THE GABRIEL-ROITER MEASURE FOR $\widetilde{\mathbb{A}}_n$ II

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ABSTRACT. Let Q be a tame quiver of type $\widetilde{\mathbb{A}}_n$ and $\mathrm{Rep}\,(Q)$ the category of finite dimensional representations over an algebraically closed field. A representation is simply called a module. We study the number of the GR submodules. It will be shown that only finitely many (central) Gabriel-Roiter measures have no direct predecessors. The quivers Q, whose central part contains no preinjective modules, will also be characterized.

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1. INTRODUCTION

Let Λ be an artin algebra and $\mathrm{mod}\,\Lambda$ the category of finitely generated right Λ-modules. For each $M \in \mathrm{mod}\,\Lambda$, we denote by $|M|$ the length of M. The symbol $\subset$ is used to denote proper inclusion. The Gabriel-Roiter (GR for short) measure $\mu(M)$ for a Λ-module M was defined in [12] by induction as follows:

$$\mu(M) = \begin{cases} 0, & \text{if } M = 0; \\ \max_{N\subset M}\{\mu(N)\}, & \text{if } M \text{ is decomposable;} \\ \max_{N\subset M}\{\mu(N)\} + \frac{1}{2^{|M|}}, & \text{if } M \text{ is indecomposable.} \end{cases}$$

(In later discussion, we will use the original definition for our convenience, see [11] or section 2.1 below.) The so-called Gabriel-Roiter submodules of an indecomposable module are defined to be the indecomposable proper submodules with maximal GR measure.

Using Gabriel-Roiter measure , Ringel obtained a partition of the module category for any artin algebra of infinite representation type [11, 12]: there are infinitely many GR measures μ_i and μ^i with i natural numbers, such that

$$\mu_1 < \mu_2 < \mu_3 < \dots \quad \dots < \mu^3 < \mu^2 < \mu^1$$

and such that any other GR measure μ satisfies $\mu_i < \mu < \mu^j$ for all i, j. The GR measures μ_i (resp. μ^i) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module is called a take-off (resp. central, landing) module if its GR measure is a take-off (resp. central, landing) measure.

To calculate the GR measure of a given indecomposable module, it is necessary to know the GR submodules. Thus it is interesting to know the number of the isomorphism classes of the GR submodules for a given indecomposable module. It was conjectured that for a representation-finite algebra (over an algebraically closed field), each indecomposable module has at most three GR

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submodules. In [5], we have proved the conjecture for representation-finite hereditary algebras. In this paper, we will start to study the GR submodules of string modules. In particular, we will show in section 3 that each string module, which contains no band submodules, has at most (up to isomorphism) two GR submodules. As an application, we show for a tame quiver Q of type $\widetilde{\mathbb{A}}_n$ that if a regular module has precisely (up to isomorphism) two GR submodules, then one of the two GR inclusions is an irreducible monomorphism. A description of the numbers of GR submodules will also be presented there.

Let μ, μ' be two GR measures for Λ. We call μ' a **direct successor** of μ if, first, $\mu < \mu'$ and second, there does not exist a GR measure μ'' with $\mu < \mu'' < \mu'$. The so-called **Successor Lemma** in [12] states that any Gabriel-Roiter measure μ different from μ^1 has a direct successor. There is no 'Predecessor Lemma'. For example, the minimal central measure (if exists) has no direct predecessor. It is clear that any GR measure over a representation-finite artin algebra has a direct predecessor. We may ask the following question: does the number of GR measures having no direct predecessors relate to the representation type of artin algebras? More precisely, does a representation-infinite (hereditary) algebra (over an algebraically closed field) is of tame type imply that there are only finitely many GR measures having no direct predecessors and vice versa? In section 4, we will study the direct predecessors of the GR measures of tame quivers Q of type $\widetilde{\mathbb{A}}_n$, and show that only finitely many GR measures have no direct predecessors.

It was shown in [11] that all landing modules are preinjective modules in the sense of Auslander-Smalø [2]. However, not all preinjective modules are landing modules in general. It is interesting to study the preinjective module, which are in central part. In section 5, We will show that for a tame quiver Q of type $\widetilde{\mathbb{A}}_n$, if there is a preinjective central module, then there are actually infinitely many ones. However, it is possible that the central part does not contain any preinjective module. We are going to characterize the tame quivers of type $\widetilde{\mathbb{A}}_n$ with this property. In particular, we show that the quiver Q of type $\widetilde{\mathbb{A}}_n$ is equipped with a sink-source orientation if and only if any indecomposable preinjective module is either a landing module or a take-off module.

2. Preliminaries and known results

2.1. **Gabriel-Roiter measure.** We fist recall the original definition of Gabriel-Roiter measure [11, 12]. Let $\mathbb{N}_1$=$\{1, 2, \ldots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets of $\mathbb{N}_1$. A total order on $\mathcal{P}(\mathbb{N}_1)$ can be defined as follows: if I,J are two different subsets of $\mathbb{N}_1$, write $I < J$ if the smallest element in $(I\backslash J) \cup (J\backslash I)$ belongs to J. Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J\backslash I$, we have $a < b$. We say that J **starts with** I if $I = J$ or $I \ll J$.

Let Λ be an artin algebra and $\operatorname{mod}\Lambda$ be the category of finite generated (right) Λ-modules. For each $M \in \operatorname{mod}\Lambda$, let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \ldots, |M_t|\}$, where $M_1 \subset M_2 \subset \ldots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure** of M. If M is an indecomposable Λ-module, we call an inclusion $T \subset M$ with T indecomposable a **GR inclusion** provided $\mu(M) = \mu(T) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(T)$. In this case, we call T a **GR submodule** of M.

Remark. We have seen in Introduction a different way to define the Gabriel-Roiter measure. These two definitions (orders) can be identified. In fact, for each $I = \{a_i|i\} \in \mathcal{P}(\mathbb{N}_1)$, let $\mu(I) = \sum_i \frac{1}{2^{a_i}}$. Then $I < J$ if and only if $\mu(I) < \mu(J)$.

Let's denote by $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{I}$ the collection of indecomposable modules, which are in take-off part, central part and landing part, respectively. We present one result concerning Gabriel-Roiter measures, which will be used later on. For more basic properties we refer to [11, 12].

Proposition 2.1. *Let Λ an artin algebra and $X \subset M$ be a GR inclusion. Then*

1) *If $\mu(X) < \mu(Y) < \mu(M)$, then $|Y| > |Z|$.*
2) *There is an irreducible monomorphism $X \to Y$ with Y indecomposable and an epimorphism $Y \to M$.*

The first statement is a direct consequence of the definition. For a proof of the second statement, we refer to [5](Proposition 3.2).

