ON GENERALISATIONS OF LOSEV-MANIN MODULI SPACES FOR CLASSICAL ROOT SYSTEMS

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In memory of Eckart Viehweg

Abstract. Losev and Manin introduced fine moduli spaces $\overline{L}_n$ of stable n-pointed chains of projective lines. The moduli space $\overline{L}_{n+1}$ is isomorphic to the toric variety $X(A_n)$ associated with the root system A_n, which is part of a general construction to associate with a root system R of rank n an n-dimensional smooth projective toric variety $X(R)$. In this paper we investigate generalisations of the Losev-Manin moduli spaces for the other families of classical root systems.

Introduction

In [LM00] Losev and Manin introduced fine moduli spaces $\overline{L}_n$ of stable n-pointed chains of projective lines. These Losev-Manin moduli spaces are similar to the moduli spaces $\overline{M}_{0,n+2}$, but whereas $\overline{M}_{0,n+2}$ parametrises trees of projective lines with $n+2$ marked points that are not allowed to coincide, the moduli space $\overline{L}_n$ parametrises chains of projective lines with two poles and n marked points that may coincide.

The Losev-Manin moduli space $\overline{L}_{n+1}$ has the structure of an n-dimensional smooth projective toric variety such that the boundary divisors parametrising reducible curves correspond to the torus invariant divisors; it coincides with the toric variety $X(A_n)$ associated with the root system A_n. This is part of a general construction to associate with a root system R of rank n an n-dimensional smooth projective toric variety $X(R)$ ([Kl85], [Pr90]). In the introduction to [LM00] the authors asked about generalisations of the moduli spaces $\overline{L}_n$ for the other families of classical root systems. In the present paper we address this problem.

Concerning the family of root systems of type B we present a variant of the Losev-Manin moduli problem by considering chains of projective lines of odd length with an involution permuting the two poles having one marked point s_0 invariant under the involution and n pairs of marked points $s_i^{\pm}$ that are interchanged by the involution. We show that these pointed curves admit a fine moduli space $\overline{L}_n^{0,\pm}$ which is isomorphic to the toric variety $X(B_n)$ such that the boundary divisors of the moduli space get identified with the torus invariant divisors.

It is well known that for the Losev-Manin moduli spaces, as for the moduli spaces $\overline{M}_{0,n}$, the universal curve over $\overline{L}_{n+1}$ is the next moduli space $\overline{L}_{n+2}$ together with a natural forgetful morphism $\overline{L}_{n+2} \to \overline{L}_{n+1}$. In [BB11] we developed functorial properties of the toric varieties $X(R)$ with respect to maps of root systems and observed that this morphism $\overline{L}_{n+2} = X(A_{n+1}) \to \overline{L}_{n+1} = X(A_n)$ is induced by the inclusion of root systems $A_n \to A_{n+1}$. Furthermore, the $n+1$ sections $X(A_n) \to X(A_{n+1})$ come from projections of root systems $A_{n+1} \to A_n$ along the $n+1$ additional pairs of opposite roots in A_{n+1} not contained in A_n.

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All this generalises to the family of root systems of type B: the morphism $X(B_{n+1}) \to X(B_n)$ coming from the inclusion of root systems $B_n \to B_{n+1}$ is flat and its fibres have the structure of chains of projective lines of odd length. The $2n+1$ additional pairs of opposite roots in B_{n+1} give $2n+1$ sections. There is a symmetry of B_{n+1} fixing B_n which induces an involution I of $X(B_{n+1})$ over $X(B_n)$ such that the sections are grouped into n pairs of sections $s_i^\pm$ interchanged by the involution and one section s_0 invariant under the involution. We show that $X(B_{n+1}) \to X(B_n)$ together with these sections and the involution I forms the universal family over the fine moduli space $\overline{L}_n^{0,\pm} = X(B_n)$. On the other hand, we will see that the toric varieties $X(R_n)$ for $R = C, D$ do not form fine moduli spaces of pointed reduced curves having $X(R_{n+1}) \to X(R_n)$ as universal family.

In the case of root systems of type C the morphism $X(C_{n+1}) \to X(C_n)$ is flat with one-dimensional fibres having the structure of $2n$-pointed chains of projective lines of odd length with involution except that over a certain torus invariant divisor nonreduced components occur. On the one hand we can consider families of pointed curves as in the B_n-case but without the section s_0 and thereby allowing an additional involution as isomorphism. This gives rise to a toric Deligne-Mumford stack $\mathcal{X}(C_n)$ which is an orbifold having the toric variety $X(C_n)$ as coarse moduli space with stacky points over the divisor determined by the nonreduced fibres. On the other hand we can describe $X(C_n)$ as a fine moduli space $\overline{L}_n^{\pm}$ of $2n$-pointed chains of projective lines of odd and even length with involution with each of the marked points corresponding to a pair of opposite roots in $C_{n+1} \backslash C_n$ that defines a projection $C_{n+1} \to C_n$. The universal family arises from $X(C_{n+1}) \to X(C_n)$ by contracting the nonreduced components in the fibres.

In the case of the remaining family of root systems of type D the morphism $X(D_{n+1}) \to X(D_n)$ is not flat. There are 2-dimensional fibres that occur over closures of certain torus orbits of codimension 2, over the other points as fibres we have $2n$-pointed chains of projective lines with involution.

We observe that in the cases of all families of root systems $R = A, B, C, D$ the torus fixed points of $X(R_n)$ correspond to pointed curves having the form of the Dynkin diagram for the root system R_{n+1}.

Outline of the paper. In the first sections 1–5 we deal with the case of root systems of type B. In section 1 we formulate a moduli problem of $(2n+1)$-pointed chains of projective lines called B_n-curves, which is a variant of the Losev-Manin moduli problem. In section 2 we collect some facts about the toric varieties $X(B_n)$ associated with root systems of type B. Section 3 is about the morphism $X(B_{n+1}) \to X(B_n)$, which, together with its sections and the involution, forms a flat family of B_n-curves, and in section 4 we prove that the toric variety $\overline{L}_n^{0,\pm} = X(B_n)$ is a fine moduli space of B_n-curves with universal family $X(B_{n+1}) \to X(B_n)$. To show that the moduli functor of B_n-curves is isomorphic to the functor of the toric variety $X(B_n)$ we use the description of the functor of toric varieties associated with root systems given in [BB11, 1.3]; our proof is a variation of our new proof of the respective statement for root systems of type A given in [BB11, 3.3]. In section 5 we present some results on the (co)homology of the spaces $\overline{L}_n^{0,\pm} = X(B_n)$, giving descriptions similar to the case of the Losev-Manin moduli spaces $\overline{L}_{n+1} = X(A_n)$.

In the remaining sections 6 and 7 the cases of the root systems of type C and D are investigated.

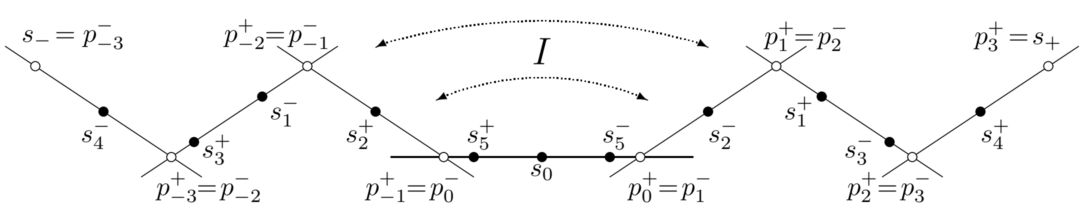
1. Pointed chains of projective lines with involution

Definition 1.1. A *chain of projective lines* of length m over an algebraically closed field K is a projective curve $C = C_1 \cup \ldots \cup C_m$ over K such that every irreducible component C_j of C is a projective line with poles p_j^-, p_j^+ and these components intersect as follows: different components C_i and C_j intersect only if $|i - j| = 1$ and in this case C_j, C_{j+1} intersect transversally in the single point $p_j^+ = p_{j+1}^-$. For $p_1^- \in C_1$ and $p_m^+ \in C_m$ we write s_- and s_+. Two chains of projective lines (C, s_-, s_+) and (C', s'_-, s'_+) are called isomorphic if there is an isomorphism $\varphi\colon C \to C'$ such that $\varphi(s_-) = s'_-$, $\varphi(s_+) = s'_+$.

Definition 1.2. A *chain of projective lines with involution* (C, I, s_-, s_+) is a chain of projective lines together with an isomorphism $I\colon C \to C$ such that $I^2 = id_C$ and $I(s_-) = s_+$. In this case we use the following notation: if the chain has odd length denote by (C_0, p_0^-, p_0^+) the central component; denote by (C_j, p_j^-, p_j^+), $(C_{-j}, p_{-j}^-, p_{-j}^+)$ the pairs of I-conjugate components (i.e. $I(C_j) = C_{-j}$, $I(p_j^-) = p_{-j}^+$, $I(p_j^+) = p_{-j}^-$) such that $p_j^+ = p_{j+1}^-$, $p_{-j}^- = p_{-(j+1)}^+$ and in case of odd lenght $p_0^+ = p_1^-$, $p_0^- = p_{-1}^+$ whereas in case of even length $p_{-1}^+ = p_1^-$. In particular, we have $s_- = p_{-m}^-$, $s_+ = p_m^+$ if the chain has length $2m$ or $2m+1$. Two chains of projective lines with involution (C, I, s_-, s_+) and (C', I', s'_-, s'_+) are called isomorphic if there is an isomorphism of chains of projective lines $\varphi\colon (C, s_-, s_+) \to (C', s'_-, s'_+)$ such that $\varphi \circ I = I' \circ \varphi$.

In the following we are concerned with certain compactifications of the algebraic torus $(2\mathbb{G}_m)^n$ parametrising n pairs of points of the form $(z, \frac{1}{z})$ in $(\mathbb{G}_m, 1) \subset (\mathbb{P}^1, 0, \infty, 1)$, i.e. pairs of points which are interchanged by the involution of $\mathbb{P}^1$ that fixes the point 1 and interchanges the two poles 0 and ∞. These compactifications, which will be associated with root systems, parametrise isomorphism classes of certain pointed chains of projective lines with an involution. We now define the type of pointed curve which will be relevant in the case of root systems of type B.

Definition 1.3. A $(2n{+}1)$-*pointed chain of projective lines with involution* $(C, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$ is a chain of projective lines with involution (C, I, s_-, s_+) of odd length together with (possibly coinciding) marked points $s_0, s_i^\pm \in C$ different from the poles such that $I(s_0) = s_0$, $I(s_i^-) = s_i^+$. Two $(2n{+}1)$ - pointed chains of projective lines with involution $(C, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$ and $(C', I', s'_-, s'_+, s'_0, {s'_1}^\pm, \ldots, {s'_n}^\pm)$ are called isomorphic if there is an isomorphism $\varphi\colon (C, I, s_-, s_+) \to (C', I', s'_-, s'_+)$ of the underlying chains of projective lines with involution such that $\varphi(s_0) = s'_0$, $\varphi(s_j^\pm) = {s'_j}^\pm$. A $(2n{+}1)$-pointed chain of projective lines with involution $(C, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$ is called *stable* if each component of C contains at least one of the points $s_0, s_j^\pm$. A B_n-*curve* over an algebraically closed field K is a stable $(2n+1)$-pointed chain of projective lines over K.



Definition 1.4. Let Y be a scheme. A *B_n-curve over Y* is a collection $(\pi\colon C \to Y, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$, where C is a scheme, π is a flat proper morphism of schemes, $I\colon C \to C$ an involution over Y and $s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm\colon Y \to C$ are sections such that for any geometric point y of Y the collection $(C_y, I_y, (s_-)_y, (s_+)_y, (s_0)_y, (s_1^\pm)_y, \ldots, (s_n^\pm)_y)$ is a B_n-curve over y. An isomorphism of B_n-curves over Y is an isomorphism of Y-schemes that is compatible with the involution and the sections. We define the *moduli functor of B_n-curves* as the functor

$$\begin{array}{rccl} \underline{\overline{L}_n^{0,\pm}}: & (\mathit{schemes})^\circ & \to & (\mathit{sets}) \\ & Y & \mapsto & \{B_n\text{-}\mathit{curves\ over}\ Y\}\ /\sim \end{array}$$

that associates to a scheme Y the set of isomorphism classes of B_n-curves over Y and to a morphism of schemes the map obtained by pulling back B_n-curves.

We will show in section 4 that a fine moduli space of B_n-curves $\overline{L}_n^{0,\pm}$ exists and that it is isomorphic to the toric variety associated with the root system B_n.

2. Toric varieties $X(B_n)$

For a root system R of rank n we have the n-dimensional smooth projective toric variety $X(R)$ associated with the fan that consists of the Weyl chambers of the root system and their faces ([Kl85], [Pr90], see also [BB11, 1.1]). Here we consider the particular case of root systems of type B.

Let E be an n-dimensional Euclidean space with basis $u_1, \ldots, u_n$. The root system B_n in E consists of the following $2n^2$ roots:

$$\pm u_i \ \mathit{for}\ i \in \{1, \ldots, n\}; \quad \pm(u_i + u_j), \pm(u_i - u_j)\ \mathit{for}\ i, j \in \{1, \ldots, n\}, i < j.$$

The root lattice $M(B_n) \cong \mathbb{Z}^n$ of the root system B_n is the lattice in E generated by $u_1, \ldots, u_n$. The following is a set of simple roots:

$$u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n.$$

The Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts by $u_i \mapsto \pm u_i$ and by permuting the u_i. So there are $2^n n!$ sets of simple roots, these are of the form $\varepsilon_1 u_{i_1} - \varepsilon_2 u_{i_2}, \varepsilon_2 u_{i_2} - \varepsilon_3 u_{i_3}, \ldots, \varepsilon_{n-1} u_{i_{n-1}} - \varepsilon_n u_{i_n}, \varepsilon_n u_{i_n}$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_1, \ldots, \varepsilon_n$. For later use we list linear relations between positive roots of B_n.

