

# The Auslander Algebra of $k[T]/T^t$

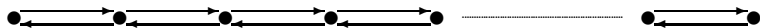
Lutz Hille

Universität Münster

Thurnau, March 19, 2012

# 1. The Auslander algebra of $k[T]/T^t$

$$\mathcal{A}_t := \text{End}\left(\bigoplus_{i=1}^t k[T]/T^i\right)$$



irreducible maps  $\alpha_i : k[T]/T^i \rightarrow k[T]/T^{i+1}$  and  $\beta_i : k[T]/T^i \rightarrow k[T]/T^{i-1}$

relations  $\beta_1\alpha_1 = 0$  and  $\alpha_{i+1}\beta_{i+1} = \beta_i\alpha_i$

have natural embeddings/projections  $k[T]/T^i \rightarrow k[T]/T^j$  given by a polynomial in  $T$

# $\Delta$ -filtered modules

$\mathcal{F}(\Delta)$  is defined as set of all  $\mathcal{A}_t$ -modules satisfying

$$\{M \mid \text{pd } M \leq 1\}$$

$$\{M \mid M(\alpha) \text{ is injective}\}$$

$$\{M \mid M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \dots M_{t-1} \longrightarrow M_t \text{ is a flag in } M_t\}$$

$$\{M \mid \text{soc } M \simeq S(t)^a\}$$

$$\{M \mid M \text{ admits a filtration by } \Delta - \text{modules}\}$$

# some particular modules and the covering

line modules  $\Delta(i)$

projective modules  $P(i)$

projective and injective module  $P(t)$

certain  $\Delta$  – filtered submodules of  $P(t)$  :  $\Delta(I), I \subseteq \{1, 2, \dots, t\}$

## Lemma (Brüstle, H., Ringel, Röhrle)

*For  $I \subseteq J$  we have  $\text{Ext}^1(\Delta(I), \Delta(J)) = 0 = \text{Ext}^1(\Delta(J), \Delta(I))$ .*

## 2. Actions of parabolic groups

To each  $\Delta$ -filtered module  $M$  we associate

1. a flag in  $M_t$  with  $\Delta$ -dimension vector  $d = (\dim M_1, \dim M_2 - \dim M_1, \dots, \dim M_t - \dim M_{t-1})$ , its stabilizer is a parabolic subgroup  $P(d)$  in a General Linear Group
2. an endomorphism  $f := \alpha_{t-1}\beta_{t-1}$  of  $M_t$  that strictly preserves the flag:  $f(M_i) \subseteq M_{i-1}$ , it is an element in Lie algebra of the unipotent radical of  $P(d)$
3. orbits of  $P(d)$  on its unipotent radical coincide with isomorphism classes of  $\Delta$ -filtered modules
4. the unique dense orbit for the action of  $P(d)$  on its unipotent radical (that exists by Richardson result and coincides with the generic nilpotent class) corresponds to a  $\Delta$ -filtered module  $M$  of  $\Delta$ -dimension vector  $d$  with  $\text{Ext}^1(M, M) = 0$
5. Consequence: for each  $d$  exists a unique  $\Delta$ -filtered module  $M(d)$  of  $\Delta$ -dimension vector  $d$  with  $\text{Ext}^1(M(d), M(d)) = 0$

### 3. $\Delta$ -filtered tilting modules

For each  $\sigma \in S_t$  we define a module

$$T(\sigma) = \bigoplus_{i=1}^t \Delta(\sigma(1), \sigma(2), \dots, \sigma(i))$$

with  $\text{Ext}^1(T(\sigma), T(\sigma)) = 0$ .

#### Theorem (BHRR)

*The modules  $T(\sigma)$  form a complete set of tilting modules (up to isomorphism and multiple summands) that are  $\Delta$ -filtered  $\mathcal{A}_t$ -modules.*

new proof using the volume of a tilting module

open: classify all tilting modules, exceptional sequences and spherical objects

observe: all tilting modules have an endomorphism algebra isomorphic to  $\mathcal{A}_t$

## 4. Exceptional sequences and spherical modules

A sequence of modules  $\varepsilon = (E(1), \dots, E(t))$  is a *full exceptional sequence* if

$\text{End}(E(i)) = k$ ;  $\text{Ext}^q((E(i), E(i)) = 0$  for  $q > 0$ ; and

$\text{Ext}^q((E(j), E(i)) = 0$  for all  $j > i$ ;

A module  $S$  is *spherical* if  $\text{Ext}^q(S, S) = k$  for  $q = 0, 2$  and  $= 0$  else, and  $\text{Hom}(S, M) \simeq \text{Ext}^2(M, S)^*$  (and vice versa). Such a spherical module (or even object in the derived category) defines an additional automorphism  $T_S$  of the derived category.

