Mirror symmetry of singularities

(joint work with Atsushi Takahashi)

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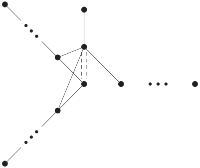
Arnold's strange duality 1

14 exceptional unimodular hypersurface singularities in $\mathbb{C}^3$ related to Schwarz triangular groups $\Gamma(\alpha_1, \alpha_2, \alpha_3) \subset \mathrm{PSL}(2;\mathbb{R})$

- Dolgachev numbers $\mathrm{Dol}(X) = (\alpha_1, \alpha_2, \alpha_3)$, $\frac{\pi}{\alpha_1}, \frac{\pi}{\alpha_2}, \frac{\pi}{\alpha_3}$ angles of hyperbolic triangle
- Gabrielov numbers $\mathrm{Gab}(X) = (\gamma_1, \gamma_2, \gamma_3)$,



Coxeter-Dynkin diagram

Arnold's strange duality: $X \leftrightarrow X^*$

- $\mathrm{Dol}(X) = \mathrm{Gab}(X^*)$
- $\mathrm{Gab}(X) = \mathrm{Dol}(X^*)$

Arnold's strange duality 2

Name	$\mathrm{Dol}(X)$	$\mathrm{Gab}(X)$	Dual
E_{12}	$2, 3, 7$	$2, 3, 7$	E_{12}
E_{13}	$2, 4, 5$	$2, 3, 8$	Z_{11}
E_{14}	$3, 3, 4$	$2, 3, 9$	Q_{10}
Z_{11}	$2, 3, 8$	$2, 4, 5$	E_{13}
Z_{12}	$2, 4, 6$	$2, 4, 6$	Z_{12}
Z_{13}	$3, 3, 5$	$2, 4, 7$	Q_{11}
Q_{10}	$2, 3, 9$	$3, 3, 4$	E_{14}
Q_{11}	$2, 4, 7$	$3, 3, 5$	Z_{13}
Q_{12}	$3, 3, 6$	$3, 3, 6$	Q_{12}
W_{12}	$2, 5, 5$	$2, 5, 5$	W_{12}
W_{13}	$3, 4, 4$	$2, 5, 6$	S_{11}
S_{11}	$2, 5, 6$	$3, 4, 4$	W_{13}
S_{12}	$3, 4, 5$	$3, 4, 5$	S_{12}
U_{12}	$4, 4, 4$	$4, 4, 4$	U_{12}

E.-Wall extension

8 bimodal series $\longleftrightarrow$ 8 triangle ICIS in $\mathbb{C}^4$

quasihomogeneous heads related to
quadrilateral groups $\Gamma[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$

Series	Head	$\mathrm{Dol}(X)$	$\mathrm{Gab}(X)$	Dual
$J_{3,k}$	$J_{3,0}$	$2,2,2,3$	$2,3,10$	J'_9
$Z_{1,k}$	$Z_{1,0}$	$2,2,2,4$	$2,4,8$	J'_{10}
$Q_{2,k}$	$Q_{2,0}$	$2,2,2,5$	$3,3,7$	J'_{11}
$W_{1,k}$	$W_{1,0}$	$2,3,2,3$	$2,6,6$	K'_{10}
$W^\sharp_{1,k}$		$2,2,3,3$	$2,5,7$	L_{10}
$S_{1,k}$	$S_{1,0}$	$2,3,2,4$	$3,5,5$	K'_{11}
$S^\sharp_{1,k}$		$2,2,3,4$	$3,4,6$	L_{11}
$U_{1,k}$	$U_{1,0}$	$2,3,3,3$	$4,4,5$	M_{11}

Invertible polynomials 1

- A quasihomogeneous polynomial f in n variables is *invertible*

  $$:\Longleftrightarrow f(x_1,\ldots,x_n)=\sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$$

  for some coefficients $a_i \in \mathbb{C}^*$ and for a matrix $E=(E_{ij})$ with non-negative integer entries and with $\det E \neq 0$.

  Ex.: $f(x,y,z)=x^6y+y^3+z^2$, $E=\begin{pmatrix}6&1&0\\0&3&0\\0&0&2\end{pmatrix}$
- For simplicity: $a_i=1$ for $i=1,\ldots,n$, $\det E>0$.
- An invertible quasihomogeneous polynomial f is *non-degenerate* if it has (at most) an isolated critical point at the origin in $\mathbb{C}^n$.

