Mutation classes of quivers with constant number of arrows and derived equivalences

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Motivation

The *BGP reflection* is an operation on *quivers* defined at specific vertices: *sinks* and *sources*.

Given a quiver Q and a sink/source s in Q, it produces a new quiver, denoted $\sigma_s(Q)$.

It has the following properties:

- Combinatorially,
  Q and $\sigma_s(Q)$ have the same number of arrows.
- Algebraically,
  the *path algebra* of Q and that of $\sigma_s(Q)$ are *derived equivalent*.

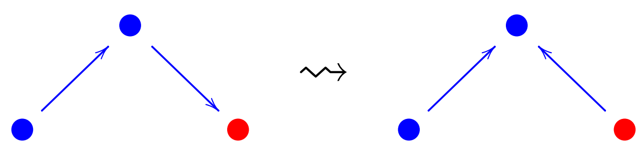
Reflection [Bernstein-Gelfand-Ponomarev 1973]

A *quiver* Q is a directed graph.

A vertex s of Q is a ...
... *sink*, if there are no arrows starting at s;
... *source*, if there are no arrows ending at s.

Let s be a sink or a source in Q.
The *BGP reflection* of Q at s is a new quiver $\sigma_s(Q)$,
obtained from Q by inverting all the arrows incident to s.

Example.



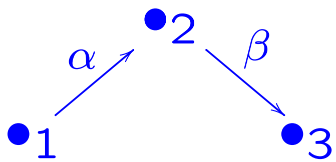
Path algebras of quivers

K – field (or commutative ring), Q – quiver

The *path algebra* KQ is the K-algebra

- spanned by all paths in Q,
- with multiplication given by composition of paths.

Example.

$Q =$ 

$$KQ = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$e_1, e_2, e_3, \alpha, \beta, \alpha\beta$

$\alpha \cdot \beta = \alpha\beta \quad \beta \cdot \alpha = 0$

Derived equivalence

R – ring, $\text{Mod}\, R$ – the category of (right) R-modules.

The *derived category* $\mathcal{D}(\text{Mod}\, R)$ is obtained from the category of complexes of R-modules by formally inverting all the quasi-isomorphisms. It is a *triangulated category*.

A *quasi-isomorphism* is a morphism of complexes $f : K \to L$ inducing isomorphisms $H^i f : H^i K \xrightarrow{\sim} H^i L$ on the cohomology for all $i \in \mathbb{Z}$.

Two rings R, S are *Morita equivalent* if $\text{Mod}\, R \simeq \text{Mod}\, S$. They are *derived equivalent* if $\mathcal{D}(\text{Mod}\, R) \simeq \mathcal{D}(\text{Mod}\, S)$.

Motivation (revisited)

The *BGP reflection* is an operation on *quivers* defined at specific vertices: *sinks* and *sources*.

Given a quiver Q and a sink/source s in Q, it produces a new quiver, denoted $\sigma_s(Q)$.

It has the following properties:

- Combinatorially,
  Q and $\sigma_s(Q)$ have the same number of arrows.
- Algebraically,
  the *path algebra* of Q and that of $\sigma_s(Q)$ are *derived equivalent* [BGP 1973, Happel]

Generalizations to arbitrary vertices

Reflection can only be done at sinks or sources.

Combinatorially, it is generalized to arbitrary vertices in the form of *mutation* of quivers [Fomin-Zelevinsky 2002].

Algebraically, path algebras of quivers are generalized by *Jacobian algebras* of *quivers with potential* [Derksen-Weyman-Zelevinsky 2008], for which there is also a notion of mutation.

However, for these mutations,

- The number of arrows is not always preserved,
- The Jacobian algebras may not be derived equivalent.

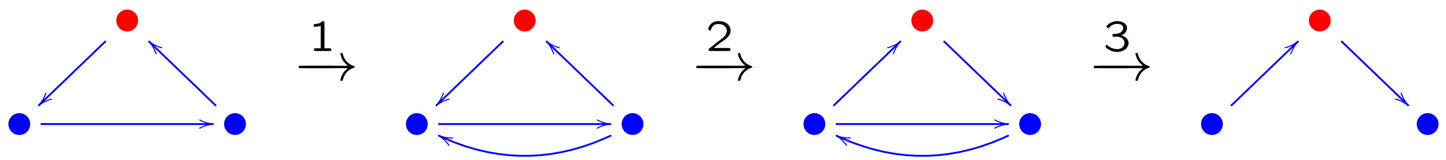
Quiver mutation [Fomin-Zelevinsky]

Q – quiver without *loops* () and *2-cycles* (),
k – any vertex in Q.

