Losev-Manin moduli spaces and toric varieties associated to root systems

Mark Blume

Universität Münster, SPP 1388
currently: Universität Hamburg, GRK 1670

Darstellungstheorie Schwerpunkttagung 2012

Losev-Manin moduli spaces

Theorem (Losev, Manin)

There is a fine moduli space

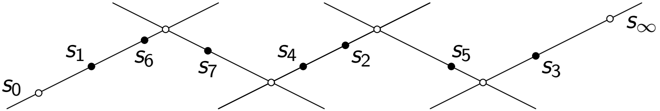
$$\overline{L}_n = \textit{fine moduli space of stable } n\textit{-pointed chains of projective lines}$$

Stable n-pointed chain of projective lines $(C, s_0, s_\infty, s_1, \ldots, s_n)$:

– C projective curve, irreducible components isomorphic to $\mathbb{P}^1$, transversal intersection at poles, shape: chain, s_0, s_∞ poles of outer components

– $s_1, \ldots, s_n$ marked points, different from poles of components, not necessarily distinct

– stability: at least one marked point on each component (therefore no nontrivial isomorphisms)



Properties of $\overline{L}_n$

– $\overline{L}_n$ is $(n-1)$-dimensional smooth projective toric variety (over $\mathbb{C}$), contains dense open torus $L_n = \{(\mathbb{P}^1, 0, \infty, s_1, \ldots, s_n)\} \subset \overline{L}_n$:

$$\begin{aligned} L_n &= (n \text{ points in } \mathbb{P}^1 \setminus \{0,\infty\})/\text{automorphisms of } (\mathbb{P}^1, 0, \infty) \\ &= (\mathbb{C}^*)^n/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-1} \end{aligned}$$

– S_n-operation by permutation of marked points.

– There are natural toric morphisms $\overline{L}_{n+1} \to \overline{L}_n$ defined by forgetting one of the marked points and stabilisation, $\overline{L}_{n+1} \to \overline{L}_n$ (with sections) forms the universal n-pointed chain.

– $\overline{L}_n$ related to similar moduli space

$\overline{M}_{0,n+2} =$ *moduli space of stable* $(n+2)$*-pointed trees of projective lines*

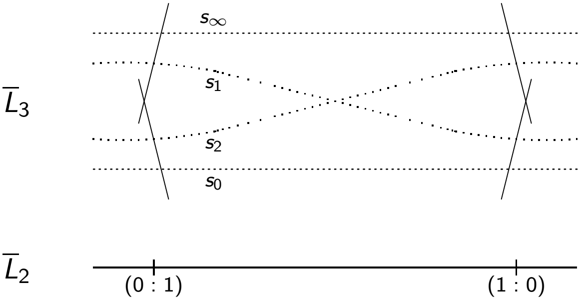
Birational morphism $\overline{M}_{0,n+2} \to \overline{L}_n$ after choice of two of the $n+2$ marked points which become s_0, s_∞.

Examples

– $\overline{L}_1 = pt.$

– $\overline{L}_2 = \mathbb{P}^1$

– $\overline{L}_3$ del Pezzo surface, $\mathbb{P}^2$ blown up in the 3 torus fixed points



Toric varieties associated with root systems

R root system

$M(R) = \langle R \rangle_{\mathbb{Z}}$ root lattice
$N(R) = M(R)^*$ dual lattice

For a set of simple roots $S \subset R$: Weyl chamber

$$\sigma_S = S^\vee = \{v \in N(R)_{\mathbb{Q}} : \forall\, \alpha \in S \colon \langle \alpha, v \rangle \geq 0\} \subset N(R)_{\mathbb{Q}}$$

Definition

$\Sigma(R)$ fan of Weyl chambers
$X(R)$ toric variety for $\Sigma(R)$

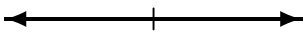
Properties and examples

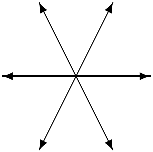
– $X(R)$ smooth projective toric variety (over $\mathbb{C}$) of dimension $\dim X(R) = \operatorname{rk} R$

