Losev-Manin moduli spaces and toric varieties associated to root systems

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Theorem (Losev, Manin)

There is a fine moduli space

\[ \overline{L}_n = \text{fine moduli space of stable } n\text{-pointed chains of projective lines} \]
Stable $n$-pointed chain of projective lines $(C, s_0, s_\infty, s_1, \ldots, s_n)$:

- $C$ projective curve, irreducible components isomorphic to $\mathbb{P}^1$, transversal intersection at poles, shape: chain, $s_0, s_\infty$ poles of outer components

- $s_1, \ldots, s_n$ marked points, different from poles of components, not necessarily distinct

- stability: at least one marked point on each component (therefore no nontrivial isomorphisms)
Properties of $\overline{L}_n$

- $\overline{L}_n$ is $(n-1)$-dimensional smooth projective toric variety (over $\mathbb{C}$), contains dense open torus $L_n = \{(\mathbb{P}^1, 0, \infty, s_1, \ldots, s_n)\} \subset \overline{L}_n$:

$$L_n = (\text{n points in } \mathbb{P}^1 \setminus \{0, \infty\})/\text{automorphisms of } (\mathbb{P}^1, 0, \infty)$$

$$= (\mathbb{C}^*)^n/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-1}$$

- $S_n$-operation by permutation of marked points.

- There are natural toric morphisms $\overline{L}_{n+1} \to \overline{L}_n$ defined by forgetting one of the marked points and stabilisation, $\overline{L}_{n+1} \to \overline{L}_n$ (with sections) forms the universal $n$-pointed chain.

- $\overline{L}_n$ related to similar moduli space

$$\overline{M}_{0,n+2} = \text{moduli space of stable } (n+2)\text{-pointed trees of projective lines}$$

Birational morphism $\overline{M}_{0,n+2} \to \overline{L}_n$ after choice of two of the $n+2$ marked points which become $s_0, s_\infty$. 
Examples

- \( \overline{L}_1 = \text{pt.} \)

- \( \overline{L}_2 = \mathbb{P}^1 \)

- \( \overline{L}_3 \) del Pezzo surface, \( \mathbb{P}^2 \) blown up in the 3 torus fixed points
Toric varieties associated with root systems

\( R \) root system

\[ M(R) = \langle R \rangle_{\mathbb{Z}} \text{ root lattice} \]
\[ N(R) = M(R)^* \text{ dual lattice} \]

For a set of simple roots \( S \subset R \): Weyl chamber

\[ \sigma_S = S^\vee = \{ v \in N(R)_\mathbb{Q} : \forall \alpha \in S : \langle \alpha, v \rangle \geq 0 \} \subset N(R)_\mathbb{Q} \]

**Definition**

\( \Sigma(R) \) fan of Weyl chambers

\( X(R) \) toric variety for \( \Sigma(R) \)
Properties and examples

– $X(R)$ smooth projective toric variety (over $\mathbb{C}$) of dimension $\dim X(R) = \text{rk } R$

– Operation of Weyl group $W(R)$

Examples: fans for $R = A_1, A_2$

$\Sigma(A_1) \quad \Sigma(A_2)$

Proposition

(Functorial property) Let $R, R'$ be root systems. Linear maps $\mu : M(R')_\mathbb{Q} \to M(R)_\mathbb{Q}$ such that $\mu(R') \subseteq \{ a\alpha : \alpha \in R, a \in \mathbb{Z} \}$ induce toric morphisms $X(\mu) : X(R) \to X(R')$. 
\(X(R)\) as moduli space

**Theorem**

There is an isomorphism

\[
\overline{L}_n \cong X(A_{n-1})
\]

Steps of proof:

– compare the moduli functor

\[
\overline{L}_n : (\text{schemes})^\circ \to (\text{sets})
\]

\[
Y \mapsto \left\{ \left( \pi : C \to Y, s_0, s_\infty, s_1, \ldots, s_n \right) \mid \pi \text{ proper flat, } s_0, s_\infty, s_i \text{ sections, geom. fibres stable n-pointed chains} \right\} / \sim
\]

to the functor of the toric variety \(\text{Mor}(\cdot, X(A_{n-1}))\).