Lemma 2.1. *Let B_n^+ be the set of positive roots of B_n corresponding to the set of simple roots $u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n$ and put $\beta_{ij} = u_i - u_j$, $\gamma_{ij} = u_i + u_j$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$. Then $B_n^+ = \{u_1, \ldots, u_n\} \cup \{\beta_{ij} \mid i < j\} \cup \{\gamma_{ij} \mid i \neq j\}$ and the tripels of positive roots $\alpha, \beta, \gamma \in B_n^+$ satisfying $\alpha + \beta = \gamma$ are the following:*

$$\begin{array}{rl} \beta_{ij} + u_j = u_i & (i, j \in \{1, \ldots, n\},\ i < j) \\ u_i + u_j = \gamma_{ij} & (i, j \in \{1, \ldots, n\},\ i \neq j) \\ \beta_{ij} + \beta_{jk} = \beta_{ik} & (i, j, k \in \{1, \ldots, n\},\ i < j < k) \\ \beta_{ij} + \gamma_{jk} = \gamma_{ik} & (i, j, k \in \{1, \ldots, n\},\ i < j,\ k \neq i, j) \end{array}$$

Let $N(B_n)$ be the lattice dual to the root lattice $M(B_n)$ and $v_1, \dots, v_n$ the basis of $N(B_n)$ dual to $u_1, \dots, u_n$. The fan $\Sigma(B_n)$ is defined as the fan of Weyl chambers in $N(B_n)$, i.e. its maximal cones are the Weyl chambers $\sigma_S = S^\vee = \{v \in N(B_n)_\mathbb{Q} | \forall \alpha \in S\colon \langle \alpha, v\rangle \geq 0\}$ for sets of simple roots S of the root system B_n and all cones arise as faces of these. For the set of simple roots $S = \{u_1 - u_2, u_2 - u_3, \dots, u_{n-1} - u_n, u_n\}$ has the dual basis $v_1, v_1 + v_2, \dots, v_1 + \dots + v_n$ of $N(B_n)$, the Weyl chamber σ_S is equal to $\langle v_1, v_1 + v_2, \dots, v_1 + \dots + v_n\rangle_{\mathbb{Q}_{\geq 0}}$. All Weyl chambers, i.e. all maximal cones of the fan $\Sigma(B_n)$, arise as translates of σ_S under the action of the Weyl group on $N(B_n)_\mathbb{Q}$, thus they are generated by sets of elements of the form $\varepsilon_1 v_{i_1}, \varepsilon_1 v_{i_1} + \varepsilon_2 v_{i_2}, \dots, \varepsilon_1 v_{i_1} + \dots + \varepsilon_n v_{i_n}$ for orderings $i_1, \dots, i_n$ of the set $\{1, \dots, n\}$ and signs $\varepsilon_i \in \{\pm 1\}$. The fan $\Sigma(B_n)$ has $3^n - 1$ one-dimensional cones generated by the elements of the form $\varepsilon_1 v_{i_1} + \dots + \varepsilon_k v_{i_k}$ for $k \in \{1, \dots, n\}$. These are via $v_B := \sum_{\varepsilon_i i \in B} \varepsilon_i v_i \leftrightarrow B$ in bijection with the set $\mathcal{B}$ of all subsets $\emptyset \neq B \subset \{\pm 1, \dots, \pm n\}$ such that $B \cap \{i, -i\} \neq \{i, -i\}$ for $i = 1, \dots, n$. The one-dimensional cones for a family of such sets $B^{(1)}, \dots, B^{(k)}$ form a higher dimensional cone whenever they can be ordered such that $B^{(i_1)} \subsetneq \dots \subsetneq B^{(i_k)}$.

We have the n-dimensional smooth projective toric variety $X(B_n)$ associated with the fan $\Sigma(B_n)$ with respect to the lattice $N(B_n)$. As usual, any element $u \in M(B_n)$ defines a character of the open dense torus $T(B_n)$ (resp. a rational function on $X(B_n)$) denoted by x^u. The toric variety $X(B_n)$ has the following covering by affine spaces. For any set S of simple roots we have the maximal cone $\sigma_S = S^\vee$ and the chart $U_S = \operatorname{Spec} \mathbb{Z}[\sigma_S^\vee \cap M(B_n)] \cong \mathbb{A}^n$, for example if $S = \{u_1 - u_2, u_2 - u_3, \dots, u_{n-1} - u_n, u_n\}$ then $\mathbb{Z}[\sigma_S^\vee \cap M(B_n)] = \mathbb{Z}[\frac{x_1}{x_2}, \dots, \frac{x_{n-1}}{x_n}, x_n]$. The Weyl group $W(B_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts on $X(B_n)$, it permutes these affine charts.

By [BB11, 1.2] the closures of torus orbits in $X(B_n)$ are isomorphic to products $X(B_{n_1}) \times X(A_{n_2}) \times \dots \times X(A_{n_k})$. The torus invariant divisor for the one-dimensional cone generated by $\varepsilon_1 v_{i_1} + \dots + \varepsilon_k v_{i_k}$ is isomorphic to $X(B_{n-k}) \times X(A_{k-1})$, in particular for $\varepsilon_1 v_{i_1} + \dots + \varepsilon_n v_{i_n}$ we have a divisor isomorphic to $X(A_{n-1})$.

3. The universal curve

We construct a B_n-curve over $X(B_n)$, which later turns out to be the universal curve over the moduli space of B_n-curves, by using functorial properties of the toric varieties associated with root systems developed in [BB11, 1.2]. We fix the following notations for roots of B_n and B_{n+1}: $\beta_{ij} = u_i - u_j$, $\gamma_{ij} = u_i + u_j$ for $i, j \in \{1, \dots, n\}$, $i \neq j$ and $\alpha_i^+ = u_{n+1} + u_i$, $\alpha_i^- = u_{n+1} - u_i$ for $i \in \{1, \dots, n\}$.

Construction 3.1. (The universal B_n-curve.)
Consider the root subsystem $B_n \subset B_{n+1}$ consisting of the roots in the subspace spanned by $u_1, \dots, u_n$. The inclusion of root systems $B_n \subset B_{n+1}$ determines a proper surjective morphism $X(B_{n+1}) \to X(B_n)$.

There are $2n + 1$ additional pairs of opposite roots, the pairs $\pm\alpha_i^+$ and $\pm\alpha_i^-$ for $i \in \{1, \dots, n\}$ and the pair $\pm u_{n+1}$. Each of these defines a projection onto the root subsystem $B_n \subset B_{n+1}$ in the sense of [BB11, 1.2], thus the pairs $\pm\alpha_i^+$ and $\pm\alpha_i^-$

define sections $s_i^+, s_i^-\colon X(B_n) \to X(B_{n+1})$ and an additional section $s_0\colon X(B_n) \to X(B_{n+1})$ is given by the projection with kernel generated by u_{n+1}.

Further, we have two sections $s_\pm\colon X(B_n) \to X(B_{n+1})$ which are inclusions of $X(B_n)$ into $X(B_{n+1})$ as torus invariant divisors (cf. [BB11, Prop. 1.9, Rem. 1.11]) corresponding to the one-dimensional cones of the fan $\Sigma(B_{n+1})$ generated by $\pm v_{n+1}$. Locally, the image of s_- (resp. s_+) is given by the equations $x^{-\alpha_i^\pm} = 0$, $x^{-u_{n+1}} = 0$ (resp. $x^{\alpha_i^\pm} = 0$, $x^{u_{n+1}} = 0$) on the affine charts of $X(B_{n+1})$ corresponding to the sets of positive roots containing $-\alpha_i^\pm, -u_{n+1}$ (resp. $\alpha_i^\pm, u_{n+1}$).

There is an involution I of $X(B_{n+1})$ over $X(B_n)$ corresponding to the involution of B_{n+1} which fixes $B_n \subset B_{n+1}$ determined by the linear map $u_i \mapsto u_i$ for $i \in \{1, \ldots, n\}$ and $u_{n+1} \mapsto -u_{n+1}$. This element of the Weyl group $W(B_{n+1})$ is the reflection determined by the root $\pm u_{n+1}$. The section s_0 is invariant under I, whereas for each $i \in \{1, \ldots, n\}$ the sections s_i^+ and s_i^- and also s_- and s_+ are interchanged.

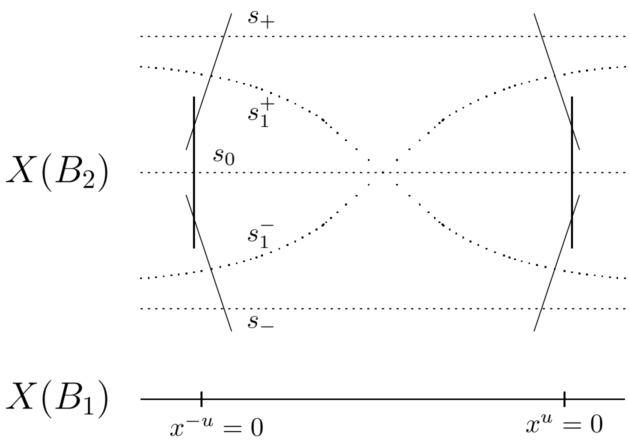
Proposition 3.2. *The collection* $(X(B_{n+1}) \to X(B_n), I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$ *of construction 3.1 is a B_n-curve over $X(B_n)$.*

Proof. The morphism $X(B_{n+1}) \to X(B_n)$ is proper. We can show that it is flat by considering the covering of $X(B_{n+1})$ and $X(B_n)$ by affine toric charts similar as in the case of root systems of type A (see [BB11, Prop. 3.7]).

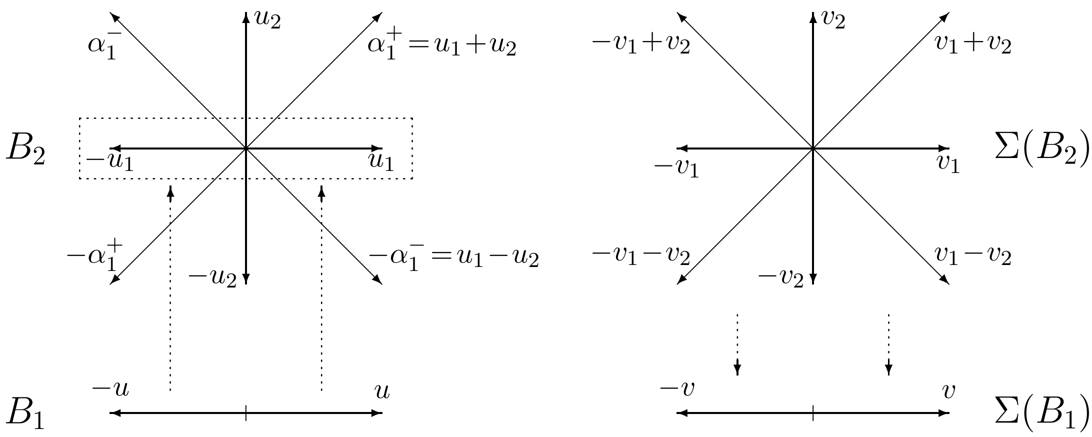
That any fibre is a B_n-curve follows from the results below. Remark 3.5 describes the universal curve in terms of equations, proposition 3.7 shows that such equations define a B_n-curve. It only remains to show that any B_n-data arises as in proposition 3.7 from a B_n-curve. This is done in lemma 3.8. □

Definition 3.3. We call the object $(X(B_{n+1}) \to X(B_n), I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm)$ of construction 3.1 the universal B_n-curve over $X(B_n)$.

Example 3.4. We picture the universal curve $X(B_2)$ over $X(B_1) \cong \mathbb{P}^1$ with the sections $s_-, s_+, s_0, s_1^\pm$. The generic fibre is a $\mathbb{P}^1$, whereas the fibres over the two torus fixed points $x^{-u} = 0$ and $x^u = 0$ of $X(B_1)$ are chains consisting of three projective lines.



This universal curve is constructed using the root system B_2 with its root subsystem $B_1 = \{\pm u_1\}$ and the corresponding map of fans $\Sigma(B_2) \to \Sigma(B_1)$.



By [BB11, 1.2] pairs of opposite roots $\{\pm\alpha\}$ in a root system R give rise to morphisms $X(R) \to \mathbb{P}^1$. We write $\mathbb{P}^1_{\{\pm\alpha\}}$ for the corresponding copy of $\mathbb{P}^1$ with homogeneous coordinates $z_\alpha, z_{-\alpha}$ such that the rational function x^α on $X(R)$ is the pull-back of $\frac{z_\alpha}{z_{-\alpha}}$. Further, the collection of these morphisms for all pairs of opposite roots $\{\pm\alpha\}$ in R, i.e. root subsystems isomorphic to A_1, defines a closed embedding $X(R) \to \prod_{\{\pm\alpha\}\subseteq R} \mathbb{P}^1_{\{\pm\alpha\}} =: P(R)$. By [BB11, 1.3] the equations defining the image of $X(R)$ in $P(R)$ come from root subsystems of type A_2 in R or equivalently linear relations between positive roots of R.