– Operation of Weyl group $W(R)$

Examples: fans for $R = A_1, A_2$

$\Sigma(A_1)$





$\Sigma(A_2)$

Proposition

(Functorial property) Let R, R' be root systems. Linear maps $\mu : M(R')_{\mathbb{Q}} \to M(R)_{\mathbb{Q}}$ such that $\mu(R') \subseteq \{a\alpha : \alpha \in R, a \in \mathbb{Z}\}$ induce toric morphisms $X(\mu)\colon X(R) \to X(R')$.

$X(R)$ as moduli space

Theorem

There is an isomorphism
$$\overline{L}_n \;\cong\; X(A_{n-1})$$

Steps of proof:

– compare the moduli functor

$$\begin{array}{rccl} \overline{L}_n : & (\text{schemes})^\circ & \to & (\text{sets}) \\ & Y & \mapsto & \left\{ \begin{array}{l} (\pi\colon C \to Y, s_0, s_\infty, s_1, \ldots, s_n) \\ \pi \textit{ proper flat, } s_0, s_\infty, s_i \textit{ sections,} \\ \textit{geom. fibres stable } n\textit{-pointed chains} \end{array} \right\} / \sim \end{array}$$

to the functor of the toric variety $\mathrm{Mor}(\,\cdot\,, X(A_{n-1}))$.

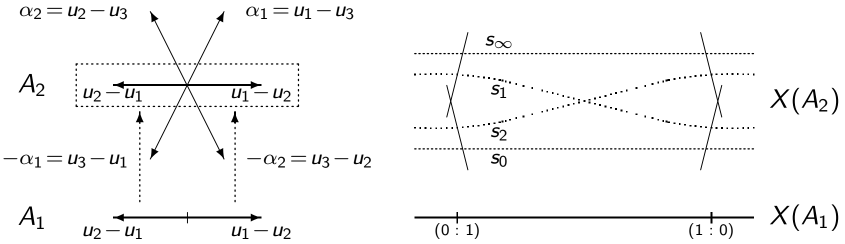
– find appropriate description of functor of $X(A_{n-1})$

– construct universal curve over $X(A_{n-1})$ in terms of maps of root systems:

$A_{n-1} \to A_n$ gives rise to proper surjective toric morphism $X(A_n) \to X(A_{n-1})$

projections $A_n \to A_{n-1}(\subset A_n)$ along pairs of opposite roots $\{\pm\alpha_i\} \subseteq A_n \setminus A_{n-1}$ give rise to sections $s_i \colon X(A_{n-1}) \to X(A_n)$

Example:



Theorem

$X(A_{n-1})$ with universal curve $X(A_n) \to X(A_{n-1})$ is a fine moduli space of n-pointed chains of projective lines.

Functor of $X(R)$

R root system

Use closed embedding

$$X(R) \quad \to \quad \prod_{A_1 \cong R' \subseteq R} X(R') \;\cong\; \prod_{\{\pm\alpha\} \subseteq R} \mathbb{P}^1$$

determined by surjection of root systems $\prod_{A_1 \cong R' \subseteq R} R' \to R$, and functor of $\mathbb{P}^1$ to obtain

$$X(R)(Y) \;=\; \left\{ \begin{array}{l} (\mathscr{L}_{\{\pm\beta\}}, \{t_\beta, t_{-\beta}\})_{\{\pm\beta\} \subseteq R} \textit{ such that} \\ t_\alpha t_\beta t_{-\gamma} = t_{-\alpha} t_{-\beta} t_\gamma \textit{ for all root subsystems} \\ A_2 \cong \{\pm\alpha, \pm\beta, \pm(\gamma = \alpha + \beta)\} \subseteq R \end{array} \right\} / \sim$$