– find appropriate description of functor of \(X(A_{n-1})\)
construct universal curve over $X(A_{n-1})$ in terms of maps of root systems:

$A_{n-1} \to A_n$ gives rise to proper surjective toric morphism $X(A_n) \to X(A_{n-1})$

projections $A_n \to A_{n-1}( \subset A_n)$ along pairs of opposite roots

$\{ \pm \alpha_i \} \subset A_n \setminus A_{n-1}$ give rise to sections $s_i : X(A_{n-1}) \to X(A_n)$

Example:

\[
\begin{align*}
\alpha_2 &= u_2 - u_3 \\
\alpha_1 &= u_1 - u_3 \\
- \alpha_1 &= u_3 - u_1 \\
- \alpha_2 &= u_3 - u_2
\end{align*}
\]

Theorem

$X(A_{n-1})$ with universal curve $X(A_n) \to X(A_{n-1})$ is a fine moduli space of $n$-pointed chains of projective lines.
Functor of $X(R)$

$R$ root system

Use closed embedding

$$X(R) \rightarrow \prod_{A_1 \cong R' \subseteq R} X(R') \cong \prod_{\{\pm \alpha\} \subseteq R} \mathbb{P}^1$$

determined by surjection of root systems $\prod_{A_1 \cong R' \subseteq R} R' \rightarrow R$, and functor of $\mathbb{P}^1$ to obtain

$$X(R)(Y) = \left\{ (\mathcal{L}\{\pm \beta\}, \{t_\beta, t_{-\beta}\})\{\pm \beta\} \subseteq R \text{ such that } t_\alpha t_\beta t_{-\gamma} = t_{-\alpha} t_{-\beta} t_\gamma \text{ for all root subsystems } A_2 \cong \{\pm \alpha, \pm \beta, \pm (\gamma = \alpha + \beta)\} \subseteq R \right\} / \sim$$
Comparison with moduli functor

\[ X(A_{n-1})(Y) \]
\[ \left( \mathcal{L}_{\{\pm \beta}\}, \{t_\beta, t_{-\beta}\}\right) \{\pm \beta\} \subseteq A_{n-1} \]

\[ \longleftrightarrow \{ \text{stable n-pointed chains over } Y \} \]
\[ \left( L_{\{\pm \beta\}}, \{t_\beta, t_{-\beta}\}\right) \{\pm \beta\} \subseteq A_{n-1} \mapsto \nabla \subseteq \prod_{\{\pm \alpha\} \subseteq A_n \setminus A_{n-1}} \mathbb{P}^1_Y \]

def. by homogeneous equations
\[ t_{\beta_{ij}} z_{\alpha_j} z_{-\alpha_i} = t_{-\beta_{ij}} z_{-\alpha_j} z_{\alpha_i} \]
for \( \beta_{ij} = \alpha_i - \alpha_j \)
\( (\beta_{ij} \in A_{n-1}, \alpha_i, \alpha_j \in A_n \setminus A_{n-1}) \)

and extra equations for the sections

\[ (t_{\beta_{12}} : t_{-\beta_{12}}) = (1 : 0) \]
\[ (t_{\beta_{12}} : t_{-\beta_{12}}) = (a : b) \]
\[ (t_{\beta_{12}} : t_{-\beta_{12}}) = (0 : 1) \]
Generalisations to other root systems

Root systems of type $B$: consider stable $(2n + 1)$-pointed chains with involution $(C, l, s_-, s_+, s_0, s_{1}^{\pm}, \ldots, s_{n}^{\pm})$, where $C$ chain of projective lines, $s_-, s_+$ poles of outer components, $l : C \to C$ involution such that $l(s_-) = s_+$, $2n + 1$ sections $s_0, s_i^{\pm}$ that satisfy $l(s_0) = s_0$, $l(s_i^-) = s_i^+$.

**Theorem**

$X(B_n)$ with universal curve $X(B_{n+1}) \to X(B_n)$ is the fine moduli space $\overline{L}_n^{0,\pm}$ of $(2n + 1)$-pointed chains with involution.

