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ON EQUIVARIANT BIJECTIONS RELATIVE TO THE DEFINING CHARACTERISTIC

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Abstract. This paper is a contribution to the general program introduced by Isaacs, Malle and Navarro to prove the McKay conjecture in the representation theory of finite groups. We develop new methods for dealing with simple groups of Lie type in the defining characteristic case. Using a general argument based on the representation theory of connected reductive groups with disconnected center, we show that the inductive McKay condition holds if the Schur multiplier of the simple group has order 2. As a consequence, the simple groups $\mathrm{P}\Omega_{2m+1}(p^n)$ and $\mathrm{PSp}_{2m}(p^n)$ are "good" for $p>2$ and the simple groups $E_7(p^n)$ are "good" for $p>3$ in the sense of Isaacs, Malle and Navarro. We also describe the action of the diagonal and field automorphisms on the semisimple and the regular characters.

1. Introduction

Let G be a finite group, p a prime dividing the order of G and P a Sylow p-subgroup of G. The McKay conjecture asserts that the number $|\operatorname{Irr}_{p'}(G)|$ of irreducible complex characters of G of degree not divisible by p coincides with the number $|\operatorname{Irr}_{p'}(N_G(P))|$ of irreducible complex characters of the normalizer $N_G(P)$ of degree not divisible by p.

In [16], Isaacs, Malle and Navarro reduced the proof of the McKay conjecture to a question about finite simple groups. They were able to prove that the McKay conjecture is true for all finite groups if every finite non-abelian simple group is "good" for all prime numbers p; see [16, Section 10] for a precise formulation.

Malle has shown that all simple groups not of Lie type and all simple groups of Lie type with exceptional Schur multiplier are "good" for all primes p [20]. Furthermore, Malle [21], [19] and Späth [23] proved important results for simple groups G of Lie type and primes p different from the defining characteristic. The case where G is a simple group of Lie type and p the defining characteristic was considered by several authors. It was shown in [16] that the simple groups $\mathrm{PSL}_2(q)$, ${}^2B_2(2^{2n+1})$, ${}^2G_2(3^{2n+1})$ and in [12], [13] that the simple groups ${}^2F_4(2^{2n+1})$ and ${}^3D_4(2^n)$ are "good" for the defining characteristic. A uniform treatment of simple groups of Lie type with trivial Schur multiplier and cyclic outer automorphism group in the case of defining characteristic was obtained in [4]. In particular, the results in [4] include that the simple groups $G_2(q)$, $F_4(q)$ and $E_8(q)$ are "good" for the defining characteristic.

In the present paper, we consider certain simple groups of Lie type with Schur multiplier of order 2 in the defining characteristic case. More precisely, our main result is the following

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Theorem 1.1. *Let $\mathbf{G}$ be a simple and simply-connected algebraic group defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$. Let $\mathbf{Z}$ be the center of $\mathbf{G}$ and let W denote its Weyl group (relative to an F-stable maximal torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$). Suppose that*

- *the finite group $X = \mathbf{G}^F / \mathrm{Z}(\mathbf{G}^F)$ is simple,*
- *the group $\mathbf{G}^F$ is the universal cover of X,*
- *the automorphism induced by F on W is trivial,*
- *the prime p is good for $\mathbf{G}$ and*
- *the center $\mathbf{Z}$ has order 2.*

Then the finite simple group X is "good" for the prime p.

A crucial step in the proof of Theorem 1.1 is to show the existence of $\mathcal{A}$-equivariant bijections between certain subsets of $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F)$, where $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}^F$ and $\mathcal{A}$ is a group of outer automorphisms of $\mathbf{G}^F$ stabilizing $\mathbf{B}^F$. We had to solve several problems which do not show up in the case where the Schur multiplier is trivial:

First, we had to consider the action of the diagonal and field automorphisms of $\mathbf{G}^F$ on $\mathrm{Irr}_{p'}(\mathbf{G}^F)$. Under the assumptions of Theorem 1.1, the set $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ is the set of semisimple characters of $\mathbf{G}^F$, and these characters are up to signs the duals (with respect to Alvis-Curtis duality) of the irreducible constituents of the Gelfand-Graev characters of $\mathbf{G}^F$. When the center $\mathbf{Z}$ of $\mathbf{G}$ is connected, we can label the semisimple characters of $\mathbf{G}^F$ by the set of semisimple conjugacy classes of $\mathbf{G}^{*F^*}$, where $(\mathbf{G}^*, F^*)$ is a pair dual to $(\mathbf{G}, F)$. When the center of $\mathbf{G}$ is not connected, there is a similar, but more complicated parametrization. It depends on some additional parameters whose choice is not canonical, and it is *a priori* a difficult problem to describe the action of the automorphisms of $\mathbf{G}^F$ on these characters with respect to this labelling. To solve this problem (see Subsection 3.3), we use the theory of Gelfand-Graev characters for connected reductive groups with disconnected center developed by Digne-Lehrer-Michel in [9] and [10].

Second, we had to find a suitable parametrization of $\mathrm{Irr}_{p'}(\mathbf{B}^F)$. When the center $\mathbf{Z}$ of $\mathbf{G}$ is connected, the set of orbits of $\mathbf{T}^F$ on the linear characters of the unipotent radical of $\mathbf{B}^F$ can be parametrized by subsets of the set of simple roots, and Clifford theory leads to a particularly nice parametrization of $\mathrm{Irr}_{p'}(\mathbf{B}^F)$. When $\mathbf{Z}$ is not connected, Clifford theory still applies, but the parametrization becomes more complicated. To solve this problem, we had to introduce, as above, additional parameters whose choice is not canonical.

Third, the above bijections have to be compatible with linear characters of the center $\mathbf{Z}^F$ of $\mathbf{G}^F$. To prove the existence of such bijections we use counting arguments based on the norm map $N_{F'^m/F'} : \mathbf{Z}^{F'^m} \to \mathbf{Z}^{F'}$, where m is some positive integer and $F' : \mathbf{G} \to \mathbf{G}$ a Frobenius map inducing some field automorphism of $\mathbf{G}^F$. One of the difficulties we had to face is, that, in general, the norm map is not surjective (because the center of $\mathbf{G}$ is not connected).

Finally, in the situation of Theorem 1.1, the cohomology condition occurring in the definition of "good" becomes non-trivial and has to be treated. In particular, we had to study extensions of the characters of p'-degree of $\mathbf{G}^F$ and $\mathbf{B}^F$.

Some of our results on equivariant bijections are also true in the more general context where the group of diagonal automorphisms has prime order (possibly > 2).

This might be helpful in proving that the remaining simple groups of Lie type are "good".

We point out that we prove Theorem 1.1 using general methods (essentially the theory of Deligne-Lusztig and the theory of Gelfand-Graev characters for connected reductive groups with disconnected center). Theorem 1.1 applies to finite simple groups of Lie type B_m, C_m and E_7 in the defining characteristic. So we get as consequence:

Corollary 1.2. *Let p be a prime and m, n positive integers. The following simple groups are "good" for the prime p:*

(a) $\mathrm{P}\Omega_{2m+1}(p^n)$ *if* $p > 2$ *and* $m \geq 2$,
(b) $\mathrm{PSp}_{2m}(p^n)$ *if* $p > 2$ *and* $m \geq 2$,
(c) $E_7(p^n)$ *if* $p > 3$.

Independently of this work, J. Maslowski obtains in his PhD thesis partial results on the inductive McKay condition for classical groups in defining characteristic. Note that his approach relies on the natural matrix representations of these groups and is completely different from ours; see [22].

This paper is organized as follows. In Section 2, we introduce the notation and general setup. In Sections 3 and 4, we describe a parametrization of the sets of irreducible characters of p'-degree of $\mathbf{G}^F$ and $\mathbf{B}^F$, respectively, and study the action of field and diagonal automorphisms on these sets of characters. Section 5 is at the heart of this paper. We prove the existence of bijections between the sets of irreducible characters of p'-degree of $\mathbf{G}^F$ and $\mathbf{B}^F$ which are equivariant with respect to field and diagonal automorphisms, and which are compatible with the central characters of $\mathbf{Z}^F$. In Section 6, we prove our main result. A large part of this section is devoted to the proof of the cohomology condition.

2. Notation and Setup

In this section, we introduce the general setup and notation which will be used throughout this paper.

2.1. **Group theoretical setup.** Le $\mathbf{G}$ be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic $p > 0$ with q elements and corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$. We do not assume that the center $\mathbf{Z}$ of $\mathbf{G}$ is connected. As a general reference for the representation theory of finite groups of Lie type with disconnected center, we refer to [1].

The algebraic group $\mathbf{G}$ can be embedded in a connected reductive group $\widetilde{\mathbf{G}}$ with an $\mathbb{F}_q$-rational structure obtained by extending F to $\widetilde{\mathbf{G}}$, such that the center of $\widetilde{\mathbf{G}}$ is connected and the groups $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ have the same derived subgroup (see for example [8, p. 139] for this construction). Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\widetilde{\mathbf{T}}$ for the unique F-stable torus of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$. Fix an F-stable Borel subgroup $\widetilde{\mathbf{B}}$ of $\widetilde{\mathbf{G}}$ containing $\widetilde{\mathbf{T}}$ and $\mathbf{B}$. Write $\mathbf{U}$ for the unipotent radical of $\mathbf{B}$ and let $\Delta = \{\alpha_i \mid i \in I\}$ be the set of simple roots defined by $\mathbf{B}$ and $\mathbf{T}$, and let Φ be the root system of $\mathbf{G}$ relative to $\mathbf{T}$. We write Φ^+ for the set of positive roots defined by $\mathbf{B}$. Note that $\mathbf{U}$ is the unipotent radical of $\widetilde{\mathbf{B}}$. Furthermore, Φ can be identified with the root system of $\widetilde{\mathbf{G}}$ relative to $\widetilde{\mathbf{T}}$, and Δ can be identified with the set of simple roots of $\widetilde{\mathbf{G}}$ defined by $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{B}}$. Let $W \simeq \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group of $\mathbf{G}$. So, W is isomorphic to the Weyl group of $\widetilde{\mathbf{G}}$. Note that, since $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ are F-stable, F induces an automorphism of W. Throughout this paper, we assume that F acts trivially on W.

For $\alpha \in \Phi$, we write $\mathbf{X}_\alpha$ for the unipotent subgroup of $\mathbf{G}$ corresponding to α. That is, $\mathbf{X}_\alpha$ is the minimal non-trivial closed unipotent subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$, such that $\mathbf{T}$ acts on $\mathbf{X}_\alpha$ by α, where α is viewed as a character of $\mathbf{T}$. Recall that $\mathbf{X}_\alpha$ and $(\overline{\mathbb{F}}_p, +)$ are isomorphic as algebraic groups. Fix an automorphism $x_\alpha : \overline{\mathbb{F}}_p \to \mathbf{X}_\alpha$ for every $\alpha \in \Phi$. Since F acts trivially on W, we can choose it in such a way that for every $t \in \overline{\mathbb{F}}_p$, we have ${}^F x_\alpha(t) = x_\alpha(t^q)$. In particular, $\mathbf{X}_\alpha$ is F-stable. We have $\mathbf{U} = \prod_{\alpha\in\Phi^+} \mathbf{X}_\alpha$. Put

$$\mathbf{U}_0 = \prod_{\alpha\in\Phi^+,\ \alpha\notin\Delta} \mathbf{X}_\alpha. \tag{1}$$

Then $\mathbf{U}_0$ is a normal subgroup of $\mathbf{U}$ and the quotient $\mathbf{U}_1 = \mathbf{U}/\mathbf{U}_0$ is abelian. Note that $\mathbf{U}_0$ is the derived subgroup of $\mathbf{U}$; see [8, 14.17]. Moreover, there is an F- and $\mathbf{T}$-equivariant isomorphism of algebraic varieties

$$\mathbf{U}_1 \simeq \prod_{\alpha\in\Delta} \mathbf{X}_\alpha, \tag{2}$$

and in the following, we will identify the left and right hand side of (2) via this isomorphism. Note that the groups $\mathbf{U}$ and $\mathbf{U}_0$ are F-rational and

$$\mathbf{U}_1^F = \prod_{\alpha\in\Delta} \mathbf{X}_\alpha^F. \tag{3}$$

2.2. **Character theoretical notation.** For a finite group H, we write $\mathrm{Irr}(H)$ for the set of complex irreducible characters of H and $\langle \cdot, \cdot \rangle_H$ for the usual scalar product on the space of class functions. If ζ is an irreducible character of a normal subgroup N of H, we define $\mathrm{Irr}(H|\zeta) := \{\chi \in \mathrm{Irr}(H) \mid \langle \mathrm{Res}^H_N(\chi), \zeta \rangle_N \neq 0\}$. Note that, if N is central in H, then $\chi \in \mathrm{Irr}(H|\zeta)$ if and only if $\mathrm{Res}^H_N(\chi)$ is a multiple of ζ.