Invertible polynomials 2

- f is quasihomogeneous, i.e. there exist weights $q_1, \ldots, q_n \in \mathbb{Q}$ such that

$$f(\lambda^{q_1} x_1, \ldots, \lambda^{q_n} x_n) = \lambda f(x_1, \ldots, x_n) \text{ for all } \lambda \in \mathbb{C}^*.$$

- Weights $(q_1, \ldots, q_n)$ defined by

$$E \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

- Kreuzer-Skarke: A non-degenerate invertible polynomial f is a (Thom-Sebastiani) sum of
  - $x_1^{p_1} x_2 + x_2^{p_2} x_3 + \ldots + x_{m-1}^{p_{m-1}} x_m + x_m^{p_m}$ (chain type; $m \geq 1$);
  - $x_1^{p_1} x_2 + x_2^{p_2} x_3 + \ldots + x_{m-1}^{p_{m-1}} x_m + x_m^{p_m} x_1$ (loop type; $m \geq 2$).

Berglund-Hübsch transpose

- The Berglund-Hübsch transpose f^T is

$$f^T(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}} \, .$$

Ex.: $E^T = \begin{pmatrix} 6 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $f^T(x,y,z) = x^6 + xy^3 + z^2$

Diagonal symmetries

- *Group of diagonal symmetries* G_f of f

$$G_f = \left\{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \;\middle|\; \begin{array}{c} f(\lambda_1 x_1, \ldots, \lambda_n x_n) \\ = f(x_1, \ldots, x_n) \end{array} \right\}$$

  finite group

- $g_0 = (e^{2\pi i q_1}, \ldots, e^{2\pi i q_n}) \in G_f$
  exponential grading operator,
  $G_0 := \langle g_0 \rangle \subset G_f$.
- Berglund-Henningson: $G \subset G_f$ subgroup

$$G^T := \mathrm{Hom}(G_f/G, \mathbb{C}^*) \quad \textit{dual group}$$

- $(G^T)^T = G$
- $G_f^T = \{1\}$
- $G_0^T = G_{f^T} \cap \mathrm{SL}_n(\mathbb{C})$

Objective

General assumption:
$n = 3$, $f(x, y, z)$ non-degenerate invertible polynomial such that $f^T(x, y, z)$ is also non-degenerate, both have critical point at 0

Aim:

- [ET, Compositio Math. 147 (2011)]

  $$(f, G_f) \longleftrightarrow (f^T, \{1\})$$

  $\Rightarrow$ Arnold's strange duality ($X = \{f = 0\}$, $G_f = G_0$)
- [ET, arXiv: 1103.5367, Int. Math. Res. Not.]
  Generalization:

  $$\begin{array}{ccc} G_0 \subset G \subset G_f & & \{1\} \subset G^T \subset G_0^T \\ (f, G) & \longrightarrow & (f^T, G^T) \end{array}$$

  $\Rightarrow$ Part of E.-Wall extension

Dolgachev numbers 1

Assumption: $G_0 \subset G \subset G_f$

$$\{1\} \longrightarrow G \longrightarrow \widehat{G} \longrightarrow \mathbb{C}^* \longrightarrow 1$$

Consider quotient stack

$$\mathcal{C}_{(f,G)} := \left[f^{-1}(0) \backslash \{0\} \Big/ \widehat{G} \right]$$

Deligne–Mumford stack (smooth projective curve with finite number of isotropic points)

Definition

Dolgachev numbers: $A_{(f,G)} = (\alpha_1, \ldots, \alpha_r)$
orders of isotropy groups of G,
$g_{(f,G)} := \mathrm{genus}\,[\mathcal{C}_{(f,G)}]$

Theorem

$G = G_f \Rightarrow g_{(f,G)} = 0$, $r \leq 3$.

Dolgachev numbers 2

$A_{(f,G_f)} = (\alpha_1', \alpha_2', \alpha_3')$, α_i' order of isotropy of point P_i.
Notation: $u * v := \underbrace{(u, \dots, u)}_{v \text{ times}}$

Theorem

$H_i \subset G_f$ minimal subgroup with $G \subset H_i$, $\mathrm{Stab}(P_i) \subset H_i$, $i = 1, 2, 3$. Then

$$A_{(f,G)} = \left(\frac{\alpha_i'}{|H_i/G|} * |G_f/H_i|, i = 1, 2, 3 \right),$$

where one omits numbers equal to 1.