2.2. Let Q be a tame quiver of type $\widetilde{\mathbb{A}}_{n,n\geq 1}$ and $\operatorname{Rep}(Q)$ the category of finite dimensional representations over an algebraically closed field. We simply call the representations in $\operatorname{Rep}(Q)$ modules. We briefly recall some notations and refer to [1, 10] for details. If X is a quasi-simple module, then there is a unique sequence $X = X_1 \to X_2 \to \ldots \to X_r \to \ldots$ of irreducible monomorphisms. Thus any indecomposable regular module M is of the form $M \cong X_i$ with X a quasi-simple module (quasi-socle of M) and i a natural number (quasi-length of M). The rank of an indecomposable regular module M is the minimal positive integer r such that $\tau^r M = M$. A regular component (tube) of the Auslander-Reiten quiver of Q is called exceptional if its rank (the rank of any quasi-simple module on it) $r > 1$. If X is quasi-simple of rank r, then the dimension vector $\underline{\dim}\, X_r = \delta = \sum_{i=1}^{r} \tau^r X$, where δ is the minimal positive imaginary root, i.e. δ is a dimension vector with $\delta_\nu = 1$ for each $\nu \in Q$. Let $|\delta| = \sum \delta_\nu = n+1$. A quasi-simple module of rank 1 will be called a homogeneous simple module. We denote by H_i an indecomposable homogeneous module with quasi-length i. We denote by $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{I}$ the collection of indecomposable preprojective, regular and preinjective modules, respectively.

We collect some known facts in the following proposition, which will be quite often used in our later discussion. The proofs can be found in [7] (or [8] and [9] for tame quivers of type $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$).

Proposition 2.2. *Let Q be a tame quiver of type $\widetilde{\mathbb{A}}_n$.*

1) *Let $\iota : T \subset M$ be a GR inclusion.*
   a) *If $M \in \mathcal{P}$, then ι is an irreducible monomorphism.*
   b) *If $M \in \mathcal{R}$ is a quasi-simple module, then $T \in \mathcal{P}$.*
   c) *If $M = X_i \in \mathcal{R}$ with X quasi-simple and $i > 1$, then $T \in \mathcal{P}$ or $T \cong X_{i-1}$.*
   d) *If $M \in \mathcal{I}$, then $T \in \mathcal{R}$.*
2) *If $X \in \mathcal{P}$, then $X \in \mathcal{T}$ and $\mu(X) < \mu(H_1)$.*
3) *$\mu(H_1)$ is a central measure and $\mu(M) > \mu(H_1)$, if $M \in \mathcal{I}$ satisfies $\underline{\dim}\, M > \delta$.*
4) *Let X be a quasi-simple module of rank $r > 1$. Then*
   a) *If $\mu(X_r) < \mu(H_1)$, then $\mu(X_i) < \mu(H_j)$ for all $i \geq 1$ and $j \geq 1$.*
   b) *If $\mu(X_r) \geq \mu(H_1)$, then X_{i-1} is the unique (up to isomorphism) GR submodule of X_i. If in addition $r > 1$, then $\mu(X_i) > \mu(H_j)$ for all $i > r$ and $j \geq 1$.*
5) *Let $\mathbb{T}$ be a regular tube of rank $r > 1$. Then there is a quasi-simple module X on $\mathbb{T}$ such that $\mu(X_r) \geq \mu(H_1)$.*
6) *Let S be a quasi-simple module of rank r which is simple. Then $\mu(S_r) < \mu(H_1)$. Thus $\mu(S_j) < \mu(H_1)$ for all $j \geq 1$.*

7) *Let $M \in \mathcal{I} \setminus \mathcal{T}$ and X_i be a GR submodule of M for some quasi-simple X. Then $\mu(M) > \mu(X_j)$ for all $j \geq 1$.*

Lemma 2.3. *Let Q be a tame quiver of type $\widetilde{\mathbb{A}}_n$. Then for every indecomposable preinjective module M, there is, in each regular tube, precisely one quasi-simple module X such that* $\mathrm{Hom}\,(X, M) \neq 0$. *In particular, up to isomorphism, each indecomposable preinjective module contains in each regular tube at most one GR submodule.*

Proof. Let $M = \tau^s I_\nu$, where I_ν is an indecomposable injective module corresponding to a vertex $\nu \in Q$. It is obvious that there is a quasi-simple module X on a given regular tube such that $\mathrm{Hom}\,(X, I_\nu) \neq 0$. Thus $\mathrm{Hom}\,(\tau^s X, M) \neq 0$. Assume that X and Y are non-isomorphic quasi-simple modules on the same tube such that $\mathrm{Hom}\,(X, M) \neq 0 \neq \mathrm{Hom}\,(Y, M)$. Then $\mathrm{Hom}\,(\tau^{-s} X, I_\nu) \neq 0 \neq \mathrm{Hom}\,(\tau^{-s} Y, I_\nu)$. Thus $(\underline{\dim}\,\tau^{-s} X)_\nu \neq 0 \neq (\underline{\dim}\,\tau^{-s} Y)_\nu$, which is impossible since $1 = \delta_\nu \geq (\underline{\dim}\,\tau^{-s} X)_\nu + (\underline{\dim}\,\tau^{-s} Y)_\nu$. □

This lemma and Proposition 2.2 imply that each indecomposable preinjective module has at most one GR submodule in each regular component.

3. THE NUMBER OF GR SUBMODULES

As we have mentioned in Introduction, the number of GR submodules of a given indecomposable modules is conjectured being bounded by three for representations-finite algebras. In this section, we will show that this number is always bounded by two for representation-finite string algebras. We will also describe the numbers of GR submodules for tame quivers Q of type $\widetilde{\mathbb{A}}_n$.

3.1. **String modules.** We first recall the definition of string modules. For details, we refer to [3]. Let Λ be a string algebra . We denote by $s(C)$ and $e(C)$ the starting and the ending vertices of a given string C, respectively. Let $C = c_n c_{n-1} \cdots c_2 c_1$ be a string, the corresponding string module $M(C)$ is defined as follows: let $u_i = s(c_{i+1})$ for $1 \leq i \leq n-1$ and $u_n = e(c_n)$. For a vertex $\nu \in Q$, let $I_\nu = \{i | u_i = \nu\} \subset \{0, 1, \ldots, n\}$. Then the vector space associated to ν is $M(C)_v$ satisfies $\dim M(C)_\nu = |I_\nu|$ and has $z_i, i \in I_\nu$ as basis. If c_i is an arrow β, define $\beta(z_{i-1}) = z_i$ for all $1 \leq i \leq n$. If c_i is an inverse of an arrow β, define $\beta(z_i) = z_{i-1}$ for all $1 \leq i \leq n$. Note that the indecomposable string modules are uniquely determined strings up to $C \sim C^{-1}$.

If $C = E\beta F$ is a string with β an arrow. Then the string module $M(E)$ is a submodule of $M(C)$: let E be of length n and F be of length m. then C has length $n+m+1$. If $M(C)$ is given by $n+m+2$ vectors $z_0, z_1, \ldots, z_{n+m+2}$, it is obvious that the spaces determined by vectors $z_0, z_1, \ldots, z_n$ is a submodule, which is $M(E)$. The corresponding factor module is $M(F)$. If $C = E\beta^{-1}F$ is a string with β an arrow, we may obtain similarly an indecomposable submodule $M(F)$ with factor module $M(E)$.