The *mutation* of Q at k, denoted $\mu_k(Q)$, is a new quiver obtained from Q as follows:

1. For any pair $i \xrightarrow{\alpha} k \xrightarrow{\beta} j$, add new arrow $i \xrightarrow{[\alpha\beta]} j$,
2. Invert the incoming and outgoing arrows at k,
3. Remove a maximal set of 2-cycles.

Example.

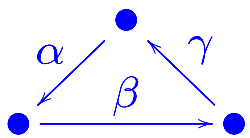


Quivers with potential (QP)

A *potential* W is a linear combination of cycles in KQ.

The *Jacobian algebra* $\mathcal{P}(Q,W)$ is the quotient of KQ by the ideal generated by all the directional derivatives of W.

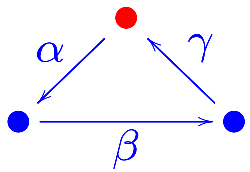
Example.

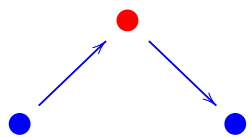


$W = \alpha\beta\gamma \quad \mathcal{P}(Q,W) = KQ/(\alpha\beta, \beta\gamma, \gamma\alpha)$

$Q \rightsquigarrow \mu_k(Q)$ admits a "good" extension to *QP mutation* $(Q,W) \rightsquigarrow \mu_k(Q,W)$ [Derksen-Weyman-Zelevinsky 2008].

Example.

(

$, W = \alpha\beta\gamma$) $\rightsquigarrow$ ( $, W' = 0$)

Discussion

Two properties that a (QP) mutation may possibly have:

- Combinatorially, Q and $\mu_k(Q)$ have the same number of arrows.
- Algebraically, the Jacobian algebra of (Q, W) and that of $\mu_k(Q, W)$ are derived equivalent.

In order to have these properties for *arbitrary quivers* we had to restrict to *specific vertices* (sinks/sources).

If we want to have these properties for mutations at *arbitrary vertices* we need to restrict to *specific quivers*.

This motivates the following two problems . . .

Two Problems

Combinatorial problem: Find all *quivers* such that:

(C) Performing arbitrary sequences of *mutations* does not change their *number of arrows*.

Algebraic Problem: Find all *QPs* such that:

(A) Performing arbitrary sequences of *QP mutations* does not change the *derived equivalence* class of their Jacobian algebras.

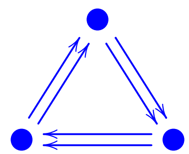
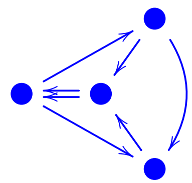
In this talk, we will . . .

- . . . find all solutions of **(C)**.
- . . . show that any solution of **(C)** $\rightsquigarrow$ solution of **(A)**.

Some quivers with property **(C)**

Combinatorial problem: Find all *quivers* such that:

(C) Performing arbitrary sequences of *mutations* does not change their *number of arrows*.

No. of vertices	≤ 2	3	4	5
Quivers satisfying **(C)**	All			None

Some QPs with property **(A)**

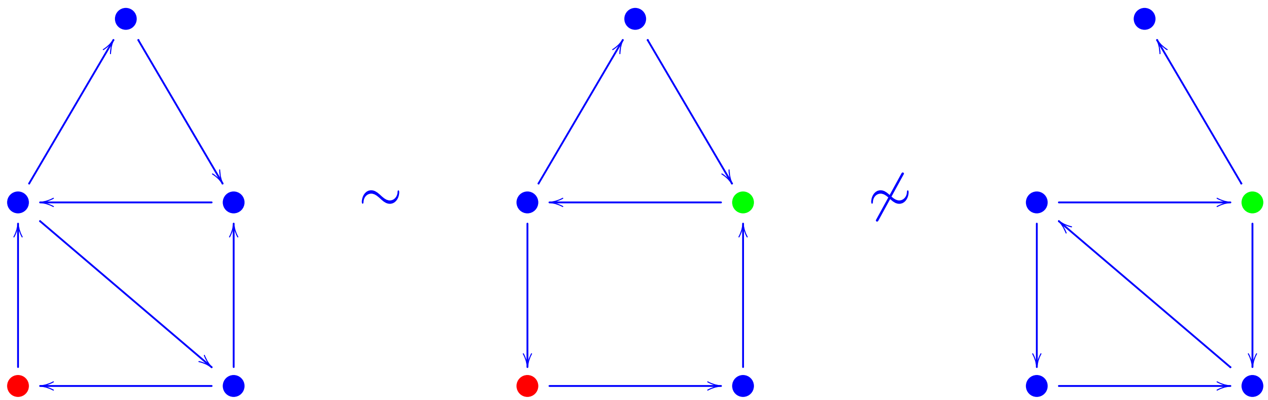
Algebraic Problem: Find all *QPs* such that:

