What is a graded Representation Theory?

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- Typical question: [Standard : Simple] $=?$
- Typical answer: $[\Delta_x : L_y] = P_{x,y}(1)$ for some polynomial $P_{x,y} \in \mathbb{Z}[q]$.
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In the nicest cases, the representations in question are just the modules over a ring A. Then one may hope to better understand these formulas as follows:

- Find a $\mathbb{Z}$-grading $A = \bigoplus_{i \in \mathbb{Z}} A^i$ on A.
- Find $\mathbb{Z}$-gradings on Δ_x and L_y making them into graded A-modules $\Delta_x^{\mathbb{Z}}$ and $L_y^{\mathbb{Z}}$ such that in the category of graded A-modules we have

$$\sum_{i \in \mathbb{Z}} [\Delta_x^{\mathbb{Z}} : L_y^{\mathbb{Z}} \langle i \rangle] \, q^i = P_{x,y}(q)$$

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In less nice cases, the category of representations in question is still equivalent to the category of modules over a ring. Then try to do the same and call the category of graded modules a

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Graded versions able to interpret the relevant Kazhdan-Lusztig polynomials are known to exist among others for the following representation categories:

- Category $\mathcal{O}$.
- Some blocks of admissible representations of reductive groups over local fields. (I expect they exist in general.)
- Finite dimensional representations of quantum groups and small quantum groups at roots of unity (with the exception of some small cases).
- Some finite dimensional representations of simple Lie-superalgebras.

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One reason to study those is their inherent beauty: Quite often these graded versions turn out to be Koszul, that is, for simple objects we have

$$\mathrm{Ext}^i(L_x^{\mathbb{Z}}, L_y^{\mathbb{Z}}\langle j\rangle) = 0 \text{ unless } i + j = 0.$$

One might view this as a graded analogon to semisimplicity. This occurs for example for graded versions of $\mathcal{O}$ and graded versions of restricted enveloping algebras.

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One might view this as a graded analogon to semisimplicity. This occurs for example for graded versions of $\mathcal{O}$ and graded versions of restricted enveloping algebras.

Another reason is, that on the level of graded representation categories sometimes new symmetries may become visible. For example, for the graded version $\mathcal{O}_0^{\mathbb{Z}}$ of the principal block $\mathcal{O}_0$ of category $\mathcal{O}$, there is an autoequivalence of triangulated categories

$$\begin{array}{rrcl} K: & \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_0^{\mathbb{Z}}) & \xrightarrow{\sim} & \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_0^{\mathbb{Z}}) \\ & L_x \text{ simples} & \mapsto & P_{w_\circ x} \text{ projectives} \\ & \nabla_x \text{ dual Vermas} & \mapsto & \Delta_{w_\circ x} \text{ Vermas} \end{array}$$

with the property $K(M\langle n\rangle) \cong (KM)[-n]\langle -n\rangle$. This explains the rather mysterious identity

$$\sum_i \dim \mathrm{Ext}^i(L_x, \nabla_y) = [\Delta_{w_\circ y} : L_{w_\circ x}]$$

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An third reason is, that the graded versions of category $\mathcal{O}$ also lead to categorifications of representations of quantum groups, the action of q being categorified as a grading shift. From there one gets to tangle invariants, knot invariants, and the like.

The main reason (for me) is, that the graded versions of representation categories are necessary to formulate an upgrading of local Langlands philosophy to an equivalence of categories. This is still largely conjectural and generalizes the strange symmetry on $\mathrm{Der}^{\mathrm{b}}(\mathcal{O}_0^{\mathbb{Z}})$ above, which we should in fact already better have written

$$K : \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g})) \xrightarrow{\sim} \mathrm{Der}^{\mathrm{b}}(\mathcal{O}_0^{\mathbb{Z}}(\mathfrak{g}^\wedge))$$

and which I like to interpret after some reworking as a derived equivalence

$$\left\{ \begin{array}{c} \text{Graded} \\ \text{representation} \\ \text{theory} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Mixed Geometry} \\ \text{of corresponding} \\ \text{Langlands parameters} \end{array} \right\}$$

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