

# Representation Theoretic Approach to Superbosonisation

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Joint work with Alexander Alldridge

## Motivation – Physics

- ▶ Hermitian operators  $H$  on  $\mathbb{C}^n$  are drawn from a random probability distribution  $d\mu(H)$ .
- ▶ Assume some invariance of the distribution under conjugation by some group  $G_{\mathbb{R}}$ .
- ▶ Interested in calculating average spectral correlation functions.
- ▶ Supersymmetry method expresses this as a superintegral of some  $G$ -invariant function.
- ▶ **Superbosonisation** simplifies this integral to one over a Lie supergroup, useful for large  $n$  asymptotics.

# The Superbosonisation Formula

$$\int_{V_{\mathbb{R}}} Q^* f \cdot dV_{\mathbb{R}} = \text{const} \int_{H.o} f(g) \text{Ber}^n(g) dg \quad n \geq p$$

## Super Vector Space

- ▶  $V = \underline{\text{Hom}}(\mathbb{C}^n, \mathbb{C}^{p|q}) \oplus \underline{\text{Hom}}(\mathbb{C}^{p|q}, \mathbb{C}^n)$  a super vector space.
- ▶ Hermitian conjugate  $\dagger : \underline{\text{Hom}}(\mathbb{C}^n, \mathbb{C}^{p|q}) \rightarrow \underline{\text{Hom}}(\mathbb{C}^{p|q}, \mathbb{C}^n)$ .
- ▶  $V_{\mathbb{R}}$  is the graph of  $\dagger$ .

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### Berezin Form

- ▶  $V_{\mathbb{R}}$  is a real linear supermanifold.
- ▶ So can define  $dV_{\mathbb{R}}$  — the standard Berezin form on  $V_{\mathbb{R}}$  and integrate over this supermanifold.
- ▶  $x_1, \dots, x_m$  basis of  $V_{\mathbb{R},\bar{0}}$  and  $\xi_1, \dots, \xi_n$  basis of  $V_{\mathbb{R},\bar{1}}$ , then Berezin module generated by

$$dx_1 dx_2 \dots dx_m \frac{\partial}{\partial \xi_1} \dots \frac{\partial}{\partial \xi_n}.$$

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## Maps

- ▶ The map  $Q$  is defined as

$$Q: \begin{array}{ll} V & \longrightarrow W \\ (A, B) & \longmapsto AB \end{array}$$

where  $A$  and  $B$  are written as matrices and  $W := \underline{\text{End}}(\mathbb{C}^{p|q})$ .

- ▶  $f$  is a superfunction on  $W$  that is of Schwartz class along  $W_{\bar{0}, \mathbb{R}}$ .
- ▶  $Q^* f$  is also Schwartz along  $V_{\bar{0}, \mathbb{R}}$ .

## The Superbosonisation Formula

$$\int_{V_{\mathbb{R}}} Q^* f \cdot dV_{\mathbb{R}} = \text{const} \int_{H.o} f(g) \text{Ber}^n(g) dg \quad n \geq p$$

### Supergroup

- ▶  $H.o$  is the orbit of a Lie supergroup  $H$  through the identity of  $W_{\mathbb{R}}$ . It is a *cs* form, i.e. a ‘real form’ where we only take a real form of the bosonic part.
- ▶ For the unitary ensemble, it is defined by the Harish-Chandra superpair  $\left( \frac{\text{GL}_p(\mathbb{C})}{U(p)} \times U(q), \mathfrak{gl}(p|q) \right)$ .
- ▶  $dg$  is the normalised Haar measure on  $H.o$ .

## The Superbosonisation Formula

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### Berezinian

- ▶  $\text{Ber}^n : H.o \rightarrow \mathbb{C}$  is the  $n^{\text{th}}$  power of the superdeterminant (Berezinian) of an automorphism.
- ▶ For an invertible matrix

$$\begin{aligned} \text{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(A) \det(D - CA^{-1}B)^{-1} \\ &= \det(A - BD^{-1}C) \det(D)^{-1}. \end{aligned}$$

## The Superbosonisation Formula

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### The Formula

- ▶ (Littelmann–Sommers–Zirnbauer 2008) The superbosonisation formula states that the two integrals are equal up to some constant.
- ▶ Only works for  $n \geq p$  — there is a different formula for  $n < p$ .