(A) Performing arbitrary sequences of *QP mutations* does not change the *derived equivalence* class of their Jacobian algebras.

(Q, W) satisfying **(A)**		$\dim \mathcal{P}(Q, W)$
≤ 2 vertices	All	$< \infty$
acyclic, ≥ 3 vertices	None	$< \infty$
3-Calabi-Yau [Keller-Yang 2011]	All	∞

No a-priori relation between properties **(C)** and **(A)**

Consider mutations at the red and green vertices:



The Jacobian algebras are cluster-tilted of Dynkin type D_5

[Bastian-Holm-L. 2010]

Quivers with finite mutation class

If a quiver satisfies **(C)**, then its *mutation class* is finite.

Theorem. [Felikson-Shapiro-Tumarkin 2008]
The connected quivers whose mutation class is finite are:

- Those with ≤ 2 vertices; or
- The quivers arising from *surface triangulations*; or
- The 11 *exceptional quivers* $E_{6,7,8}$, $\widehat{E}_{6,7,8}$, $\widehat{\widehat{E}}_{6,7,8}$, $X_{6,7}$ (and members in their mutation classes).

Remark. No exceptional quiver satisfies property **(C)**.

Quivers from surface triangulations

A *marked bordered surface* is a pair (S, M) consisting of:

- a compact, connected, oriented surface S (possibly with boundary),
- a finite set $M \subset S$ of *marked points*, containing at least one point on each boundary component of S.

(S, M) is *unpunctured* if $M \subset \partial S$.

Facts. [Fomin-Shapiro-Thurston 2008]

triangulation	$\rightsquigarrow$	*adjacency quiver*
flip	$\rightsquigarrow$	mutation
all triangulations of (S, M)	$\rightsquigarrow$	finite mutation class

Solution to the combinatorial problem

Theorem [L]. The connected quivers with property

(C) Performing arbitrary sequences of *mutations* does not change their *number of arrows*.

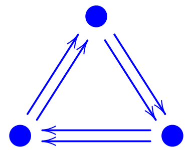
are:

1. Quivers with ≤ 2 vertices;
2. Adjacency quivers of triangulations of a surface *without boundary* and *one puncture*; ($\rightsquigarrow \mathcal{Q}_{g,0}$)
3. Adjacency quivers of triangulations of a surface *with boundary* and exactly *one marked point* on each boundary component (no punctures). ($\rightsquigarrow \mathcal{Q}_{g,b}$)

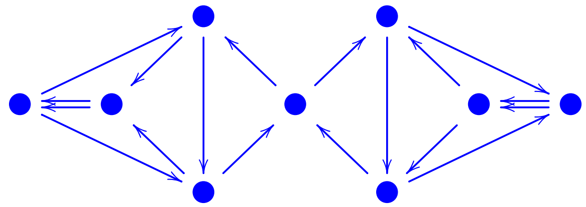
The mutation classes are therefore parameterized by the *genus* g and the number of *boundary* components b.