Furthermore, let $\mathrm{Irr}_{p'}(H)$ be the set of all $\chi \in \mathrm{Irr}(H)$ such that $\chi(1)$ is not divisible by p, and similarly, $\mathrm{Irr}_{p'}(H|\zeta)$ the set of all $\chi \in \mathrm{Irr}(H|\zeta)$ such that $\chi(1)$ is not divisible by p.

2.3. **Semisimple and regular characters.** Let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ and let $(\widetilde{\mathbf{G}}^*, F^*)$ be a pair dual to $(\widetilde{\mathbf{G}}, F)$ in the sense of [7, Section 4.3]. The natural embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ induces a surjective homomorphism $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ commuting with F^*, which induces a surjective homomorphism $i^* : \widetilde{\mathbf{G}}^{*F^*} \to \mathbf{G}^{*F^*}$, see [1, 2.D, 2.F]. Let $\mathbf{T}^*$ be an F^*-stable maximal torus of $\mathbf{G}^*$ in duality with $\mathbf{T}$. Note that there is a natural anti-isomorphism between W and the Weyl group of $\mathbf{G}^*$ relative to $\mathbf{T}^*$, and we will identify the sets of elements of these two Weyl groups via this anti-isomorphism. For $w \in W$, we write $\mathbf{T}_w$ for a F-stable maximal torus of $\mathbf{G}$ obtained from $\mathbf{T}$ by twisting with w. Write $\mathbf{T}^*_w$ for an maximal torus in duality with $\mathbf{T}_w$. Recall that, for $w \in W$, there is an isomorphism $\mathbf{T}_w^{*F^*} \to \mathrm{Irr}(\mathbf{T}_w^F)$.

For $s \in \mathbf{T}_w^{*F^*}$, we can define the corresponding Deligne-Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_w}(s)$ as follows. Using the above isomorphism, we associate to $s \in \mathbf{T}_w^{*F^*}$ the linear character $\theta_s \in \mathrm{Irr}(\mathbf{T}_w^F)$ and put $R_{\mathbf{T}_w}^{\mathbf{G}}(s) = R_{\mathbf{T}_w}^{\mathbf{G}}(\theta_s)$, where $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta_s)$ is the Deligne-Lusztig character corresponding to $\theta_s \in \mathrm{Irr}(\mathbf{T}_w^F)$. For more details on the construction and properties of Deligne-Lusztig characters, we refer to [7, Section 7] or [8].

For a semisimple element $s \in \mathbf{G}^{*F^*}$, let $W^\circ(s) \subseteq W$ be the Weyl group of $C^\circ_{\mathbf{G}^*}(s)$. We define

$$\rho_s = \frac{1}{|W^\circ(s)|} \sum_{w \in W^\circ(s)} R_{\mathbf{T}_w}^{\mathbf{G}}(s), \tag{4}$$

$$\chi_s = \frac{\varepsilon_{\mathbf{G}} \varepsilon_{C^\circ_{\mathbf{G}^*}(s)}}{|W^\circ(s)|} \sum_{w \in W^\circ(s)} \varepsilon(w) R_{\mathbf{T}_w}^{\mathbf{G}}(s), \tag{5}$$

where ε is the sign character of W and $\varepsilon_{\mathbf{G}} = (-1)^{\mathrm{rk}_{\mathbb{F}_q}(\mathbf{G})}$. Here $\mathrm{rk}_{\mathbb{F}_q}(\mathbf{G})$ is the $\mathbb{F}_q$-rank of $\mathbf{G}$; see [8, 8.3]. Note that χ_s and ρ_s only depend on the semisimple class of s in $\mathbf{G}^{*F^*}$ and that the class functions ρ_s and χ_s are multiplicity free characters of $\mathbf{G}^F$, see [1, 15.11]. Moreover, if $\widetilde{s}$ denotes a semisimple element of $\widetilde{\mathbf{G}}^F$ such that $i^*(\widetilde{s}) = s$, then we have

$$\chi_s = \mathrm{Res}^{\widetilde{\mathbf{G}}^F}_{\mathbf{G}^F}(\chi_{\widetilde{s}}) \quad \text{and} \quad \rho_s = \mathrm{Res}^{\widetilde{\mathbf{G}}^F}_{\mathbf{G}^F}(\rho_{\widetilde{s}}).$$

The irreducible constituents of ρ_s and χ_s are the so-called semisimple and the regular characters of $\mathbf{G}^F$, respectively.

To obtain a better understanding of these characters, we now describe the Gelfand-Graev characters of $\mathbf{G}^F$. Fix $\phi_0 \in \mathrm{Irr}(\mathbf{U}_1^F)$ such that $\phi_0|_{\mathbf{X}_\alpha^F}$ is non-trivial for all $\alpha \in \Delta$. The corresponding linear character of $\mathbf{U}^F$, obtained by inflation and also denoted by ϕ_0 in the following, is called regular. As explained in [8, 14.28], the set of $\mathbf{T}^F$-orbits of regular characters of $\mathbf{U}^F$ is in bijection with $H^1(F, \mathbf{Z})$ as follows. For $z \in H^1(F,\mathbf{Z})$, we choose $t_z \in \mathbf{T}$ such that $t_z^{-1}F(t_z) \in z$. Then the regular character $\phi_z = {}^{t_z}\phi_0$ of $\mathbf{U}^F$ is a representative for the corresponding $\mathbf{T}^F$-orbit. For $z \in H^1(F,\mathbf{Z})$, we define the corresponding Gelfand-Graev character of $\mathbf{G}^F$ by

$$\Gamma_z = \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{U}^F}(\phi_z).$$

Thanks to [8, 14.49], the constituents of Γ_z (for any $z \in H^1(F,\mathbf{Z})$) are the regular characters of $\mathbf{G}^F$. More precisely, for every $z \in H^1(F,\mathbf{Z})$, the multiplicity free characters χ_s and Γ_z have exactly one irreducible constituent in common, denoted by $\chi_{s,z}$, such that

$$\Gamma_z = \sum_{s \in \mathcal{S}} \chi_{s,z},$$

where $\mathcal{S}$ is a set of representatives for the semisimple conjugacy classes of $\mathbf{G}^{*F^*}$; see [8, 14.49]. Let $D_{\mathbf{G}}$ be the Alvis-Curtis duality functor [8, 8.8], defined for $g \in \mathbf{G}^F$ and $\chi \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$ by

$$D_{\mathbf{G}}(\chi)(g) = \sum_{\mathbf{P} \supseteq \mathbf{B}} (-1)^{r(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}(\chi)(g), \tag{6}$$

where the summation is over the set of F-stable parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$ and where $\mathbf{L}$ is an F-stable Levi complement of $\mathbf{P}$, $r(\mathbf{P})$ is the semisimple $\mathbb{F}_q$-rank of $\mathbf{P}$, $R_{\mathbf{L}}^{\mathbf{G}}$ denotes the Harish-Chandra induction and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ the Harish-Chandra restriction (i.e., the adjoint functor of $R_{\mathbf{L}}^{\mathbf{G}}$); see [8, 4.6]. For every semisimple element $s \in \mathbf{G}^{*F}$ and $z \in H^1(F, \mathbf{Z})$, define

$$\rho_{s,z} = \varepsilon_{\mathbf{G}} \varepsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} D_{\mathbf{G}}(\chi_{s,z}).$$

Note that the characters $\rho_{s,z}$ are the semisimple characters of $\mathbf{G}^F$ and $\{\rho_{s,z} \mid z \in H^1(F, \mathbf{Z})\}$ is the set of constituents of ρ_s. Moreover, we have

$$D_{\mathbf{G}}(\Gamma_z) = \sum_{s \in \mathcal{S}} \varepsilon_{\mathbf{G}} \varepsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} \rho_{s,z}.$$

Let $\widetilde{\mathbf{T}}$ be the maximal F-stable torus of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$. Then the group $\widetilde{\mathbf{T}}^F$ acts on $\mathbf{G}^F$ by conjugation. Note that the induced outer automorphisms of $\mathbf{G}^F$ obtained in this way are the diagonal automorphisms of $\mathbf{G}^F$. The group generated by the diagonal automorphisms of $\mathbf{G}^F$ will be denoted by D in the following. Write $\widetilde{\mathbf{G}}^F(s)$ for the inertia subgroup of $\rho_{s,1}$ in $\widetilde{\mathbf{G}}^F$ and $A_{\mathbf{G}^*}(s) := \mathrm{C}_{\mathbf{G}^*}(s) / \mathrm{C}_{\mathbf{G}^*}(s)^\circ$. Then $|\widetilde{\mathbf{G}}^F / \widetilde{\mathbf{G}}^F(s)| = |A_{\mathbf{G}^*}(s)^{F^*}|$, see [1, 11.E, 15.13]. In particular, the character ρ_s has $|A_{\mathbf{G}^*}(s)^{F^*}|$ irreducible constituents and they form a single $\widetilde{\mathbf{T}}^F$-orbit.

2.4. **The norm map.** Let $\mathbf{H}$ be an abelian algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius map F'. For any positive integer m we define

$$N_{F'^m/F'} : \mathbf{H}^{F'^m} \to \mathbf{H}^{F'},\ h \mapsto hF'(h) \cdots F'^{m-1}(h). \tag{7}$$

For a class function θ of $\mathbf{H}^{F'}$, we set $N^*_{F'^m/F'}(\theta) := \theta \circ N_{F'^m/F'}$. So, for $\theta \in \mathrm{Irr}(\mathbf{H}^{F'})$, the map $N^*_{F'^m/F'}(\theta)$ is an F'-stable irreducible character of $\mathbf{H}^{F'^m}$.

Lemma 2.1. *If $N_{F'^m/F'}$ is surjective, then*

$$\mathrm{Irr}_{F'}(\mathbf{H}^{F'^m}) = \{N^*_{F'^m/F'}(\theta) \mid \theta \in \mathrm{Irr}(\mathbf{H}^{F'})\},$$

where $\mathrm{Irr}_{F'}(\mathbf{H}^{F'^m})$ is the set of F'-stable linear characters of $\mathbf{H}^{F'^m}$. Moreover, for all generalized characters θ, $\theta' \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{H}^{F'})$, one has

$$\langle N^*_{F'^m/F'}(\theta), N^*_{F'^m/F'}(\theta') \rangle_{\mathbf{H}^{F'^m}} = \langle \theta, \theta' \rangle_{\mathbf{H}^{F'}}.$$

Proof. By [15, 6.32], one has $|\,\mathrm{Irr}_{F'}(\mathbf{H}^{F'^m})| = |\mathbf{H}^{F'}|$ and, if $N_{F'^m/F'}$ is surjective, then the map $N^*_{F'^m/F'} : \mathrm{Irr}(\mathbf{H}^{F'}) \to \mathrm{Irr}(\mathbf{H}^{F'^m})$, $\theta \mapsto \theta \circ N_{F'^m/F'}$ is injective. Since $N^*_{F'^m/F'}(\theta) \in \mathrm{Irr}_{F'}(\mathbf{H}^{F'^m})$, the first equality follows. Now, let θ, $\theta' \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{H}^{F'})$.

