful;
- ❸ $j^* i_* = 0$;
- ❹ $\forall X \in \mathcal{D}$, there are triangles given by the (co)units of the adjunctions:

$$i_* i^! X \rightarrow X \rightarrow j_* j^* X \rightarrow i_* i^! X[1] \quad , \quad j_! j^* X \rightarrow X \rightarrow i_* i^* X \rightarrow j_! j^* X[1] .$$

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where B is also a f.d. piecewise hereditary and $\text{End}_{\mathcal{D}^b(A)}(X)$ is a f.d. skew-field over $\mathbb{K}$.

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Theorem (Beilinson-Bernstein-Deligne'82)

Given $\mathcal{R}$ and $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$, $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ t -structures in $\mathcal{X}$ and $\mathcal{Y}$, then

$$\mathcal{D}^{\leq 0} := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}^{\leq 0}, i^*Z \in \mathcal{Y}^{\leq 0}\}$$

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Theorem (Bondarko'10)

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Lemma

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t -structure with length heart in $\mathcal{D}^b(R)$. Then $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \cong \text{mod}(S)$, where S is a f.d. directed algebra over $\mathbb{K}$.

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Theorem (Liu-V.'11)

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t -structure with length heart in $\mathcal{D}^b(R)$. Then there is a recollement of $\mathcal{D}^b(R)$ by derived module categories such that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is glued with respect to this recollement.

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- *$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is glued with respect to this recollement from t -structures $(j^* \mathcal{D}^{\leq 0}, j^* \mathcal{D}^{\geq 0})$ and $(i^* \mathcal{D}^{\leq 0}, i^! \mathcal{D}^{\geq 0})$*

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Theorem (Keller-Vossieck'88)

Q Dynkin quiver. The bounded t -structures in $\mathcal{D}^b(\mathbb{K}Q)$ are in bijection with basic silting objects in $\mathcal{D}^b(\mathbb{K}Q)$.

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Assume $\text{gldim}(R) < \infty$ ($\Rightarrow \mathcal{K}^b(\text{proj}(R)) \cong \mathcal{D}^b(R)$).

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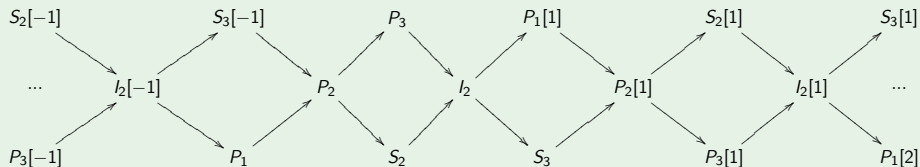
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- **co- t -structure**: $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}) \rightsquigarrow$ **silting**: M s.t. $\text{add}(M) = \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$.
- **t -structure**: $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \rightsquigarrow$ **co- t -structure**: $(({}^{\perp}\mathcal{D}^{\leq 0})[1], \mathcal{D}^{\leq 0})$.
- **silting**: $M \rightsquigarrow$ **t -structure**:
 $\mathcal{D}_M^{\leq 0} = \{X \in \mathcal{D}^b(R) : \text{Hom}(M, X[i]) = 0, \forall i > 0\}$
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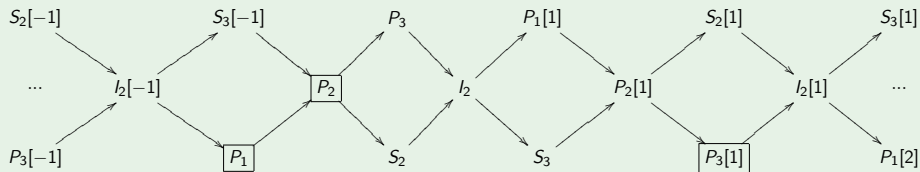
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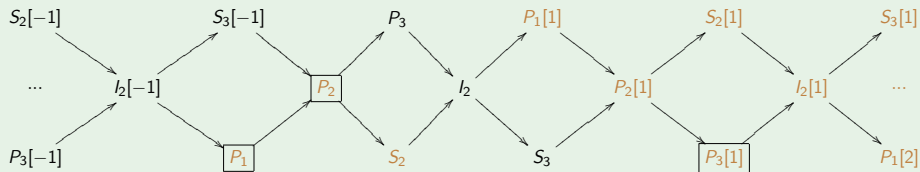
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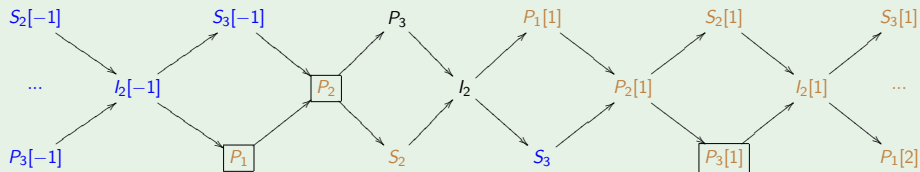