Comparison with moduli functor

$$
\begin{array}{rcl}
X(A_{n-1})(Y) & \longleftrightarrow & \{\textit{stable } n\textit{-pointed chains over } Y\} \\
(\mathscr{L}_{\{\pm\beta\}}, \{t_\beta, t_{-\beta}\})_{\{\pm\beta\}\subseteq A_{n-1}} & \mapsto & C \ \subseteq \ \prod_{\{\pm\alpha\}\subseteq A_n\setminus A_{n-1}} \mathbb{P}^1_Y
\end{array}
$$

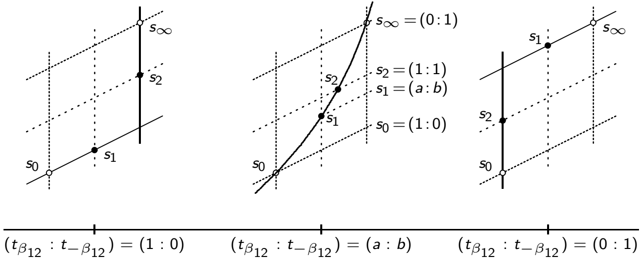
def. by homogeneous equations

$t_{\beta_{ij}} z_{\alpha_j} z_{-\alpha_i} = t_{-\beta_{ij}} z_{-\alpha_j} z_{\alpha_i}$

for $\beta_{ij} = \alpha_i - \alpha_j$

$(\beta_{ij} \in A_{n-1},\ \alpha_i, \alpha_j \in A_n \setminus A_{n-1})$

and extra equations for the sections



Generalisations to other root systems

Root systems of type B: consider stable $(2n+1)$-pointed chains with involution $(C, I, s_-, s_+, s_0, s_1^\pm, \dots, s_n^\pm)$, where C chain of projective lines, s_-, s_+ poles of outer components, $I\colon C \to C$ involution such that $I(s_-) = s_+$, $2n+1$ sections $s_0, s_i^\pm$ that satisfy $I(s_0) = s_0$, $I(s_i^-) = s_i^+$.

Theorem

$X(B_n)$ with universal curve $X(B_{n+1}) \to X(B_n)$ is the fine moduli space $\overline{L}_n^{0,\pm}$ of $(2n+1)$-pointed chains with involution.

Root systems of type C: consider $2n$-pointed chains with involution. $X(C_{n+1}) \to X(C_n)$ also has nonreduced fibres, but we have

Theorem

$X(C_n)$ is the fine moduli space $\overline{L}_n^{\pm}$ of $2n$-pointed chains with involution.

Root systems of type D: fibres of $X(D_{n+1}) \to X(D_n)$ may have higher dimensional components.

The stacks $\overline{\mathcal{L}}_n$

Consider chains of projective lines (C, s_0, s_∞) with subschemes $S \subset C$ finite of degree n, not meeting the poles and intersection points of components, but meeting every component. Call (C, s_0, s_∞, S) a degree-n-pointed chain.

An n-pointed chain $(C, s_0, s_\infty, s_1, \ldots, s_n)$ gives rise to a degree-n-pointed chain (C, s_0, s_∞, S) by forgetting the labels of the marked points.

(C, s_0, s_∞, S) may have nontrivial automorphisms:
- automorphism group of (C, s_0, s_∞): $(\mathbb{C}^*)^l$, l length of chain
- automorphism group of (C, s_0, s_∞, S): finite subggroup of $(\mathbb{C}^*)^l$
- for example $(\mathbb{P}^1, 0, \infty, S)$, S given by $z_0^k - z_1^k = 0$, has automorphism group μ_k.

$$\overline{\mathcal{L}}_n = \textit{moduli stack of degree-n-pointed chains } (C, s_0, s_\infty, S)$$

Category $\overline{\mathcal{L}}_n$ of degree-n-pointed chains:

- objects: $(C \to Y, s_0, s_\infty, S)$, Y scheme
- morphisms: cartesian diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

compatible with the additional structure.