3.2. **'covering' of a string module.** Let $C = c_n c_{n-1} \cdots c_2 c_1$ be a string. We associate to C a Dynkin quiver $\mathbb{A}_{n+1}$ as follows: the vertices are u_i, and there is an arrow $u_{i-1} \xrightarrow{\alpha_i} u_i$ if c_i is an arrow, and an arrow $u_i \xrightarrow{\alpha_i} u_{i-1}$ in case c_i is an inverse of an arrow. Let $M(C)$ be the string module and $M_{\mathbb{A}}(C)$ be the unique sincere indecomposable representation over $\mathbb{A}_{n+1}$. For each string submodule N of $M(C)$, we may similarly construct an indecomposable module $N_{\mathbb{A}}(C)$, which is a submodule of $M_{\mathbb{A}}(C)$. By the description of string modules and the morphisms among them [4], we actually obtain in this way a functor:

$$\mathcal{F} : \operatorname{sub} M_Q(C) \longrightarrow \operatorname{sub} M(C),$$

where sub M denotes the full subcategory consisting of all submodules of M. The following properties are direct consequences of the description of string modules:

- $\mathcal{F}$ is faithful, but not full in general.
- $\mathcal{F}(X) \cong \mathcal{F}(Y)$ does not imply $X \cong Y$.
- The submodules of $M(C)$, which are not of the form $\mathcal{F}(X)$, are the band modules.
- $\mathcal{F}$ preserves indecomposables, monomorphisms and lengths.
- $\mathcal{F}$ preserves the GR submodules and thus the GR measures.

For a Dynkin quiver of type $\mathbb{A}$, we have shown in [5] the following result:

Proposition 3.1. *Let $\mathbb{A}_{n+1}$ be a Dynkin quiver. Then each indecomposable module has at most two GR submodules and each factor of a GR inclusion is a uniserial module.*

As a consequence of this proposition and the properties of $\mathcal{F}$, we have

Theorem 3.2. *Let Λ be a string algebra and $M(C)$ be a string module containing no band submodules. Then $M(C)$ contains, up to isomorphism, at most two GR submodules and the factors of the GR inclusions are uniserial modules.*

Corollary 3.3. *If Λ is a representation-finite string algebra, then each indecomposable module has up to isomorphism at most two GR submodules and the GR factors are uniserial.*

3.3. Now we assume that Q is a tame quiver of type $\widetilde{\mathbb{A}}_n$. Then every indecomposable regular module with rank $r > 1$ is string module containing no band submodules, thus has at most two GR submodules, up to isomorphism.

Proposition 3.4. *If an exceptional regular module has precisely two GR submodules, then one of the GR inclusions is an irreducible map.*

Proof. Let $M(C)$ be an exceptional regular module with $C = c_m \cdots c_2 c_1$, which has precisely (up to isomorphism) two GR submodules. Then the module $M_{\mathbb{A}}(C)$ also has two GR submodules, which are actually given by the irreducible monomorphism $X \rightarrow M_Q(C)$ and $Y \rightarrow M_Q(C)$. By definition of $M_{\mathbb{A}}(C)$, we may identify the arrows α_i or its inverse in $\mathbb{A}_{m+1}$with c_i in the string C of $\widetilde{\mathbb{A}}_n$. We may assume that X is determine by string E and $M_{\mathbb{A}}(C)$ is determined by $F\alpha^{-1}E$, where F is a composition of arrows or a trivial path and α is an arrow. Then under the above identification, we have $C = F\alpha^{-1}E$. Let $M(C) \rightarrow M'$ be the unique irreducible monomorphism with M' determined by a string $F'\beta^{-1}F\alpha^{-1}E$, where F' is a compositions of arrows or a trivial path and β is an arrow. Thus either the ending vertex $e(F)$ is a sink, or F is a trivial path. Again by the description of irreducible monomorphism in $\widetilde{\mathbb{A}}_n$, we have that the inclusion $\mathcal{F}(X) \rightarrow M(C)$ is still an irreducible map. □

Remark. Let Q be a tame quiver of type $\widetilde{\mathbb{A}}_n$ and M be a non-simple indecomposable module. Let $gr(M)$ denote the number of GR submodules (up to isomorphism) of M.

1) If M is preprojective, each GR inclusion $X \subset M$ is namely an irreducible map. In particular, $gr(M) \leq 2$ since every irreducible map to M is a monomorphism (Proposition 2.2(1)).

2) If M is a quasi-simple module of rank $r > 1$, then $gr(M) = 1$ since M is uniserial.
3) If M is regular of rank $r > 1$, then $gr(M) \leq 2$, and one of the GR inclusion is irreducible in case $gr(M) = 2$.
4) If M is non-quasi-simple regular module, and if $\mu(X_r) \geq \mu(H_1)$, then $gr(M) = 1$ and the unique GR inclusion is an irreducible map.
5) If M is preinjective, then M contains, up to isomorphism, at most one GR submodule in each regular component. Thus $gr(M) \leq 3$ if we identify the parameter $\lambda \in k \setminus \{0\}$ of homogeneous (band) modules where k is the ground field. In this sense, $gr(M) = 3$ implies the dimension vector of each GR submodule of M is δ.
6) A homogeneous simple module H_1 may contains more GR submodules. For example, if n is odd and $Q = \widetilde{\mathbb{A}}_n$ is with sink-source orientation (see [8] example 3). In this case, the GR measure of a homogeneous simple modules is $\mu(H_1) = \{1, 3, 5, \ldots, n, n+1\}$. There are $\frac{n+1}{2}$ indecomposable preprojective modules with length n and they are all non-isomorphic GR submodules of H_1. In general, $gr(H_1)$ is bounded by the number of the indecomposable summands of the projective cover of H_1.

4. Direct predecessor

Given an artin algebra. Recall that a GR measure J is called a direct successor of I if, first, $I < J$ and second, there does not exist a GR measure J' with $I < J' < J$. It is easily seen that if J is the direct successor of I, then J is a take-off (resp. central, landing) measure if and only if so is I. Let I^1 be the largest GR measure, i.e. the GR measure of an indecomposable injective module with maximal length. It was proved in [12] that any Gabriel-Roiter measure I different from I^1 has a direct successor. However, there are GR measures, which does not admit a direct predecessor. By the construction of the take-off measures and the landing measures in [11], the GR measures having no direct predecessors are central measures. From now on, we fix a tame quiver Q of type $\widetilde{\mathbb{A}}_n$.

4.1. The following proposition gives a GR measure possessing no direct predecessor. The proof uses Proposition 2.2.

Proposition 4.1. *The GR measure $\mu(H_1)$ of a homogeneous quasi-simple module H_1 has no direct predecessor.*

Proof. For the purpose of a contradiction, we assume that $\mu(M)$ is the direct predecessor of $\mu(H_1)$ for some indecomposable module M. Since $\mu(H_1)$ is a central measure, so is $\mu(M)$. It follows that M is not preprojective. Let Y be a GR submodule of H_1. Since Y is preprojective, $\mu(Y) < \mu(M) < \mu(H_1)$ and thus $|M| > |H_1|$. If M is preinjective, then there is a monomorphism $H_1 \to M$ because $|M| > |H_1|$, and hence $\mu(H_1) < \mu(M)$. This contradiction implies that M is a regular module. Assume that $M = X_i$ for some quasi-simple module X of rank $r > 1$. Because $|M| > |H_1|$, we have $i > r$. Therefore, $\mu(X_r) < \mu(M) < \mu(H_1)$. It follows that $\mu(M) < \mu(X_j) < \mu(H_1)$ for all $j > i$. This is a contradiction. □