Representative quivers in $\mathcal{Q}_{g,0}$ for $g = 1, 2, 3, 4$

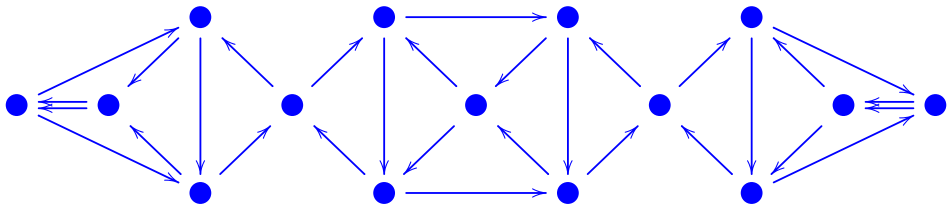
$(1, 0)$



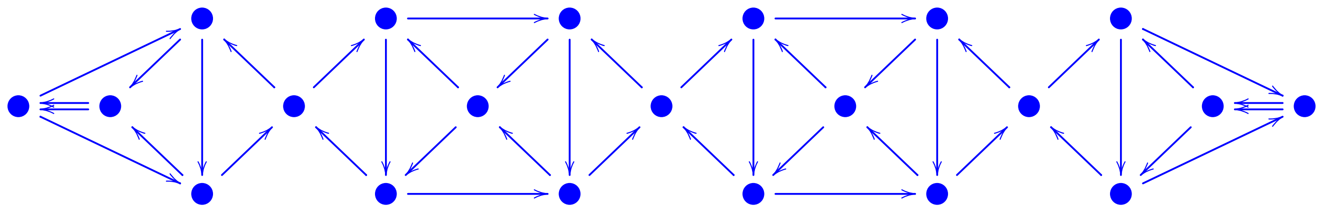
$(2, 0)$



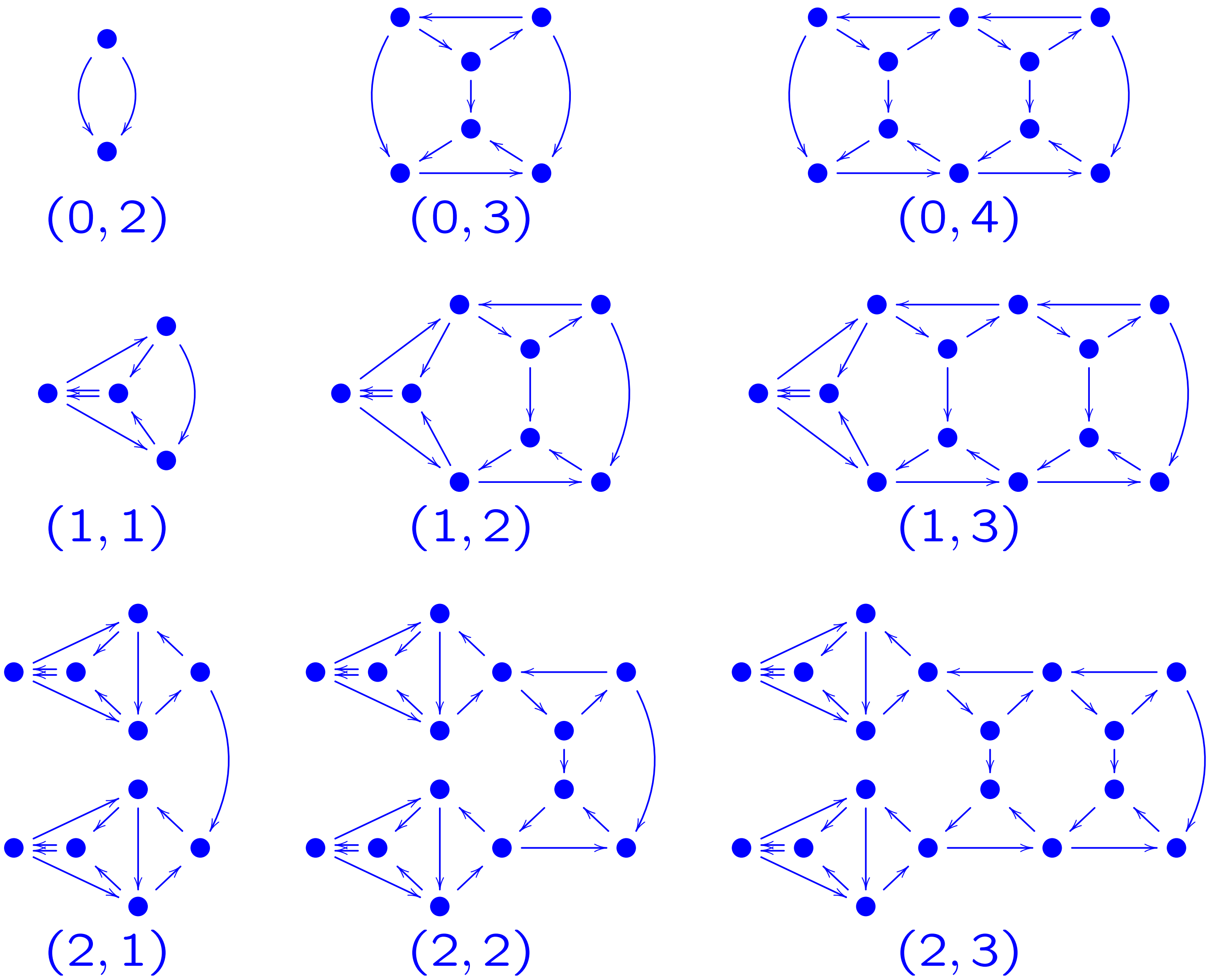
$(3, 0)$



$(4, 0)$



Representative quivers for some $\mathcal{Q}_{g,b}$ $(b \geq 1)$



Numerical properties of $\mathcal{Q}_{g,0}$, $\mathcal{Q}_{g,b}$

	No. of vertices	No. of arrows
$\mathcal{Q}_{g,0}$	$6g-3$	$12g-6$
$\mathcal{Q}_{g,b}$	$6(g-1)+4b$	$12(g-1)+7b$

Moreover, for any quiver in $\mathcal{Q}_{g,0}$, at any vertex:

$$(\text{in-degree}, \text{out-degree}) = (2,2).$$

Corollary [L]. Let $n > 1$. There is a connected quiver on n vertices with property **(C)** if and only if $n \not\equiv 1, 5 \pmod 6$.

Potentials on adjacency quivers

Two kinds of oriented cycles in adjacency quivers:

- 3-cycles corresponding to *internal triangles*,
- cycles around *punctures*,

give rise to a potential $W = W_{\triangle} + W_P$. [Labardini 2009]

In the *unpunctured* case, $W = W_{\triangle}$ and the Jacobian algebra is finite-dimensional *gentle* [ABCP 2010].

When there is only *one puncture* and *no boundary*, we also set $W = W_{\triangle}$, ignoring the term W_P.
The Jacobian algebra is infinite-dimensional and *locally gentle* in the sense of [Bessenrodt-Holm 2008].

Results on the algebraic problem

Theorem [L]. Any QP in $\mathcal{Q}_{g,b}$ $(b \geq 1)$ has property