Proposition

The category $\overline{\mathcal{L}}_n$ is a category fibred in groupoids over the category of schemes. It forms a stack over the fpqc site of schemes ($\Rightarrow$ stack also in fppf, étale topology) with representable, finite diagonal. Over fields of characteristic 0 the diagonal is unramified.

Operation of the symmetric group and relation to $\overline{L}_n$

Proposition

There is a morphism $\overline{L}_n \to \overline{\mathcal{L}}_n$, $(C, s_0, s_\infty, s_1, \ldots, s_n) \mapsto (C, s_0, s_\infty, \sum_i s_i)$. It is faithfully flat and finite of degree $n!$.

Operation of $S_n = W(A_{n-1})$ on $\overline{L}_n$ by permuting the sections of n-pointed chains.

Quotient morphism factors as $\overline{L}_n \to \overline{\mathcal{L}}_n \to \overline{L}_n / S_n$.

$\overline{\mathcal{L}}_n \to \overline{L}_n / S_n$ is coarse moduli space.

Substack of irreducible pointed chains

Substack $[\mathbb{A}^{n-1}/\mu_n] = \{(\mathbb{P}^1, s_0, s_\infty, S)\} \subset \overline{\mathcal{L}}_n$ of irreducible curves:

$$\begin{array}{rcl} (\mathbb{P}^1, s_0, s_\infty, S) & \longleftrightarrow & \textit{polynomials } \prod_{i=1}^n (y - s_i),\ s_i \in \mathbb{C}^*, \textit{ up to} \\ & & \textit{multiplication by a factor } \lambda \in \mathbb{C}^* \\ & \longleftrightarrow & \textit{polynomials } x^n + a_{n-1}x^{n-1} + \ldots + a_1 x_1 + 1, \\ & & a_i \in \mathbb{C}, \textit{ up to multiplication of } x \textit{ by} \\ & & \textit{an } n\textit{-th root of unity} \end{array}$$

$[\mathbb{A}^{n-1}/\mu_n]$ contains dense open torus $(\mathbb{C}^*)^{n-1}$ (nonzero coefficients a_i).

Points of $(\mathbb{C}^*)^{n-1}$ given by $(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \in (\mathbb{C}^*)^{2(n-1)}$ up to equivalence
$(a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}) \sim (\kappa_1 a_1, \ldots, \kappa_{n-1} a_{n-1}, \lambda_1 b_1, \ldots, \lambda_{n-1} b_{n-1})$
for $\kappa_i \in K^*$ and $\lambda_i = \kappa_i^2/(\kappa_{i-1}\kappa_{i+1})$ (putting $\kappa_0 = \kappa_n = 1$).

Compactification:

$$\begin{array}{ccccc} (\mathbb{C}^*)^{n-1} & \subset & [\mathbb{A}^{n-1}/\mu_n] & \subset & \overline{\mathcal{L}}_n \\ a_i, b_j \neq 0 & & \textit{allow } a_i = 0 & & \textit{allow } a_i, b_j = 0 \textit{ (certain subsets)} \end{array}$$

$\overline{\mathcal{L}}_n$ as toric stack

Theorem

There is an isomorphism of stacks

$$\overline{\mathcal{L}}_n \;\cong\; \mathcal{Y}(A_{n-1})$$

where $\mathcal{Y}(A_{n-1})$ is a toric orbifold associated to the Cartan matrix A_{n-1}.

Toric orbifolds

Stacky fan (N, Σ, β), where N lattice, Σ fan in N, $\beta\colon \mathbb{Z}^{\Sigma(1)} \to N$ map of lattices such that images of the standard base vectors generate the one-dimensional cones $\Sigma(1)$, gives rise to a toric orbifold:
Exact sequence of abelian groups

$$0 \longrightarrow M = N^* \xrightarrow{\beta^*} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\beta^\vee} DG(\beta) \longrightarrow 0$$

Exact sequence of diagonalisable group schemes

$$1 \longrightarrow G \longrightarrow T_{\Sigma(1)} \longrightarrow T_M \longrightarrow 1$$

Open subset $U \subset \mathbb{A}^{\Sigma(1)}$ defined in terms of Σ.
Operation of $T_{\Sigma(1)}$ on $\mathbb{A}^{\Sigma(1)}$ induces operation on U.

