Hodge Theory and Real Groups
(Joint work with Wilfried Schmid)

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Introduction

Let us consider a very general situation. Let G be a group acting on the space X. In the language of Klein we have specified a geometry. We can linearize the situation and obtain a representation of G on $\mathrm{Func}(X)$. The goal or representation theory (and harmonic analysis) is twofold:

i) Classify all the irreducible (and indecomposable) representations of G

ii) Decompose $\mathrm{Func}(X)$ into irreducible (indecomposable) representations.

To make things more definite let us consider the (simplest) situation when $\mathrm{Func}(X) = L^2(X)$ and let our group be real reductive group $G_{\mathbb{R}}$.

This leads us to the main problem discussed in this lecture:

Problem

Determine all the irreducible unitary representations of $G_{\mathbb{R}}$.

Geometry

The $G_{\mathbb{R}}$-representation $L^2(X)$ is in general very large. To produce irreducible representations we proceed as follows. We write

$K_{\mathbb{R}}$ for the maximal compact subgroup of $G_{\mathbb{R}}$

G, K for the complexifications of $G_{\mathbb{R}}$, $K_{\mathbb{R}}$

X the flag variety of G: the complex projective variety with transitive algebraic G-action, such that as set,

$$\begin{aligned} X &= \text{collection of all Borel subalgebras } \mathfrak{b} \subset \mathfrak{g} \\ &\cong G/B \quad (B = \text{any particular Borel subgroup of } G) \end{aligned}$$

For $\lambda \in \mathrm{H}^2(X, \mathbb{C})$ we have, by taking cohomology,

$$\begin{aligned} \{K - \text{equivariant sheaves on } X\}_\lambda &\leftrightarrow \{G_{\mathbb{R}} - \text{equivariant sheaves on } X\}_\lambda \\ &\longleftrightarrow \{G_{\mathbb{R}} - \text{representations with infinitesimal character } \lambda\} \end{aligned}$$

This construction says nothing about unitarity, so far.

The orbit method

An old idea of Kirilov and Kostant is the orbit method. Instead of taking the flag manifold X for G we consider a coadjoint orbit $\mathcal{O}$ of $G_{\mathbb{R}}$ in the dual lie algebra $\mathfrak{g}^*$. It is a symplectic manifold.

Principle

Quantizing the coadjoint orbits yields all the irreducible unitary representations of $G_{\mathbb{R}}$.

Unfortunately, quantization is not an exact science .... . Same goes for the opposite process of passing to the classical limit. Fortunately, in our situation we already have the "quantum" objects, the representations. Furthermore, as will be explained, the passage to the classical limit can be made a precise procedure in our case.

The Langlands program

Perhaps the most important *current* context for these ideas is the case when $X = \Gamma \backslash G_{\mathbb{R}}$ with Γ and arithmetic subgroup. This case constitutes the Langlands program. It prescribes a deep relation between constituents of such a decomposition between different groups, called the Langlands functoriality. Functoriality preserves Hecke eigenvalues, a source of deep information.
The description of the representations of local groups is also part of the program and amounts in our case to

$$\{W_{\mathbb{R}} \to {}^L G\} \longleftrightarrow \{\text{Irreducible representations of } G_{\mathbb{R}}\}$$

The lefthand side is usually referred to as Langlands parameters. In our situation the parameters amount to Hodge Theory.

Remark

The $W_{\mathbb{R}}$ can be replaced be other gadgets. The most general is the Langlands group and then we get all the automorphic forms.

Let X be a compact Kähler manifold. Then it's cohomology has the following decomposition

$$\mathrm{H}^n(X) = \oplus_{p+q=n} \mathrm{H}^{p,q}(X) \qquad \mathrm{H}^{p,q}(X) = \mathrm{H}^q(X, \Omega_X^p).$$

We make the following general definition.

Definition

A complex Hodge structure consists of a complex vector space V and a direct sum decomposition $V = \oplus_{p+q=n} V^{p,q}$.

Furthermore, we make the following:

Definition

A polarization of a complex Hodge structure consists of a non-degenerate Hermitian pairing $Q : V \times V \to \mathbb{C}$ such that the spaces $V^{p,q}$ are orthogonal with respect to Q and $(-1)^p(2\pi i)^n$ is positive definite on $V^{p,q}$.

Hodge Theory, continued

A polarized Hodge structure is a Hodge structure equipped with a polarization. A polarization is unique up to a positive real scalar. We have the following alternate description of a Hodge structure which is more suitable for generalization. A Hodge structure consists of a vector space V together with two decreasing filtrations F^p and $\bar{F}^q$ which are n-complementary, i.e., the following two equivalent conditions hold:

$$\text{For every } p \in \mathbb{Z} \text{ we have } \quad F^p \oplus \bar{F}^{n-p+1} \; = \; V$$

It is related to the first definition via

$$V \; = \; \oplus_{p \in \mathbb{Z}} F^p \cap \bar{F}^{n-p} \, .$$

Mixed Hodge structures

We also have the notion of a complex mixed Hodge structure. It consists of a complex vector space V, two decreasing filtrations F^p and $\bar{F}^q$, and an increasing filtration W_k.