Remark 3.5. Consider $X(B_{n+1})$ and $X(B_n)$ as embedded $X(B_{n+1}) \subseteq P(B_{n+1})$, $X(B_n) \subseteq P(B_n)$. Then the morphism $X(B_{n+1}) \to X(B_n)$ is induced by the projection onto the subproduct $P(B_{n+1}) \to P(B_n)$. The subvarieties $X(B_{n+1})$ (resp. $X(B_n)$) are determined by the homogeneous equations $z_\alpha z_\beta z_{-\gamma} = z_{-\alpha} z_{-\beta} z_\gamma$ for roots α, β, γ such that $\alpha + \beta = \gamma$, i.e. root subsystems of type A_2 in B_{n+1} (resp. B_n).
If we first consider the product $P(B_{n+1})$ and the equations coming from root subsystems of type A_2 in B_n, we have

$$\begin{aligned} P(B_{n+1}/B_n)_{X(B_n)} &= \big(\textstyle\prod_{A_1\cong R\subseteq B_{n+1}\setminus B_n} \mathbb{P}^1_R\big)_{X(B_n)} \\ &= \big(\textstyle\prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^+\}} \times \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^-\}} \times \mathbb{P}^1_{\{\pm u_{n+1}\}}\big)_{X(B_n)} \end{aligned}$$

Therein, $X(B_{n+1})$ is the closed subvariety given by the equations corresponding to root subsystems of type A_2 in B_{n+1} which are not contained in B_n. We choose the set of positive roots B^+_{n+1} corresponding to the set of simple roots $\{u_{n+1} - u_1, u_1 - u_2, u_2 - u_3, \dots, u_{n-1} - u_n, u_n\}$. Then $B^+_{n+1} \setminus B^+_n = \{u_{n+1}, \alpha_1^\pm, \dots, \alpha_n^\pm\}$ and we can write these equations as follows

$$t_\beta z_{\alpha_2} z_{-\alpha_1} = t_{-\beta} z_{-\alpha_2} z_{\alpha_1} \qquad \begin{array}{l} \textit{for } \alpha_1, \alpha_2 \in \{u_{n+1}, \alpha_1^\pm, \dots, \alpha_n^\pm\} \\ \textit{such that } \beta = \alpha_1 - \alpha_2 \textit{ is a root of } B_n \end{array} \tag{1}$$

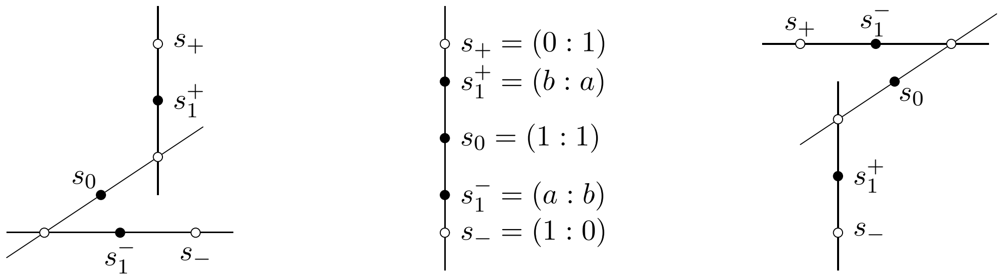
where $t_{\pm\beta}$ are the homogeneous coordinates of $\mathbb{P}^1_{\{\pm\beta\}}$ (consider $X(B_n)$ as embedded in $P(B_n)$) or equivalently the two generating sections of the line bundle $\mathscr{L}_{\{\pm\beta\}}$ being part of the universal data on $X(B_n)$ as defined in [BB11, 1.3].
The sections $s_i^\pm \colon X(B_n) \to X(B_{n+1})$ for $i \in \{1, \dots, n\}$ are given by the additional equations $z_{\alpha_i^\pm} = z_{-\alpha_i^\pm}$, the section s_0 by $z_{u_{n+1}} = z_{-u_{n+1}}$. The sections s_- (resp. s_+)

are given by $z_{-u_{n+1}} = 0$, $z_{-\alpha_i^\pm} = 0$ for $i = 1, \dots, n$ (resp. $z_{u_{n+1}} = 0$, $z_{\alpha_i^\pm} = 0$ for $i = 1, \dots, n$).

Example 3.6. The universal B_1-curve $X(B_2) \subset (\mathbb{P}^1_{\{\pm\alpha_1^+\}} \times \mathbb{P}^1_{\{\pm\alpha_1^-\}} \times \mathbb{P}^1_{\{\pm u_2\}})_{X(B_1)}$ over $X(B_1)$ is given by the homogeneous equations

$$t_{u_1} z_{u_2} z_{-\alpha_1^+} = t_{-u_1} z_{-u_2} z_{\alpha_1^+}, \quad t_{u_1} z_{\alpha_1^-} z_{-u_2} = t_{-u_1} z_{-\alpha_1^-} z_{u_2}$$

where $(\mathscr{L}_{\{\pm u_1\}}, \{t_{u_1}, t_{-u_1}\})$ is the universal B_1-data on $X(B_1) \cong \mathbb{P}^1$. We picture the B_1-curves defined by these equations for $(t_{u_1} : t_{-u_1}) = (0:1), (a:b), (1:0)$. If $(t_{u_1} : t_{-u_1}) = (a:b) \neq (0:1), (1:0)$ we have a projective line, we draw its projection onto $\mathbb{P}^1_{\{\pm u_2\}}$ and write the sections in terms of the homogeneous coordinates z_{u_2}, z_{-u_2}. Over the two torus fixed points of $X(B_1)$ the curve is a chain of three projective lines in $\mathbb{P}^1_{\{\pm\alpha_1^+\}} \times \mathbb{P}^1_{\{\pm\alpha_1^-\}} \times \mathbb{P}^1_{\{\pm u_2\}}$.



$(t_{u_1} : t_{-u_1}) = (1:0)$ $\quad (t_{u_1} : t_{-u_1}) = (a:b)$ $\quad (t_{u_1} : t_{-u_1}) = (0:1)$

By remark 3.5 the universal B_n-curve over $X(B_n)$ can be embedded into a product $P(B_{n+1}/B_n)_{X(B_n)} \cong (\mathbb{P}^1)^{2n+1}_{X(B_n)}$ and the embedded curve is given by equations (1) determined by the universal B_n-data. We show that any B_n-curve C over a field can be embedded into a product $(\mathbb{P}^1)^{2n+1}$ and extract B_n-data such that C is described by the same equations as the universal curve.

We fix the following notation: given a B_n-curve $(C \to Y, I, s_-, s_+, s_0, s_1^\pm, \dots, s_n^\pm)$ we associate with the sections s_0, s_i^-, s_i^+ the roots $u_{n+1}, \alpha_i^-, \alpha_i^+$ of $B_{n+1}^+ \setminus B_n^+ = \{u_{n+1}, \alpha_1^\pm, \dots, \alpha_n^\pm\}$ (cf. remark 3.5 and construction 3.1); we will write α_s for the positive root associated with the section s and conversely s_α for the section associated with the root.

Proposition 3.7. *Let $(C, I, s_-, s_+, s_0, s_1^\pm, \dots, s_n^\pm)$ be a B_n-curve over a field. For any $s \in \{s_0, s_1^\pm, \dots, s_n^\pm\}$ let $z_{\alpha_s}, z_{-\alpha_s} \in H^0(C, \mathcal{O}_C(s))$ be a basis of $H^0(C, \mathcal{O}_C(s))$ such that $z_{-\alpha_s}(s_-) = 0$, $z_{\alpha_s}(s_+) = 0$, $z_{-\alpha_s}(s) = z_{\alpha_s}(s) \neq 0$ (cf. remark 3.5 for this choice). We will write $\mathbb{P}^1_{\{\pm\alpha_s\}}$ for $\mathbb{P}(H^0(C, \mathcal{O}_C(s)))$. Then by*

$$(t_\beta : t_{-\beta}) = (z_{-\alpha_2}(s_1) : z_{\alpha_2}(s_1))$$

if $\beta = \alpha_1 - \alpha_2$ is a root of B_n and α_1, α_2 are roots corresponding to distinct marked points $s_1, s_2 \in \{s_0, s_1^\pm, \dots, s_n^\pm\}$, we can define B_n-data $(t_\beta : t_{-\beta})_{\{\pm\beta\} \subseteq B_n}$ and the morphism

$$C \to \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^+\}} \times \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^-\}} \times \mathbb{P}^1_{\{\pm u_{n+1}\}} = P(B_{n+1}/B_n)$$

is an isomorphism onto the closed subvariety $C' \subseteq P(B_{n+1}/B_n)$ determined by the homogeneous equations

$$t_\beta z_{\alpha_2} z_{-\alpha_1} = t_{-\beta} z_{-\alpha_2} z_{\alpha_1} \qquad \begin{array}{l} \text{for } \alpha_1, \alpha_2 \in \{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} \\ \text{such that } \beta = \alpha_1 - \alpha_2 \text{ is a root of } B_n \end{array} \tag{2}$$

Furthermore, C' together with the marked points s'_0 resp. $s'^{\pm}_i$ defined by the additional equations $z_{u_{n+1}} = z_{-u_{n+1}}$ resp. $z_{\alpha_i^\pm} = z_{-\alpha_i^\pm}$, the poles s'_- resp. s'_+ defined by $z_{-u_{n+1}} = 0$, $z_{-\alpha_i^\pm} = 0$ $(i = 1, \ldots, n)$ resp. $z_{u_{n+1}} = 0$, $z_{\alpha_i^\pm} = 0$ $(i = 1, \ldots, n)$ and the involution I' given by $\mathbb{P}^1_{\{\pm\alpha_i^+\}} \leftrightarrow \mathbb{P}^1_{\{\pm\alpha_i^-\}}$, $z_{\alpha_i^+} \leftrightarrow z_{-\alpha_i^-}$ and $\mathbb{P}^1_{\{\pm u_{n+1}\}} \leftrightarrow \mathbb{P}^1_{\{\pm u_{n+1}\}}$, $z_{u_{n+1}} \leftrightarrow z_{-u_{n+1}}$ is a B_n-curve and $(C, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm) \to (C', I', s'_-, s'_+, s'_0, s'^{\pm}_1, \ldots, s'^{\pm}_n)$ an isomorphism of B_n-curves.

Proof. The data $(t_\beta : t_{-\beta})$ is defined as position of a marked point s_1 relative to another marked point s_2 of C if $\beta = \alpha_{s_1} - \alpha_{s_2}$. We also write s_1/s_2 for this data. We have the following cases:

$$\begin{array}{rclcl} \beta_{ij} & = & \alpha_i^+ - \alpha_j^+ & = & \alpha_j^- - \alpha_i^- \\ \gamma_{ij} & = & \alpha_i^+ - \alpha_j^- & = & \alpha_j^+ - \alpha_i^- \\ u_i & = & \alpha_i^+ - u_{n+1} & = & u_{n+1} - \alpha_i^- \end{array}$$

Note that because of the symmetry of the B_n-curve with respect to the involution I we have for the corresponding data $s_i^+/s_j^+ = s_j^-/s_i^-$, $s_i^+/s_j^- = s_j^+/s_i^-$, $s_i^+/s_0 = s_0/s_i^-$, so the data $(t_\beta : t_{-\beta})_{\{\pm\beta\}\subseteq B_n}$ is well defined.

The rest of the proof is similar to the proof of [BB11, Prop. 3.12]. To check that $(t_\beta : t_{-\beta})_{\{\pm\beta\}\subseteq B_n}$ is B_n-data, we have to check the equations $t_\beta t_\gamma t_{-\delta} = t_{-\beta} t_{-\gamma} t_\delta$ for the linear relations $\beta + \gamma = \delta$ given in lemma 2.1; these equations can be written in the form $s_1/s_2 \cdot s_2/s_3 = s_1/s_3$ for some sections s_1, s_2, s_3. □

We will continue to use the notations $s'/s = (t_\beta : t_{-\beta})$ for $\beta = \alpha_{s'} - \alpha_s$, we have $s_-/s = (0 : 1)$ and $s_+/s = (1 : 0)$ (points s', s_-, s_+ with respect to the coordinates $(z_{-\alpha_s} : z_{\alpha_s})$).

Lemma 3.8. *Any B_n-data over a field arises as B_n-data extracted from a B_n-curve by the method of proposition 3.7.*

Proof. Let $(t_\beta : t_{-\beta})_{\{\pm\beta\}\subseteq B_n}$ be B_n-data over a field.

We can define an ordering $\prec$ on the set of positive roots $\{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} = B_{n+1}^+ \setminus B_n^+$: for distinct α, α' define $\alpha' \prec \alpha$ (resp. $\alpha' \preceq \alpha$) if $(t_\beta : t_{-\beta}) = (0 : 1)$ (resp. $(t_\beta : t_{-\beta}) \neq (1 : 0)$) for $\beta = \alpha' - \alpha$. This defines a decomposition $\{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} = P_{-m} \sqcup \ldots \sqcup P_m$ into nonempty equivalence classes such that $\alpha' \prec \alpha \iff \alpha' \in P_{k'}$, $\alpha \in P_k$ for $k' < k$. We have the symmetries $u_{n+1} \prec \alpha_i^\pm \iff \alpha_i^\mp \prec u_{n+1}$ and $\alpha_i^{\varepsilon_i} \prec \alpha_j^{\varepsilon_j} \iff \alpha_j^{-\varepsilon_j} \prec \alpha_i^{-\varepsilon_i}$ and these symmetries imply $u_{n+1} \in P_0$ and $\alpha_i^+ \in P_k \iff \alpha_i^- \in P_{-k}$.

Now it is easy to construct a B_n-curve such that the B_n-data extracted from it by the method of proposition 3.7 is the given B_n-data by taking a chain of projective lines of length $2m+1$ with involution (C, I, s_-, s_+) (see definition 1.2) and choosing suitable marked points satisfying $s_0 \in C_0$ and $s_i^\pm \in C_k \iff \alpha_i^\pm \in P_k$. □

Let C be a B_n-curve over a field. It decomposes into irreducible components $C = C_{-m} \cup \ldots \cup C_m$ with $s_- \in C_{-m}$, $s_+ \in C_m$. The decomposition

$$\{0, \pm 1, \ldots, \pm n\} = P_{-m} \sqcup \ldots \sqcup P_m$$

such that $0 \in P_0$ and $\varepsilon i \in P_k \iff s_i^\varepsilon \in C_k$, we will call the combinatorial type of the B_n-curve (or of the corresponding B_n-data) over a field. We will also write this in the form $s_{i_1}^{-\varepsilon_1} \ldots s_{i_l}^{-\varepsilon_l} | \ldots | s_{i_1}^{\varepsilon_1} \ldots s_{i_l}^{\varepsilon_l}$ with the sections for the different sets P_k separated by the symbol "$|$" starting on the left with P_{-m}. Considering the fibres of the universal B_n-curve resp. the universal B_n-data, these combinatorial types determine a stratification of $X(B_n)$ which coincides with the stratification of this toric variety into torus orbits.