Toric orbifold

$$\mathcal{X}(\Sigma) = [U/G]$$

Toric orbifolds associated to Cartan matrices

Toric orbifold $\mathcal{Y}(A_n)$ defined via stacky fan $\boldsymbol{\Upsilon}(A_n) = (N, \Upsilon(A_n), \beta)$:

$N = \mathbb{Z}^n$

$\beta \colon \mathbb{Z}^{2n} \to \mathbb{Z}^n$ given by $n \times 2n$ matrix $(-C(A_n)\ I_n)$ where $C(A_n)$ is Cartan matrix of root system A_n

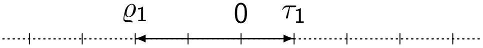
1-dimensional cones $\varrho_1, \ldots, \varrho_n, \tau_1, \ldots, \tau_n$ of $\Upsilon(A_n)$ generated by columns of matrix

maximal cones generated by $\{\varrho_i : i \notin I\} \cup \{\tau_i : i \in I\}$ for subsets $I \subseteq \{1, \ldots, n\}$

Examples

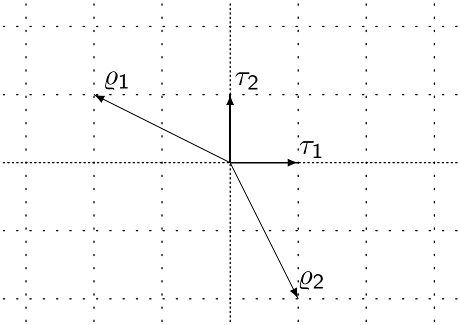
- $\overline{\mathcal{L}}_2 \cong \mathcal{Y}(A_1) \cong \mathbb{P}(1,2)$, stacky fan $\boldsymbol{\Upsilon}(A_1)$ arising from matrix $\begin{pmatrix} -2 & 1 \end{pmatrix}$:

$\boldsymbol{\Upsilon}(A_1)$



- $\overline{\mathcal{L}}_3 \cong \mathcal{Y}(A_2)$, stacky fan $\boldsymbol{\Upsilon}(A_2)$ arising from matrix $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$:

$\boldsymbol{\Upsilon}(A_2)$



Generalisations to other root systems

Theorem

Let $\overline{\mathscr{L}}^{\pm}_{n,+}$ be the main component of the moduli stack of stable degree-$2n$-pointed chains of projective lines with involution. There is an isomorphism of stacks $\overline{\mathscr{L}}^{\pm}_{n,+} \cong \mathcal{Y}(C_n)$.

Theorem

Let $\overline{\mathscr{L}}^{0,\pm}_{n}$ be the moduli stack of stable degree-$(2n+1)$-pointed chains of projective lines with involution. There is an isomorphism of stacks $\overline{\mathscr{L}}^{0,\pm}_{n} \cong \mathcal{Y}(B_n)^{\mathrm{can}}$.

References

- V. Batyrev, M. Blume, *The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces*, Tohoku Math. J. 63 (2011), arXiv:0911.3607.
- V. Batyrev, M. Blume, *On generalisations of Losev-Manin moduli spaces for classical root systems*, Pure and Applied Mathematics Quarterly 7 (2011), 1053–1084, arXiv:0912.2898.
- M. Blume, *Toric orbifolds associated to Cartan matrices*. arXiv:1110.2761.
- A. Losev, Yu. Manin, *New Moduli Spaces of Pointed Curves and Pencils of Flat Connections*, Michigan Math. J. 48 (2000), 443–472, arXiv:math/0001003.