## The Superbosonisation Formula

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### Our Setup

- ▶ Only consider the case  $G = \text{GL}_n(\mathbb{C})$  with its action on  $V$  given by a

$$\left. \begin{array}{l} \text{contragradient action on } \underline{\text{Hom}}(\mathbb{C}^n, \mathbb{C}^{p|q}) \\ \text{natural action on } \underline{\text{Hom}}(\mathbb{C}^{p|q}, \mathbb{C}^n) \end{array} \right\} g \cdot (x, y) = (xg^{-1}, gy).$$

- ▶ We are interested in integrating a suitable class of  $G$ -invariant functions on  $V_{\mathbb{R}}$  (Note that  $Q^* f$  is automatically  $G$ -invariant).

## The Superbosonisation Formula

$$\int_{V_{\mathbb{R}}} Q^* f \cdot dV_{\mathbb{R}} = \text{const} \int_{H.o} f(g) \text{Ber}^n(g) dg \quad n \geq p$$

“... bosonisation”?

- ▶ If  $p = 0$  then LHS is a fermionic integral and RHS is

$$\text{const} \int_{U(q)} f(u) \det^{-n}(u) du$$

a bosonic integral.

- ▶ If  $p = 0, q = 1$  then we have Cauchy integral formula

$$\frac{1}{n!} f^{(n)}(0) = \int_{U(1)} u^{-n} f(u) du.$$

## The Superbosonisation Formula

$$\int_{V_{\mathbb{R}}} Q^* f \cdot dV_{\mathbb{R}} = \text{const} \int_{H.o} f(g) \text{Ber}^n(g) dg \quad n \geq p$$

### Representation Theory?

- ▶ Construct a representation of  $\mathfrak{g} := \mathfrak{gl}(n)$  on  $\mathbb{C}[V]$  via the Heisenberg–Clifford algebra and Lie superalgebra  $\mathfrak{spo}$ .
- ▶ Find the Howe dual  $\mathfrak{g}' := \mathfrak{z}_{\mathfrak{spo}}(\mathfrak{g})$  with a representation of  $\mathbb{C}[V]^{G'}$ .
- ▶ Construct a Verma module isomorphic to  $\mathbb{C}[W]$ .
- ▶ Use basic multiplicity statements to conclude that integrals are the same (up to a constant).

## Heisenberg–Clifford Algebra

- ▶  $V \oplus V^*$  has a canonical skew-supersymmetric form

$$s : (V \oplus V^*) \otimes (V \oplus V^*) \longrightarrow \mathbb{C}$$

$$((u, \varphi), (v, \psi)) \longmapsto \varphi(v) - (-1)^{|u||v|} \psi(u)$$

which satisfies  $s(a, b) = -(-1)^{|a||b|} s(b, a)$  for homogeneous  $a$  and  $b$ .

- ▶ Let  $\mathfrak{hc}(V \oplus V^*)$  be abelian Lie superalgebra generated by  $V \oplus V^*$ .
- ▶ Consider a central extension  $\mathfrak{hc}_c(V \oplus V^*) := \mathfrak{hc}(V \oplus V^*) \oplus \mathbb{C}\mathbf{1}$  with bracket given by

$$[v + \lambda\mathbf{1}, w + \mu\mathbf{1}] = s(v, w)\mathbf{1}.$$

## Definition

The **Heisenberg–Clifford algebra** is

$$\mathrm{HC}(V \oplus V^*) := \frac{U(\mathfrak{hc}_c(V \oplus V^*))}{(\mathbf{1} - 1)}$$

where  $U(\mathfrak{hc}_c(V \oplus V^*))$  is the universal envelope and  $1$  is the unit inside the universal envelope.

- ▶ Note that if  $V$  is purely even (resp. purely odd) then  $\mathrm{HC}(V \oplus V^*)$  is the Weyl algebra (resp. Clifford algebra).