Proposition 4.2. *Let $M \in \mathcal{I} \backslash \mathcal{T}$. If $\mu(N)$ is the direct predecessor of $\mu(M)$ for some indecomposable module N, then $N \in \mathcal{I}$ and $|N| > |M|$.*

Proof. Since $\mu(N)$ is not a take-off measure, N is not preprojective. Assume for a contradiction that $N = Y_j$ is regular for some quasi-simple module Y. Let X_i be a GR submodule of M for some quasi-simple module X and some $i \geq 1$. Then $\mu(M) > \mu(X_t)$ and thus $\mu(X_t) \neq \mu(Y_j)$ for all $t \geq 1$ by 2.2(7). Therefore $\mu(X_i) < \mu(Y_j) < \mu(M)$. It follows that $|Y_j| > |M|$ and $\mu(M) < \mu(Y_{j+1})$ since $\mu(N) = \mu(Y_j)$ is a direct predecessor of $\mu(M)$. Notice that a GR submodule T of Y_{j+1} is either Y_j or a preprojective module. In particular $\mu(T) < \mu(M) < \mu(Y_{j+1})$. Thus $|M| > |Y_{j+1}|$. This contradicts $|Y_j| > |M|$. Therefore, N is preinjective. □

4.2. Proposition 2.2(4) tells that the GR measure $\mu(X_r)$ for a quasi-simple module X of rank r is important when comparing the GR measures of regular modules X_i and those of homogeneous modules H_j. Namely, there is a similar result that can be used to compare the GR measures of two non-homogeneous regular modules.

Lemma 4.3. *Let X, Y be quasi-simple modules with rank r and s, respectively. Assume that $\mu(X_r) \geq \mu(H_1)$.*

1) *If $\mu(X_r) > \mu(Y_s)$, then $\mu(X_i) > \mu(Y_j)$ for all $i \geq r,\ , j \geq 1$.*
2) *If $\mu(X_i) = \mu(Y_j)$ for some $i \geq 2r$. Then $r = s$ and $\mu(X_t) = \mu(Y_t)$ for every $t \geq r$.*
3) *If $\mu(X_{2r}) > \mu(Y_{2s})$, then $\mu(X_i) > \mu(Y_j)$ for all $i \geq 2r, j \geq 1$.*

Proof. 1) If $\mu(Y_s) < \mu(H_1)$, then $\mu(Y_j) < \mu(H_1)$ for all $j \geq 1$. Thus we may assume that $\mu(Y_s) \geq \mu(H_1)$. Since for each $j \geq s$, $\mu(Y_j)$ starts with $\mu(Y_s)$ and $|Y_s| = |X_r| = |\delta|$, we have $\mu(X_r) > \mu(Y_j)$.

2) It is clear that $r = 1$ if and only if $s = 1$. Now we assume $r > 1$. Since $\mu(X_r) \geq \mu(H_1)$, we have $\mu(Y_s) \geq \mu(H_1)$. Thus $j \geq 2s$ and

$$\begin{aligned}\mu(Y_j) &= \mu(Y_s) \cup \{|Y_{s+1}|, |Y_{s+2}|, \ldots, |Y_{2s}|, |Y_{2s+1}|, \ldots, |Y_j|\} \\ &= \mu(X_r) \cup \{|X_{r+1}|, |X_{r+2}|, \ldots, |X_{2r}|, |X_{2r+1}|, \ldots, |X_i|\} \quad = \quad \mu(X_i).\end{aligned}$$

Because $|X_r| = |Y_s| = |\delta|$ and $|X_{2r}| = |Y_{2s}| = 2|\delta|$, we obtain that $r = s$, $\mu(X_r) = \mu(Y_s)$ and $\mu(X_{2r}) = \mu(Y_{2s})$. Note that

$$|X_{r+l}| - |X_{r+l-1}| = |Y_{r+l}| - |Y_{r+l-1}|$$

for all $l \geq 1$. It follows $\mu(X_t) = \mu(Y_t)$ for all $t \geq r = s$.

3) follows similarly. □

Corollary 4.4. *Let X be a quasi-simple module of rank r such that $\mu(X_r) \geq \mu(H_1)$. If M is an indecomposable module such that $\mu(M) = \mu(X_i)$ for some $i \geq 2r$, then M is a regular module.*

Proof. Assume for a contradiction that M is preinjective. Let Y_t be a GR submodule of M for some quasi-simple module Y of rank s. Then $\mu(M) > \mu(Y_j)$ for all $j \geq 1$ by Proposition 2.2(7). Thus $Y \ncong X$ and $t \geq 2s$ since $|M| = |X_i| > 2|\delta|$. It follows that $\mu(Y_s) \geq \mu(H_1)$. Notice that $\mu(Y_t) = \mu(X_{i-1})$. Therefore, $r = s$ and $\mu(Y_{t+1}) = \mu(X_i)$ by above lemma. This contradicts $|Y_{t+1}| > |M| = |X_i|$ (Proposition 2.1). Thus M is regular. □

4.3. We have seen in Proposition 2.2(4) that the irreducible maps $H_1 \to H_2 \to H_3 \to \ldots$ are GR inclusions. This is not only the case. Namely, in [7] ([8] for general cases) we proved that $\mu(H_{i+1})$ is the direct successor of $\mu(H_i)$ for each $i \geq 1$. Let X be a quasi-simple of rank $r > 1$. It is possible [for example, if $\mu(X_r) \geq \mu(H_1)$] that all irreducible maps $X_r \to X_{r+1} \to X_{r+2} \to \ldots$ are GR inclusions. However, it is not true in general that $\mu(X_{j+1})$ is the direct successor of $\mu(X_j)$ for all $j \geq r$ (Example

4 in [7]). The following proposition tells if $\mu(X_r) \geq \mu(H_1)$ and if $\mu(X_{j+1})$ is not the direct successor of $\mu(X_j)$, then $j < 2r$.

Proposition 4.5. *Let X be a quasi-simple module of rank r such that $\mu(X_r) \geq \mu(H_1)$. Then $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for each $j \geq 2r$.*

Proof. We may assume $r > 1$. We first show that there does not exist an indecomposable regular module M such that $\mu(M)$ lies between $\mu(X_j)$ and $\mu(X_{j+1})$ for any $j \geq 2r$. For the purpose of a contradiction, we assume that there exists a $j \geq 2r$ and an indecomposable regular module M with $\mu(X_j) < \mu(M) < \mu(X_{j+1})$. We may assume that $|M|$ is minimal. Then $|M| > |X_{j+1}| > 2|\delta|$, since X_j is a GR submodule of X_{j+1}. Let $M = Y_i$ for some quasi-simple module Y of rank $s > 1$. It follows that $\mu(Y_s) \geq \mu(H_1)$ and $i > 2s$. Therefore, Y_{i-1} is a GR submodule of Y_i and

$$\mu(Y_{i-1}) \leq \mu(X_j) < \mu(M) = \mu(Y_i) < \mu(X_{j+1})$$

by minimality of M. This implies $\mu(Y_{i-1}) = \mu(X_j)$, since otherwise $|X_j| > |M| > |X_{j+1}|$, which is a contradiction. Observe that $i - 1 \geq 2s$ and $j \geq 2r$. Then Lemma 4.3 implies $\mu(X_t) = \mu(Y_t)$ for all $t \geq r = s$. This contradicts the assumption $\mu(X_j) < \mu(M) = \mu(Y_i) < \mu(X_{j+1})$. Therefore, there are no indecomposable regular modules M satisfying $\mu(X_j) < \mu(M) < \mu(X_{j+1})$ for any $j \geq 2r$.