(A) Performing arbitrary sequences of *QP mutations* does not change the *derived equivalence* class of the Jacobian algebras.

Moreover, the Jacobian algebras corresponding to a class $\mathcal{Q}_{g,b}$ form a *complete* derived equivalence class of finite-dimensional algebras.

Remark. Explicit descriptions of complete derived equivalence classes of algebras are quite rare since it is hard to control all possible tilting complexes.

Other instances where such description is possible include the Brauer tree algebras with fixed numerical parameters [König-Zimmermann, Rickard].

Results on the algebraic problem

Theorem [L]. (continued)

- Any QP in $\mathcal{Q}_{g,0}$ satisfies **(A)**. Its Jacobian algebra is infinite dimensional and *not 3-Calabi-Yau*.
- If a QP arising from a triangulation of an *unpunctured* surface satisfies **(A)**, then it must belong to one of the classes $\mathcal{Q}_{g,b}$ $(b \geq 1)$.
- A QP in an *exceptional* mutation class satisfies **(A)** if and only if it belongs to the class of $\widehat{\widehat{E}}_6$ or X_6. In these cases the Jacobian algebras are finite-dimensional.

Corollary. For any quiver Q satisfying **(C)** there is a naturally associated potential W such that (Q, W) satisfies **(A)**.

The key proposition

Relating the combinatorial and algebraic properties for a large class of quivers with potentials.