## The Symplectic Orthogonal Lie Superalgebra

### Definition

The Lie superalgebra  $\mathfrak{spo}(V \oplus V^*, s)$  is defined as a subalgebra of  $\mathfrak{gl}(V \oplus V^*)$  consisting of  $X$  such that

$$s(Xv, w) + s(v, Xw) = 0$$

for all  $v, w \in V \oplus V^*$ .

- ▶  $\mathrm{HC}(V \oplus V^*)$  is filtered – the filtration descends from that on the tensor algebra of  $V \oplus V^*$  in  $U(\mathfrak{hc}_c)$ .
- ▶ As usual, we obtain the associated graded algebra

$$\mathrm{gr} \mathrm{HC}(V \oplus V^*) = \bigoplus_{i=0}^{\infty} \mathrm{gr}_i \mathrm{HC}(V \oplus V^*).$$

## Lemma

$\mathrm{gr}_2\mathrm{HC}(V \oplus V^*) \cong \mathfrak{spo}(V \oplus V^*)$  as Lie superalgebras, where the bracket on the left hand side is the super commutator of endomorphisms. Further, the isomorphism is given by  $ab \mapsto [ab, -]$ .

## Representation of $\mathfrak{spo}(V \oplus V^*)$

- ▶ As usual, construct a representation of  $\mathrm{HC}(V \oplus V^*)$  on  $\mathbb{C}[V]$ .
- ▶ It is the symmetrised extension of

$$\begin{aligned} v^* \cdot f &= v^* f \\ v \cdot f &= \partial_v f \end{aligned}$$

where  $v \in V$ ,  $v^* \in V^*$  and  $f \in \mathbb{C}[V]$ .

- ▶ This representation restricts to a representation of  $\mathfrak{spo}(V \oplus V^*)$ .

## Representation of $\mathfrak{gl}(n)$

- ▶  $G$  acts linearly on  $V$  and this defines a faithful representation of  $\mathfrak{g} = \mathfrak{gl}(n)$  on  $V$ .
- ▶ We can embed  $\mathfrak{g}$  into  $\mathfrak{spo}(V \oplus V^*)$ .

## Howe dual pairs

### Definition

Lie superalgebras  $\mathfrak{g}$  and  $\mathfrak{g}'$  form a **Howe dual pair** if they are mutual centralisers in  $\mathfrak{spo}(V \oplus V^*)$ .

### Lemma

*With  $\mathfrak{g}$  embedded into  $\mathfrak{spo}(V \oplus V^*)$  as above, the Howe dual to  $\mathfrak{g}$  is given by  $\mathfrak{g}' := \mathfrak{gl}(2p|2q)$ .*

## Theorem (Howe)

*If  $\mathfrak{g}$  and  $\mathfrak{g}'$  form a Howe dual pair in  $\mathfrak{spo}(V \oplus V^*)$  then the representation  $\mathbb{C}[V]$  decomposes into a multiplicity free representation of  $\mathfrak{g} \times \mathfrak{g}'$ .*

- ▶ More precisely,

$$\mathbb{C}[V] = \bigoplus_i (\sigma_i \otimes \tau_i)$$

with  $\sigma_i$  and  $\tau_i$  irreducible representations of  $\mathfrak{g}$  and  $\mathfrak{g}'$  respectively such that  $\sigma_i \neq \sigma_j$  and  $\tau_i \neq \tau_j$  for  $i \neq j$ .

- ▶ In particular, we have that  $\mathbb{C}[V]^G$  is an irreducible  $\mathfrak{g}'$  representation.