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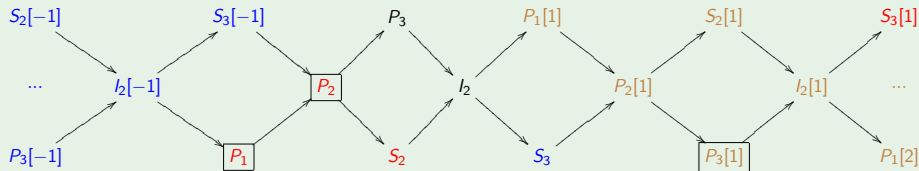
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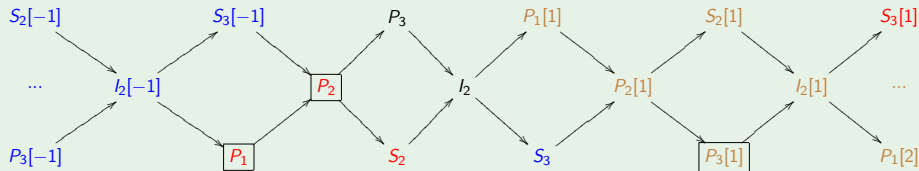
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Heart: $\mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0} \cong \text{End}_{\mathcal{D}^b(R)}(M) \cong \mathbb{K}A_2 \times \mathbb{K}$, $A_2 = 1 \longrightarrow 2$

Silting objects

Example (t-structure in $\mathcal{D}^b(\mathbb{K}A_3)$, $A_3 = 1 \longrightarrow 2 \longrightarrow 3$)



Silting object: $M = P_1 \oplus P_2 \oplus P_3[1]$

Aisle: $\mathcal{D}_M^{\leq 0}$

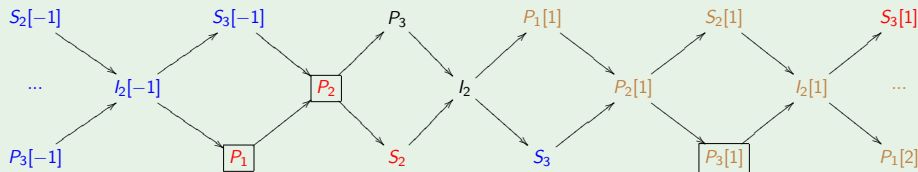
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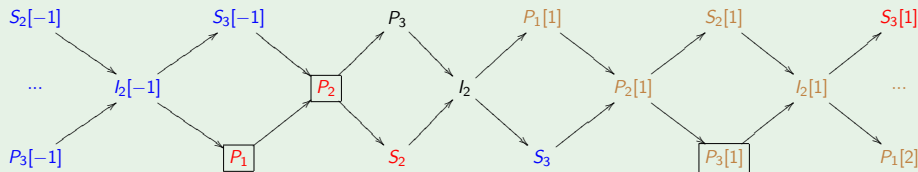
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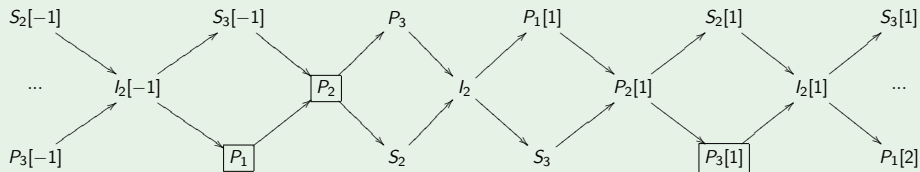
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In general:

- The heart of $(\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0})$ is $\text{End}_{\mathcal{D}^b(R)}(M)$;
- M is tilting if and only if $M \in \mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0}$.

Silting objects

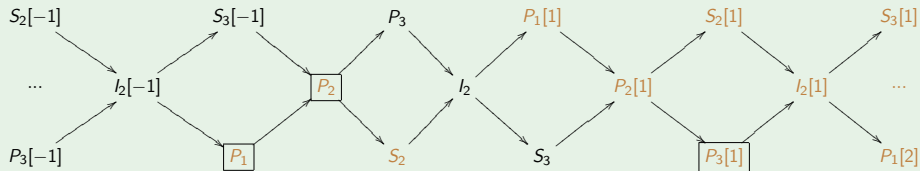
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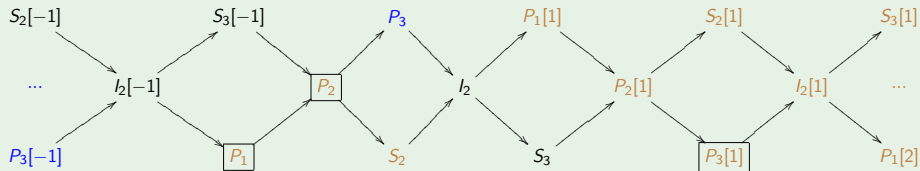