Proposition [L]. For (Q, W) arising from a triangulation of a marked *unpunctured* surface and a vertex k in Q, TFAE:

(i) (in-degree, out-degree) of k is not $(1, 1)$;

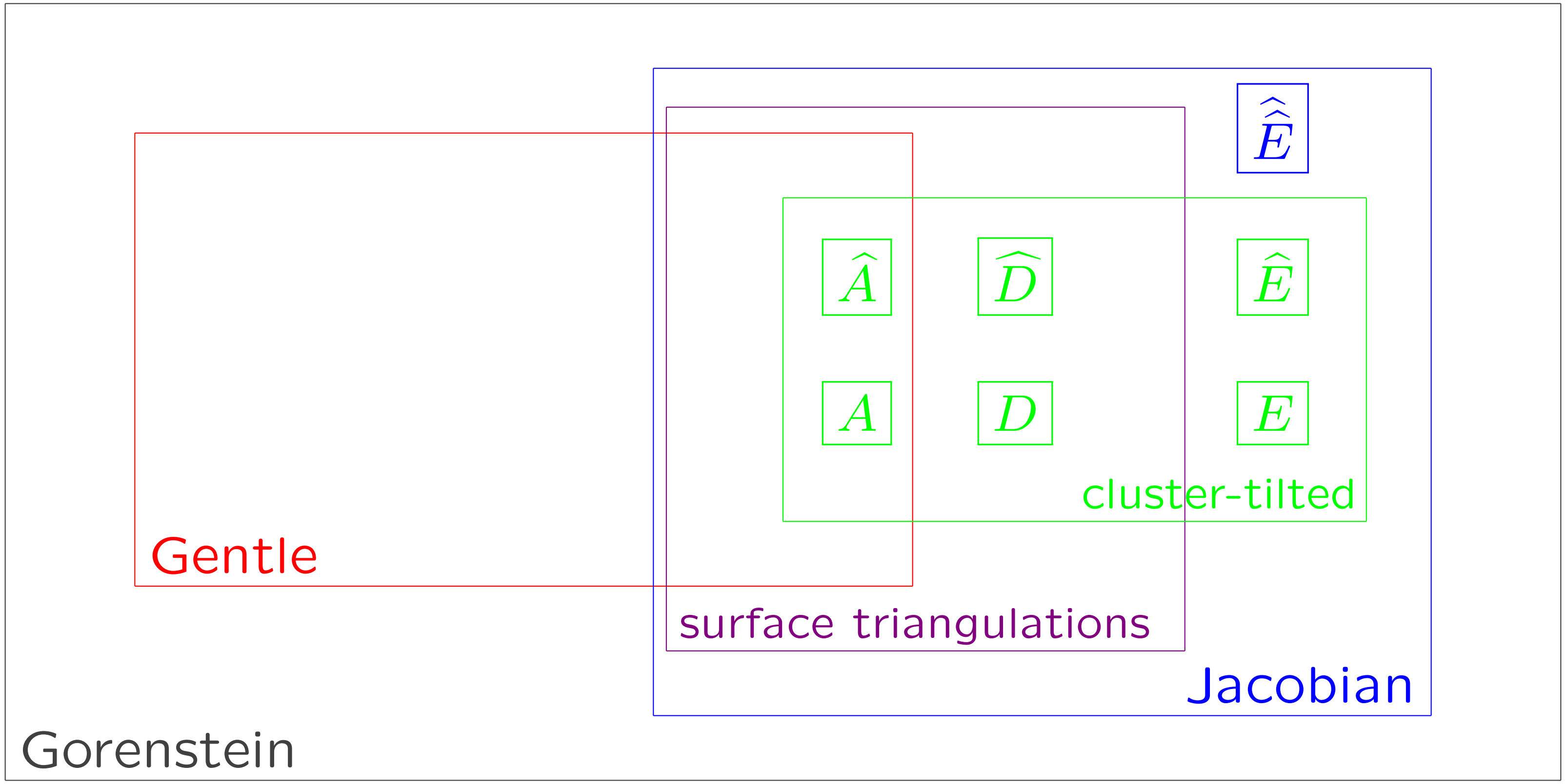
(ii) Q and $\mu_k(Q)$ have the *same number of arrows*;

(iii) $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_k(Q, W))$ are *derived equivalent*;