Then

$$\begin{aligned}
\langle N^*_{F'^m/F'}(\theta), N^*_{F'^m/F'}(\theta')\rangle_{\mathbf{H}^{F'}} &= \frac{1}{|\mathbf{H}^{F'^m}|}\sum_{h\in\mathbf{H}^{F'^m}} \theta\circ N_{F'^m/F'}(h)\,\overline{\theta'\circ N_{F'^m/F'}(h)} \\
&= \frac{1}{|\mathbf{H}^{F'^m}|}\sum_{k\in\mathbf{H}^{F'}}\sum_{h\in N^{-1}_{F'^m/F'}(k)} \theta(k)\,\overline{\theta'(k)} \\
&= \frac{1}{|\mathbf{H}^{F'^m}|}\sum_{k\in\mathbf{H}^{F'}} |N^{-1}_{F'^m/F'}(k)|\theta(k)\,\overline{\theta'(k)} \\
&= \frac{1}{|\mathbf{H}^{F'^m}|}\sum_{k\in\mathbf{H}^{F'}} \frac{|\mathbf{H}^{F'^m}|}{|\mathbf{H}^{F'}|}\theta(k)\,\overline{\theta'(k)} \\
&= \frac{1}{|\mathbf{H}^{F'}|}\sum_{k\in\mathbf{H}^{F'}} \theta(k)\,\overline{\theta'(k)} \\
&= \langle \theta, \theta'\rangle_{\mathbf{H}^{F'}}.
\end{aligned}$$

This yields the claim. □

Note that if $\mathbf{H}$ is connected, then $N_{F'^m/F'}$ is surjective. Indeed, if $y \in \mathbf{H}^{F'}$, then the Lang-Steinberg theorem implies that there is $x \in \mathbf{H}$ with $y = x^{-1}F'^m(x)$. Then the element $x^{-1}F'(x)$ lies in $\mathbf{H}^{F'^m}$ and $N_{F'^m/F'}(x^{-1}F'(x)) = y$.

2.5. **Semisimple characters and central characters.** As in Subsection 2.1, let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ (with Frobenius map F) and let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ as above. Note that for every positive integer m, the map F^m is a Frobenius map on $\mathbf{G}$ defining a rational structure over $\mathbb{F}_{q^m}$, and $(\mathbf{G}^*, F^{*m})$ is in duality with $(\mathbf{G}, F^m)$. Moreover, if s is an F^*-stable semisimple element of $\mathbf{G}^*$ contained in an F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ and if $(\mathbf{T}, \theta)$ is a pair in duality with $(\mathbf{T}^*, s)$, then $(\mathbf{T}, N^*_{F^m/F}(\theta))$ is in duality with $(\mathbf{T}^*, s)$ with respect to the Frobenius map F^m.

Lemma 2.2. *Fix a positive integer m and let $s \in \mathbf{G}^{*F^*}$ be a semisimple element contained in the F^*-stable maximal torus $\mathbf{T}^*$. Let ρ_s (resp. $\rho_s^{[m]}$) be the corresponding sum of semisimple irreducible characters of $\mathbf{G}^F$ (resp. of $\mathbf{G}^{F^m}$). If $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective, then one has*

$$\operatorname{Res}^{\mathbf{G}^{F^m}}_{\mathbf{Z}^{F^m}}(\rho_s^{[m]}) = \rho_s^{[m]}(1)\cdot N^*_{F^m/F}(\nu),$$

where ν is a linear character of $\mathbf{Z}^F$ such that $\operatorname{Res}^{\mathbf{G}^F}_{\mathbf{Z}^F}(\rho_s) = \rho_s(1)\cdot\nu$.

Proof. We use the notation from Subsections 2.3 and 2.4. By the construction of $\widetilde{\mathbf{G}}$, we have $\mathbf{Z} = Z(\mathbf{G}) \subseteq Z(\widetilde{\mathbf{G}})$ and so $\mathbf{Z}^{F^m} \subseteq Z(\widetilde{\mathbf{G}})^{F^m} = Z(\widetilde{\mathbf{G}}^{F^m})$, see also [1, 6.2]. Thus, there is some $\nu \in \operatorname{Irr}(\mathbf{Z}^{F^m})$ such that

$$\operatorname{Res}^{\mathbf{G}^{F^m}}_{\mathbf{Z}^{F^m}}(\rho_s^{[m]}) = \operatorname{Res}^{\widetilde{\mathbf{G}}^{F^m}}_{\mathbf{Z}^{F^m}}(\rho_{\widetilde{s}}^{[m]}) = \rho_{\widetilde{s}}^{[m]}(1)\cdot\nu = \rho_s^{[m]}(1)\cdot\nu,$$

where $\widetilde{s}$ in an F^*-stable element of $\widetilde{\mathbf{G}}$ satisfying $i^*(\widetilde{s}) = s$. For $w \in W^\circ(s)$, let $\theta_s^{[m]}$ be the linear character of $\mathbf{T}_w^{F^m}$ associated with s. Thanks to [1, 9.D], we have

$$\operatorname{Res}^{\mathbf{T}_w^{F^m}}_{\mathbf{Z}^{F^m}}(\theta_s^{[m]}) = \nu.$$

Furthermore, s is F^*-stable. Thus, $\theta_s^{[m]}$ is F-stable (see the proof of [4, 1.1]), implying ν is F-stable. Since $N_{F^m/F}: \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective, Lemma 2.1 implies there is a linear character ν_0 of $\mathbf{Z}^F$ satisfying $N^*_{F^m/F}(\nu_0) = \nu$. Write θ_s for the linear character of $\mathbf{T}_w^F$ associated with s. By the remarks preceding Lemma 2.2, we have $N^*_{F^m/F}(\theta_s) = \theta_s^{[m]}$. Let $z_0 \in \mathbf{Z}^F$. Then there is $z \in \mathbf{Z}^{F^m}$ with $N_{F^m/F}(z) = z_0$. It follows

$$\begin{aligned}
\nu_0(z_0) &= \nu_0 \circ N_{F^m/F}(z) \\
&= \nu(z) \\
&= \mathrm{Res}^{T_w^{F^m}}_{Z^{F^m}}(\theta_s^{[m]})(z) \\
&= \theta_s^{[m]}(z) \\
&= \theta_s \circ N_{F^m/F}(z) \\
&= \theta_s(z_0).
\end{aligned}$$

In particular, we have $\mathrm{Res}^{\mathbf{T}_w^F}_{\mathbf{Z}^F}(\theta_s) = \nu_0$ and [1, 9.D] implies

$$\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{Z}^F}(\rho_s) = \rho_s(1) \cdot \nu_0,$$

as required. □

2.6. **Central products.** We recall some general facts about characters of central products. If N is a normal subgroup of a finite group G, then we can associate to every G-invariant irreducible character χ of N an element $[\chi]_{G/N}$ of the cohomology group $H^2(G/N, \mathbb{C}^\times)$ of G/N; see [15, 11.7] for more details. If $G = HK$ is a central product with $Z = H \cap K$ and ν a linear character of Z, then for $\chi_H \in \mathrm{Irr}(H|\nu)$ and $\chi_K \in \mathrm{Irr}(K|\nu)$, one defines

$$(\chi_H \cdot \chi_K)(hk) = \chi_H(h)\chi_K(k) \tag{8}$$

for all $h \in H$ and $k \in K$. Note that $\chi_H \cdot \chi_K$ is a well-defined irreducible character of HK and every irreducible character of HK has this form; see [16, 5.1].

3. Action of automorphisms on semisimple characters of finite reductive groups

Let $F' : \mathbf{G} \to \mathbf{G}$ be a Frobenius map of $\mathbf{G}$ commuting with F such that $\mathbf{T}$ and $\mathbf{B}$ are F'-stable. In particular, for $\alpha \in \Phi$, $F'(\mathbf{X}_\alpha)$ is a non-trivial minimal closed unipotent subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$. Write $F'(\alpha) \in \Phi$ for the corresponding root. Moreover, we suppose that for every $\alpha \in \Phi$, there is a non-negative integer $m \geq 0$ such that for all $t \in \overline{\mathbb{F}}_p$ we have $F'(x_\alpha(t)) = x_{F'(\alpha)}(t^{p^m})$. Note that $\mathbf{U}$ and $\mathbf{U}_1$ are F'-stable (because F and F' commute), so $F'(\Phi^+) = \Phi^+$ and $F'(\Delta) = \Delta$.

In this section, we study the action of F' on $\mathrm{Irr}_{p'}(\mathbf{G}^F)$. In a first step, we show that $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ is exactly the set of semisimple irreducible characters of $\mathbf{G}^F$. Then, we determine the action of F' on these semisimple characters. As an intermediate result, we obtain the action of F' on the set of regular irreducible characters of $\mathbf{G}^F$.

3.1. **F'-stable linear characters of $\mathbf{U}_1^F$.** We assume the setup from Section 2. For $J \subseteq \Delta$, let $\widetilde{\omega}_J$ be the set of characters of $\mathbf{U}_1$ of the form $\prod_{\alpha \in J} \phi_\alpha$ where ϕ_α is a non-trivial irreducible character of $\mathbf{X}_\alpha^F$.

Lemma 3.1. *Let J be an F'-stable subset of Δ. Then $\widetilde{\omega}_J$ contains an F'-stable character.*

Proof. Let Λ be the set of F'-orbits on J. For $\lambda \in \Lambda$, fix $\alpha_\lambda \in \lambda$ and write r_λ for a non-negative integer such that $F'^{r_\lambda}(\alpha_\lambda) = \alpha_\lambda$. Then F'^{r_λ} is an automorphism of $\mathbf{X}_{\alpha_\lambda}^F$ (because F and F' commute). Since $F'^{r_\lambda}(x_{\alpha_\lambda}(1)) = x_{\alpha_\lambda}(1)$, it follows from [15, Theorem (6.32)] that $\mathbf{X}_{\alpha_\lambda}^F$ has an F'-stable character. Fix such a character $\phi_\lambda \in \mathrm{Irr}(\mathbf{X}_{\alpha_\lambda}^F)$. For $0 \leq i \leq r_\lambda - 1$, the map $F'^i : \mathbf{X}_{\alpha_\lambda}^F \to \mathbf{X}_{F'^i(\alpha_\lambda)}^F$ is a group isomorphism, and hence induces a natural bijection $\mathrm{Irr}(\mathbf{X}_{\alpha_\lambda}^F) \to \mathrm{Irr}(\mathbf{X}_{F'^i(\alpha_\lambda)}^F)$. We write ${}^{F'^i}\phi_\lambda$ for the image of ϕ_λ under this bijection. Therefore, the character

$$\phi = \prod_{\lambda \in \Lambda} \prod_{i=0}^{r_\lambda - 1} {}^{F'^i}\phi_{\alpha_\lambda}$$

is in $\widetilde{\omega}_J$ and is F'-stable. □

Remark 3.2. *Note that Lemma 3.1 is also true for a non-split Frobenius F.*

3.2. **Action on the semisimple and regular characters.** In this subsection, we study the action of F' on the set of semisimple and the set of regular irreducible characters of $\mathbf{G}^F$.

In the following, we say that the prime p is nonsingular if

- if $p = 2$, then $\mathbf{G}$ has no simple component of type B_n, C_n, F_4 or G_2.
- if $p = 3$, then $\mathbf{G}$ has no simple component of type G_2.