Proposition 3.9. *Over the torus orbit in $X(B_n)$ corresponding to the one-dimensional cone generated by $\varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_k} v_{i_k}$ we have the combinatorial type*

$$s_{i_1}^{\varepsilon_{i_1}} \cdots \ s_{i_k}^{\varepsilon_{i_k}} | s_0 s_{i_{k+1}}^{\pm} \cdots \ s_{i_n}^{\pm} | s_{i_1}^{-\varepsilon_{i_1}} \cdots \ s_{i_k}^{-\varepsilon_{i_k}}$$

Proof. The universal B_n-data over each point of the closure of the orbit corresponding to a generator of a one-dimensional cone generated by v has the property $(t_\beta : t_{-\beta}) = (0 : 1)$ if $\langle \beta, v \rangle > 0$ (see [BB11, Rem. 1.21]). For $v = \varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_k} v_{i_k}$ this in particular implies $s_{i_l}^{\varepsilon_{i_l}} / s_0 = s_0 / s_{i_l}^{-\varepsilon_{i_l}} = (t_{\varepsilon_{i_l} u_{i_l}} : t_{-\varepsilon_{i_l} u_{i_l}}) = (0 : 1) = s_- / s_0$. $\square$

4. Moduli space of B_n-curves

In this section we show that there is a fine moduli space of B_n-curves $\overline{L}_n^{0,\pm}$ which is isomorphic to the toric variety $X(B_n)$ by constructing an isomorphism between the moduli functor of B_n-curves and the functor of $X(B_n)$. For the second functor we use the description in [BB11, 1.3] in terms of B_n-data.

To relate B_n-curves to B_n-data we consider an embedding of arbitrary B_n-curves over a scheme Y into a product $(\mathbb{P}^1)_Y^{2n+1}$ that generalises the embedding in proposition 3.7 to the relative situation. The main tool are the following contraction morphisms (cf. [BB11, 3.3]): for a subset $\{s_1, \ldots, s_l\}$ of the sections of a pointed chain of projective lines C there is a line bundle $\mathcal{O}_C(s_1 + \ldots + s_l)$ on C and a morphism $C \to C_{\{s_1,\ldots,s_l\}} \subseteq \mathbb{P}_Y(\pi_* \mathcal{O}_C(s_1 + \ldots + s_l))$ such that the morphisms $C_y \to (C_{\{s_1,\ldots,s_l\}})_y$ on the fibres are isomorphisms on the components containing one of the sections $s_i(y)$ and contract all other components (see [BB11, Constr. 3.15]).

We will make use of the particular cases of contraction with respect to one section onto a $\mathbb{P}^1$-bundle, with respect to two sections onto an A_1-curve and with respect to three sections onto an A_2-curve; we will apply [BB11, Constr. 3.16; Lemma 3.17 and 3.18].

We associate with the sections $s_0, s_i^\pm$ the roots $u_{n+1}, \alpha_i^\pm$ as we did before proposition 3.7. For a B_n-curve $(C \to Y, I, s_-, s_+, s_1^\pm, \ldots, s_n^\pm)$ we denote the contraction morphisms with respect to one section s_0, s_i^- resp. s_i^+ by $p_0 \colon C \to (\mathbb{P}^1_{\{\pm u_{n+1}\}})_Y$, $p_i^- \colon C \to (\mathbb{P}^1_{\{\pm \alpha_i^-\}})_Y$ resp. $p_i^+ \colon C \to (\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$, where $(\mathbb{P}^1_{\{\pm u_{n+1}\}})_Y$, $(\mathbb{P}^1_{\{\pm \alpha_i^-\}})_Y$ resp. $(\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$ is a copy of $\mathbb{P}^1_Y$ with homogeneous coordinates $z_{u_{n+1}}, z_{-u_{n+1}}$ resp.

$z_{\alpha_i^-}, z_{-\alpha_i^-}$ resp. $z_{\alpha_i^+}, z_{-\alpha_i^+}$ such that in these coordinates s_-, s_+, s_0 resp. s_-, s_+, s_i^- resp. s_-, s_+, s_i^+ become the $(1:0), (0:1), (1:1)$-section of $\mathbb{P}^1_Y$.

Theorem 4.1. *There exists a fine moduli space $\overline{L}_n^{0,\pm}$ of B_n-curves isomorphic to the toric variety $X(B_n)$ with universal family $X(B_{n+1}) \to X(B_n)$.*

Proof. We show that the moduli functor of B_n-curves $\overline{L}_n^{0,\pm}$ as defined in section 1 is isomorphic to the functor F_{B_n} of the toric variety $X(B_n)$ as described in [BB11, 1.3].

Let Y be a scheme. For B_n-data on Y we construct a B_n-curve C over Y via equations in $P(B_{n+1}/B_n)_Y$ as in remark 3.5 with the given B_n-data on Y replacing the universal B_n-data on $X(B_n)$. This is a B_n-curve: any B_n-data is pull-back of the universal B_n-data on $X(B_n)$, so the constructed curve is pull-back of the universal B_n-curve over $X(B_n)$.

In the other direction, given a B_n-curve over Y we extract B_n-data. For each pair of distinct sections $s_1, s_2 \in \{s_0, s_1^\pm, \ldots, s_n^\pm\}$ we have a contraction morphism $C \to C_{\{s_1,s_2\}}$ onto an A_1-curve over Y. From $(C_{\{s_1,s_2\}}, s_1, s_2)$ we extract A_1-data $(\mathscr{L}_{\{1,2\}}, \{t_{1,2}, t_{2,1}\})$ as in [BB11, Constr. 3.16]: we put $\mathscr{L}_{\{\pm\beta\}} := \mathscr{L}_{\{1,2\}}$, $t_\beta := t_{1,2}$, $t_{-\beta} := t_{2,1}$ for $\beta = \alpha_{s_1} - \alpha_{s_2}$ (then $(t_\beta : t_{-\beta})$ measures the position of s_1 relative to s_2, we write this as s_1/s_2). We have the following cases:

$$\begin{array}{rclcl}
\beta_{ij} & = & \alpha_i^+ - \alpha_j^+ & = & \alpha_j^- - \alpha_i^- \\
\gamma_{ij} & = & \alpha_i^+ - \alpha_j^- & = & \alpha_j^+ - \alpha_i^- \\
u_i & = & \alpha_i^+ - u_{n+1} & = & u_{n+1} - \alpha_i^-
\end{array}$$

Because of the symmetry of the B_n-curve with respect to the involution we have for the corresponding data $s_i^+/s_j^+ = s_j^-/s_i^-$, $s_i^+/s_j^- = s_j^+/s_i^-$, $s_i^+/s_0 = s_0/s_i^-$, so the data $(\mathscr{L}_{\{\pm\beta\}}, \{t_\beta, t_{-\beta}\})_{\{\pm\beta\}\subseteq B_n}$ is well defined.

We show that the data obtained this way is B_n-data. Let β, γ, δ be positive roots of B_n such that $\beta + \gamma = \delta$. We have to verify that the collection $\{(\mathscr{L}_{\{\pm\beta\}}, \{t_\beta, t_{-\beta}\}), (\mathscr{L}_{\{\pm\gamma\}}, \{t_\gamma, t_{-\gamma}\}), (\mathscr{L}_{\{\pm\delta\}}, \{t_\delta, t_{-\delta}\})\}$ satisfies $t_\beta t_\gamma t_{-\delta} = t_{-\beta} t_{-\gamma} t_\delta$, which means that it is A_2-data. By lemma 2.1 we have the following cases:

$$\begin{array}{rl}
\beta_{ij} + u_j = u_i & (i,j \in \{1,\ldots,n\},\ i<j) \\
u_i + u_j = \gamma_{ij} & (i,j \in \{1,\ldots,n\},\ i \neq j) \\
\beta_{ij} + \beta_{jk} = \beta_{ik} & (i,j,k \in \{1,\ldots,n\},\ i<j<k) \\
\beta_{ij} + \gamma_{jk} = \gamma_{ik} & (i,j,k \in \{1,\ldots,n\},\ i<j,\ k \neq i,j)
\end{array}$$

In each of these cases we can write $\beta = \alpha_{s_1} - \alpha_{s_2}$, $\gamma = \alpha_{s_2} - \alpha_{s_3}$ for three distinct sections $s_1, s_2, s_3 \in \{s_0, s_1^\pm, \ldots, s_n^\pm\}$. Then these equations can be interpreted as relations between the relative positions of pairs of sections in a set of three sections, we write this as $s_1/s_2 \cdot s_2/s_3 = s_1/s_3$:

$$\begin{array}{ccc}
\beta_{ij} + u_j = u_i, & \beta_{ij} = \alpha_i^+ - \alpha_j^+, u_j = \alpha_j^+ - u_{n+1}, & s_i^+/s_j^+ \cdot s_j^+/s_0 = s_i^+/s_0 \\
u_i + u_j = \gamma_{ij}, & u_i = \alpha_i^+ - u_{n+1}, u_j = u_{n+1} - \alpha_j^-, & s_i^+/s_0 \cdot s_0/s_j^- = s_i^+/s_j^- \\
\beta_{ij} + \beta_{jk} = \beta_{ik}, & \beta_{ij} = \alpha_i^+ - \alpha_j^+, \beta_{jk} = \alpha_j^+ - \alpha_k^+, & s_i^+/s_j^+ \cdot s_j^+/s_k^+ = s_i^+/s_k^+ \\
\beta_{ij} + \gamma_{jk} = \gamma_{ik}, & \beta_{ij} = \alpha_i^+ - \alpha_j^+, \gamma_{jk} = \alpha_j^+ - \alpha_k^-, & s_i^+/s_j^+ \cdot s_j^+/s_k^- = s_i^+/s_k^-
\end{array}$$

We have a contraction morphism $C \to C_{\{s_1,s_2,s_3\}}$ over Y onto an A_2-curve $C_{\{s_1,s_2,s_3\}}$ over Y. The data $\{(\mathscr{L}_{\{\pm\beta\}}, \{t_\beta, t_{-\beta}\}), (\mathscr{L}_{\{\pm\gamma\}}, \{t_\gamma, t_{-\gamma}\}),$

$(\mathscr{L}_{\{\pm\delta\}}, \{t_\delta, t_{-\delta}\})\}$ coincides with the data extracted from this A_2-curve and is A_2-data by [BB11, Lemma 3.18].

Both constructions commute with base-change and thus define morphisms of functors $F_{B_n} \to \underline{\overline{L}_n^{0,\pm}}$ and $\underline{\overline{L}_n^{0,\pm}} \to F_{B_n}$. As in the proof of [BB11, Thm. 3.19] one shows that they are inverse to each other. □

Remark 4.2. The moduli space $\overline{L}_n^{0,\pm}$ embeds naturally into $\overline{L}_{2n+1}$. A morphism $\overline{L}_n^{0,\pm} \to \overline{L}_{2n+1}$ is given by considering a B_n-curve with sections $s_1^-, \ldots, s_n^-, s_0, s_n^+, \ldots, s_1^+$ as an A_{2n}-curve with sections $s_1, \ldots, s_{n+1}, \ldots, s_{2n+1}$. This corresponds to the toric morphism $X(B_n) \to X(A_{2n})$ given by the projection of root systems $A_{2n} \to B_n$ mapping $u_i - u_{n+1} \mapsto u_i$, $u_{2n+2-i} - u_{n+1} \mapsto -u_i$ $(i = 1, \ldots, n)$ with kernel generated by $u_i + u_{2n+2-i} - 2u_{n+1}$ $(i = 1, \ldots, n)$.

5. (Co)homology of $\overline{L}_n^{0,\pm} = X(B_n)$

We show that the (co)homology of the moduli space $\overline{L}_n^{0,\pm} = X(B_n)$ over the complex numbers has a description similar to that of the (co)homology of the Losev-Manin moduli spaces $\overline{L}_n = X(A_n)$ (cf. [BB11, 2.2]).

The torus invariant divisors of $\overline{L}_n^{0,\pm} = X(B_n)$ correspond to elements of the set $\mathcal{B}$ (see section 2 and prop. 3.9). Here, as in the case of the toric varieties $X(A_n)$, all primitive collections consist of two elements corresponding to non comparable sets $B, B' \in \mathcal{B}$. As usual the integral cohomology is torsion free and confined to the even degrees and standard methods from toric geometry (see e.g. [Dan, (10.8)]) give:

Proposition 5.1. *For the cohomology ring of the toric variety $X(B_n)$ over the complex numbers we have*

$$H^*(X(B_n), \mathbb{Z}) \;\cong\; \mathbb{Z}[\,l_B : B \in \mathcal{B}\,]/(R_1 + R_2)$$

where R_1 is the ideal generated by the elements $r_i = \sum_{i\in B} l_B - \sum_{-i\in B} l_B$ for $i = 1, \ldots, n$ and R_2 the ideal generated by the elements $r_{B,B'} = l_B l_{B'}$ for $B, B' \in \mathcal{B}$ such that $B \not\subseteq B'$, $B' \not\subseteq B$.

We proceed by determining the Betti numbers and the Poincaré polynomial and obtain the following closed formula which is an analogue to [LM00, (2.3)].