Root systems of type $C$: consider $2n$-pointed chains with involution. $X(C_{n+1}) \to X(C_n)$ also has nonreduced fibres, but we have

**Theorem**

$X(C_n)$ is the fine moduli space $\overline{L}_n^{\pm}$ of $2n$-pointed chains with involution.

Root systems of type $D$: fibres of $X(D_{n+1}) \to X(D_n)$ may have higher dimensional components.
The stacks $\mathcal{L}_n$

Consider chains of projective lines $(C, s_0, s_\infty)$ with subschemes $S \subset C$ finite of degree $n$, not meeting the poles and intersection points of components, but meeting every component. Call $(C, s_0, s_\infty, S)$ a degree-$n$-pointed chain.

An $n$-pointed chain $(C, s_0, s_\infty, s_1, \ldots, s_n)$ gives rise to a degree-$n$-pointed chain $(C, s_0, s_\infty, S)$ by forgetting the labels of the marked points.

$(C, s_0, s_\infty, S)$ may have nontrivial automorphisms:
- automorphism group of $(C, s_0, s_\infty)$: $(\mathbb{C}^*)^l$, $l$ length of chain
- automorphism group of $(C, s_0, s_\infty, S)$: finite subgroup of $(\mathbb{C}^*)^l$
- for example $(\mathbb{P}^1, 0, \infty, S)$, $S$ given by $z_0^k - z_1^k = 0$, has automorphism group $\mu_k$.

$\mathcal{L}_n = \text{moduli stack of degree-}n\text{-pointed chains} (C, s_0, s_\infty, S)$
Category $\mathcal{L}_n$ of degree-$n$-pointed chains:
- objects: $(C \rightarrow Y, s_0, s_\infty, S)$, $Y$ scheme
- morphisms: cartesian diagrams

\[
\begin{array}{ccc}
C' & \longrightarrow & C \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

compatible with the additional structure.

**Proposition**

The category $\mathcal{L}_n$ is a category fibred in groupoids over the category of schemes. It forms a stack over the fpqc site of schemes (⇒ stack also in fppf, étale topology) with representable, finite diagonal. Over fields of characteristic 0 the diagonal is unramified.
Operation of the symmetric group and relation to $\overline{L}_n$

**Proposition**

There is a morphism $\overline{L}_n \to \overline{L}_n$, $(C, s_0, s_\infty, s_1, \ldots, s_n) \mapsto (C, s_0, s_\infty, \sum_i s_i)$. It is faithfully flat and finite of degree $n!$.

Operation of $S_n = W(A_{n-1})$ on $\overline{L}_n$ by permuting the sections of $n$-pointed chains.

Quotient morphism factors as $\overline{L}_n \to \overline{L}_n \to \overline{L}_n/S_n$.

$\overline{L}_n \to \overline{L}_n/S_n$ is coarse moduli space.
Substack of irreducible pointed chains

Substack \( \mathbb{A}^{n-1}/\mu_n \) = \{\((\mathbb{P}^1, s_0, s_\infty, S)\)\} \subset \overline{\mathcal{L}}_n \) of irreducible curves:

\[(\mathbb{P}^1, s_0, s_\infty, S) \leftrightarrow \text{polynomials } \prod_{i=1}^n (y - s_i), s_i \in \mathbb{C}^*, \text{ up to multiplication by a factor } \lambda \in \mathbb{C}^* \]

\[(\mathbb{P}^1, s_0, s_\infty, S) \leftrightarrow \text{polynomials } x^n + a_{n-1}x^{n-1} + \ldots + a_1x_1 + 1, a_i \in \mathbb{C}, \text{ up to multiplication of } x \text{ by an } n\text{-th root of unity} \]

\(\mathbb{A}^{n-1}/\mu_n \) contains dense open torus \((\mathbb{C}^*)^{n-1}\) (nonzero coefficients \(a_i\)).