Lemma 3.3. *We keep the notation of Subsection 2.1. If p is a nonsingular prime for $\mathbf{G}$, then the set of irreducible p'-characters of $\mathbf{G}^F$ is the set of semisimple irreducible characters of $\mathbf{G}^F$.*

Proof. Let $(\widetilde{\mathbf{G}}, F)$ be as above. Note that p is a nonsingular prime for $\mathbf{G}$ if and only if p is a nonsingular prime for $\widetilde{\mathbf{G}}$. In particular, the simple components of $\widetilde{\mathbf{G}}^F$ are not one of the groups $B_m(2)$, $G_2(2)$, $G_2(3)$, $F_4(2)$, ${}^2B_2(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$. Moreover, Since the center of $\widetilde{\mathbf{G}}$ is connected, the proof of [4, Lemma 5] also holds for $\widetilde{\mathbf{G}}^F$, and so

$$\mathrm{Irr}_{p'}(\widetilde{\mathbf{G}}^F) = \{\rho_{\widetilde{s}} \mid \widetilde{s} \in \widetilde{S}\},$$

where $\widetilde{S}$ is a set of representatives for the semisimple classes of $\widetilde{\mathbf{G}}^{*F^*}$. In particular, the semisimple characters of $\mathbf{G}^F$, which are the constituents of the restriction of $\rho_{\widetilde{s}}$, are p'-characters of $\mathbf{G}^F$.

Conversely, let $\widetilde{\chi}$ be a non semisimple irreducible character of $\widetilde{\mathbf{G}}^F$, that is p divides $\widetilde{\chi}(1)$, and let γ be an irreducible constituent of $\mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\widetilde{\chi})$. By [15, 11.29], we have $\widetilde{\chi}(1) = m \cdot \gamma(1)$ where m is an integer dividing $|\widetilde{\mathbf{G}}^F / \mathbf{G}^F|$. Since $|\widetilde{\mathbf{G}}^F / \mathbf{G}^F|$ is not divisible by p, it follows that $\gamma(1)$ is a multiple of p. Thus

$$\mathrm{Irr}_{p'}(\mathbf{G}^F) = \{\rho_{s,z} \mid s \in \mathcal{S},\, z \in H^1(F, \mathbf{Z})\}.$$

□

Remark 3.4. *Note that, in Lemma 3.3 we do not need to suppose that p is a good prime for $\mathbf{G}$. Moreover, if $\mathbf{G}$ is simple and p is singular for $\mathbf{G}$ (the prime p is said singular for $\mathbf{G}$ if it is not nonsingular), then the p'-characters of $\mathbf{G}^F$ are well-known, see for example* [4, Remark 2]*. In particular, the groups $B_m(2)$ are treated in* [5].

Let $\mathcal{S}$ be a set of representatives for the semisimple classes of $\mathbf{G}^{*F^*}$. Since F' stabilizes $\widetilde{\omega}_\Delta$, Lemma 3.1 implies that there is an F'-stable character in $\widetilde{\omega}_\Delta$. We fix such a character $\phi_0 \in \widetilde{\omega}_\Delta$ and use it for the construction of the Gelfand-Graev characters as described in Subsection 2.3.

Lemma 3.5. *For $w \in W$, let $\mathbf{T}_w$ denote the F-stable maximal torus of $\mathbf{G}$ obtained from $\mathbf{T}$ by twisting with w. Then for all semisimple elements s of $\widetilde{\mathbf{G}}^F$ and $w \in W$, one has*

$$F'\left(R^{\mathbf{G}}_{\mathbf{T}_w}(s)\right) = R^{\mathbf{G}}_{\mathbf{T}_{F'(w)}}\left(F'^{*-1}(s)\right).$$

Proof. In the proof of [4, Proposition 1], it is shown that

$$F'\left(R^{\mathbf{G}}_{\mathbf{T}_w}(s)\right) = R^{\mathbf{G}}_{F'(\mathbf{T}_w)}\left(F'^{*-1}(s)\right).$$

Since F and F' commute, the maximal torus $F'(\mathbf{T}_w)$ is F-stable. We claim that $F'(\mathbf{T}_w)$ is obtained from $\mathbf{T}$ by twisting with $F'(w)$. There is $x \in \mathbf{G}$ such that

$$x^{-1}F(x) = n_w \quad \text{and} \quad \mathbf{T}_w = x\mathbf{T}x^{-1}, \tag{9}$$

where $n_w \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ and $n_w\mathbf{T} = w$. Let $n_{F'(w)} := F'(n_w)$. Since $\mathbf{T}$ is F'-stable, $n_{F'(w)} \in N_{\mathbf{G}}(\mathbf{T})$ and $n_{F'(w)}\mathbf{T} = F'(w)$ and equation (9) implies

$$F'(\mathbf{T}_w) = F'(x)\mathbf{T}F'(x)^{-1},$$

and

$$F'(x)^{-1}F(F'(x)) = F'(x^{-1}F(x)) = F'(n_w) = n_{F'(w)}.$$

This yields the claim. $\square$

Theorem 3.6. *We make the same assumptions as in Lemma 3.5. Additionally, we suppose that F' acts trivially on the root system Φ. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. For all $s \in \mathcal{S}$ and $z \in H^1(F, \mathbf{Z})$, one has*

$$F'(\rho_{s,z}) = \rho_{F'^{*-1}(s),F'(z)} \quad \textit{and} \quad F'(\chi_{s,z}) = \chi_{F'^{*-1}(s),F'(z)}.$$

Proof. For $s \in \mathcal{S}$, let A_s be the set of constituents of χ_s. For $z \in H^1(F, \mathbf{Z})$, write A_z for the set of constituents of Γ_z. We have

$$A_s \cap A_z = \{\chi_{s,z}\}.$$

Let ϕ_z be a regular linear character of $\mathbf{U}^F$ such that $\Gamma_z = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\phi_z)$. Since F and F' commute, we have $F'(\mathbf{U}^F) = \mathbf{U}^F$. It follows

$$F'(\Gamma_z) = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(F'(\phi_z)).$$

Let $t_z \in \mathbf{T}$ such that $t_z^{-1}F(t_z) \in z$. Then, for $u \in \mathbf{U}^F$, one has

$$\begin{aligned} F'(\phi_z)(u) &= \phi_0(t_z^{-1}F'^{-1}(u)t_z) \\ &= \phi_0(F'^{-1}(F'(t_z)^{-1}uF'(t_z))) \\ &= {}^{F'(t_z)}\phi_0(u), \end{aligned}$$

because $F'(\phi_0) = \phi_0$. Furthermore,

$$F'(t_z)^{-1}F(F'(t_z)) = F'(t_z^{-1}F(t_z)) \in F'(z).$$

Thus, ${}^{F'(t_z)}\phi_0 = {}^{t_{F'(z)}}\phi_0$ and

$$F'(\Gamma_z) = \Gamma_{F'(z)}. \tag{10}$$

Now, Lemma 3.5 implies

$$F'(\chi_s) = \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}}{|W^\circ(s)|}\sum_{w\in W^\circ(s)} \varepsilon(w) R^{\mathbf{G}}_{\mathbf{T}_{F'(w)}}(F'^{*-1}(s)).$$

Since $(\mathbf{T}_{F'(w)})^* = \mathbf{T}^*_{F'^{*-1}(w)}$ (see [7, 4.3.2]), it follows

$$F'(\chi_s) = \chi_{F'^{*-1}(s)}. \tag{11}$$

Relations (10) and (11) say that

$$F'(A_s) = A_{F'^{*-1}(s)} \quad \text{and} \quad F'(A_z) = A_{F'(z)}.$$

Therefore

$$\begin{aligned} \{F'(\chi_{s,z})\} &= F'(\{\chi_{s,z}\}) = F'(A_s \cap A_z) \\ &= F'(A_s) \cap F'(A_z) = A_{F'^{*-1}(s)} \cap A_{F'(s)} \\ &= \{\chi_{F'^{*-1}(s),F'(z)}\}. \end{aligned}$$

Thus

$$F'(\chi_{s,z}) = \chi_{F'^{*-1}(s),F'(z)}. \tag{12}$$

Since F' acts trivially on the set of roots, F' stabilizes every F-stable parabolic subgroup $\mathbf{P}$ containing $\mathbf{B}$ and every F-stable Levi complement of $\mathbf{P}$. Because F and F' commute, F' also stabilizes $\mathbf{P}^F$ and $\mathbf{L}^F$. Hence, we have $F'(R^{\mathbf{G}}_{\mathbf{L}}(\phi)) = R^{\mathbf{G}}_{\mathbf{L}}(F'(\phi))$ for every $\phi \in \mathrm{Irr}(\mathbf{L}^F)$. To prove that $F'({}^*R^{\mathbf{G}}_{\mathbf{L}}(\chi)) = {}^*R^{\mathbf{G}}_{\mathbf{L}}(F'(\chi))$ for every $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ it is sufficient to show that we have $\langle F'({}^*R^{\mathbf{G}}_{\mathbf{L}}(\chi)), F'(\psi)\rangle_{\mathbf{L}^F} = \langle {}^*R^{\mathbf{G}}_{\mathbf{L}}(F'(\chi)), F'(\psi)\rangle_{\mathbf{L}^F}$ for every $\psi \in \mathrm{Irr}(\mathbf{L}^F)$. We have

$$\begin{aligned} \langle F'({}^*R^{\mathbf{G}}_{\mathbf{L}}(\chi)), F'(\psi)\rangle_{\mathbf{L}^F} &= \langle {}^*R^{\mathbf{G}}_{\mathbf{L}}(\chi), \psi\rangle_{\mathbf{L}^F} \\ &= \langle \chi, R^{\mathbf{G}}_{\mathbf{L}}(\psi)\rangle_{\mathbf{G}^F} \\ &= \langle F'(\chi), F'(R^{\mathbf{G}}_{\mathbf{L}}(\psi)\rangle_{\mathbf{G}^F} \\ &= \langle F'(\chi), R^{\mathbf{G}}_{\mathbf{L}}(F'(\psi))\rangle_{\mathbf{G}^F} \\ &= \langle {}^*R^{\mathbf{G}}_{\mathbf{L}}(F'(\chi)), F'(\psi)\rangle_{\mathbf{L}^F}. \end{aligned}$$

Using the definition of the duality functor $D_{\mathbf{G}}$ in (6), we deduce that $F'(D_{\mathbf{G}}(\chi)) = D_{\mathbf{G}}(F'(\chi))$. Applying $D_{\mathbf{G}}$ to equation (12), we get

$$F'(\rho_{s,z}) = \rho_{F'^{*-1}(s),F'(z)}.$$

This yields the claim. □

3.3. **Automorphisms and Jordan decomposition.** In this subsection, we consider how the action of a Frobenius map F' commuting with F on the set of regular characters and the set of semisimple characters behaves with respect to the Jordan decomposition of characters.

For every semisimple $s \in \mathbf{G}^{*F^*}$, let $\mathcal{E}(\mathbf{G}^F, s) \subseteq \mathrm{Irr}(\mathbf{G}^F)$ be the corresponding Lusztig series. Note that the Lusztig series give rise to a partition of $\mathrm{Irr}(\mathbf{G}^F)$; see [1, 11.2]. Moreover, For every semisimple element $s \in \mathbf{G}^{*F^*}$, there is a bijection

$$\psi_s : \mathcal{E}(C(s)^{F^*}, 1) \to \mathcal{E}(\mathbf{G}^F, s),$$

where $C(s) = \mathrm{C}_{\mathbf{G}^*}(s)$; see [8, 13.23]. When the centralizer $C(s)$ is not connected, the set $\mathcal{E}(C(s)^{F^*}, 1)$ is defined as the set of constituents of $\mathrm{Ind}^{C(s)^{F^*}}_{C(s)^{\circ F^*}}(\phi)$

where ϕ runs through $\mathcal{E}(C(s)^{\circ F^*},1)$. Note that the trivial character $1_{C(s)^\circ}$ and the Steinberg character $\mathrm{St}_{C(s)^\circ}$ of $C(s)^{\circ F^*}$ (we identify these two characters when $C(s)$ is a torus) extend to $C(s)^{F^*}$. So, [15, 6.17] implies that the extensions of these characters are labelled by the irreducible characters of $C(s)^{F^*}/C(s)^{\circ F^*} \simeq A_{\mathbf{G}^*}(s)^{F^*}$. For $\epsilon \in \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$, let 1_ϵ and St_ϵ be the corresponding extension of $1_{C(s)^\circ}$ and $\mathrm{St}_{C(s)^\circ}$, respectively. Put $\mathcal{B}_s = \{1_\epsilon, \mathrm{St}_\epsilon \mid \epsilon \in \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*}\}$.