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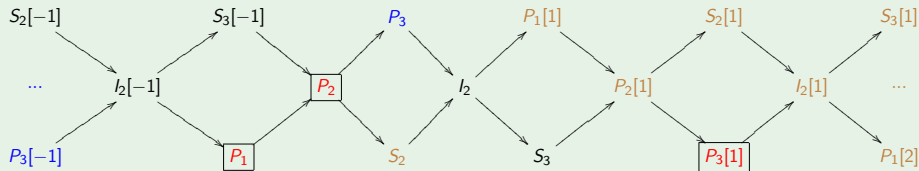
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$$\text{Coheart: } \mathcal{D}_{\geq 0}^M \cap \mathcal{D}_{\leq 0}^M \cong \text{add}(M)$$

Glueing silting

Theorem (Liu-V.-Yang'12)

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- $\mathcal{R}$ recollement of triangulated category $\mathcal{D}$;

$$(\mathcal{R}) : \quad \mathcal{Y} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{X} ;$$

- $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ silting objects;
- $(\mathcal{X}_{\geq 0}, \mathcal{X}_{\leq 0})$ and $(\mathcal{Y}_{\geq 0}, \mathcal{Y}_{\leq 0})$ corresponding *co-t-structures* in $\mathcal{X}$ and $\mathcal{Y}$.

Then

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Then the glued co-t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ corresponds to the silting $Z = i_* Y \oplus K_X$, where K_X is defined by

$$i_* \beta_{\geq 1} i^! j_! X \longrightarrow j_! X \longrightarrow K_X \longrightarrow (i_* \beta_{\geq 1} i^! j_! X)[1]$$

($\beta_{\geq 1}$ is a (non-functorial) choice of truncation for $(\mathcal{Y}_{\geq 0}, \mathcal{Y}_{\leq 0})$ in $\mathcal{Y}$).

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Idea of proof

Glueing silting

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- *[BBD'82] $\rightsquigarrow$ functor $j_{!*}$ (7th/intermediate functor) $\rightsquigarrow$ glueing simples in the hearts of **t-structures**;*
- *$\{\text{Summands of glued silting}\} \leftrightarrow \{\text{"simples" of glued **coheart**}\};$*
- *"Triangulated" description of 7th functor (using truncations);*
- *Apply it to "simples" of the coheart of $\mathcal{X}$;*
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Remark

Glueing silting objects means glueing compatibly with the glueing of **co-t-structures**.

Glueing silting

Remark

$\mathcal{R}$ recollement of $\mathcal{D}^b(R)$, R of *finite global dimension*;

- There is Serre functor in $\mathcal{D}^b(R)$;
- Recollements can be reflected (Jørgensen);

$\rightsquigarrow$ *Glueing of silting can be done, in this setting, compatibly with the glueing of t-structures.*

This corresponds to glueing with respect to certain co-t-structures via a reflected recollement.

Glueing tilting

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Recall that, in $\mathcal{D}^b(R)$ a silting is tilting if and only if it lies in the heart of the corresponding t-structure.

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Theorem (Liu-V.-Yang'12)

- $\mathcal{R}$ recollement of $\mathcal{D}^b(R)$ by $\mathcal{D}^b(C)$ and $\mathcal{D}^b(B)$

$$(\mathcal{R}) : \quad \mathcal{D}^b(B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D}^b(R) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{D}^b(C) ;$$

- X and Y be *tilting* objects of $\mathcal{D}^b(C)$ and $\mathcal{D}^b(B)$.

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Then $Z = i_* Y \oplus K_X$ is tilting if and only if

- (a) $\mathrm{Hom}_{\mathcal{D}^b(B)}(Y, i^* j_* X[k]) = 0$ for all $k < -1$;
- (b) $\mathrm{Hom}_{\mathcal{D}^b(B)}(i^* j_* X, Y[k]) = 0$ for all $k < 0$;
- (c) $\mathrm{Hom}_{\mathcal{D}^b(B)}(i^* j_* X, i^* j_* X[k]) = 0$ for all $k < -1$.

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Then $Z = i_* Y \oplus K_X$ is tilting if and only if $\exists X'_{-1}, X'_0, X'_1, X'_2 \in \text{mod}(B)$:

- X'_2 projective;
- $i^* j_* X \cong X'_{-1}[1] \oplus X'_0 \oplus X'_1[-1] \oplus X'_2[-2]$;
- $\text{Hom}_B(X_m, X_n) = 0$ whenever $n - m \geq 2$.

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Thank you for your attention.