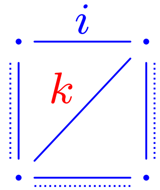
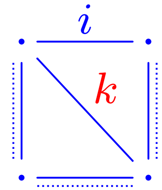
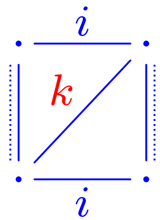
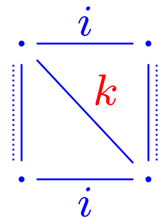
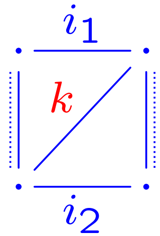
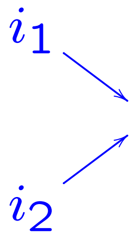
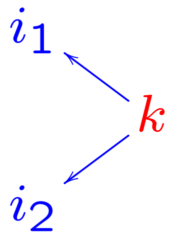
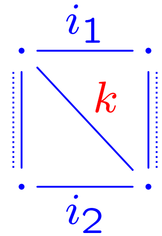
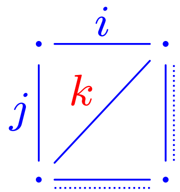
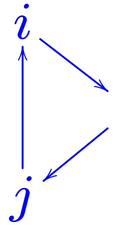
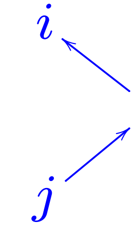
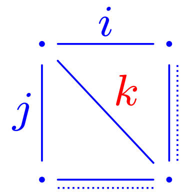
(iv) The QP mutation of (Q, W) at k is *good*.

Application [L]. Derived equivalence classification of the gentle algebras arising from surface triangulations.

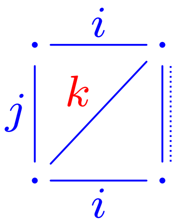
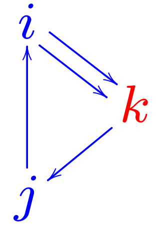
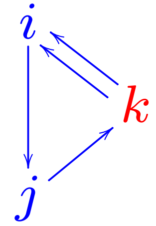
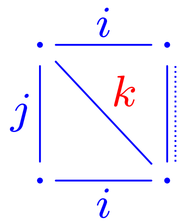
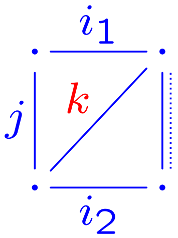
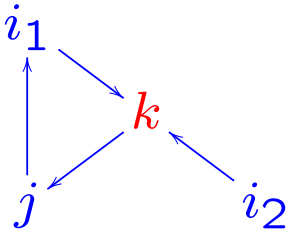
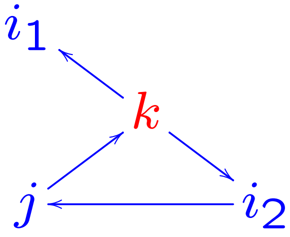
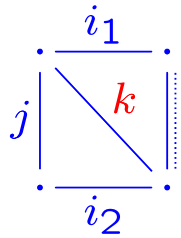
A map of finite-dimensional algebras



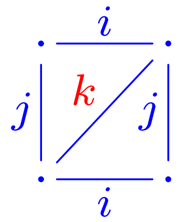
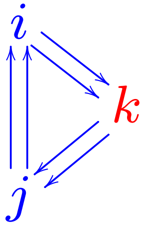
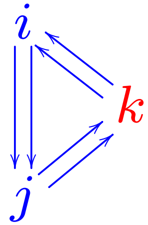
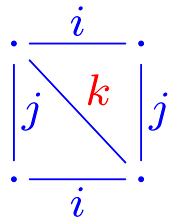
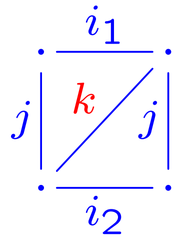
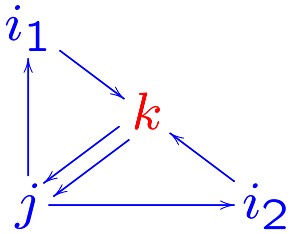
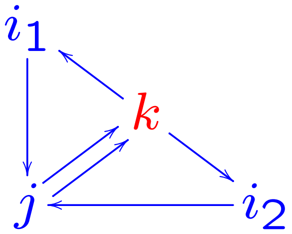
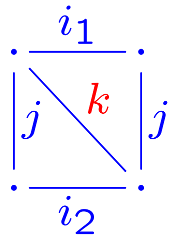
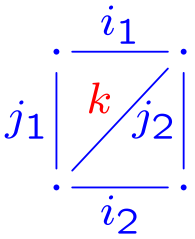
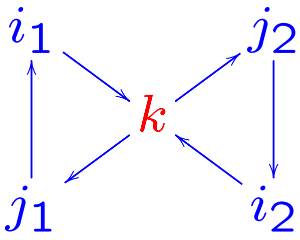
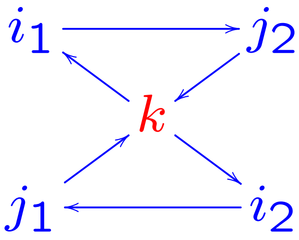
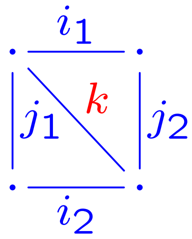
Neighborhoods of k, valency ≤ 2