Now, we will deduce from Theorem 3.6 that ψ_s can be chosen such that $\psi_s|_{\mathcal{B}_s}$ is compatible with the action of a Frobenius map $F' : \mathbf{G} \to \mathbf{G}$ commuting with F. This can be obtained as follows. Let $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ be the embedding as above. Put $\ker'(i^*) = \ker(i^*) \cap [\widetilde{\mathbf{G}}^*, \widetilde{\mathbf{G}}^*]$ and let $\mathcal{Z}(\mathbf{G})$ be the component group of the center $\mathbf{Z}$ of $\mathbf{G}$. Recall that there is an isomorphism $\hat{\omega} : \mathcal{Z}(\mathbf{G}) \to \mathrm{Irr}(\ker'(i^*))$ which induces an isomorphism $\hat{\omega}^0 : H^1(F,\mathbf{Z}) \to \mathrm{Irr}(\ker'(i^*)^{F^*})$; see [1, (4.11)]. Moreover, for every semisimple $s \in \mathbf{G}^{*F^*}$, we define an injective homomorphism

$$\varphi_s : A_{\mathbf{G}^*}(s) \to \ker'(i^*),\ g \mapsto \widetilde{g}\,\widetilde{s}\,\widetilde{g}^{-1}\,\widetilde{s}^{-1}, \tag{13}$$

where $\widetilde{g}$ and $\widetilde{s}$ are elements of $\widetilde{\mathbf{G}}$ such that $i^*(\widetilde{g}) = g$ and $i^*(\widetilde{s}) = s$. This morphism induces a surjective morphism $\hat{\varphi}_s : \mathrm{Irr}(\ker'(i^*)) \to \mathrm{Irr}(A_{\mathbf{G}^*}(s))$. Note that $\hat{\varphi}_s$ induces a surjective morphism from $\mathrm{Irr}(\ker'(i^*)^{F^*})$ to $\mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$. Composing this last morphism with $\hat{\omega}^0$, we obtain a surjective map $\hat{\omega}_s^0 : H^1(F,\mathbf{Z}) \to \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$; see [1, 8.A] for more details. The Frobenius map F' induces an automorphism on $H^1(F,\mathbf{Z})$, because F and F' commute. Moreover, F'^* induces an isomorphism from $A_{\mathbf{G}^*}(F'^{*-1}(s))$ to $A_{\mathbf{G}^*}(s)$. Thus, by dualizing, we obtain the following isomorphism

$$F'^* : \mathrm{Irr}\left(A_{\mathbf{G}^*}(s)\right) \to \mathrm{Irr}\left(A_{\mathbf{G}^*}(F'^{*-1}(s))\right),\ \phi \mapsto \phi \circ F'^*.$$

Now, consider the diagram:

$$\begin{array}{ccc} H^1(F,\mathbf{Z}) & \xrightarrow{\hat{\omega}_s^0} & \mathrm{Irr}(A_{\mathbf{G}^*}(s)^{F^*}) \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F'^*} \\ H^1(F,\mathbf{Z}) & \xrightarrow{\hat{\omega}^0_{F'^{*-1}(s)}} & \mathrm{Irr}(A_{\mathbf{G}^*}(F'^{*-1}(s))^{F^*}) \end{array} \tag{14}$$

Fix $z \in H^1(F,\mathbf{Z})$ and $g \in A_{\mathbf{G}^*}(F'^{*-1}(s))^{F^*}$. Equation (13) implies

$$\varphi_s(F'^*(g)) = F'^*\left(\varphi_{F'^{*-1}(s)}(g)\right).$$

Then one has

$$\begin{aligned} F'^*\left(\hat{\omega}_s^0(z)\right)(g) &= \hat{\omega}_s^0(z)(F'^*(g)) \\ &= \hat{\omega}^0(z) \circ \varphi_s(F'^*(g)) \\ &= \hat{\omega}^0(z)\left(F'^*\left(\varphi_{F'^{*-1}(s)}(g)\right)\right) \\ &= \omega^0\left(F'^*(\varphi_{F'^{*-1}(s)}(g))\right)(z) \\ &= \omega^0\left(\varphi_{F'^{*-1}(s)}(g)\right)(F'(z)) \end{aligned}$$

Here, ω^0 is defined as in [1, 4.11], and the last equality comes from [1, 4.10]. It follows that

$$F'^*\left(\hat{\omega}_s^0(z)\right) = \hat{\omega}^0_{F'^{*-1}(s)}\left(F'(z)\right).$$

Hence, diagram (14) is commutative. Now, we define ψ_s on $\mathcal{B}_s$ by setting

$$\psi_s(1_{\hat{\omega}_s^0(z)}) = \rho_{s,z} \quad \text{and} \quad \psi_s(\mathrm{St}_{\hat{\omega}_s^0(z)}) = \chi_{s,z}.$$

Note that, by [1, 11.13], we have $\rho_{s,z} = \rho_{s,z'}$ (resp. $\chi_{s,z} = \chi_{s,z'}$) if and only if $z'z^{-1} \in \ker(\hat{\omega}_s^0)$. Thus $\psi_s|_{\mathcal{B}_s}$ is well-defined. Put

$$\Psi : \bigcup_s \mathcal{B}_s \to \mathrm{Irr}(\mathbf{G}^F),\ 1_{\hat{\omega}_s^0(z)} \mapsto \rho_{s,z},\ \mathrm{St}_{\hat{\omega}_s^0(z)} \mapsto \chi_{s,z},$$

where s runs through a set of representatives for the geometric conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$. The commutativity of diagram (14) and Theorem 3.6 imply that Ψ is F'-equivariant, as required.

4. Characters of Borel subgroups

We continue to use the setup from Section 2. Additionally, we assume that $\mathbf{G}$ is simple and that $\widetilde{\mathbf{G}}^F$ is not one of the groups $B_m(2), C_m(2), G_2(2), G_2(3), F_4(2)$. Let $F' : \mathbf{G} \to \mathbf{G}$ be a Frobenius map of $\mathbf{G}$ commuting with F such that $\mathbf{T}$ and $\mathbf{B}$ are F'-stable as in Section 3. In this section, we consider the action of $\widetilde{\mathbf{T}}^F$ and the Frobenius morphism F' on the set of irreducible p'-characters of the Borel subgroup $\mathbf{B}^F$.

4.1. **p'-characters of the Borel subgroup and automorphisms.** Let $\mathbf{U}_0$ be the subgroup defined in equation (1) and $\mathbf{U}_1 = \mathbf{U}/\mathbf{U}_0$, see also equation (2). Set

$$\mathbf{B}_0 = \mathbf{U}_1 \rtimes \mathbf{T}.$$

The group $\mathbf{B}_0$ is an algebraic group with a rational structure over $\mathbb{F}_q$ given by the Frobenius map F. Since $\widetilde{\mathbf{G}}^F$ is not one of the groups listed at the beginning of this section, [17, Lemma 7] implies that $\mathbf{U}_0^F$ is the derived subgroup of $\mathbf{U}^F$, so the sets $\mathrm{Irr}_{p'}(\mathbf{B}^F)$ and $\mathrm{Irr}(\mathbf{B}_0^F)$ are $\mathcal{A}$-equivalent, where $\mathcal{A}$ denotes the set of automorphisms of $\mathbf{B}_0^F$ leaving $\mathbf{U}_1^F$ and $\mathbf{T}^F$ invariant.

Let $\widetilde{\Omega}$ and Ω be the sets of $\widetilde{\mathbf{T}}^F$-orbits and $\mathbf{T}^F$-orbits on $\mathrm{Irr}(\mathbf{U}_1^F)$, respectively. For $J = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_l}\} \subseteq \Delta$, set

$$\begin{aligned}\mathbf{U}_J &:= \{x_{\alpha_{j_1}}(u_1)x_{\alpha_{j_2}}(u_2)\cdots x_{\alpha_{j_l}}(u_l) \,|\, u_1, \ldots, u_l \in \overline{\mathbb{F}}_q\} \text{ and}\\ \mathbf{U}_J^* &:= \{x_{\alpha_{j_1}}(u_1)x_{\alpha_{j_2}}(u_2)\cdots x_{\alpha_{j_l}}(u_l) \,|\, u_1, \ldots, u_l \in \overline{\mathbb{F}}_q^\times\}.\end{aligned}$$

Let $\mathrm{Irr}^*(\mathbf{U}_J^F)$ be the set of all $\chi \in \mathrm{Irr}(\mathbf{U}_J^F)$ such that $\mathrm{Res}^{\mathbf{U}_J^F}_{\mathbf{X}_\alpha^F}(\chi)$ is non trivial for all $\alpha \in J$. By extending every $\phi \in \mathrm{Irr}(\mathbf{U}_J^F)$ trivially, we can identify $\mathrm{Irr}^*(\mathbf{U}_J^F)$ with a subset of $\mathrm{Irr}(\mathbf{U}_1^F)$ in a natural way. With this identification, we have

$$\mathrm{Irr}(\mathbf{U}_1^F) = \bigsqcup_{J \subseteq \Delta} \mathrm{Irr}^*(\mathbf{U}_J^F), \tag{15}$$

and each $\mathrm{Irr}^*(\mathbf{U}_J^F)$ is invariant under the action of $\widetilde{\mathbf{T}}^F$, $\mathbf{T}^F$ and F. In the proof of [3, 5.1] it is shown that the irreducible characters of $\mathbf{B}_0^F$ (or equivalently, the p'-characters of $\mathbf{B}^F$) can be labelled as follows. We can parametrize the elements of $\widetilde{\Omega}$ by the subsets of Δ. More precisely, for $J \subseteq \Delta$, the corresponding $\widetilde{\mathbf{T}}^F$-orbit is $\mathrm{Irr}^*(\mathbf{U}_J^F)$. Fix $\phi_J \in \mathrm{Irr}^*(\mathbf{U}_J^F)$ and choose $t_j \in \widetilde{\mathbf{T}}^F$ such that the characters $\phi_{J,j} = {}^{t_j}\phi_J$ form a set of representatives for the $\mathbf{T}^F$-orbits of $\mathrm{Irr}^*(\mathbf{U}_J^F)$, where we choose $t_1 = 1$. Then the $\mathbf{T}^F$-orbit of $\phi_{J,j}$ will be denoted by $\omega_{J,j}$. Since $\omega_{J,j}$ and $\omega_{J,1}$ are conjugate by an element of $\widetilde{\mathbf{T}}^F$, the size $|\omega_{J,j}|$ does not depend on j.

Furthermore

$$\operatorname{Irr}^*(\mathbf{U}_J^F) = \bigsqcup_{j=1}^{i(J)} \omega_{J,j} \quad \text{where } i(J) = |\operatorname{Irr}^*(\mathbf{U}_J^F)|/|\omega_{J,1}|. \tag{16}$$

Remark 4.1. *Note that if $\mathbf{L}_J$ denotes the standard Levi subgroup corresponding to J, the proof of* [3, 5.2] *shows that there is an element $t_{z_j} \in \mathbf{T}$ for some $z_j \in H^1(F, Z(\mathbf{L}_J))$ satisfying $t_{z_j}^{-1}F(t_{z_j}) \in z_j$ and*

$$\phi_{J,j} = {}^{t_j}\phi_J = {}^{t_{z_j}}\phi_J. \tag{17}$$

Let $I_{\phi_{J,j}} = \mathbf{U}^F \rtimes \mathrm{C}_{\mathbf{T}^F}(\phi_{J,j})$ be the inertia subgroup of $\phi_{J,j}$ in $\mathbf{B}_0^F$. Note that $\phi_{J,j}$ extends to $I_{\phi_{J,j}}$ by setting, for $u \in \mathbf{U}^F$ and $t \in \mathrm{C}_{\mathbf{T}^F}(\phi_{J,j})$,

$$\hat{\phi}_{J,j}(ut) = \phi_{J,j}(u). \tag{18}$$

Thus, the irreducible characters of $\operatorname{Irr}(\mathbf{B}_0^F)$ are the characters $\operatorname{Ind}_{I_{\phi_{J,j}}}^{\mathbf{B}_0^F}(\hat{\phi}_{J,j} \otimes \psi)$ for J and j as above and $\psi \in \operatorname{Irr}(\mathrm{C}_{\mathbf{T}^F}(\phi_{J,j}))$.