Now we assume that M is an indecomposable preinjective module such that $\mu(X_j) < \mu(M) < \mu(X_{j+1})$. Let Y_i be a GR submodule of M for some quasi-simple module Y and $i \geq 1$. Then $Y_i \ncong X_t$ for any $t > 0$ by Proposition 2.2(7). Comparing the lengths, we have $\mu(Y_i) \geq \mu(X_j)$. Thus Proposition 2.2(7) implies $\mu(X_j) < \mu(Y_{i+1}) < \mu(M) < \mu(X_{j+1})$. Therefore, we get an indecomposable regular module Y_{i+1} with GR measure lying between $\mu(X_j)$ and $\mu(X_{j+1})$, which is a contradiction. The proof is completed. □

4.4. Let X be a quasi-simple module of rank r such that $\mu(X_r) \geq \mu(H_1)$. For a given $i \geq 2r$, let $\mu_{i,1} > \mu_{i,2} > \ldots > \mu_{i,t_i}$ be all different GR measures of the form $\mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\}$ and $a_{i,j} \neq |X_{i+1}|$ for any $1 \leq j \leq t_i$. Notice that there are only finitely many such $\mu_{i,j}$ for each given i.

Lemma 4.6. 1) *$a_{i,j} < |X_{i+1}|$ for all $1 \leq j \leq t_i$ and $a_{i,j} < a_{i,l}$ if $j < l$.*

2) *$\mu_{i,j} > \mu(X_t)$ for all $1 \leq j \leq t_i, t \geq 1$.*

3) *$\mu_{i,j} > \mu_{l,h}$ if $i < l$.*

4) *If M is an indecomposable module such that $\mu(M) = \mu_{i,j}$, then $M \in \mathcal{I}$.*

Proof. 1) If $a_{i,j} > |X_{i+1}|$, then

$$\mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\} < \mu(X_i) \cup \{|X_{i+1}|\} = \mu(X_{i+1}).$$

This contradicts $\mu(X_{i+1})$ is a direct successor of $\mu(X_i)$ (Proposition 4.5). Thus $a_{i,j} < |X_{i+1}|$.

2) follows from 1) and the fact that $X_{2r} \subset X_{2r+1} \subset \ldots \subset X_t \subset X_{t+1} \subset \ldots$ is a sequence of GR inclusions.

3) If $i < l$, then

$$\begin{aligned} \mu_{l,h} &= \mu(X_l) \cup \{a_{l,h}\} \\ &= \mu(X_i) \cup \{|X_{i+1}|, \ldots, |X_l|, a_{l,h}\} \\ &< \mu(X_i) \cup \{a_{i,j}\} \\ &= \mu_{i,j} \end{aligned}$$

4) If M is not preinjective, then M is regular, say $M = Y_t$ for some quasi-simple module Y of rank s. Thus $t > 2s$ since $|M| > |X_i| > 2|\delta|$, and $\mu(Y_s) \geq \mu(H_1)$. In particular, Y_{t-1} is a GR submodule of Y_t and $\mu(Y_{t-1}) = \mu(X_i) < \mu(X_{i+1}) < \mu(M) = \mu(Y_t)$. This is a contradiction since $\mu(Y_t)$ is also a direct successor of $\mu(Y_{t-1})$. □

Proposition 4.7. *The sequence of GR measures*

$$\ldots < \mu_{i+1,2} < \mu_{i+1,1} < \mu_{i,t_i} < \ldots < \mu_{i,j+1} < \mu_{i,j} < \ldots < \mu_{i,2} < \mu_{i,1}$$

is a sequences of direct predecessors.

Proof. Let M be an indecomposable module such that

$$\mu(X_i) \cup \{a_{i,j+1}\} = \mu_{i,j+1} < \mu(M) < \mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\}.$$

Then $\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}$ with $a_{i,j} < b_1 \leq a_{i,j+1} < |X_{i+1}|$. By the choices of $\mu_{i,j}$, we have $m \geq 2$ and $b_1 = a_{i,j+1}$. This implies M contains a submodule N with $\mu(N) = \mu(X_i) \cup \{a_{i,j+1}\}$, which is thus a preinjective module by above lemma. However, an indecomposable preinjective module can not be a submodule of any other indecomposable module. We therefore get a contradiction.

Now let M be an indecomposable module such that

$$\mu(X_i) < \mu(X_i) \cup \{|X_{i+1}|, a_{i+1,1}\} = \mu_{i+1,1} < \mu(M) < \mu_{i,t_i} = \mu(X_i) \cup \{a_{i,t_i}\}.$$

It follows that $\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}$. By definition of μ_{i,t_i}, we have $b_1 = |X_{i+1}| < a_{i+1,1} < |X_{i+2}|$ and $m \geq 2$. From $b_2 \leq a_{i+1,1}$ and the definition of $\mu_{i+1,1}$, we obtain that $b_2 = a_{i+1,1}$ and $m \geq 3$. Therefore, M contains an indecomposable preinjective module N with GR measure $\mu(X_i) \cup \{|X_{i+1}|, a_{i+1,1}\}$ as a submodule, which is impossible.

The proof is completed. □

Remark. We should note that some segment of the sequence of the GR measures in this proposition may not exist. In this case, we can still show as in the proof that for example, $\mu_{j,1}$ is a direct predecessor of μ_{i,t_i} for some $j \geq i+2$.

Remark. Assume that $\mu_{i,j}$, constructed as above, are not landing measures (For example, X is a homogeneous simple module H_1. See section 5). Since each GR measure different from I^1 has a direct successor, We may construct direct successors starting from $\mu_{i,1}$ for a fixed i. Let $\mu(M)$ be the direct successor of $\mu_{i,1}$. If M is preinjective, then $|M| < |\mu_{i,1}| = a_{i,1}$ by Proposition 4.2. Thus after taking finitely many direct successors, we obtain a regular measure (meaning that it is a GR measure of an indecomposable regular module). Proposition 4.2 tells that all direct successors starting with this regular measure are still regular ones. One the other direction, if there are infinitely many preinjective modules containing X_i as GR submodules, then the sequence $\mu_{i,j}$ is infinite (This does occur in some case, see section 5. Thus we obtain a sequence of GR measures indexed by integers $\mathbb{Z}$.

4.5. Let's fix a tame quiver Q of type $\widetilde{\mathbb{A}}_n$. There are always GR measures having no direct successors, for example, $\mu(H_1)$ (Proposition 4.1). We are going to show that the number of GR measures possessing no direct predecessors is always finite.