Using universal extensions we construct tilting modules from exceptional sequences.

## 5. Tilting bundles on rational surfaces and universal extensions

### Theorem (H., Markus Perling)

*Let  $X$  be a rational surface then  $X$  admits a tilting bundle.*

The category of coherent sheaves is never equivalent to a category of finite dimensional modules over some finite dimensional algebra, only the derived categories are.

Certain subcategories, that are of interest for a better understanding of spherical twists, are equivalent.

Example (with David Ploog): Take a sequence  $D_i$  of  $(-2)$ -curves in  $X$  and consider the subcategory generated by  $\mathcal{O}$ ,  $\mathcal{O}(D_1)$ ,  $\mathcal{O}(D_1 + D_2)$ , ...,  $\mathcal{O}(D_1 + \dots + D_{t-1})$ . The category of coherent sheaves in this subcategory is equivalent to the category of modules over  $\mathcal{A}_t$  (the  $\Delta$ -filtered modules correspond to the locally free sheaves).

# Universal extensions

We start with an exceptional sequence  $\varepsilon$  so that all higher Ext-groups (that is  $\text{Ext}^2$  and higher) vanish. Then we define universal extensions/coextensions for each pair  $(E, F)$

$$0 \longrightarrow \text{Ext}^1(E, F)^* \otimes F \longrightarrow \overline{E} \longrightarrow E \longrightarrow 0$$

$$0 \longrightarrow F \longrightarrow \overline{E} \longrightarrow \text{Ext}^1(E, F)^* \otimes E \longrightarrow 0$$

Recursively we can construct many tilting modules from one full exceptional sequence.

Starting with  $\varepsilon = (\Delta(1), \dots, \Delta(t))$  we get ALL  $\Delta$ -filtered tilting modules  $T(\sigma)$ .

## 6. The classification of all exceptional sequences and all spherical modules

### Theorem (H., David Ploog)

*Any exceptional or spherical module satisfies either  $M_i = 0$  or  $M_i = k$ . Such a module (where  $M$  is not trivial) is exceptional precisely if it is indecomposable with  $M_t = k$ . Such a module is spherical precisely if it is indecomposable with  $M_t = 0$ . Any full exceptional sequence of  $\mathcal{A}_t$ -modules defines a filtration of the projective and injective module  $P(t)$ .*

*Any total universal extension of any full exceptional sequence is isomorphic to  $P(t)$ .*

We have even a conjecture how to classify exceptional/spherical objects in the derived category.

Conjecture (with David Ploog): Any tilting module is a universal extension of a full exceptional sequence (classified above).

## 7. Further applications: generic $k[T]$ -module homomorphisms (joint with Dieter Vossieck)

$$f : M = \bigoplus (k[T]/T^i)^{a_i} \longrightarrow \bigoplus (k[T]/T^i)^{b_i}$$

Question: Is there a generic homomorphism  $f$ ?

- the classification of all  $f$  is a wild problem
- (Vossieck) there is always a generic one
- determine the generic one (just as a matrix)
- the problem is equivalent to the Theorem [BHRR]

## 8. Further applications: the complement of the Richardson orbit (joint with Karin Baur)

We consider the complement of the Richardson orbit in the Lie algebra of the unipotent radical

$$\mathfrak{p}_u(d) \setminus \text{Richardson orbit}$$

and decompose it into irreducible components.

## 9. Further applications: nilpotent class representations and irreducible component

We associate to each  $\mathcal{A}_t$ -module a set of nilpotent classes:

$$M_i, f_i = \beta_i \alpha_i \text{ or } f_i = \alpha_{i-1} \beta_{i-1}.$$

Now we fix the nilpotent class of each  $f_i$  and can determine the irreducible components for the space of all modules with this fixed nilpotent class.

We can determine for example the irreducible components in  $\mathfrak{p}_u(d) \cap C(\lambda)$ .