We now will describe more precisely the group $\mathrm{C}_{\mathbf{T}^F}(\phi_{J,j})$. As usual, we write $(X(\mathbf{T}), Y(\mathbf{T}), \Phi, \Phi^\vee)$ for the root datum corresponding to $(\mathbf{G}, \mathbf{T})$ in the sense of [8, Theorem 3.17]. In particular, $X(\mathbf{T})$ is the character group and $Y(\mathbf{T})$ the cocharacter group of $\mathbf{T}$. Choose $\mathbb{Z}$-bases $b = \{b_1, \dots, b_r\}$ and $b' = \{b_1', \dots, b_r'\}$ of $X(\mathbf{T})$ and $Y(\mathbf{T})$ respectively, such that b and b' are dual to each other with respect to the natural pairing, see [7, Section 1.9]. By [7, Proposition 3.1.2], we have $\mathbf{T} \simeq Y \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q^\times$ as abelian groups. Every element of $Y \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q^\times$ can be written uniquely as $\sum_{i=1}^r b_i' \otimes t_i$ with $t_i \in \overline{\mathbb{F}}_q^\times$ and we write $(t_1, t_2, \dots, t_r)$ for the corresponding element of $\mathbf{T}$. Note that $|b| = |b'| = |\Delta|$ because $\mathbf{G}$ is simple.

Since $\Phi \subset X(\mathbf{T})$, we can write every element of Φ as a $\mathbb{Z}$-linear combination of b and can define a matrix $A = (a_{ij}) \in \mathbb{Z}^{r \times r}$ as follows: Let the ith row of A consist of the coefficients of the simple root α_i written as a linear combination of b. For a simple root $\alpha_i \in \Delta$, the action of a torus element $t = (t_1, t_2, \dots, t_r) \in \mathbf{T}$ on $\mathbf{X}_{\alpha_i}$ is given by

$${}^t x_{\alpha_i}(u) = x_{\alpha_i}\left(u \cdot \prod_{j=1}^r t_j^{a_{ij}}\right). \tag{19}$$

Fix $J = \{\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_l}\} \subseteq \Delta$ as above. For $x \in \mathbf{U}_J^*$, we have

$$\mathbf{T}_J := \operatorname{Stab}_{\mathbf{T}}(x) = \left\{ t \in \mathbf{T} \mid \prod_{j=1}^r t_j^{a_{j_k,j}} = 1 \text{ for } k = 1, \dots, l \right\}. \tag{20}$$

In particular, the stabilizers in $\mathbf{T}$ of all $x \in \mathbf{U}_J^*$ coincide. Note that the stabilizer $\mathbf{T}_J$ is an F-stable diagonalizable group. According to [24, 13.2.5(1)], $\mathbf{T}_J^\circ$ is a split subtorus of $\mathbf{T}$ and $\mathbf{T}_J = \mathbf{T}_J^\circ \times H_J$, where H_J is a finite group isomorphic to the torsion group of $X(\mathbf{T}_J)$. Note that $\mathbf{T}_J^\circ$ and H_J are F-stable and $\mathbf{T}_J^F = \mathbf{T}_J^{\circ F} \times H_J^F$.

Lemma 4.2. *With the above notation, for $J \subseteq \Delta$, the group $\mathbf{T}_J^F$ is the centralizer in $\mathbf{T}^F$ of all irreducible characters in $\operatorname{Irr}^*(\mathbf{U}_J^F)$. Write $\phi_{J,j}$ for the character of $\operatorname{Irr}^*(\mathbf{U}_J^F)$ in the $\mathbf{T}^F$-orbit $\omega_{J,j}$ constructed from ϕ_J as in equation (17), and let $i(J)$*

be as in equation (16). Set

$$\chi_{J,j,\psi} = \mathrm{Ind}_{\mathbf{U}^F \rtimes \mathbf{T}_J^F}^{\mathbf{B}_0^F}(\hat{\phi}_{J,j} \otimes \psi),$$

where $\hat{\phi}_{J,j}$ is the extension of $\phi_{J,j}$ defined in equation (18). Then, one has

$$\mathrm{Irr}(\mathbf{B}_0^F) = \left\{\chi_{J,j,\psi} \mid J \subseteq \Delta,\, 1 \leq j \leq i(J),\, \psi \in \mathrm{Irr}(\mathbf{T}_J^F)\right\}.$$

Proof. This is just Clifford theory. □

Remark 4.3. *Note that $\mathbf{T}_J$ is the center of the Levi subgroup $\mathbf{L}_J$. In particular, one has $H_J \simeq \mathcal{Z}(\mathbf{L}_J)$, where $\mathcal{Z}(\mathbf{L}_J) = Z(\mathbf{L}_J)/Z(\mathbf{L}_J)^\circ$.*

Remark 4.4. *We will make some choices in the labelling of the irreducible characters of $\mathbf{B}_0^F$ given in Lemma 4.2. In the following, if $J \subseteq \Delta$ is F'-stable (implying that F' acts on $\mathrm{Irr}^*(\mathbf{U}_J^F)$), the character ϕ_J of $\mathrm{Irr}^*(\mathbf{U}_J^F)$ used for the parametrization of $\omega_{J,j}$ in Lemma 4.2 will be chosen F'-stable, which is possible by Lemma 3.1.*

Lemma 4.5. *For $J \subseteq \Delta$, $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$ and $i = i(J)$ as in equation (16), the set*

$$D_{J,\psi,i} := \{\chi_{J,j,\psi} \in \mathrm{Irr}(\mathbf{B}_0^F) \mid 1 \leq j \leq i\}$$

is a $\widetilde{\mathbf{T}}^F$-orbit of $\mathrm{Irr}(\mathbf{B}_0^F)$ of size i and all $\widetilde{\mathbf{T}}^F$-orbits of $\mathrm{Irr}(\mathbf{B}_0^F)$ of size i arise in this way.

Proof. Fix $t \in \widetilde{\mathbf{T}}^F$ and let j be the integer such that ${}^t\phi_J$ and $\phi_{J,j}$ are $\mathbf{T}^F$-conjugate. Note that $\mathbf{U}^F \rtimes \mathbf{T}_J^F$ is $\widetilde{\mathbf{T}}^F$-invariant, implying that for $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$

$${}^t\chi_{J,1,\psi} = \mathrm{Ind}_{\mathbf{U}^F \rtimes \mathbf{T}_J^F}^{\mathbf{B}_0^F}\left({}^t\hat{\phi}_J \otimes {}^t\psi\right).$$

Furthermore, ${}^t\hat{\phi}_J = \widehat{{}^t\phi}_J$ and $\mathbf{T}_J^F$ is the inertia subgroup of ${}^t\phi_J$ in $\mathbf{T}^F$ (because $\mathbf{T}_J^F$ is the inertia subgroup of any character of $\mathbf{U}_J^F$). Moreover, $\mathbf{T}_J^F$ and $\widetilde{\mathbf{T}}^F$ commute, so ${}^t\psi = \psi$ and then

$${}^t\chi_{J,1,\psi} = \chi_{J,j,\psi}.$$

Conversely, ${}^{t_j}\chi_{J,1,\psi} = \chi_{J,j,\psi}$ and the result follows. □

The following lemma describes the action of the Frobenius morphism F' on the set of $\widetilde{\mathbf{T}}^F$-orbits on $\mathrm{Irr}(\mathbf{B}_0^F)$.

Lemma 4.6. *We assume that the convention of Remark 4.4 holds. Let $F' : \mathbf{G} \to \mathbf{G}$ be a Frobenius map commuting with F such that $\mathbf{T}$ and $\mathbf{B}$ are F'-stable. For an F'-stable $J \subseteq \Delta$, an integer $i = i(J)$ as in equation (16) and $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$, one has*

$$F'(D_{J,\psi,i}) = D_{J,F'(\psi),i}.$$

More precisely, for every $1 \leq j \leq i$, we have

$$F'(\chi_{J,j,\psi}) = \chi_{J,j',F'(\psi)},$$

where j' is such that $z_{j'} = F'(z_j)$ for $z_j \in H^1(F', Z(\mathbf{L}_J))$ as in Remark 4.1.

Proof. Note first that F' permutes the $\widetilde{\mathbf{T}}^F$-orbits of $\mathrm{Irr}(\mathbf{B}_0^F)$ because $F'(\widetilde{\mathbf{T}}^F) = \widetilde{\mathbf{T}}^F$. Let $\chi_{J,j,\psi} \in D_{J,\psi,i}$. Then F' acts on $\mathrm{Irr}(\mathbf{U}_J^F)$ and fixes ϕ_J (see Remark 4.4). In particular, $\mathbf{T}_J^F$ is F'-stable. Let $t_{z_j} \in \mathbf{T}$ be such that $\phi_{J,j} = {}^{t_{z_j}}\phi_J$ and $z_j \in H^1(F, Z(\mathbf{L}_J))$. Since ϕ_J is F'-stable, one has

$$F'(\phi_{J,j}) = {}^{F'(t_{z_j})}\phi_J = {}^{t_{F'(z_j)}}\phi_J = \phi_{J,j'},$$

where j' is such that $F'(z_j) = z_{j'}$. It follows

$$F'(\chi_{J,j,\psi}) = \mathrm{Ind}_{\mathbf{U}^F\mathbf{T}_J^F}^{\mathbf{B}^F}\left(\hat{\phi}_{J,j'} \otimes F'(\psi)\right) = \chi_{J,j',F'(\psi)},$$

because $F'(\hat{\phi}_{J,j}) = \hat{\phi}_{J,j'}$ The result follows. □

4.2. **Characters of the Borel subgroup and central characters.**

Lemma 4.7. *Let $\nu \in \mathrm{Irr}(Z(\mathbf{G}^F))$ and write $d := |Z(\mathbf{G}^F)|$. Then*

$$|\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)| = \frac{1}{d}|\mathrm{Irr}_{p'}(\mathbf{B}^F)|.$$

Proof. Write $\mathbf{Z} := Z(\mathbf{G})$. With this notation, we have

$$\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu) = \{\chi \in \mathrm{Irr}_{p'}(\mathbf{B}^F) \mid \mathrm{Res}_{\mathbf{Z}^F}^{\mathbf{B}^F}(\chi) = \chi(1)\cdot\nu\}$$

and $\mathrm{Irr}_{p'}(\mathbf{B}^F) = \sqcup_{\nu\in\mathrm{Irr}(\mathbf{Z}^F)}\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. Let $\nu, \nu' \in \mathrm{Irr}(\mathbf{Z}^F)$. We can extend ν, ν' to linear characters of $\mathbf{B}^F$ and denote these extensions also by ν and ν', respectively. The map $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu) \to \mathrm{Irr}_{p'}(\mathbf{B}^F|\nu')$, $\chi \mapsto \chi\cdot\overline{\nu}\cdot\nu'$ is a bijection, where $\overline{\nu}$ is the complex-conjugate of ν. So, $|\mathrm{Irr}_{p'}(\mathbf{B}^F)| = |\mathbf{Z}^F|\cdot|\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)|$ and the claim follows. □

Lemma 4.8. *Let J be a subset of Δ, $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$ and $\chi_{J,j,\psi} \in \mathrm{Irr}(\mathbf{B}_0^F)$ as Lemma 4.2. For $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, one has*

$$\langle\,\chi_{J,j,\psi}, \mathrm{Ind}_{\mathbf{Z}^F}^{\mathbf{B}_0^F}(\nu)\,\rangle_{\mathbf{B}_0^F} \neq 0 \iff \langle\,\mathrm{Res}_{\mathbf{Z}^F}^{\mathbf{T}_J^F}(\psi), \nu\,\rangle_{\mathbf{Z}^F} \neq 0.$$

Proof. By the definition of induced characters [15, (5.1)], we have

$$\mathrm{Res}_{\mathbf{Z}^F}^{\mathbf{B}_0^F}(\chi_{J,j,\psi}) = \chi_{J,j,\psi}(1)\cdot\mathrm{Res}_{\mathbf{Z}^F}^{\mathbf{T}_J^F}(\psi)$$

and the claim follows from Frobenius reciprocity. □

5. Equivariant bijections

We use the notation and setup from the previous sections. In particular, we have an embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ where $\widetilde{\mathbf{G}}$ is a connected reductive group defined over $\mathbb{F}_q$ with connected center and Frobenius map F, such that $i(\mathbf{G})$ is the derived subgroup of $\widetilde{\mathbf{G}}$. Let $\widetilde{\mathbf{T}}$ be the F-stable maximal torus of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$ and $\widetilde{\mathbf{B}}$ a Borel subgroup of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$ and $\mathbf{B}$. The quotient $\widetilde{\mathbf{T}}^F/\mathbf{T}^F = \mathbf{T}'^F$ acts on $\mathbf{G}^F$. Let D be the subgroup of $\mathrm{Out}(\mathbf{G}^F)$ induced by this action. We will assume throughout this whole section that the following hypothesis is satisfied.