## Triangular Decomposition

- ▶ Recall  $\mathfrak{spo}(V \oplus V^*) \cong \text{gr}_2 \text{HC}(V \oplus V^*) \cong S^2(V \oplus V^*)$ . This induces a grading

$$\mathfrak{spo}_- = S^2(V), \quad \mathfrak{spo}_0 = V \otimes V^*, \quad \mathfrak{spo}_+ = S^2(V^*).$$

- ▶ Form a triangular decomposition

$$\mathfrak{spo}(V \oplus V^*) = \mathfrak{spo}_- \oplus \mathfrak{spo}_0 \oplus \mathfrak{spo}_+.$$

- ▶  $\mathfrak{g} \subset \mathfrak{spo}_0$  means that  $\mathfrak{g}'$  has a triangular decomposition  $\mathfrak{g}' = \mathfrak{g}'_- \oplus \mathfrak{h} \oplus \mathfrak{g}'_+$  where

$$\mathfrak{g}'_- = \mathfrak{gl}(p|q), \quad \mathfrak{h} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q), \quad \mathfrak{g}'_+ = \mathfrak{gl}(p|q).$$

## Highest Weight

- ▶ The weight space

$$\bigcap_{v \in V} \ker \partial_v = \mathbb{C}_\lambda$$

is the highest weight of the  $\mathfrak{g}'$ -representation  $\mathbb{C}[V]^G$ .

- ▶ For  $(A, B) \in \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q) = \mathfrak{h}$ , we have

$$\lambda(A, B) = \frac{n}{2} (\text{str}(A) - \text{str}(B)).$$

## Verma Module

- ▶ Let  $\mathfrak{p} := \mathfrak{h} \oplus \mathfrak{g}'_+$ , a parabolic subspace.
- ▶ Define the Verma module

$$M(\lambda) := U(\mathfrak{g}') \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda.$$

- ▶  $M(\lambda) \cong U(\mathfrak{g}'_-) \otimes \mathbb{C}_\lambda$  as vector spaces.
- ▶ Using the pairing

$$\begin{aligned} \mathfrak{g}'_- \otimes \mathfrak{g}'_+ &\longrightarrow \mathbb{C} \\ \langle x_-, x_+ \rangle &= \lambda(-\text{ad}(x_-)(x_+)) \end{aligned}$$

we construct an isomorphism ( $W := \mathfrak{g}'_+^*$ )

$$\begin{aligned} M(\lambda) &\longrightarrow \mathbb{C}[W] \\ x_1 \dots x_n \otimes 1 &\longmapsto \langle x_1, - \rangle \dots \langle x_n, - \rangle. \end{aligned}$$

## A Representation of $H_{\mathbb{C}}$

### Lemma

*The  $\mathfrak{h}$  representation on  $\mathbb{C}[W]$  integrates to a  $H_{\mathbb{C}}$  representation.  
For  $h \in H_{\mathbb{C}}$ ,  $x \in W$  and  $p \in \mathbb{C}[W]$*

$$h \cdot p(x) = p(\text{Ad}(h^{-1})x)\chi(h)$$

*Here,  $\mathfrak{h} = \text{Lie}(H_{\mathbb{C}})$ .*

- ▶ For  $h = (a, d) \in \text{GL}(p|q) \times \text{GL}(p|q)$ ,  
 $\chi(h) = \text{Ber}^{n/2}(a)\text{Ber}^{-n/2}(d)$  is the integrated character.
- ▶ Multiply representations on  $\mathbb{C}[V]^G$  and  $\mathbb{C}[W]$  by appropriate  $e^{-\text{str}}$  to have suitable decay.

## Proof

- ▶  $H = (\mathrm{GL}_p(\mathbb{C}) \times \mathrm{U}(q) \times \mathrm{U}(q), \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q))$  is cs form of  $H_{\mathbb{C}}$ .
- ▶  $\dim \mathrm{Hom}_H(e^{-\mathrm{str}}\mathbb{C}[V]^G, \chi^{-1}) = 1$ .
- ▶ Can prove

$$\int_{V_{\mathbb{R}}} - dV_{\mathbb{R}} \in \mathrm{Hom}_H(e^{-\mathrm{str}}\mathbb{C}[V]^G, \chi^{-1}).$$

- ▶ Further, can prove

$$\int_{H.o} - \mathrm{Ber}^n(g) dg \in \mathrm{Hom}_H(e^{-\mathrm{str}}\mathbb{C}[W], \chi^{-1}).$$

- ▶  $Q^*$  is  $H$ -equivariant and surjective.
- ▶ Right exactness of  $\mathrm{Hom}_H(-, \chi^{-1})$  then implies the result.

Thank you