1		$i \longrightarrow k$	μ_k^-	μ_k^+	$i \longleftarrow k$	
$2a$		$i \Longrightarrow k$	μ_k^-	μ_k^+	$i \Longleftarrow k$	
$2b$			μ_k^-	μ_k^+		
$2c$		 $a_{ji} \geq 1$	none	μ_k^-, μ_k^+	 $a_{ij} = 0$	

Neighborhoods of k, valency 3

$3a$		 $a_{ji} = 1$	μ_k^-	μ_k^+	 $a_{ij} = 1$	
$3b$		 $a_{ji_1} \geq 1$ $a_{ji_2} = 0$	μ_k^-	μ_k^+	 $a_{i_1j} = 0$ $a_{i_2j} \geq 1$	

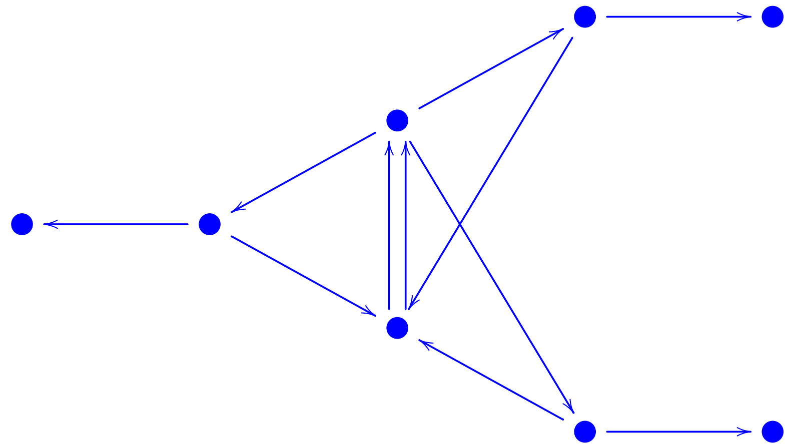
Neighborhoods of k, valency 4 (all sides are arcs)

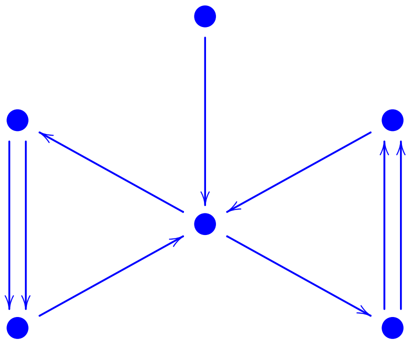
$4a$				
$4b$		 $a_{ji_1} = a_{ji_2} = 1$	 $a_{i_1j} = a_{i_2j} = 1$	
$4c$		 $a_{j_1i_1}, a_{j_2i_2} \geq 1$ $a_{j_1i_2} = a_{j_2i_1} = 0$	 $a_{i_1j_2}, a_{i_2j_1} \geq 1$ $a_{i_1j_1} = a_{i_2j_2} = 0$	

Both μ_k^- and μ_k^+ are always defined.

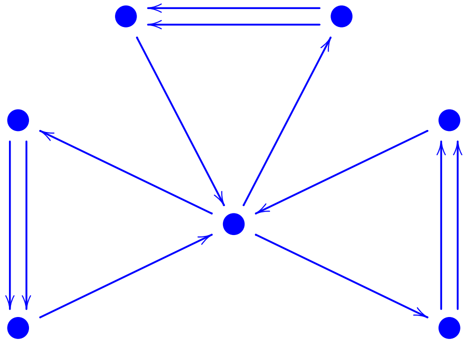
Some exceptional quivers

$\widehat{\widehat{E}}_6$





X_6



X_7