Hypothesis 5.1. *Let $\mathbf{G}$ be as above. Assume that*

- *The algebraic group $\mathbf{G}$ is simple.*
- *The prime p is nonsingular for $\mathbf{G}$.*
- *The automorphism induced by F on W is trivial.*
- *The order $d := |D|$ is a prime number.*

Remark 5.2. *Note that if Hypothesis 5.1 holds, then $d \neq p$.*

In this section, we are going to construct bijections between the sets $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ for fixed $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, which are compatible with the action of certain groups of automorphisms.

Recall that $\mathbf{G}$ is generated by the elements $x_\alpha(t)$ where $\alpha \in \Phi$ and $t \in \overline{\mathbb{F}}_p$ (because $\mathbf{G}$ is simple). Moreover, since F acts trivially on W, we can choose x_α such that

$$F(x_\alpha(t)) = x_\alpha(t^q) \quad \text{for all } \alpha \in \Phi \text{ and } t \in \overline{\mathbb{F}}_p.$$

We then define a bijective algebraic group homomorphism $F_0 : \mathbf{G} \to \mathbf{G}$ satisfying $F_0(x_\alpha(t)) = x_\alpha(t^p)$ for all $\alpha \in \Phi$ and $t \in \overline{\mathbb{F}}_p$. Note that the map F_0 defines an $\mathbb{F}_p$-rational structure on $\mathbf{G}$. Moreover, if $q = p^n$ for a positive integer n, then $F_0^n = F$.

5.1. **Automorphisms of $\mathbf{G}^F$.** Since F_0 and F commute, we have $F_0(\mathbf{G}^F) = \mathbf{G}^F$. Thus, F_0 induces an automorphism of $\mathbf{G}^F$, also denoted by F_0 in the following. We set $K = \langle F_0 \rangle \subseteq \mathrm{Out}(\mathbf{G}^F)$. Note that the finite group K has order n and is the group of field automorphisms of $\mathbf{G}^F$, see [6, 12.2].

We write $A \subseteq \mathrm{Out}(\mathbf{G}^F)$ for the subgroup of $\mathrm{Out}(\mathbf{G}^F)$ generated by K and D. Note that, by construction, the groups $\mathbf{T}^F$, $\widetilde{\mathbf{T}}^F$, $\mathbf{B}^F$, $\widetilde{\mathbf{B}}^F$, $\mathbf{U}^F$, $\mathbf{U}_1^F$ are K-stable, D-stable and then A-stable. The group A acts on $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F)$. For every subgroup $H \subseteq \mathrm{Out}(\mathbf{G}^F)$, we denote by $\mathcal{O}_H$ and $\mathcal{O}'_H$ the set of H-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F)$, respectively.

In this whole Section 5, let ν denote a linear character of $\mathbf{Z}^F$ and let A_ν be the subgroup of A fixing ν. Note that D acts trivially on $\mathbf{Z}^F$. Then

$$A_\nu = D \rtimes K_\nu, \tag{21}$$

where K_ν is the subgroup of field automorphisms fixing ν. Then the group A_ν acts on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. For every subgroup H of A_ν, let $\mathcal{O}_{H,\nu}$ and $\mathcal{O}'_{H,\nu}$ be the set of H-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$, respectively.

Lemma 5.3. *The group K acts on the sets $\mathcal{O}_D$ and $\mathcal{O}'_D$. And the group K_ν acts on the sets $\mathcal{O}_{D,\nu}$ and $\mathcal{O}'_{D,\nu}$.*

Proof. The first statement follows from $D \trianglelefteq A$, see [11, 2.5.14]. The second statement is then clear. □

5.2. **A D-equivariant bijection respecting central characters.** The following lemma describes a sufficient condition for the existence of a D-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. In the subsequent remark we then show that this condition is satisfied if the relative McKay conjecture is true for $\widetilde{\mathbf{G}}^F$ and $\mathbf{G}^F$ at the prime p.

Lemma 5.4. *Suppose that Hypothesis 5.1 is satisfied and assume that*

$$|\mathrm{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)| = |\mathrm{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\nu)| \text{ and } |\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)| = |\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)|.$$

Then there is a D-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$.

Proof. Fix $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$. Since $|D| = d$ is prime, the D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ have size 1 or d. For $i \in \{1, d\}$, let $N_i(\nu)$ (resp. $N'_i(\nu)$) be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ (resp. $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$) of size i. Hence, we have

$$|\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)| = |N_1(\nu)| + d|N_d(\nu)| \quad \text{and} \quad |\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)| = |N'_1(\nu)| + d|N'_d(\nu)|.$$

Let $\omega \in \mathcal{O}_{D,\nu}$. Then there is a semisimple element s of $\mathbf{G}^{*F^*}$, such that ω is the set of the constituents of ρ_s. Let $\widetilde{s}$ be a semisimple element of $\widetilde{\mathbf{G}}^{*F^*}$ satisfying $i^*(\widetilde{s}) = s$. Then $\rho_{\widetilde{s}} \in \operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)$ and every $\widetilde{s}'$ restricting to ρ_s also lies in $\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)$. Let $\chi \in \omega$ be an irreducible constituent of ρ_s. Since $\mathbf{T}'^F$ is abelian of order $q-1$ and ρ_s is multiplicity free, Clifford theory [15, Problem (6.2)] implies

$$|\operatorname{Irr}(\widetilde{\mathbf{G}}^F|\rho_s)| = \frac{q-1}{|A_{\mathbf{G}^*}(s)^{F^*}|}.$$

Similarly, the results in Section 4 on the action of $\widetilde{\mathbf{T}}^F$ and $\mathbf{T}^F$ on $\operatorname{Irr}(\mathbf{U}_1^F)$ imply that, for every $\omega \in \mathcal{O}'_{D,\nu}$, there are exactly $(q-1)/|\omega|$ irreducible characters $\widetilde{\chi} \in \operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}|\nu)$ such that ω is the set of constituents of $\operatorname{Res}^{\widetilde{\mathbf{B}}^F}_{\mathbf{B}^F}(\widetilde{\chi})$. So,

$$\begin{aligned} |\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)| &= (q-1)|N_1(\nu)| + \frac{q-1}{d}|N_d(\nu)| \quad \text{and} \\ |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\nu)| &= (q-1)|N'_1(\nu)| + \frac{q-1}{d}|N'_d(\nu)|. \end{aligned}$$

By assumption, $|\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)| = |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\nu)|$ and $|\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu)| = |\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)|$. So, we can deduce that

$$\begin{cases} (|N_1(\nu)| - |N'_1(\nu)|) + d(|N_d(\nu)| - |N'_d(\nu)|) = 0, \\ d(|N_1(\nu)| - |N'_1(\nu)|) + (|N_d(\nu)| - |N'_d(\nu)|) = 0. \end{cases}$$

We can conclude $|N_1(\nu)| = |N'_1(\nu)|$ and $|N_d(\nu)| = |N'_d(\nu)|$. Thus, there is a D-equivariant bijection between $\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)$ which can be described as follows: First, for $i \in \{1, d\}$, we choose any bijection $f_i : N_i(\nu) \to N'_i(\nu)$. For $\omega \in N_i(\nu)$, we choose any $x_\omega \in \omega$ and any $y_\omega \in f_i(\omega)$. We set $D = \langle \delta \rangle$. We define $f : \operatorname{Irr}_{p'}(\mathbf{G}^F|\nu) \to \operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)$ by

$$f(\delta(x_\omega)) = \delta(y_\omega) \quad \text{for } \omega \in \mathcal{O}_{D,\nu}.$$

So, by construction, the map f is a D-equivariant bijection. □

Remark 5.5. *Suppose $\widetilde{\mathbf{G}}^F$ satisfies the relative McKay conjecture at the prime p, that is $|\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\widetilde{\nu})| = |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\widetilde{\nu})|$ holds for all $\widetilde{\nu} \in \operatorname{Irr}(Z(\widetilde{\mathbf{G}}^F))$. Then, for every $\nu \in \operatorname{Irr}(\mathbf{Z}^F)$, one has*

$$|\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)| = |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\nu)|.$$

Proof. If $\operatorname{Ind}_{\mathbf{Z}^F}^{Z(\widetilde{\mathbf{G}}^F)}(\nu) = \sum_{k=1}^r \widetilde{\nu}_i$, then

$$\begin{aligned} |\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\nu)| &= \sum_{k=1}^r |\operatorname{Irr}_{p'}(\widetilde{\mathbf{G}}^F|\widetilde{\nu}_k)| \\ &= \sum_{k=1}^r |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\widetilde{\nu}_k)| \\ &= |\operatorname{Irr}_{p'}(\widetilde{\mathbf{B}}^F|\nu)|. \end{aligned}$$

□

5.3. **Central characters and automorphisms of $\mathbf{G}^F$.** In this subsection, we study the action of the field automorphisms on the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$. In fact, we consider the action of F on the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^{F^m}|\mu)$ for positive integers m and $\mu \in \mathrm{Irr}(\mathbf{Z}^{F^m})$. The following remark will be used to parametrize F-stable semisimple conjugacy classes of $\mathbf{G}^{*F^{*m}}$.

Remark 5.6. *Suppose that F and F' are Frobenius maps on $\mathbf{G}$ which commute. Let s be an F^*- and F'^*-stable element of $\mathbf{G}^*$. For $\alpha \in H^1(F^*, A_{\mathbf{G}^*}(s))$, we choose $g_\alpha \in \mathbf{G}^*$ such that the class of $g_\alpha^{-1}F^*(g_\alpha)$ in $H^1(F^*, A_{\mathbf{G}^*}(s))$ equals α. Let $s_\alpha := g_\alpha s g_\alpha^{-1}$. Then $s_\alpha \in \mathbf{G}^{*F^*}$ and*

$$[s]_{\mathbf{G}^*} \cap \mathbf{G}^{*F^*} = \bigsqcup_{\alpha \in H^1(F^*, A_{\mathbf{G}^*}(s))} [s_\alpha]_{\mathbf{G}^{*F^*}}.$$

Moreover, since s is F'^*-stable, one has $F'^*(s_\alpha) = F'^*(g_\alpha)sF'^*(g_\alpha)^{-1}$, implying that $F'^*(\alpha)$ is equal to the class of $F'^*(g_\alpha^{-1}F^*(g_\alpha)) = F'^*(g_\alpha^{-1})F^*(F'^*(g_\alpha))$ in $H^1(F^*, A_{\mathbf{G}^*}(s))$, because F^* and F'^* commute. We then deduce that $F'^*(s_\alpha)$ and $s_{F'^*(\alpha)}$ are $\mathbf{G}^{*F^*}$-conjugate.