Lemma 4.8. *Let X be a quasi-simple of rank $r > 1$. Assume that there is an $i \geq 1$ such that $X_i \in \mathcal{C}$ is a central module. Then there is an $i_0 \geq i$ such that $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for each $j \geq i_0$.*

Proof. By Proposition 4.5, we may assume that $\mu(X_r) < \mu(H_1)$. Since X_i is a central module, X_j is the unique, up to isomorphism, GR submodule of X_{j+1} for every $j \geq i$. We first show that there is a j_0 such that there does not exist a regular module with GR measure μ satisfying $\mu(X_j) < \mu < \mu(X_{j+1})$ for any $j \geq j_0$.

Let Y be a quasi-simple module of rank s such that $\mu(X_j) < \mu(Y_l) < \mu(X_{j+1})$ for some $j \geq i \geq r$ and $l \geq 1$. In this case, Y_l is a GR inclusion of Y_{l+1} since Y_l is a central module. Comparing the lengths, we have $\mu(Y_{l+1}) < \mu(X_{j+1})$, and similarly $\mu(Y_h) < \mu(X_{j+1})$ for all $h \geq 1$. Now replace j by some $j' > j$ and repeat the above consideration. Since there are only finitely many quasi-simple modules Z such that $\mu(Z_{r_Z}) \leq \mu(H_1)$, where r_Z is the rank of Z, we may obtain an index j_0 such that a GR measure μ of an indecomposable regular module satisfies either $\mu < \mu(X_{j_0})$ or $\mu > \mu(X_j)$ for all $j \geq 1$.

Fix the above chosen j_0. Assume that there is an indecomposable preinjective module M such that $\mu(X_j) < \mu(M) < \mu(X_{j+1})$ for some $j \geq j_0$. Then $\mu(M)$ starts with $\mu(X_j)$ and thus there is an indecomposable submodule N of M in a GR filtration of M such that $\mu(N) = \mu(X_j)$. Note that N is a regular module and thus $N = Y_l$ for some $l \geq 1$. If $X_j \cong N$, then $\mu(M) > \mu(X_j)$ for all $j \geq 0$, a contradiction. Therefore, $X_j \ncong N$. It follows that $\mu(X_j) = \mu(N) < \mu(Y_{l+1}) < \mu(M) < \mu(X_{j+1})$, which contradicts the choice of j_0. We can finish the proof by taking $i_0 = j_0$. □

Corollary 4.9. *Only finitely many GR measures of regular modules have no direct predecessors.*

Proof. Let X be a quasi-simple module of rank $r > 1$. If $\mu(X[r]) \geq \mu(H_1)$, then for every $i > 2r$, $\mu(X_i)$ has a direct predecessor $\mu(X_{i-1})$ (Proposition 4.5). Thus we may assume that $\mu(X_r) < \mu(H_1)$. If every X_i is a take-off module, then $\mu(X_i)$ has direct predecessor by definition. If there is an index $i \geq 1$ such that X_j are central modules for all $j \geq i$, then there is an index $i_0 \geq i$ such that $\mu(X_j)$ is a direct predecessor of $\mu(X_{j+1})$ for every $j \geq i_0$. Therefore, there are only finitely many GR measures of indecomposable regular modules having no direct predecessor. □

Theorem 4.10. *Only finitely many GR measures have no direct predecessors.*

Proof. By previous discussions, it is sufficient to show that all but finitely many GR measures of preinjective modules have no direct predecessors. Let M be an indecomposable preinjective module. Since there are only finitely many isomorphism classes of indecomposable preinjective modules with length smaller than $2|\delta|$, we may assume that $|M| > 2|\delta|$. Thus a GR submodule of M is X_i for some quasi-simple X of rank $r \geq 1$ and some $i \geq 2r$, and $\mu(X_r) \geq \mu(H_1)$. Without loss of generality, we may also assume that there are GR measures μ starting with $\mu(X_i)$ and $\mu < \mu(M)$. (Namely, if such a μ does not exist, we may replace M by an indecomposable preinjective module M' with $|M'| > |M| + |\delta|$. Then the GR submodule of M' is $Y_{i'}$ with $Y \ncong X$. By this way, we may finally find an integer d such that all indecomposable preinjective modules with length greater than d contain $Z_l, l \geq 2r_Z$ as GR submodules for some fixed quasi-simple module Z. Thus there are infinitely many indecomposable preinjective modules with GR measures starting with $\mu(Z_l), l \geq 2r_Z$.) Then Proposition 4.7 ensures the existence of the direct predecessor of $\mu(M)$. □

5. preinjective Central modules

In [11], it was proved that all landing modules are preinjective in the sense of Auslander and Smalø [2]. There may exist infinitely many preinjective central modules. In this section, we are going to study the preinjective modules and the central part. Throughout this section, let Q be a fixed tame quiver of type $\widetilde{\mathbb{A}}_n$.

5.1. We first describe the landing modules.

Proposition 5.1. *Let M be an indecomposable preinjective module. Then either $M \in \mathcal{L}$ or $\mu(M) < \mu(X)$ for some indecomposable regular module X.*

Proof. Assume that $\mu(M) > \mu(X)$ for all regular modules $X \in \mathcal{R}$. Let μ_1 be the direct successor of $\mu(M)$ and $\mathcal{A}(\mu_1)$ the collection of indecomposable modules with GR measure μ_1. It follows that $\mathcal{A}(\mu_1)$ contains only preinjective modules. Let $Y^1 \in \mathcal{A}(\mu_1)$ and $X^1 \to Y^1$ be a GR inclusion. Since $X^1 \in \mathcal{R}$, we have $\mu(X^1) < \mu(M) < \mu(Y^1) = \mu_1$. Thus $|M| > |Y^1|$. Let μ_2 be the direct successor of μ_1 and $Y^2 \in \mathcal{A}(\mu_2)$. As above we obtain $|Y^1| > |Y^2|$. Repeating this procedure, we get a sequence of indecomposable preinjective modules $M = Y^0, Y^1, Y^2, \ldots, Y^n, \ldots$ such that $\mu(Y^i)$ is the direct successor of $\mu(Y^{i-1})$ and $|Y^i| < |Y^{i-1}|$. Because the lengths decrease, there is some $j < \infty$ such that $\mu(Y^j)$ has no direct successor. It follows that $\mu(Y^j) = I^1$ and $\mu(M)$ is a landing measure. □

Corollary 5.2. *Let M be an indecomposable module. Then $\mu(M) > \mu(X)$ for all regular module X if and only if M is a landing module.*

Proposition 5.3. *If M, N are landing modules, then $\mu(M) < \mu(N)$ if and only if $|M| > |N|$.*

Proof. Assume that $\mu(M) < \mu(N)$. Let X be a GR submodule of N. Since X is a regular modules, we have $\mu(X) < \mu(M) < \mu(N)$ and thus $|M| > |N|$. □

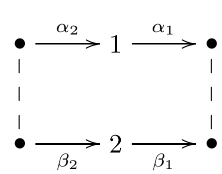
Proposition 5.4. *Assume that there is a regular tube of rank $r > 1$. Then almost all landing modules contain only exceptional regular modules as GR submodules.*