Proposition 5.2. *Let $p_{X(B_n)}(t) = \sum_{i=0}^n \beta_{2i}(X(B_n))t^i$ be the Poincaré polynomial of $X(B_n)$ with $\beta_{2i}(X(B_n)) = \operatorname{rk} H^{2i}(X(B_n), \mathbb{Z})$ the Betti numbers. Then we have*

$$\sum_{n=0}^{\infty} \frac{p_{X(B_n)}(t)}{n!} y^n \;=\; e^{y(t-1)} \frac{t-1}{t - e^{2y(t-1)}} \;\in\; \mathbb{Z}[t][[y]]$$

Proof. We have $p_{X(B_n)}(t) = \sum_{m=0}^n d_m(B_n)(t-1)^{n-m}$ (see [Ful, p. 92] or [Dan, (10.8)]; this can be shown in different ways, one possibility is by counting points over finite fields as in [LM00]) with $d_m(B_n) =$ *number of $(n-m)$-dim. torus orbits of* $X(B_n) =$

number of m-dim. cones of $\Sigma(B_n)$. Inserting this into $\sum_{n=0}^{\infty}\frac{p_{X(B_n)}(t)}{n!}y^n$ and interchanging summation by n and m, we get

$$\sum_{n=0}^{\infty}\frac{p_{X(B_n)}(t)}{n!}y^n \;=\; \sum_{m=0}^{\infty}\frac{1}{(t-1)^m}\sum_{n=m}^{\infty}\frac{d_m(B_n)}{n!}(t-1)^n y^n$$

The number $d_m(B_n)$ can be calculated as

$$\tfrac{1}{n!}d_m(B_n) \;=\; \sum_{(a_0,a_1,\ldots,a_m)}\frac{1}{a_0!}\frac{2^{a_1}}{a_1!}\cdots\frac{2^{a_m}}{a_m!}$$

where the sum runs over sequences $a_0\in\mathbb{Z}_{\geq 0}$, $a_1\in\mathbb{Z}_{>0}$, …, $a_m\in\mathbb{Z}_{>0}$ such that $\sum_i a_i = n$ (note that any family $B^{(m)}\subsetneq\ldots\subsetneq B^{(1)}$ of elements of $\mathcal{B}$ corresponding to an m-dimensional cone of $\Sigma(B_n)$ determines such a partition by $a_m = |B^{(m)}|$, $a_{m-1} = |B^{(m-1)}| - |B^{(m)}|$, … , $a_0 = n - |B^{(1)}|$, in addition we have orderings and signs). Making use of the fact that $\frac{1}{n!}d_m(B_n)$ coincides with the coefficient of x^n in the power series $e^x(e^{2x}-1)^m$, we obtain

$$\sum_{n=0}^{\infty}\frac{p_{X(B_n)}(t)}{n!}y^n \;=\; e^{y(t-1)}\sum_{m=0}^{\infty}\frac{1}{(t-1)^m}(e^{2y(t-1)}-1)^m$$

which yields the result. □

In particular we have $\chi(X(B_n)) = 2^n n!$ (this reflects the fact that we have $2^n n!$ maximal cones), $\beta_2(X(B_n)) = 3^n - n - 1$ (corresponding to the fact that we have $3^n - 1$ one-dimensional cones) and for the first Poincaré polynomials

$$p_{X(B_1)}(t) = t+1,\quad p_{X(B_2)}(t) = t^2+6t+1,\quad p_{X(B_3)}(t) = t^3+23t^2+23t+1$$
$$p_{X(B_4)}(t) = t^4+76t^3+230t^2+76t+1$$
$$p_{X(B_5)}(t) = t^5+237t^4+1682t^3+1682t^2+237t+1$$
$$p_{X(B_6)}(t) = t^6+722t^5+10543t^4+23548t^3+10543t^2+722t+1$$

The ring $\mathbb{Z}[\,l_B : B\in\mathcal{B}\,]/R_2$ is the Stanley-Reisner ring for the triangulation of the $(n-1)$-dimensional sphere determined by the fan $\Sigma(B_n)$. It is a Cohen-Macaulay ring and the elements $r_1,\ldots,r_n$ that generate R_1 form a regular sequence. The calculation of the Poincaré polynomial of a toric variety in [Dan, (10.8)] in terms of the numbers of cones of dimension $d = 1,\ldots,n$ only depends on the Hilbert-Poincaré series of the Stanley-Reisner ring of the fan and the fact that the quotient by an ideal generated by a regular sequence is taken. In [Re01] a ring has been defined by taking the same Stanley-Reisner ring (over a field) but instead of R_1 an ideal generated by a different regular sequence, so by construction this ring has the same Poincaré polynomial as the cohomology ring of $X(B_n)$.

The $\mathbb{Z}$-module $\mathbb{Z}[\,l_B : B\in\mathcal{B}\,]/(R_1+R_2)$ is generated by the classes of square-free monomials (see [Dan, (10.7.1)]). We can restrict to monomials each of which has only factors corresponding to one-dimensional faces of one maximal cone. Such a monomial $\prod_{i=1}^{m} l_{B^{(i)}}$ corresponds to an m-dimensional face of the respective maximal cone and on the other hand to a collection $B^{(m)}\subsetneq\ldots\subsetneq B^{(1)}$ of elements of $\mathcal{B}$. We denote the $\mathbb{Z}$-submodule of $\mathbb{Z}[\,l_B : B\in\mathcal{B}\,]$ generated by these monomials by G. There is the canonical isomorphism of $\mathbb{Z}$-modules $G/U\cong\mathbb{Z}[\,l_B : B\in\mathcal{B}\,]/(R_1+R_2)$ where $U = (R_1+R_2)\cap G$. As usual, the module G/U can be identified with the homology module $H_*(X(B_n),\mathbb{Z})$. The monomial $\prod_{i=1}^m l_{B^{(i)}}$ then corresponds to the class of the orbit closure for the cone determined by the collection $B^{(m)} \subsetneq \ldots \subsetneq B^{(1)}$, in particular the monomials of G of degree m generate $H_{2(n-m)}(X(B_n),\mathbb{Z})$.

The maximal cones of the fan $\Sigma(B_n)$ correspond to collections $B^{(n)} \subsetneq \ldots \subsetneq B^{(1)}$ of elements of $\mathcal{B}$ and these correspond to so called signed permutations, that is elements of the Weyl group $W(B_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n =: S_n^{\pm}$. A signed permutation $w \in S_n^{\pm}$ corresponds via $(w(1),\ldots,w(n))$ to a sequence of distinct elements in $\{\pm 1,\ldots,\pm n\}$ for any i not containing both $-i$ and i. For a collection $B^{(n)} \subsetneq \ldots \subsetneq B^{(1)}$ of elements of $\mathcal{B}$ the corresponding signed permutation $\sigma \in S_n^{\pm}$ is given by $\{w(k)\} = B^{(k)} \backslash B^{(k+1)}$ for $k = 1,\ldots,n$ (put $B^{(n+1)} = \emptyset$). The descent set of a signed permutation $w \in S_n^{\pm}$ is the set (put $w(0) = 0$)

$$\operatorname{Desc}(w) = \{k \in \{1,\ldots,n\} \mid w(k-1) > w(k)\}$$

For any $w \in S_n^{\pm}$ we define a monomial in G by

$$l^w = \textstyle\prod_{k \notin \operatorname{Desc}(w)} l_{\{w(k),\ldots,w(n)\}}$$

this way we have defined $2^n n!$ distinct monomials.

Proposition 5.3. *The classes of the monomials l^w for $w \in S_n^{\pm}$ form a basis of the homology module $G/U = H_*(X(B_n),\mathbb{Z})$. The module of relations U is generated by the elements*

$$r_{i,j}((B^{(h)})_h, k) = \Big(\sum_{\substack{i \in B \\ j \notin B}} l_B - \sum_{\substack{j \in B \\ i \notin B}} l_B \Big) \textstyle\prod_{h=1}^m l_{B^{(h)}}$$

(sums over sets $B^{(k+1)} \subsetneq B \subsetneq B^{(k)}$) for collections $B^{(m)} \subsetneq \ldots \subsetneq B^{(1)}$, $m \geq 1$ of elements of $\mathcal{B}$ and $k \in \{1,\ldots,m\}$, $i,j \in B^{(k)} \setminus B^{(k+1)}$ (put $B^{(m+1)} = \emptyset$), $i \neq j$, and by the elements

$$r_i((B^{(h)})_h) = \Big(\textstyle\sum_{i \in B} l_B - \sum_{-i \in B} l_B \Big) \prod_{h=1}^m l_{B^{(h)}}$$

(sums over sets $B \in \mathcal{B}$ such that $B^{(1)} \subsetneq B$ if $m \geq 1$) for collections $B^{(m)} \subsetneq \ldots \subsetneq B^{(1)}$, $m \geq 0$ of elements of $\mathcal{B}$ and $i \in \{1,\ldots,n\}$ such that $-i, i \notin B^{(1)}$ if $m \geq 1$.

Proof. We observe that the given relations are contained in U. We have $2^n n!$ monomials l^w, this number coincides with the rank of G/U. Thus it remains to show that every monomial in G via the given relations is equivalent to a linear combination of the monomials l^w.

For a monomial $\prod_{k=1}^m l_{B^{(k)}}$ corresponding to a collection $B^{(m)} \subsetneq \ldots \subsetneq B^{(1)}$, $m \geq 1$ we define the number $d(\prod_{k=1}^m l_{B^{(k)}}) := |\{k \in \{1,\ldots,m\} \mid \min P_{k-1} > \max P_k\}| \in \mathbb{Z}_{\geq 0}$ in terms of the associated partition $P_m = B^{(m)}$, $P_{m-1} = B^{(m-1)} \setminus B^{(m)}$, $\ldots$, $P_1 = B^{(1)} \setminus B^{(2)}$, $P_0 = \{0,\pm 1,\ldots,\pm n\} \setminus \{\pm i \,|\, i \in B^{(1)} \text{ or } -i \in B^{(1)}\}$. The monomials $y \in G$ satisfying $d(y) = 0$ are exactly the monomials of the form l^w. We define the following ordering $\prec$ of the monomials of G: take the partition $(P_k)_{k=0,\ldots,m}$ associated with a monomial and consider the sequence that arises by taking the sets $P_m,\ldots,P_1$ in this order and by ordering the elements of each P_k according to their size, on these sequences we take the lexicographic order.

We show that every monomial in G modulo U is equivalent to a linear combination of the monomials l^w, $w \in S_n^{\pm}$ by showing that every monomial $y \in G$ with $d(y) > 0$ modulo a relation is equivalent to a linear combination of monomials y' with $y \prec y'$. In fact, let $B^{(m)} \subsetneq \ldots \subsetneq B^{(1)}$, $m \geq 1$ be a collection of elements of $\mathcal{B}$ (put $B^{(m+1)} := \emptyset$) with associated partition $(P_k)_{k=0,\ldots,m}$ such that the corresponding monomial $y = \prod_{k=1}^m l_{B^{(k)}}$ satisfies $d(y) > 0$. Take $k \in \{1, \ldots, m\}$ such that $i := \min P_{k-1} > \max P_k =: j$. If $k \in \{2, \ldots, m\}$ then

$$r_{i,j}((B^{(h)})_{h\neq k}, k-1) = \Big(\sum_{\substack{i\in B\\ j\notin B}} l_B - \sum_{\substack{j\in B\\ i\notin B}} l_B \Big) \prod_{h\neq k} l_{B^{(h)}}$$

(sums over sets B such that $B^{(k+1)} \subsetneq B \subsetneq B^{(k-1)}$) is a relation that contains y as the unique monomial minimal with respect to $\prec$. If $k = 1$ then

$$r_{-j}((B^{(h)})_{h\neq 1}) = \Big(\sum_{-j\in B} l_B - \sum_{j\in B} l_B \Big) \prod_{h=2}^m l_{B^{(h)}}$$

(sums over sets $B \in \mathcal{B}$ such that $B^{(2)} \subsetneq B$) is such a relation. □

The proposition implies that the Betti numbers of $X(B_n)$ coincide with the number of signed permutations with prescribed number of descents, for this see also [DL94, Section 4], [St94]. Our basis of $H_*(X(B_n), \mathbb{Z})$ coincides with the basis given in [Kl85], [Kl95] in the general case of toric varieties associated with root systems (see the following remark).

Remark 5.4. In [Kl85] a basis of the homology $H_*(X(R), \mathbb{Z})$ is constructed as follows. For a fixed set of simple roots $S \subset R$ and the corresponding Weyl chamber $\sigma_S = S^\vee$ consider for each $w \in W(R)$ the face $\sigma_w \subseteq w\sigma_S$ given as the intersection of those walls of $w\sigma_S$ that separate σ_S and $w\sigma_S$, i.e. we have the intersection of $w\sigma_S$ with those subspaces $(w\alpha)^\perp$, $\alpha \in S$, for which $w\alpha$ is a negative root. The cycles corresponding to the family of cones $(\sigma_w)_{w\in W(R)}$ form a basis of $H_*(X(R), \mathbb{Z})$.

In our case we may choose the set of simple roots $S = \{u_n - u_{n-1}, \ldots, u_2 - u_1, u_1\} \subset B_n$; the corresponding Weyl chamber is generated by $v_n, v_{n-1} + v_n, \ldots, v_1 + \ldots + v_n$. Then for $w \in W(B_n) = S_n^{\pm}$ we have $w(u_k - u_{k-1})$ *is negative* $\Longleftrightarrow w(k-1) > w(k)$ for $k \in \{2, \ldots, n\}$ and $w(u_1)$ *is negative* $\Longleftrightarrow 0 > w(1)$. So, each root $\alpha \in S$ such that $w\alpha$ is negative corresponds to an element of $\mathrm{Desc}(w)$. Since $(w(u_k - u_{k-1}))^\perp \cap w\sigma_S$ is generated by $\{w(v_n), \ldots, w(v_1 + \ldots + v_n)\} \setminus \{w(v_k + \ldots + v_n)\}$ and $(w(u_1))^\perp \cap w\sigma_S$ by $\{w(v_n), \ldots, w(v_2 + \ldots + v_n)\}$, it follows that σ_w is generated by $\{v_{\{w(k),\ldots,w(n)\}} \mid k \notin \mathrm{Desc}(w)\}$ and the class of the respective torus invariant cycle corresponds to the monomial l^w.

6. Root systems of type C

Consider an n-dimensional Euclidean space E with basis $u_1, \ldots, u_n$. The root system C_n in E consists of the $2n^2$ roots:

$$\pm 2u_i \ \textit{for}\ i \in \{1, \ldots, n\}; \quad \pm(u_i + u_j), \pm(u_i - u_j) \ \textit{for}\ i, j \in \{1, \ldots, n\}, i < j.$$

The following is a set of simple roots:

$$u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, 2u_n.$$

Let $M(C_n)$ be the root lattice. The Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts by $u_i \mapsto \pm u_i$ and by permuting the u_i. So there are $2^n n!$ sets of simple roots, these are of the form $\varepsilon_1 u_{i_1} - \varepsilon_2 u_{i_2}, \varepsilon_2 u_{i_2} - \varepsilon_3 u_{i_3}, \ldots, \varepsilon_{n-1} u_{i_{n-1}} - \varepsilon_n u_{i_n}, 2\varepsilon_n u_{i_n}$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_1, \ldots, \varepsilon_n$.