- $e_{\mathrm{st}}(\mathcal{C}_{(f,G)}) = 2 - 2g_{(f,G)} + \sum_{i=1}^{r}(\alpha_i - 1)$
  stringy Euler number

Gabrielov numbers 1

Assumption: $\{1\} \subset G \subset G_f \cap \mathrm{SL}_3(\mathbb{C})$
$g \in G$ order r

$$\begin{aligned} g &= \mathrm{diag}(e^{2\pi i a_1/r}, e^{2\pi i a_2/r}, e^{2\pi i a_2/r}) \quad \text{with } 0 \leq a_i < r. \\ \mathrm{age}(g) &:= \frac{1}{r}(a_1 + a_2 + a_3) \in \mathbb{Z} \\ j_G &:= |\{g \in G \,|\, \mathrm{age}(g) = 1, g \text{ fixes only } 0\}| \end{aligned}$$

Theorem

$f(x,y,z) - xyz \sim F(x,y,z) = x^{\gamma_1'} + y^{\gamma_2'} + z^{\gamma_3'} - axyz, a \in \mathbb{C}^*$, cusp singularity of type $T_{\gamma_1',\gamma_2',\gamma_3'}$

Definition

Gabrielov numbers of the pair $(f, \{1\})$:
$\Gamma_{(f,\{1\})} := (\gamma_1', \gamma_2', \gamma_3')$

Gabrielov numbers 2

Proposition

Above coordinate change is G-equivariant. In particular, F G-invariant.

Definition

$K_i \subset G$ maximal subgroup fixing i-th coordinate.

$$\Gamma_{(f,G)} = (\gamma_1, \ldots, \gamma_s) := \left(\frac{\gamma_i'}{|G/K_i|} * |K_i|, i = 1, 2, 3 \right),$$

where one omits numbers equal to 1.
Gabrielov numbers of the pair (f, G).

- $\mu_{(f,G)} := 2 - 2j_G + \sum_{i=1}^{s} (\gamma_i - 1)$

Geometric meaning

$X_F := F^{-1}(1)$ Milnor fibre of cusp singularity F
$\sigma : X_F \to X_F$ monodromy diffeomorphism
characteristic polynomial of $\sigma_* : H_2(X_F) \to H_2(X_F)$

$$\phi_{(F,\{1\})}(t) = (t-1)^2 \prod_{i=1}^{3} \frac{t^{\gamma_i'}-1}{t-1}$$

Action of G commutes with σ
$\to$ G-equivariant monodromy
characteristic polynomial

$$\phi_{(F,G)}(t) = (t-1)^{2-2j_G} \prod_{i=1}^{s} \frac{t^{\gamma_i}-1}{t-1}$$

$\mu_{(f,G)}$: *G-equivariant Milnor number*

Mirror symmetry

Theorem

$$A_{(f,G_f)} = \Gamma_{(f^T,\{1\})}, \quad A_{(f^T,G_{f^T})} = \Gamma_{(f,\{1\})}.$$

Corollary

Arnold's strange duality ($G_0 = G_f$).

Theorem

$G_0 \subset G \subset G_f$, $f^T(x,y,z) - xyz \sim F(x,y,z)$

$$A_{(f,G)} = \Gamma_{(f^T,G^T)}, \quad e_{\mathrm{st}}(\mathcal{C}_{(f,G)}) = \mu_{(F,G^T)}, \quad g_{(f,G)} = j_{G^T}$$

Proof.

$K_i = H_i^T$ for a suitable ordering of the isotropic points P_1, P_2, P_3. □

Categorification

$G_0 \subset G \subset G_f$, $f^T(x,y,z) - xyz \sim F(x,y,z)$

- $G = G_f$, $\mathcal{C}_{(f,G_f)}$ weighted projective line

$$\begin{tikzcd}
\mathrm{HMF}_S^{\widehat{G}_f}(f) \arrow[r, "\sim"] \arrow[d, leftrightarrow] & D^b\mathrm{Fuk}^{\to}(f^T) \arrow[d, leftrightarrow] \\
D^b\mathrm{Coh}\,\mathcal{C}_{(f,G_f)} \arrow[r, "\sim"] & D^b\mathrm{Fuk}^{\to}(F)
\end{tikzcd}$$

- $G_0 \subset G \subset G_f$

$$\begin{tikzcd}
\mathrm{HMF}_S^{\widehat{G}}(f) \arrow[r, "\sim"] \arrow[d, leftrightarrow] & D^b\mathrm{Fuk}^{\to}(f^T)//G^T \arrow[d, leftrightarrow] \\
D^b\mathrm{Coh}\,\mathcal{C}_{(f,G)} \arrow[r, "\sim"] & D^b\mathrm{Fuk}^{\to}(F)//G^T
\end{tikzcd}$$