Proof. Let M be a landing module which is thus preinjective. Thus the GR submodules of M are all regular modules. Assume that M contains homogeneous modules H_i as GR submodules. Let $\mathbb{T}$ be a regular tube of rank $r > 1$. Then there exists a quasi-simple module X on $\mathbb{T}$ such that $\mu(X_r) \geq \mu(H_1)$ (2.2(5)). Thus $\mu(H_i) < \mu(X_{r+1}) < \mu(M)$ and therefore, $|X_{r+1}| > |M|$. This implies $i = 1$ and $|M| < 2|\delta|$. □

Corollary 5.5. *Assume that there is a regular tube of rank $r > 1$. If M is an indecomposable containing homogeneous modules H_i as GR submodules for some $i \geq 2$, then M is a central module.*

5.2. We are going to classify the tame quivers Q of type $\widetilde{\mathbb{A}}_n$.

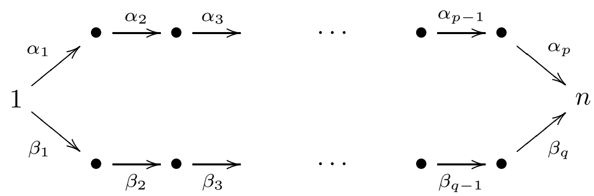
Case 1 Assume that in the quiver Q there is a clockwise path of arrows $\alpha_1\alpha_2$ and a counter clockwise path $\beta_1\beta_2$ as follows:



Let C be a string starting with α_2^{-1} and ending with β_2. Thus $s(C) = 1, e(C) = 2$. It is obvious that the string modules $M(C)$ contains both simple regular modules $S(1)$ and $S(2)$, which are in different regular components, as submodules. Thus $M(C)$ is an indecomposable preinjective module. Fix such a string C such that the length of C is large enough, i.e. $M(C)$ contains homogeneous simple

H_1 as submodules. The GR submodules of $M(C)$ is of one of the following forms $S(1)_i, S(2)_j, H_t$ for some $i,j,t \geq 1$. However, $\mu(S(1)_i) < \mu(H_1), \mu(S(2)_i) < \mu(H_1)$ for all $i \geq 0$ (2.2(6)). Thus the GR submodules of $M(C)$ are homogeneous modules. In particular, there are infinitely many indecomposable preinjective modules containing only homogeneous modules as GR submodules. Thus there are infinitely many preinjective central modules by Corollary 5.5.

As an example, we consider the following quiver $Q = \widetilde{A}_{p,q,(p+q=n)}$ with precisely one source and one sink:



There are two regular tubes $\mathbb{T}_X$ and $\mathbb{T}_Y$ consisting of string modules. The regular tube $\mathbb{T}_Y$ contains the string module Y determined by the string $\beta_q\beta_{q-1}\cdots\beta_2\beta_1$, and simple modules S corresponding to the vertices $s(\alpha_i), 2 \leq i \leq p$ as quasi-simples. The rank of Y is p. The other tube $\mathbb{T}_X$ contains string module X determined by the string $\alpha_p\alpha_{p-1}\cdots\alpha_2\alpha_1$ and simple modules S corresponding to the vertices $s(\beta_i), 2 \leq i \leq q$. The ranks of X is q. All the other regular tubes contain only band modules, and thus are homogeneous tubes.

We can easily determine the GR measures of these quasi-simple modules. Notice that any non-simple quasi-simple module (X,Y and H_1) contains $S(n)$ as the unique simple submodule. Therefore, each homogeneous simple module H_1 has GR measure $\mu(H_1) = \{1, 2, 3, \ldots, n-1, n\}$ and the GR measure for X and Y are $\mu(X) = \{1, 2, 3, \ldots, p, p+1\}$ and $\mu(Y) = \{1, 2, 3, \ldots, q, q+1\}$. It easily seen that $X_q \subset X_{q+1} \subset \ldots \subset X_j \subset \ldots$ is a chain of GR inclusions and thus $\mu(X_q) = \mu(H_1)$. Similarly, $Y_p \subset Y_{p+1} \subset \ldots \subset Y_j \subset \ldots$ is a chain of GR inclusions and $\mu(Y_p) = \mu(H_1)$.

Any non-sincere indecomposable module belongs to take-off part. This is true because the GR submodule of H_1 is a uniserial module and has GR measure $\{1, 2, 3, \ldots, n-1\}$ and a non-sincere indecomposable module has length smaller than $|\delta|$. Let $M \in \mathcal{I}$ be a sincere indecomposable preinjective module and $X \subset M$ a GR submodule. Then X is isomorphic to H_i, X_{sq} or Y_{tp} for some $i, s, t \geq 1$.

Notice that if $p \geq 2$ and $q \geq 2$, then there are infinitely many preinjective central modules by above discussion.

Case 2 $Q = \widetilde{\mathbb{A}}_{p,1}$. Let's keep the notations in the above example. By Proposition 5.4, we know that there are infinitely many landing modules containing only exceptional modules of form Y_i as GR submodules. Given an indecomposable preinjective module M and its GR submodule $Y_i, i > p$. We claim that the GR submodules of τM are homogeneous ones. Namely, if τM contains an exceptional regular module N as a GR submodule, then $N \cong Y_j$ for some $j \geq p$. In particular, both M and τM contains Y as a submodule, i.e. $\mathrm{Hom}\,(Y, M) \neq 0 \neq \mathrm{Hom}\,(Y, \tau M)$. Therefore, we have $\mathrm{Hom}\,(\tau^{-}Y, M) \neq 0 \neq \mathrm{Hom}\,(Y, M)$, which contradicts Lemma 2.3. Thus, there are infinite many indecomposable preinjective modules containing only homogeneous modules as GR submodules and hence infinitely many preinjective central modules.

Case 3 $Q \neq \widetilde{\mathbb{A}}_{p,q}$ is of the following form: all non-trivial clockwise (or counter clockwise) paths (compositions of arrows) are of length 1. In this case, all exceptional quasi-simple modules in one of the exceptional tubes are of length at least 2, and the quasi-simple modules on the other exceptional tube have length at most 2.

Let $p = \beta_t \dots \beta_2\beta_1$ be a composition of arrows in Q with maximal length. Thus there is an arrow α with ending vertex $e(\alpha) = e(p)$ and $s(\alpha)$ is a source. Let $X = M(p)$ be the string module, which is thus a quasi-simple module, say with rank r. By the maximality of p and the description of irreducible maps between string modules, we may easily deduce that the sequence of irreducible monomorphism $X = X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_r \rightarrow X_{r+1} \rightarrow \dots$ is namely a sequence of GR inclusions. Therefore

$$\mu(X_{r+1}) = \{1, 2, 3, \dots, t+1, |X_2|, |X_3|, \dots, |X_r|, |X_{r+1}|\}$$

with $|X_i| - |X_{i-1}| \geq 2$ for $2 \leq i \leq r$ and $|X_{r+1}| = |X_r| + (t+1)$.

Let Y be the string module determined by arrow α. It is also a regular quasi-simple module, say with rank s. By the description of irreducible monomorphism, we obtain that $|Y_j| = j+1$ for $j \leq t$ and $|Y_{t+1}| = t+3$. Thus

$$\mu(Y_{s+1}) \geq \{1, 2, \dots, t+1, t+3, |Y_{t+2}|, \dots, |Y_s|, |Y_{s+1}|\}$$

with $|Y_i| - |Y_{i-1}| \leq 2$ for $i \leq s$ and $|Y_{s+1}| = |Y_s| + 2$.