The vector space E^* dual to E with basis $v_1, \ldots, v_n$ dual to $u_1, \ldots, u_n$ contains the lattice $N(C_n)$ dual to $M(C_n)$. To describe the fan $\Sigma(C_n)$ in the lattice $N(C_n)$ we describe a Weyl chamber. For the set of simple roots $S = \{u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, 2u_n\}$ has the dual basis $v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{n-1}, \frac{1}{2}(v_1 + \ldots + v_n)$ of $N(C_n)$, the Weyl chamber σ_S is equal to $\langle v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{n-1}, \frac{1}{2}(v_1 + \ldots + v_n)\rangle_{\mathbb{Q}_{\geq 0}}$. All Weyl chambers are generated by collections of elements of the form $\varepsilon_1 v_{i_1}, \varepsilon_1 v_{i_1} + \varepsilon_2 v_{i_2}, \ldots, \frac{1}{2}(\varepsilon_1 v_{i_1} + \ldots + \varepsilon_n v_{i_n})$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs ε_i. There are $3^n - 1$ one-dimensional cones generated by elements of the form $\varepsilon_1 v_{i_1} + \ldots + \varepsilon_k v_{i_k}$ for $k \in \{1, \ldots, n-1\}$ or of the form $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$.

The torus invariant divisor for the one-dimensional cone generated by $\varepsilon_1 v_{i_1} + \ldots + \varepsilon_k v_{i_k}$ is isomorphic to $X(C_{n-k}) \times X(A_{k-1})$, that for $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ is isomorphic to $X(A_{n-1})$.

$X(C_{n+1})$ over $X(C_n)$. Consider the proper surjective morphism $X(C_{n+1}) \to X(C_n)$ induced by the root subsystem $C_n \subset C_{n+1}$ consisting of the roots in the subspace generated by $u_1, \ldots, u_n$. As in the B-case one shows that $X(C_{n+1})$ is flat over $X(C_n)$.

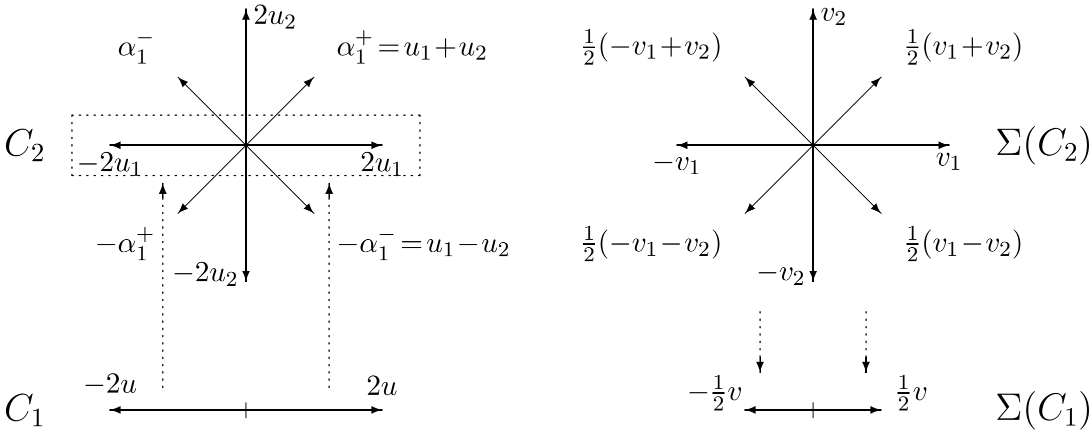
The automorphism of C_{n+1} given as the reflection for the root $\pm u_{n+1}$ fixes $C_n \subset C_{n+1}$ and induces an involution I of $X(C_{n+1})$ over $X(C_n)$. We have two sections s_-, s_+ defined as in the B-case. There are $2n+1$ additional pairs of opposite roots, the pairs $\pm\alpha_i^+ = \pm(u_{n+1} + u_i)$, $\pm\alpha_i^- = \pm(u_{n+1} - u_i)$ for $i \in \{1, \ldots, n\}$ and the pair $\pm 2u_{n+1}$. Any pair $\pm\alpha_i^+$, $\pm\alpha_i^-$ defines a projection onto the root subsystem $C_n \subset C_{n+1}$ in the sense of [BB11, 1.2], thus we have sections s_i^+ and s_i^-. The pair $\pm 2u_{n+1}$ does not define a projection of root systems $C_{n+1} \to C_n$, so it does not induce a section. However, we can consider the morphism $X(C_{n+1}) \to \mathbb{P}^1_{\{\pm 2u_{n+1}\}}$ and the preimage of the point $(1:1)$. We denote this subscheme of $X(C_{n+1})$ by S_0; it is finite flat of degree 2 over $X(C_n)$ (see below), such a subscheme we will call a double-section.

If we consider $X(C_{n+1})$ and $X(C_n)$ as embedded $X(C_{n+1}) \subseteq P(C_{n+1})$, $X(C_n) \subseteq P(C_n)$, then the morphism $X(C_{n+1}) \to X(C_n)$ is induced by the projection onto the subproduct $P(C_{n+1}) \to P(C_n)$ and $X(C_{n+1})$ is given in $P(C_{n+1}/C_n)_{X(C_n)} = \big(\prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^+\}} \times \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^-\}} \times \mathbb{P}^1_{\{\pm 2u_{n+1}\}}\big)_{X(C_n)}$ by the homogeneous equations involving the universal C_n-data on $X(C_n)$

$$z_{\alpha_i^-} z_{\alpha_i^+} z_{-2u_{n+1}} = z_{-\alpha_i^-} z_{-\alpha_i^+} z_{2u_{n+1}}, \quad i \in \{1, \ldots, n\} \tag{3}$$

$$t_\beta z_{\alpha_2} z_{-\alpha_1} = t_{-\beta} z_{-\alpha_2} z_{\alpha_1}, \quad \alpha_1, \alpha_2 \in \{\alpha_1^\pm, \ldots, \alpha_n^\pm\},\ \alpha_1 \neq \alpha_2,\ \beta = \alpha_1 - \alpha_2 \tag{4}$$

Example 6.1. We picture the inclusion of root systems $C_1 \subset C_2$ and the map of fans $\Sigma(C_2) \to \Sigma(C_1)$.



The fibres of $X(C_{n+1}) \to X(C_n)$ can be studied for example using the above description in terms of equations or by employing the description of $X(C_n)$ as quotient of $X(B_n)$ (see below). We obtain the following result, in particular the fibres over a union of torus invariant divisors are not reduced.

Proposition 6.2. *We define $D \subset X(C_n)$ to be the union of the torus invariant divisors corresponding to the one-dimensional cones of $\Sigma(C_n)$ generated by elements of the form $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$. For the structure of the fibres of the morphism $X(C_{n+1}) \to X(C_n)$ together with the involution I, the sections $s_i^\pm$ and the double-section S_0, there are the following two situations.*

Over $X(C_n) \setminus D$ the fibres are B_n-curves except that instead of the section s_0 we have a double-section S_0 which consists of the two fixed points under I. In this case the central component contains some of the sections $s_i^\pm$.

Over D the fibres are B_n-curves except that the central component is nonreduced of the form $\mathbb{P}^1_{K[\varepsilon]/\langle\varepsilon^2\rangle}$ with the double-section $S_0 \cong \operatorname{Spec} K[\varepsilon]/\langle\varepsilon^2\rangle$ concentrated in one point. The intersection of the central component with the other components locally is isomorphic to the subscheme in $\mathbb{A}^2_K = \operatorname{Spec} K[x,y]$ defined by the equation $x^2 y = 0$. All sections $s_i^\pm$ are on the other components.

In both cases the combinatorial types over the torus invariant divisors, after the appropriate modifications, are given by the description in the B-case (prop. 3.9).

$X(C_n)$ as quotient of $X(B_n)$. We investigate the description of $X(C_n)$ as a quotient $X(B_n)/\mu_2$. On the moduli side this leads to a characterisation of $X(C_n)$ as the coarse moduli space of a toric Deligne-Mumford stack. For simplicity, in this part we will work over the field of complex numbers.

On the moduli space $\overline{L}_n^{0,\pm}$ of B_n-curves we have an involution J that transforms a B_n-curve over a scheme Y to the B_n-curve with the other fixed point section with respect to the involution I as section s_0, i.e. we apply the automorphism of the central component that commutes with I (see the following remark) to the section s_0.

Remark 6.3. Let (C, I, s_-, s_+) be a chain of projective lines with involution of odd length over $\mathbb{C}$. Consider the central component (C_0, p_0^-, p_0^+) which we identify with $(\mathbb{P}^1_{\mathbb{C}}, 0, \infty)$ such that $I|_{C_0}\colon x \mapsto \frac{1}{x}$. Then there are two automorphisms of (C_0, p_0^-, p_0^+) that commute with I, namely the identity and $x \mapsto -x$, determined by the action on the fixed points $\{1, -1\}$ of $I|_{C_0}$.

Identifying $\overline{L}_n^{0,\pm}$ with $X(B_n)$, the involution J is given on the functor of B_n-data (see [BB11, 1.3]) by $(\mathscr{L}_{\{\pm u_i\}}, \{t_{u_i}, t_{-u_i}\}) \mapsto (\mathscr{L}_{\{\pm u_i\}}, \{t_{u_i}, -t_{-u_i}\})$ or equivalently $f_{\pm u_i} \mapsto -f_{\pm u_i}$ on the part corresponding to the roots $\pm u_1, \ldots, \pm u_n$, whereas the part corresponding to the other roots remains unchanged.

In both the C_n-case and the B_n-case we start with the same vector space E with basis $u_1, \ldots, u_n$. The root lattice $M(C_n)$ is a sublattice of the root lattice $M(B_n)$ of index 2 and dually $N(B_n) \subset N(C_n)$ of index 2, whereas the fan $\Sigma(C_n)$ as a set of cones in $N(C_n)_{\mathbb{Q}} = N(B_n)_{\mathbb{Q}}$ is the same as the fan $\Sigma(B_n)$. Thus, the toric variety $X(C_n)$ is the quotient of $X(B_n)$ by the involution that maps $x^{u_i} \mapsto -x^{u_i}$. This involution on $X(B_n)$ coincides with the involution J. Locally, we have quotients $\mathbb{A}^n/\mu_2$ by the action of μ_2 that changes the sign of one coordinate of $\mathbb{A}^n$. In particular, $X(B_n)$ is flat over $X(C_n)$ of degree 2. $X(C_n)$ can be considered as the μ_2-Hilbert scheme of $X(B_n)$, then $X(B_n) \to X(C_n)$ forms the universal family of μ_2-clusters, the fibres over $X(C_n) \setminus D$ consist of two points, the fibres over D are nonreduced μ_2-clusters.

Concerning the double-section $S_0 \subset X(C_{n+1})$ we obtain:

Lemma 6.4. *The scheme S_0 is isomorphic to $X(B_n)$ over $X(C_n)$.*

Proof. Let $\tilde{S}_0 \subset X(B_{n+1})$ be the fixed point subscheme of the involution I on $X(B_{n+1})$. The scheme $\tilde{S}_0$ over $X(B_n)$ consists of two components $s_0(X(B_n))$ and another copy of $X(B_n)$ such that $J\colon X(B_{n+1}) \to X(B_{n+1})$ restricts to an isomorphism between these components over $J\colon X(B_n) \to X(B_n)$. The scheme S_0 arises as quotient of $\tilde{S}_0$ by J, the section $s_0\colon X(B_n) \to \tilde{S}_0$ determines an isomorphism $X(B_n) \to S_0$ over $X(C_n) = X(B_n)/\mu_2$. $\square$

We are led to the following type of curves to be parametrised by $X(C_n)$.

Definition 6.5. (First definition of C_n-curves). A C_n-curve over a scheme Y is a collection $(\pi\colon C \to Y, I, s_-, s_+, s_1^\pm, \ldots, s_n^\pm)$ which arises from a B_n-curve over Y by omitting the section s_0.

Equivalently, we can replace the section s_0 of a B_n-curve $C \to Y$ by the subscheme $s_0(Y) \cup J(s_0(Y))$, which coincides with the fixed point subscheme of the involution I on C. The section s_0 selects one of the two components of this fixed point subscheme. Forgetting this information, the B_n-curves for points y, Jy in the moduli space $\overline{L}_n^{0,\pm} = X(B_n)$ define C_n-curves related by an isomorphism of C_n-curves. If the central component contains sections $s_i^\pm$, then two nonisomorphic B_n-curves over a field give rise to isomorphic C_n-curves. If the central component does not contain a section $s_i^\pm$, then one B_n-curve corresponds to one C_n-curve, but C_n-curves of this type have an extra automorphism that interchanges the two fixed points of I (cf. remark 6.3).

This functor of C_n-curves cannot be representable by a scheme. However, we can consider the stack of C_n-curves.

Theorem 6.6. *The category of C_n-curves forms a Deligne-Mumford stack $\mathcal{X}(C_n)$ isomorphic to the quotient stack $[X(B_n)/\mu_2]$ with the group operation given by $J\colon X(B_n) \to X(B_n)$.*

Proof. Let $\mathcal{X}(C_n)$ be the category of C_n-curves, i.e. an object of $\mathcal{X}(C_n)$ over a scheme Y is a C_n-curve C over Y, a morphism $(C \to Y) \to (C' \to Y')$ over $Y \to Y'$ is a cartesian diagram compatible with the involution I and the sections. This is a category fibred in groupoids, we show that it is equivalent as a fibred category to the Deligne-Mumford stack $[X(B_n)/\mu_2]$.

An object of $[X(B_n)/\mu_2]$ over a scheme Y is a μ_2-torsor $T \to Y$ together with a μ_2-equivariant map $T \to X(B_n)$. A morphism $(T \to Y, \alpha\colon T \to X(B_n)) \to (T' \to Y', \alpha'\colon T' \to X(B_n))$ over $Y \to Y'$ is a cartesian diagram of μ_2-torsors given by a morphism $\theta\colon T \to T'$ such that $\alpha' \circ \theta = \alpha$. We will use that the functor of $X(B_n)$ is isomorphic to the functor of B_n-curves and fix an isomorphism resp. a universal family over $X(B_n)$.