Points of \((\mathbb{C}^*)^{n-1}\) given by \((a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \in (\mathbb{C}^*)^{2(n-1)}\) up to equivalence

\[(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \sim (\kappa_1a_1, \ldots, \kappa_{n-1}a_{n-1}, \lambda_1b_1, \ldots, \lambda_{n-1}b_{n-1})\]

for \(\kappa_i \in K^*\) and \(\lambda_i = \kappa_i^2/((\kappa_{i-1}\kappa_{i+1})\) (putting \(\kappa_0 = \kappa_n = 1\)).

Compactification:

\((\mathbb{C}^*)^{n-1} \subset \mathbb{A}^{n-1}/\mu_n \subset \overline{\mathcal{L}}_n\)

\(a_i, b_j \neq 0\) allow \(a_i = 0\) \(a_i, b_j = 0\) (certain subsets)
Theorem

There is an isomorphism of stacks

\[ \mathcal{L}_n \cong \mathcal{Y}(A_{n-1}) \]

where \( \mathcal{Y}(A_{n-1}) \) is a toric orbifold associated to the Cartan matrix \( A_{n-1} \).
Toric orbifolds

Stacky fan \((N, \Sigma, \beta)\), where \(N\) lattice, \(\Sigma\) fan in \(N\), \(\beta : \mathbb{Z}^{\Sigma(1)} \to N\) map of lattices such that images of the standard base vectors generate the one-dimensional cones \(\Sigma(1)\), gives rise to a toric orbifold:

Exact sequence of abelian groups

\[
0 \longrightarrow M = N^* \overset{\beta^*}{\longrightarrow} \mathbb{Z}^{\Sigma(1)} \overset{\beta^\vee}{\longrightarrow} DG(\beta) \longrightarrow 0
\]

Exact sequence of diagonalisable group schemes

\[
1 \longrightarrow G \longrightarrow T_{\Sigma(1)} \longrightarrow T_M \longrightarrow 1
\]

Open subset \(U \subset \mathbb{A}^{\Sigma(1)}\) defined in terms of \(\Sigma\).
Operation of \(T_{\Sigma(1)}\) on \(\mathbb{A}^{\Sigma(1)}\) induces operation on \(U\).

Toric orbifold

\[
\mathcal{X}(\Sigma) = [U/G]
\]
Toric orbifolds associated to Cartan matrices

Toric orbifold $\mathcal{Y}(A_n)$ defined via stacky fan $\Upsilon(A_n) = (N, \Upsilon(A_n), \beta)$:

$N = \mathbb{Z}^n$

$\beta : \mathbb{Z}^{2n} \rightarrow \mathbb{Z}^n$ given by $n \times 2n$ matrix $(-C(A_n) \ I_n)$ where $C(A_n)$ is Cartan matrix of root system $A_n$

1-dimensional cones $\rho_1, \ldots, \rho_n, \tau_1, \ldots, \tau_n$ of $\Upsilon(A_n)$ generated by columns of matrix

maximal cones generated by $\{\rho_i : i \not\in I\} \cup \{\tau_i : i \in I\}$ for subsets $I \subseteq \{1, \ldots, n\}$
Examples

- \( \mathcal{L}_2 \cong \mathcal{Y}(A_1) \cong \mathbb{P}(1, 2) \), stacky fan \( \mathcal{Y}(A_1) \) arising from matrix \((-2 \ 1)\):

  \[
  \mathcal{Y}(A_1)
  \]

- \( \mathcal{L}_3 \cong \mathcal{Y}(A_2) \), stacky fan \( \mathcal{Y}(A_2) \) arising from matrix \(
  \begin{pmatrix}
  -2 & 1 & 1 & 0 \\
  1 & -2 & 0 & 1
  \end{pmatrix}
  \):
**Theorem**

Let $\mathcal{L}_{n,+}^{\pm}$ be the main component of the moduli stack of stable degree-$2n$-pointed chains of projective lines with involution. There is an isomorphism of stacks $\mathcal{L}_{n,+}^{\pm} \cong \mathcal{V}(C_n)$.

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**Theorem**

Let $\mathcal{L}_{n}^{0,\pm}$ be the moduli stack of stable degree-$(2n+1)$-pointed chains of projective lines with involution. There is an isomorphism of stacks $\mathcal{L}_{n}^{0,\pm} \cong \mathcal{V}(B_n)^{\text{can}}$. 
References