This proves the following lemma.

Lemma 5.6. *Let keep the notation as above. If $t > 1$, i.e. Q is not equipped with sink-source orientation, then $\mu(Y_s) \geq \mu(X_r)$ and $\mu(Y_j) > \mu(X_i)$ for $i \geq 1$ and $j > s$.*

5.3. we are going to to characterize the tame quivers Q of type $\widetilde{\mathbb{A}}_n$ such that no indecomposable preinjective modules are central modules, and show that there are always infinitely many preinjective central modules if any.

Theorem 5.7. *$\mathcal{I} \cap \mathcal{C} = \emptyset$ if and only if $\widetilde{\mathbb{A}}_n$ is equipped with sink-source orientation.*

Proof. This is clear for Kronecker quiver ($n = 1$). Thus we may assume $n \geq 2$, i.e. there exists an exceptional regular tube. Since $\mathcal{I} \cap \mathcal{C} = \emptyset$, a sincere indecomposable preinjective is always a landing module. Then the proof of Proposition 5.4 implies that there is no indecomposable preinjective modules M containing homogeneous modules $H_i, i \geq 2$ as GR submodules. Therefore, by above classification of Q, we need only to consider Case 3 and show that $\mathcal{I} \cap \mathcal{C} = \emptyset$ implies $t = 1$ (let's keep the notations in case 3). Assume for a contradiction that $t > 1$. Let S be the simple module corresponding to $s(\beta_t)$. Thus S is a quasi-simple of rank s and $\tau S \cong Y$. Let I be the (indecomposable) injective cover of S. It is obvious that $\mathrm{Hom}\,(X, I) \neq 0$. Consider indecomposable preinjective modules $\tau^{um} I$, where u is a positive integer and $m = [r, s]$ is the lowest common multiple of r and s. Since $\mathrm{Hom}\,(S, \tau^{um} I) \neq 0 \neq \mathrm{Hom}\,(X, \tau^{um} I)$, a GR submodule of $\tau^{um} I$ is either S_i or X_j. Notice that $\mu(H_1) > \mu(S_i)$ for all $i \geq 0$ since S is simple. Therefore, for u large enough, the unique GR submodule of $\tau^{um} I$ is X_j for some $j \geq 1$ because no indecomposable preinjective modules containing H_i as GR submodules for $i \geq 2$. In particular there are infinitely many preinjective modules containing GR submodules of the form $X_j, j \geq 1$. Thus we may select a GR inclusion $X_j \subset M$ such $|X_j| > |Y_{s+1}|$. Because $\mu(X_j) < \mu(Y_{s+1}) < \mu(M)$, we have $|Y_{s+1}| > |M|$. This contracts $|X_j| > |Y_{s+1}|$. Thus we have $t = 1$ and Q is of sink-source orientation.

Conversely, if $\widetilde{\mathbb{A}}_n$ is equipped with a sink-source orientation, we may see directly that $\mathcal{I} \cap \mathcal{C} = \emptyset$ (for details, see [7] Example 3). □

Theorem 5.8. $\mathcal{I} \cap \mathcal{C} \neq \emptyset \Rightarrow |\mathcal{I} \cap \mathcal{C}| = \infty$.

Proof. We have seen that an indecomposable module containing homogeneous modules $H_i, i \geq 2$ as GR submodules is a central module. Thus we may assume that there are only finitely many indecomposable preinjective module containing homogenous modules as GR submodules. Thus, we need only consider case 3. Let's keep the notations there. Then $\mathcal{I} \cap \mathcal{C} \neq \emptyset$ implies that Q is not of sink-source orientation. In particular, the length t of the longest path of arrows $\beta_t \cdots \beta_1$ is greater than 1. In particular, $\mu(Y_j) > \mu(X_i)$ for all $i \geq 1, j > s$. Again let $m = [r, s]$. By assumption, the GR submodules of $\tau^{um} I$ are of the form X_i for almost all $u \geq 1$. To avoid a contradiction as in the proof of above theorem, $\mu(\tau^{um} I)$ have to smaller than $\mu(Y_{s+1})$ for u large enough and thus almost all $\tau^{um} I$ are central modules. □

6. APPENDIX

In section 4 we showed that for a tame quiver of type $\widetilde{\mathbb{A}}_n$, there are only finitely many GR measures having no direct predecessors. Namely, this can be generalized for any tame quiver, i.e. a quiver of type $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$.

Theorem 6.1. *Let Λ be a tame quiver. Then there are only finitely many GR measures having no direct predecessors.*

The proof of this theorem is almost the same as that for $\widetilde{\mathbb{A}}_n$ case. The statements 2)–4) and 7) in Lemma 2.2 and Lemma 4.3 hold for all tame quivers [8]. Proposition 4.5 remain true. But the proof should be changed a little bit because in general, a GR submodule of a preinjective module is not necessary a regular module. The first part of the proof is valid in general cases. For the second part, we have to change as follows:

Proof. Assume that M is an indecomposable preinjective module such that $\mu(X_i) < \mu(M) < \mu(X_{i+1})$ with $|M|$ minimal. Let N be a GR submodule of M. Comparing the lengths, we have $\mu(X_i) \leq \mu(N)$. If $N = Y_j$ is regular for some quasi-simple module Y of rank s, then $\mu(M) > \mu(Y_{j+1}) > \mu(Y_j) \geq \mu(X_i)$. This contradicts the first part of the proof. If N is preinjective, then $\mu(N) = \mu(X_i)$ by the minimality of $|M|$. Thus a GR filtration of N contains a regular module Z_{2t} for a quasi-simple Z of rank t. It follows that $\mu(X_{2r}) = \mu(Z_{2t})$. Thus $\mu(M) > \mu(N) > \mu(Z_{i+1}) = \mu(X_{i+1})$ which is a contradiction. □

Lemma 4.6 is true in general. However, Proposition 4.7 should be replaced by the following one:

Proposition 6.2. 1) *There are only finitely many GR measures lying between $\mu_{i,j}$ and $\mu_{i,j+1}$.*
2) *There are only finitely many GR measures lying between μ_{i,t_i} and $\mu_{i+1,1}$.*
3) *In particular, $\mu_{i,j}$ has a direct predecessor.*

Proof. Assume that M is an indecomposable module such that $\mu(X_i) \cup \{a_{i,j+1}\} = \mu_{i,j+1} < \mu(M) < \mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\}$. Then $\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}$. By definition of $\mu_{i,j}$, we have $b_1 = a_{i,j+1}$ and $m \geq 2$, In particular, M has a GR filtration containing an indecomposable module N such that $\mu(N) = \mu(X_i) \cup \{b_1\}$, which is thus preinjective. However, there are only finitely many indecomposable modules containing a given indecomposable preinjective module as a submodule.

It follows that only finitely many GR measures starting with $\mu(N) = \mu(X_i) \cup \{b_1\}$. Therefore, the number of GR measures, which lies between $\mu_{i,j+1}$ and $\mu_{i,j}$ is finite for each $i \geq 2r$.

2) follows similarly and 3) is a direct consequence of 1) and 2). The first remark after Proposition 4.7 still works for this case. □

The remaining proof of Theorem 6.1 is similar.

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