We define a morphism of fibred categories $\Phi\colon [X(B_n)/\mu_2] \to \mathcal{X}(C_n)$. For an object $(T \to Y, \alpha\colon T \to X(B_n))$ we have a B_n-curve $B \to T$ corresponding to the equivariant morphism α such that the action of μ_2 on T is given by interchanging the two possible choices of s_0. After forgetting the section s_0, the quotient of $B \to T$ by μ_2 gives a C_n-curve $C \to Y$ using the canonical isomorphism $T/\mu_2 \cong Y$. For a morphism $(T \to Y, \alpha\colon T \to X(B_n)) \to (T' \to Y, \alpha'\colon T' \to X(B_n))$ we obtain a cartesian diagram of C_n-curves $(C \to Y) \to (C' \to Y')$.

We define a morphism of fibred categories $\Psi\colon \mathcal{X}(C_n) \to [X(B_n)/\mu_2]$. Let $C \to Y$ be a C_n-curve over Y. Consider the fixed point subscheme $T \subset C$ under I, this is a μ_2-torsor over Y. Let B be the pull-back of the C_n-curve $C \to Y$ to T, with the section s_0 defined as the diagonal of $T \times_Y T \subset B$ this is a B_n-curve and defines a μ_2-equivariant morphism $\alpha\colon T \to X(B_n)$. A morphism $(C \to Y) \to (C' \to Y')$ given by $\gamma\colon C \to C'$ determines a cartesian diagram of B_n-curves by $\gamma \times \gamma\colon B = C \times_Y T \to B' = C' \times_{Y'} T'$ over a cartesian diagram of μ_2-torsors given by $\gamma\colon T \to T'$. So the diagram formed by $\gamma\colon T \to T'$ and $T, T' \to X(B_n)$ is commutative.

The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are isomorphic to the respective identities. In the case of $\Phi \circ \Psi$ the quotient of the pull-back of a C_n-curve $C \to Y$ to $T \subset C$ is canonically isomorphic to $C \to Y$. In the case of $\Psi \circ \Phi$ the quotient of a B_n-curve $B \to T$ over a μ_2-torsor T gives a C_n-curve $C \to Y$, together these form a cartesian square. The section $s_0\colon T \to B$ determines an inclusion $T \subset C$ as fixed point subscheme with respect to I. Applying the functor Ψ we recover a B_n-curve canonically isomorphic to the original B_n-curve. □

Corollary 6.7. *The toric variety $X(C_n)$ is a coarse moduli space of C_n-curves.*

The stack $\mathcal{X}(C_n)$ is a toric Deligne-Mumford stack as introduced in [BCS04] (see also [FMN10]): we define the stacky fan $\mathbf{\Sigma}(C_n)$ as the fan $\Sigma(C_n)$ in the lattice $N(C_n)$ with the difference that we choose on the rays generated by $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ the second lattice points $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$. In comparision to the fan $\Sigma(B_n)$ the underlying lattice is finer and the toric DM stack associated with $\mathbf{\Sigma}(C_n)$ coincides with the quotient stack $[X(B_n)/\mu_2]$.

Corollary 6.8. *The stack $\mathcal{X}(C_n)$ is isomorphic to the toric Deligne-Mumford stack associated with the stacky fan $\mathbf{\Sigma}(C_n)$.*

Example 6.9. The stacky fan $\mathbf{\Sigma}(C_2)$ in the lattice $\mathbb{Z}\frac{1}{2}v \cong \mathbb{Z}$ consists of the two cones $\mathbb{Q}_{\geq 0}v, \mathbb{Q}_{\geq 0}(-v)$ with chosen lattice points $v, -v$. The associated toric DM stack is $\mathcal{X}(C_2) \cong [\mathbb{P}^1/\mu_2]$ (cf. also [FMN10, example 7.31]), it is an orbifold with two stacky points.

$X(C_n)$ as fine moduli space. We give a characterisation of $X(C_n)$ as a fine moduli space $\overline{L}_n^{\pm}$ of $2n$-pointed chains of projective lines. Here the universal curve is not $X(C_{n+1}) \to X(C_n)$, however, the universal curve and the general notion of a C_n-curve are defined naturally in terms of the inclusion of root systems $C_n \to C_{n+1}$.

We have the root subsystem $C_n \subset C_{n+1}$ in the subspace generated by the roots $u_1, \dots, u_n$. Take those pairs of opposite roots in $C_{n+1} \setminus C_n$ which define projections $C_{n+1} \to C_n$ in the sense of [BB11, 1.2]; these are $\pm\alpha_1^-, \pm\alpha_1^+, \dots, \pm\alpha_n^-, \pm\alpha_n^+$ but not $\pm 2u_{n+1}$. To each of these pairs $\pm\alpha_i^-$ and $\pm\alpha_i^+$ we associate a section s_i^- and s_i^+. The element of the Weyl group given as the reflection for the root $\pm 2u_{n+1}$ mapping $u_{n+1} \mapsto -u_{n+1}$ and $u_i \mapsto u_i$ for $i \in \{1, \dots, n\}$ is an isomorphism of C_{n+1} fixing $C_n \subset C_{n+1}$. It maps $\alpha_i^- \leftrightarrow -\alpha_i^+$. This leads us to the following definition.

Definition 6.10. (Second definition of C_n-curves). A C_n-curve over an algebraically closed field K is a chain of projective lines with involution of odd or even length with $2n$ (possibly coinciding) marked points $s_1^{\pm}, \dots, s_n^{\pm}$ different from the poles, the involution interchanging $s_i^- \leftrightarrow s_i^+$, such that every component contains at least one of the points $s_i^{\pm}$. We define a C_n-curve over an arbitrary scheme, isomorphisms of C_n-curves and the moduli functor of C_n-curves in the same way as we did in the case of B_n-curves.

Construction 6.11. Let the subscheme

$$C(C_{n+1}/C_n) \subset \Big(\prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^-\}} \times \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^+\}}\Big)_{X(C_n)}$$

be defined by the equations (4) using the universal C_n-data on $X(C_n)$. This morphism $C(C_{n+1}/C_n) \to X(C_n)$ has the sections $s_-, s_+, s_i^{\pm}$, where s_- is defined by $z_{-\alpha_i^{\pm}} = 0$ $(i = 1, \dots, n)$, s_+ is defined by $z_{\alpha_i^{\pm}} = 0$ $(i = 1, \dots, n)$ and the sections $s_i^{\pm}$ by the equations $z_{\alpha_i^{\pm}} = z_{-\alpha_i^{\pm}}$. The involution maps $\mathbb{P}^1_{\{\pm\alpha_i^-\}} \leftrightarrow \mathbb{P}^1_{\{\pm\alpha_i^+\}}$, $(z_{\alpha_i^-} : z_{-\alpha_i^-}) \leftrightarrow (z_{-\alpha_i^+} : z_{\alpha_i^+})$.

Remark 6.12. The toric variety $C(C_{n+1}/C_n)$ arises from $X(C_{n+1})$ by contracting certain torus invariant prime divisors. The fibres of $X(C_{n+1}) \to X(C_n)$ over the divisors corresponding to the rays generated by elements of the form $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ (forming D in proposition 6.2) have a central component containing none of the sections $s_i^{\pm}$. In $X(C_{n+1})$ the support of the central components of the fibers over the divisor corresponding to $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ forms a torus invariant divisor which corresponds to the ray in $\Sigma(C_{n+1})$ generated by $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$ and is isomorphic to $X(C_1) \times X(A_{n-1}) \cong \mathbb{P}^1 \times X(A_{n-1})$. We contract these divisors $\mathbb{P}^1 \times X(A_{n-1})$ to $X(A_{n-1})$ by omitting the rays in $\Sigma(C_{n+1})$ generated by elements of the form $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$, but retaining the two-dimensional cones $\left\langle \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n - v_{n+1}), \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n + v_{n+1}) \right\rangle_{\mathbb{Q}_{\geq 0}}$. On the fibers over D the central components are contracted.

Proposition 6.13. *The morphism $C(C_{n+1}/C_n) \to X(C_n)$ with the involution I and the sections $s_-, s_+, s_1^\pm, \ldots, s_n^\pm$ is a C_n-curve. The combinatorial types of the fibres over the torus orbits corresponding to one-dimensional cones are as follows:*

$$\begin{array}{ll}
\varepsilon_{i_1} v_{i_1} & s_{i_1}^{\varepsilon_1} | s_{i_2}^\pm \cdots s_{i_n}^\pm | s_{i_1}^{-\varepsilon_1} \\
\varepsilon_{i_1} v_{i_1} + \varepsilon_{i_2} v_{i_2} & s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} | s_{i_3}^\pm \cdots s_{i_n}^\pm | s_{i_2}^{-\varepsilon_2} s_{i_1}^{-\varepsilon_1} \\
\vdots & \vdots \\
\varepsilon_{i_1} v_{i_1} + \ \ldots \ + \varepsilon_{i_{n-2}} v_{i_{n-2}} & s_{i_1}^{\varepsilon_1} \cdots s_{i_{n-2}}^{\varepsilon_{n-2}} | s_{i_{n-1}}^\pm s_{i_n}^\pm | s_{i_{n-2}}^{-\varepsilon_{n-2}} \cdots s_{i_1}^{-\varepsilon_1} \\
\varepsilon_{i_1} v_{i_1} + \ \ldots\ldots \ + \varepsilon_{i_{n-1}} v_{i_{n-1}} & s_{i_1}^{\varepsilon_1} \cdots s_{i_{n-1}}^{\varepsilon_{n-1}} | s_{i_n}^\pm | s_{i_{n-1}}^{-\varepsilon_{n-1}} \cdots s_{i_1}^{-\varepsilon_1} \\
\frac{1}{2}(\varepsilon_{i_1} v_{i_1} + \ \ldots\ldots\ldots \ + \varepsilon_{i_n} v_{i_n}) & s_{i_1}^{\varepsilon_1} \cdots s_{i_n}^{\varepsilon_n} | s_{i_n}^{-\varepsilon_n} \cdots s_{i_1}^{-\varepsilon_1}
\end{array}$$

Definition 6.14. We call $C(C_{n+1}/C_n) \to X(C_n)$ together with the involution I and the sections $s_-, s_+, s_1^-, s_1^+, \ldots, s_n^-, s_n^+$ the universal C_n-curve over $X(C_n)$.

By the same procedure as in the case of root systems of type A and B we can prove the following.

Theorem 6.15. *There exists a fine moduli space $\overline{L}_n^\pm$ of C_n-curves isomorphic to the toric variety $X(C_n)$ with universal family $C(C_{n+1}/C_n) \to X(C_n)$.*

Remark 6.16. There is a natural closed embedding of the moduli spaces $\overline{L}_n^\pm = X(C_n) \to \overline{L}_{2n} = X(A_{2n-1})$ determined by considering a C_n-curve with sections $s_1^-, \ldots, s_n^-, s_n^+, \ldots, s_1^+$ as an A_{2n-1}-curve with sections $s_1, \ldots, s_{2n}$. The toric morphism $X(C_n) \to X(A_{2n-1})$ is given by the projection of root systems $A_{2n-1} \to C_n$ induced by $\bigoplus_{i=1}^{2n} \mathbb{Z} u_i \to M(C_n)$, $u_i \mapsto u_i$, $u_{2n+1-i} \mapsto -u_i$ for $i = 1, \ldots, n$. The kernel in $M(A_{2n-1})$ is generated by $u_{2n+1-i} + u_i - u_{2n+1-j} - u_j$ for some fixed j and $i \in \{1, \ldots, n\} \setminus \{j\}$. By employing this embedding we have an alternative approach to prove the above statements.

7. Root systems of type D

Consider for $n \geq 2$ an n-dimensional Euclidean space E with basis $u_1, \ldots, u_n$. The root system D_n in E consists of the $2n(n-1)$ roots

$$\pm(u_i + u_j), \pm(u_i - u_j) \ \textit{for} \ i, j \in \{1, \ldots, n\}, i < j.$$

The following is a set of simple roots:

$$u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_{n-1} + u_n.$$

The Weyl group $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ acts by $u_i \mapsto \varepsilon_i u_i$, where the ε_i are signs such that $\prod_i \varepsilon_i = 1$, and by permuting the u_i. So there are $2^{n-1} n!$ sets of simple roots, these are of the form $\varepsilon_1 u_{i_1} - \varepsilon_2 u_{i_2}, \varepsilon_2 u_{i_2} - \varepsilon_3 u_{i_3}, \ldots, \varepsilon_{n-1} u_{i_{n-1}} - \varepsilon_n u_{i_n}, \varepsilon_{n-1} u_{i_{n-1}} + \varepsilon_n u_{i_n}$

for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_1, \ldots, \varepsilon_n$ (note that $\varepsilon_n = 1$ and $\varepsilon_n = -1$ give the same set).

The vector space E^* dual to E with basis $v_1, \ldots, v_n$ dual to $u_1, \ldots, u_n$ contains the lattice $N(D_n)$ dual to the root lattice $M(D_n)$. To describe the fan $\Sigma(D_n)$ in the lattice $N(D_n)$ we determine a Weyl chamber. The set of simple roots $u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_{n-1} + u_n$ has the dual basis $v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{n-2}, \frac{1}{2}(v_1 + \ldots + v_{n-1} - v_n), \frac{1}{2}(v_1 + \ldots + v_{n-1} + v_n)$ of $N(D_n)$ which generates the corresponding Weyl chamber. There are $3^n - n2^{n-1} - 1$ one-dimensional cones generated by elements of the form $\sum_{i\in A} \varepsilon_i v_i$ for $A \subset \{1, \ldots, n\}$, $1 \le |A| \le n-2$ or of the form $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1} + \varepsilon_n v_n)$, where the ε_i are signs.

The torus invariant divisor for the one-dimensional cone generated by $\varepsilon_1 v_{i_1} + \ldots + \varepsilon_k v_{i_k}$, $1 \le k \le n-2$ is isomorphic to $X(D_{n-k}) \times X(A_{k-1})$, that for $\varepsilon_1 v_1 + \ldots + \varepsilon_{n-2} v_{n-2}$ is isomorphic to $X(A_1) \times X(A_1) \times X(A_{n-3}) \cong X(D_2) \times X(A_{n-3})$ and that for $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ is isomorphic to $X(A_{n-1})$ (see [BB11, 1.2]).