14 exceptional unimodular singularities

Name	$\alpha_1, \alpha_2, \alpha_3$	f	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{12}	$2, 3, 7$	$x^2 + y^3 + z^7$	$2, 3, 7$	E_{12}
E_{13}	$2, 4, 5$	$x^2 + y^3 + yz^5$	$2, 3, 8$	Z_{11}
E_{14}	$3, 3, 4$	$x^3 + y^2 + yz^4$	$2, 3, 9$	Q_{10}
Z_{11}	$2, 3, 8$	$x^2 + zy^3 + z^5$	$2, 4, 5$	E_{13}
Z_{12}	$2, 4, 6$	$x^2 + zy^3 + yz^4$	$2, 4, 6$	Z_{12}
Z_{13}	$3, 3, 5$	$x^2 + xy^3 + yz^3$	$2, 4, 7$	Q_{11}
Q_{10}	$2, 3, 9$	$x^3 + zy^2 + z^4$	$3, 3, 4$	E_{14}
Q_{11}	$2, 4, 7$	$x^2y + y^3z + z^3$	$3, 3, 5$	Z_{13}
Q_{12}	$3, 3, 6$	$x^3 + zy^2 + yz^3$	$3, 3, 6$	Q_{12}
W_{12}	$2, 5, 5$	$x^5 + y^2 + yz^2$	$2, 5, 5$	W_{12}
W_{13}	$3, 4, 4$	$x^2 + xy^2 + yz^4$	$2, 5, 6$	S_{11}
S_{11}	$2, 5, 6$	$x^2y + y^2z + z^4$	$3, 4, 4$	W_{13}
S_{12}	$3, 4, 5$	$x^3y + y^2z + z^2x$	$3, 4, 5$	S_{12}
U_{12}	$4, 4, 4$	$x^4 + zy^2 + yz^2$	$4, 4, 4$	U_{12}

E.-Wall extension of Arnold's strange duality

Bimodal series versus ICIS in $\mathbb{C}^4$

Series	Head	$A_{(f,G_0)}$	f	Dual
$J_{3,k}$	$J_{3,0}$	$2,2,2,3$	$x^6y + y^3 + z^2$	J'_9
$Z_{1,k}$	$Z_{1,0}$	$2,2,2,4$	$x^5y + xy^3 + z^2$	J'_{10}
$Q_{2,k}$	$Q_{2,0}$	$2,2,2,5$	$x^4y + y^3 + xz^2$	J'_{11}
$W_{1,k}$	$W_{1,0}$	$2,2,3,3$	$x^6 + y^2 + yz^2$	K'_{10}
$W^\sharp_{1,k}$				L_{10}
$S_{1,k}$	$S_{1,0}$	$2,2,3,4$	$x^5 + xy^2 + yz^2$	K'_{11}
$S^\sharp_{1,k}$				L_{11}
$U_{1,k}$	$U_{1,0}$	$2,3,3,3$	$x^3 + xy^2 + yz^3$	M_{11}

Example 1

f of table, $G = G_0 \subset G_f$ index 2.
$G_0^T \cong \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}^3$ by

$$(x, y, z) \mapsto (-x, -y, z)$$

Invariant polynomials:

$$W := y^2, \quad X := x^2, \quad Y := xy, \quad , Z := z$$

$$\left\{ \begin{array}{c} XW - Y^2 = 0 \\ f^T(W, X, Y, Z) = 0 \end{array} \right\}$$

yields equations of ICIS in $\mathbb{C}^4$ in five cases.

Example

$f(x, y, z) = x^6 y + y^3 + z^2,$
$f^T(x, y, z) = x^6 + xy^3 + z^2 = X^3 + YW + Z^2$

Example 2

$$f(x,y,z) = x^2 + xy^3 + yz^5, \ f^T(x,y,z) = x^2y + y^3z + z^5$$

$\mathcal{C}_{(f,G_0)}$ is a curve of genus two with no isotropic points.

$$f^T(x,y,z) - xyz \sim F(x,y,z) = x^5 + y^5 + z^5 - xyz$$

$$[G_f : G_0] = 5, \quad G_0^T = \mathbb{Z}/5\mathbb{Z} = \frac{1}{5}(1,3,1)$$

Example of P. Seidel.

Thank you!