Definition

A triple $(V, F^p, \bar{F}^q, W_k)$ constitutes a mixed Hodge structure if the filtrations F^p and $\bar{F}^q$ induce a Hodge structure of weight k on all the graded quotients $gr_k^W V = W_k V / W_{k-1}$. It is called polarizable if all the graded quotients can be equipped with a polarization.

A slightly non-trivial key fact is that the mixed Hodge structures form an abelian category, i.e., the morphisms which preserve the filtrations preserve them strictly.

Variation of Hodge structures and more

The Hodge structures and mixed Hodge structures can be varied over a complex manifold X. In this scenario the filtration F^p moves holomorphically and the filtration $\bar{F}^q$ anti-holomorphically. A very general formulation is as follows. We write $\mathcal{D}_X$ for the sheaf of linear differential operators on X. A mixed Hodge module consists of a holonomic regular $\mathcal{D}_X$-module $\mathcal{M}$ with a filtration $W_k\mathcal{M}$ together with a $\mathcal{D}_X$-filtration $F_p\mathcal{M}$; the latter means that $\mathcal{D}_X(k)F_p\mathcal{M} \subset F_{p+k}\mathcal{M}$. These structures have many good functorial properties and in particular we have:

$$gr_k^W\mathcal{M} \qquad \text{is semi simple}$$

Modules with just one weight are called Hodge modules. They are polarized via a non-degenerate $(\mathcal{D}_X, \mathcal{D}_{\bar{X}})$ pairing:

$$\mathcal{M} \otimes_{\mathbb{C}} \bar{\mathcal{M}} \longrightarrow C^{-\infty}(X_{\mathbb{R}})$$

The classical limit

Let us write $MHM(X)$ for the category of mixed Hodge modules on X. We have an exact faithful functor

$$\mathrm{gr} : MHM(X) \longrightarrow \mathcal{C}oh^{\mathbb{C}^*}_{CM}(T^*X)\,.$$

The right hand side stands for coherent sheaves on the cotangent bundle which are Cohen-Macauley and $\mathbb{C}^*$-equivariant. They are supported on conic Lagrangian sub varieties of T^*X.

This functor has good functorial properties.

Problem

Descibe the image of gr.

Back to representation theory

Let us recall that we have

$$\{K - \text{equivariant sheaves on } X\}_\lambda \longleftrightarrow \{G_\mathbb{R} - \text{representations}\}_{\chi_\lambda}$$

where X is the flag variety of G and $\lambda \in \mathrm{H}^2(X, \mathbb{C})$. For the purposes of classifying irreducible unitary representations it suffices to consider the situation where $\lambda \in \mathrm{H}^2(X, \mathbb{R})$ is real. In that case we can canonically lift the irreducible representations and the standard representations to the the corresponding category of polarizable mixed Hodge modules $MHM_K(X)_\lambda$.
As before, we have an exact faithful functor

$$\mathrm{gr} : MHM_K(X)_\lambda \longrightarrow \mathcal{C}oh_{CM}^{K\times\mathbb{C}^*}(T^*X)\,.$$

with good functorial properties.

Hodge structures on representations

Our work can be summarized as follows:

Theorem

For $\lambda \in \mathrm{H}^2(X, \mathbb{R})$ dominant we obtain a faithful exact functor

$$MHM_K(X)_\lambda \longrightarrow \{\textit{polarizable mixed Hodge structures}\} .$$

This result implies, in particular, the following:

Corollary

For $\lambda \in \mathrm{H}^2(X, \mathbb{R})$ the standard representations carry canonical polarizable mixed Hodge structures.

Corollary

For $\lambda \in \mathrm{H}^2(X, \mathbb{R})$ the irreducible representations carry canonical polarized Hodge structures. The polarization coincides with the unique $\mathfrak{u}_\mathbb{R}$-invariant form.

Hodge structures on representations, continued

Here $\mathfrak{u}_{\mathbb{R}}$ stands for the Lie algebra of a compact form of G. It is important to note that the decomposition

$$V = \oplus_{p+q=n} V^{p,q}$$

takes place on a dense subspace of the $G_{\mathbb{R}}$-representation. However, we have the following

Conjecture

The Hilbert space completion of the Hodge structure $V = \oplus_{p+q=n} V^{p,q}$ has a natural structure of a $G_{\mathbb{R}}$-representation.

Conclusion

The $\mathfrak{u}_{\mathbb{R}}$-invariant form can be directly related to the $\mathfrak{g}_{\mathbb{R}}$-invariant form, when the latter exists. This gives a unitarity algorithm and more .....