$X(D_{n+1})$ over $X(D_n)$. Consider the proper surjective morphism $X(D_{n+1}) \to X(D_n)$ induced by the root subsystem $D_n \subset D_{n+1}$ consisting of the roots in the subspace generated by $u_1, \ldots, u_n$. We have a projection of fans $\Sigma(D_{n+1}) \to \Sigma(D_n)$ along the subspace generated by v_{n+1}. The generic fibre is $\mathbb{P}^1$. Note that the torus invariant divisor in $X(D_{n+1})$ corresponding to $v_1 + \ldots + v_{n-1}$ is lying over the closure of the torus orbit in $X(D_n)$ of codimension 2 corresponding to the 2-dimensional cone generated by $\frac{1}{2}(v_1 + \ldots + v_{n-1} + v_n), \frac{1}{2}(v_1 + \ldots + v_{n-1} - v_n)$; here (and on the translates under the Weyl group $W(B_n)$) we have fibres of dimension 2. This implies that the morphism $X(D_{n+1}) \to X(D_n)$ is not flat.

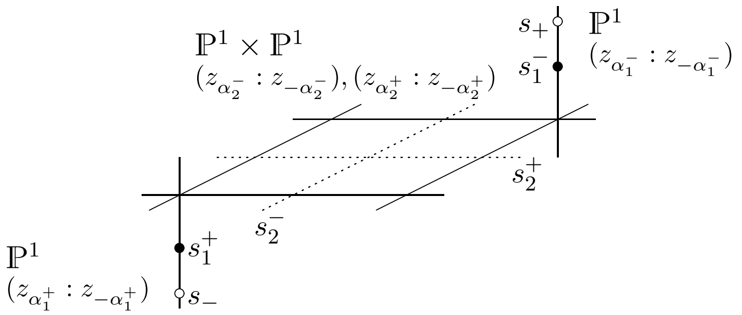
There are $2n$ additional pairs of opposite roots, the pairs $\pm\alpha_i^+ = \pm(u_{n+1} + u_i)$ and $\pm\alpha_i^- = \pm(u_{n+1} - u_i)$ for $i \in \{1, \ldots, n\}$. The projections along the subspaces generated by these do not define projections of root systems $D_{n+1} \to D_n$ in the sense of [BB11, 1.2]: we have $\alpha_i^+ - \alpha_i^- = 2u_i$, so the projection along the subspace generated by α_i^+ (resp. α_i^-) maps α_i^- (resp. α_i^+) to $2u_i$ which is not a multiple of a root of D_n. Instead we can consider the preimages of $(1:1) \in \mathbb{P}^1_{\{\pm\alpha_i^-\}}, \mathbb{P}^1_{\{\pm\alpha_i^+\}}$ with respect to the projections $X(D_{n+1}) \to \mathbb{P}^1_{\{\pm\alpha_i^-\}}, \mathbb{P}^1_{\{\pm\alpha_i^+\}}$ determined by the inclusions of root systems $\{\pm\alpha_i^-\}, \{\pm\alpha_i^+\} \subset D_{n+1}$, we denote these subschemes by s_i^-, s_i^+. As in the B and C-case we have sections s_-, s_+; further we have an involution I coming from the automorphism of D_{n+1} fixing $D_n \subset D_{n+1}$ which maps $u_{n+1} \mapsto -u_{n+1}$, $u_i \mapsto u_i$ for $i \in \{1, \ldots, n\}$ and is not an element of the Weyl group $W(D_{n+1})$.

As in the other cases we can study $X(D_{n+1})$ over $X(D_n)$ via the embedding into $P(D_{n+1}/D_n)_{X(D_n)} = \big(\prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^-\}} \times \prod_{i=1}^n \mathbb{P}^1_{\{\pm\alpha_i^+\}}\big)_{X(D_n)}$. The subscheme $X(D_{n+1}) \subset P(D_{n+1}/D_n)_{X(D_n)}$ is given by the homogeneous equations parametrised by the universal D_n-data

$$t_\beta z_{\alpha_2} z_{-\alpha_1} = t_{-\beta} z_{-\alpha_2} z_{\alpha_1} \qquad \begin{array}{l} \textit{for } \alpha_1, \alpha_2 \in \{\alpha_1^\pm, \ldots, \alpha_n^\pm\} \\ \textit{such that } \beta = \alpha_1 - \alpha_2 \textit{ is a root of } D_n \end{array}$$

We will see that over the complement of a closed subset of codimension 2 the fibres are chains of projective lines with sections $s_i^{\pm}$. Over these points we have a combinatorial type for a fibre resp. for the universal D_n-data as in the B-case (see proposition 3.9), we use the notation introduced there.

Example 7.1. $X(D_3)$ over $X(D_2)$.
The root system D_2 consists of the 4 roots $\pm u_1 \pm u_2$. It is contained in the root system D_3, this has the 8 additional roots $\pm\alpha_1^- = \pm(u_3 - u_1)$, $\pm\alpha_1^+ = \pm(u_3 + u_1)$, $\pm\alpha_2^- = \pm(u_3 - u_2)$, $\pm\alpha_2^+ = \pm(u_3 + u_2)$. Because of the isomorphism of root systems $D_2 \cong A_1 \times A_1$ we have $X(D_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The fan $\Sigma(D_2)$ has 4 one-dimensional cones generated by $\frac{1}{2}(\pm v_1 \pm v_2)$. The fan $\Sigma(D_3)$ has 14 one-dimensional cones, 6 of the form $\pm v_i$ and 8 of the form $\frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3)$. The projection $\Sigma(D_3) \to \Sigma(D_2)$ maps the generator of the one-dimensional cone $\frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3)$ to the generator of the one-dimensional cone $\frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2)$, the vector $\pm v_i$ for $i = 1, 2$ is not mapped to a one-dimensional cone of D_2 but into the interior of the 2-dimensional cone $\langle \pm v_i + v_j, \pm v_i - v_j \rangle_{\mathbb{Q}_{\geq 0}}$.
In $P(D_3/D_2)_{X(D_2)} = \left(\mathbb{P}^1_{\{\pm\alpha_1^-\}} \times \mathbb{P}^1_{\{\pm\alpha_1^+\}} \times \mathbb{P}^1_{\{\pm\alpha_2^-\}} \times \mathbb{P}^1_{\{\pm\alpha_2^+\}}\right)_{X(D_2)}$ the subscheme $X(D_3)$ is given by 4 equations parametrised by the universal D_2-data on $X(D_2)$. For each point we have D_2-data of the form $(t_{\beta_{12}} : t_{-\beta_{12}})$, $(t_{\gamma_{12}} : t_{-\gamma_{12}})$ where $\beta_{12} = u_1 - u_2$, $\gamma_{12} = u_1 + u_2$. Over the affine chart $\operatorname{Spec} \mathbb{Z}[\frac{x_1}{x_2}, x_1 x_2]$ corresponding to the cone $\langle \frac{1}{2}(v_1 - v_2), \frac{1}{2}(v_1 + v_2) \rangle_{\mathbb{Q}_{\geq 0}}$ for the set of simple roots β_{12}, γ_{12} this data has the property $(t_{\beta_{12}} : t_{-\beta_{12}}) \neq (1 : 0)$, $(t_{\gamma_{12}} : t_{-\gamma_{12}}) \neq (1 : 0)$ (see [BB11, Rem. 1.21]). We study the fibres of $X(D_3) \to X(D_2)$ over this affine chart. Over the dense torus we have a $\mathbb{P}^1$, over the torus orbit corresponding to $\frac{1}{2}(v_1 - v_2)$ (resp. $\frac{1}{2}(v_1 + v_2)$) we have chains of two $\mathbb{P}^1$ of combinatorial type $s_1^+ s_2^- | s_1^- s_2^+$ (resp. $s_1^+ s_2^+ | s_1^- s_2^-$). Over the torus fixed point corresponding to the cone $\langle \frac{1}{2}(v_1 - v_2), \frac{1}{2}(v_1 + v_2) \rangle_{\mathbb{Q}_{\geq 0}}$ we have D_2-data of the form $(t_{\beta_{12}} : t_{-\beta_{12}}) = (0 : 1)$, $(t_{\gamma_{12}} : t_{-\gamma_{12}}) = (0 : 1)$ and the fibre decomposes into three irreducible components $\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1$.



The general case can be studied using the same methods, see also the B_n-case and in particular proposition 3.9, here details will be left to the reader. We define $Z \subset X(D_n)$ to be the union of the closures of torus orbits corresponding to the 2-dimensional cones of the form $\langle \frac{1}{2}(\varepsilon_1 v_{i_1} + \ldots + \varepsilon_{n-1} v_{i_{n-1}} + \varepsilon_{i_n} v_{i_n}), \frac{1}{2}(\varepsilon_1 v_{i_1} + \ldots + \varepsilon_{n-1} v_{i_{n-1}} - \varepsilon_n v_{i_n}) \rangle_{\mathbb{Q}_{\geq 0}}$.

Proposition 7.2. *Over $X(D_n) \setminus Z$ the fibres of the morphism $X(D_{n+1}) \to X(D_n)$ are chains of projective lines of odd or even length with sections $s_i^\pm$. The combinatorial types of the fibres over the torus orbits corresponding to one-dimensional cones are as follows:*

$$\begin{array}{ll}
\varepsilon_{i_1}v_{i_1} & s_{i_1}^{\varepsilon_1} | s_{i_2}^{\pm} \cdots s_{i_n}^{\pm} | s_{i_1}^{-\varepsilon_1} \\
\varepsilon_{i_1}v_{i_1} + \varepsilon_{i_2}v_{i_2} & s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} | s_{i_3}^{\pm} \cdots s_{i_n}^{\pm} | s_{i_2}^{-\varepsilon_2} s_{i_1}^{-\varepsilon_1} \\
\vdots & \vdots \\
\varepsilon_{i_1}v_{i_1} + \ldots + \varepsilon_{i_{n-2}}v_{i_{n-2}} & s_{i_1}^{\varepsilon_1} \cdots s_{i_{n-2}}^{\varepsilon_{n-2}} | s_{i_{n-1}}^{\pm} s_{i_n}^{\pm} | s_{i_{n-2}}^{-\varepsilon_{n-2}} \cdots s_{i_1}^{-\varepsilon_1} \\
\frac{1}{2}(\varepsilon_{i_1}v_{i_1} + \ldots + \varepsilon_{i_{n-2}}v_{i_{n-2}} + \varepsilon_{i_{n-1}}v_{i_{n-1}} + \varepsilon_{i_n}v_{i_n}) & s_{i_1}^{\varepsilon_1} \cdots s_{i_n}^{\varepsilon_n} | s_{i_n}^{-\varepsilon_n} \cdots s_{i_1}^{-\varepsilon_1} \\
\frac{1}{2}(\varepsilon_{i_1}v_{i_1} + \ldots + \varepsilon_{i_{n-2}}v_{i_{n-2}} + \varepsilon_{i_{n-1}}v_{i_{n-1}} - \varepsilon_{i_n}v_{i_n}) & s_{i_1}^{\varepsilon_1} \cdots s_{i_n}^{-\varepsilon_n} | s_{i_n}^{\varepsilon_n} \cdots s_{i_1}^{-\varepsilon_1}
\end{array}$$

Over Z the fibres are 2-dimensional and decompose into irreducible components isomorphic to $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ intersecting transversally. We have a central component $\mathbb{P}^1 \times \mathbb{P}^1$ with action of I that interchanges two torus fixed points and leaves the other two fixed. Further, we have chains of $\mathbb{P}^1$ emanating from the two torus fixed points of $\mathbb{P}^1 \times \mathbb{P}^1$ interchanged by I with the sections $s_\pm$ on the outer components. Concerning the subschemes $s_i^\pm$, each of them intersects only with one component, those intersecting with one of the $\mathbb{P}^1$ locally are sections, one pair s_i^-, s_i^+ intersects with $\mathbb{P}^1 \times \mathbb{P}^1$ as $(1:1) \times \mathbb{P}^1$, $\mathbb{P}^1 \times (1:1)$.

Remark 7.3. The combinatorial type of fibres of $X(R_{n+1})$ over the torus fixed points of $X(R_n)$ can be pictured in form of the Dynkin diagram of the root system R_{n+1} such that a component $\mathbb{P}^1$ with one section corresponds to a vertex.
In the A_n-case (see [BB11]) we have a string starting with the section s_{i_1} on the component containing s_- and ending with the section $s_{i_{n+1}}$ on the component containing s_+ in the form of the Dynkin diagram for the root system A_{n+1}:

$$s_{i_1} \;\text{———}\; s_{i_2} \;\text{———}\; s_{i_3} \;\text{———}\; \cdots \;\text{———}\; s_{i_{n+1}}$$

In the B_n-case, because of the involution I, it suffices to consider the central component containing the section s_0 and one of the two chains emanating from the central component. This forms a Dynkin diagram of type B_{n+1}:

$$s_0 \;\Leftarrow\; s_{i_1}^{\varepsilon_1} \;\text{———}\; s_{i_2}^{\varepsilon_2} \;\text{———}\; \cdots \;\text{———}\; s_{i_n}^{\varepsilon_n}$$

In the C_n-case we have the double-section S_0 replacing the section s_0:

$$S_0 \;\Rightarrow\; s_{i_1}^{\varepsilon_1} \;\text{———}\; s_{i_2}^{\varepsilon_2} \;\text{———}\; \cdots \;\text{———}\; s_{i_n}^{\varepsilon_n}$$

Finally, in the D_n-case we can take the torus invariant divisors in the central component $\mathbb{P}^1 \times \mathbb{P}^1$ and their intersection with the fibres of the schemes $s_i^\pm$. Together with the other components we have a Dynkin diagram of type D_{n+1}:

$$\begin{tikzcd}[column sep=small, row sep=small]
s_{i_1}^{+} \arrow[dr, dash] & & & & \\
& s_{i_2}^{\varepsilon_2} \arrow[r, dash] & s_{i_3}^{\varepsilon_3} \arrow[r, dash] & \cdots \arrow[r, dash] & s_{i_n}^{\varepsilon_n} \\
s_{i_1}^{-} \arrow[ur, dash] & & & &
\end{tikzcd}$$

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