Lemma 5.7. *Suppose that $\mathbf{G}$ is simple and $d = |D|$ a prime number. Let $s \in \mathbf{G}^{*F^*}$ be a semisimple element such that $|A_{\mathbf{G}^*}(s)^{F^*}| = d$ and $\{s_\alpha \mid \alpha \in H^1(F^*, A_{\mathbf{G}^*}(s))\}$ be a set of representatives for the F-rational semisimple classes corresponding to s as in Remark 5.6 (we choose the notation such that $s_1 = s$). Let S_α be the set of constituents of ρ_{s_α} and $\nu_\alpha \in \mathrm{Irr}(\mathbf{Z}^F)$ the character satisfying $\langle \mathrm{Res}^{\mathbf{G}^F}_{\mathbf{Z}^F}(\chi), \nu_\alpha \rangle_{\mathbf{Z}^F} \neq 0$ for all $\chi \in S_\alpha$. Then we have $\nu_\alpha \neq \nu_{\alpha'}$ for $\alpha \neq \alpha'$.*

Proof. Since $d = |D|$ is prime, we have $|A_{\mathbf{G}^*}(s)| = |\mathbf{Z}^F| = d$; see [25, p. 166] and [2, II.4.4]. So, $|A_{\mathbf{G}^*}(s)^{F^*}| = d$ implies $A_{\mathbf{G}^*}(s)^{F^*} = A_{\mathbf{G}^*}(s)$. Following [1, Subsection 8], there is a group homomorphism

$$\omega_s^1 : H^1(F^*, A_{\mathbf{G}^*}(s)) \to \mathrm{Irr}(\mathbf{Z}^F).$$

Again, since d is a prime ω_s^1 is an isomorphism. Hence, [1, 9.14] implies $\nu_\alpha\nu_1^{-1} = \omega_s^1(\alpha)$ for all $\alpha \in H^1(F^*, A_{\mathbf{G}^*}(s))$. Since ω_s^1 is injective (even bijective), the result follows. ☐

Lemma 5.8. *Suppose that $\mathbf{G}$ is simple and $d = |D|$ a prime number. Assume that p is a good prime for $\mathbf{G}$, and fix a positive integer m and an F-stable character $\mu \in \mathrm{Irr}(\mathbf{Z}^{F^m})$. For $i \in \{1, d\}$, let*

$$N_{i,m}(\mu) = \{\rho_s \mid s \in \mathcal{S}^{[m]}, \langle \rho_s, \rho_s \rangle_{\mathbf{G}^{F^m}} = i, \langle \mathrm{Res}^{\mathbf{G}^{F^m}}_{\mathbf{Z}^{F^m}}(\rho_s), \mu \rangle_{\mathbf{Z}^{F^m}} \neq 0\},$$

where $\mathcal{S}^{[m]}$ is a set of representatives for the semisimple conjugacy classes of $\mathbf{G}^{*F^{*m}}$. Since $\mathbf{Z}^F \in \{1, d\}$, we have the following two cases:

- *Suppose $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is trivial. Then $N_{1,m}(\mu)^F = \emptyset$ for $\mu \neq 1_{\mathbf{Z}^{F^m}}$ and $N_{1,m}(1_{\mathbf{Z}^{F^m}})^F = \{\rho_s \mid s \in \mathcal{S},\, F^*([s]_{\mathbf{G}^{*F^{*m}}}) = [s]_{\mathbf{G}^{*F^{*m}}}, |A_{\mathbf{G}^*}(s)| = 1\}$, where $\mathcal{S}$ is a set of representatives for the semisimple classes of $\mathbf{G}^{*F^*}$. Moreover, for μ, $\mu' \in \mathrm{Irr}(\mathbf{Z}^{F^m})$, one has $|N_{d,m}(\mu)^F| = |N_{d,m}(\mu')^F|$.*
- *Suppose $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective. Then for every μ, $\mu' \in \mathrm{Irr}(\mathbf{Z}^{F^m})$ and $i \in \{1, d\}$, one has $|N_{i,m}(\mu)^F| = |N_{i,m}(\mu')^F|$.*

Proof. Let t be a semisimple element of $\mathbf{G}^{*F^{*m}}$ such that $F^*([t]_{\mathbf{G}^{*F^{*m}}}) = [t]_{\mathbf{G}^{*F^{*m}}}$. Then one has $F^*([t]_{\mathbf{G}^*}) = [t]_{\mathbf{G}^*}$ and by the Lang-Steinberg theorem, $[t]_{\mathbf{G}^*}$ contains an F^*-stable element s. As in Remark 5.6, let $\{s_\alpha \,|\, \alpha \in H^1(F^{*m}, A_{\mathbf{G}^*}(s))\}$ be a set of representatives for the F^{*m}-rational classes of s. In particular, one has $[t]_{\mathbf{G}^{*F^{*m}}} = [s_\alpha]_{\mathbf{G}^{*F^{*m}}}$ for some $\alpha \in H^1(F^{*m}, A_{\mathbf{G}^*}(s))$.

Suppose $i = d$ and let $C_s = \{\rho_{s_\alpha} \,|\, \langle \rho_{s_\alpha}, \rho_{s_\alpha} \rangle_{\mathbf{G}^{F^m}} = d,\ \alpha \in H^1(F^{*m}, A_{\mathbf{G}^*}(s))\}$. Therefore, Lemma 5.7 implies

$$|N_{d,m}(\mu) \cap C_s| = 1.$$

We then deduce that the number $|N_{d,m}(\mu)^F|$ is equal to the number of F^*-stable geometric semisimple classes in $\mathbf{G}^*$ whose centralizer is disconnected. In particular, this number does not depend on μ. Hence, for all F-stable μ, $\mu' \in \mathrm{Irr}(\mathbf{Z}^{F^m})$, one has

$$|N_{d,m}(\mu)^F| = |N_{d,m}(\mu')^F|.$$

Suppose now that $i = 1$. Note that if $\rho_s \in N_{1,m}(\mu)^F$, the preceding discussion implies that we can suppose $F^*(s) = s$. Fix a maximal F^{*m}-stable torus $\mathbf{T}^*$ containing s and $(\mathbf{T}, \theta_s)$ a pair dual to $(\mathbf{T}^*, s)$. Since $F^*(s) = s$, the character θ_s of $\mathbf{T}^{F^m}$ is F-stable. Moreover, Lemma 2.1 implies (because $\mathbf{T}$ is connected) that there is $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ satisfying $\theta_s = N^*_{F^m/F}(\theta)$. Now, if $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is trivial, then for every $z \in \mathbf{Z}^{F^m}$, one has $\theta_s(z) = \theta(1) = 1$. In particular, $\mathrm{Res}^{\mathbf{T}^{F^m}}_{\mathbf{Z}^{F^m}}(\theta_s) = 1_{\mathbf{Z}^{F^m}}$, implying ρ_s is over $1_{\mathbf{Z}^{F^m}}$. Thus $\rho_s \in N_{1,m}(1_{\mathbf{Z}^{F^m}})^F$. If $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective, then by Lemma 2.1, there is $\mu_0 \in \mathrm{Irr}(\mathbf{Z}^F)$ such that $\mu = N^*_{F^m/F}(\mu_0)$. Lemma 2.2 then implies that $|N_{1,m}(\mu)^F|$ equals the number N of semisimple characters ρ'_s of $\mathbf{G}^F$ such that the centralizer of s in $\mathbf{G}^*$ is connected and $\langle \mathrm{Res}^{\mathbf{G}^F}_{\mathbf{Z}^F}(\rho'_s), \mu_0 \rangle_{\mathbf{Z}^F} \neq 0$. In the proof of [3, 6.6], it is shown that

$$N = \frac{1}{|\mathbf{Z}^F|} \langle \Gamma_z, \Gamma_{z'} \rangle_{\mathbf{G}^F},$$

for some fixed $z \neq z' \in H^1(F, \mathbf{Z})$ not depending on μ_0 (note that the part of the proof of [3, 6.6] which we use requires p to be good). So, N does not depend on μ and the result follows. ☐

5.4. **Central characters and automorphisms of $\mathbf{B}^F$.** In this subsection, we study the action of the field automorphisms on the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. In fact, we consider the action of F on the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{B}^{F^m}|\mu)$ for positive integers m and $\mu \in \mathrm{Irr}(\mathbf{Z}^{F^m})$.

Lemma 5.9. *Suppose that $\mathbf{G}$ is simple and $d = |D|$ is a prime number. Fix a positive integer m and an F-stable character $\mu \in \mathrm{Irr}(\mathbf{Z}^{F^m})$. For $i \in \{1, d\}$ let $N'_{i,m}(\mu)$ be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{B}_0^F|\mu)$ of size i. Then we have the following two cases:*

- *Suppose $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is trivial. Then $N'_{1,m}(\mu)^F = \emptyset$ for $\mu \neq 1_{\mathbf{Z}^{F^m}}$ and $N'_{1,m}(1_{\mathbf{Z}})^{F^m} = \{\chi_{J,1,\psi} \,|\, \psi \in \mathrm{Irr}(\mathbf{T}_J^{F^m}),\ F(\psi) = \psi,\ |H_J^{F^m}| = 1\}$, where $\chi_{J,1,\psi}$ is the irreducible character of $\mathbf{B}_0^{F^m}$ defined in Lemma 4.2. Moreover, for μ, $\mu' \in \mathrm{Irr}(\mathbf{Z}^{F^m})$, one has $|N'_{d,m}(\mu)^F| = |N'_{d,m}(\mu')^F|$.*
- *Suppose $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective. Then for every μ, $\mu' \in \mathrm{Irr}(\mathbf{Z}^{F^m})$ and $i \in \{1, d\}$, one has $|N'_{i,m}(\mu)^F| = |N'_{i,m}(\mu')^F|$.*

Proof. Recall that the set of characters $\chi_{J,j,\psi}$ for $1 \leq j \leq i = i(J)$ is a D-orbit of $\mathrm{Irr}_{p'}(\mathbf{B}_0^{F^m})$ and

$$\mathbf{T}_J^{F^m} = \mathbf{T}_J^{\circ F^m} \times H_J^{F^m}. \tag{22}$$

Let $\mathbf{L}_J$ be the standard Levi subgroup of $\mathbf{G}^F$ corresponding to J. Then thanks to Remark 4.3, the group H_J is isomorphic to the component group $\mathcal{Z}(\mathbf{L}_J)$ and these two groups are F-equivalent. Recall that the inclusion $\mathbf{Z} \subseteq Z(\mathbf{L})$ induces an F-equivariant surjective homomorphism $h_{\mathbf{L}_J} : \mathcal{Z}(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}_J)$, see [1, 4.2]. Moreover, $\mathcal{Z}(\mathbf{G}) = \mathbf{Z}$ because $\mathbf{G}$ is simple. Then $h_{\mathbf{L}_J}$ is the projection of $\mathbf{Z}$ on H_J in the direct product $\mathbf{T}_J = \mathbf{T}_J^\circ \times H_J$. Write $\mathrm{Irr}_F(H_J^{F^m})$ for the set of F-stable linear characters of $H_J^{F^m}$. Let ψ be a linear character of $\mathbf{T}_J^{F^m}$. Then there are $\psi^\circ \in \mathrm{Irr}(\mathbf{T}_J^{\circ F^m})$ and $\psi_{H_J} \in \mathrm{Irr}(H_J^{F^m})$ satisfying $\psi = \psi^\circ \otimes \psi_{H_J}$. Let $\psi' = \psi^\circ \otimes \psi'_{H_J}$ be such that

$$\mathrm{Res}^{\mathbf{T}_J^{F^m}}_{\mathbf{Z}^{F^m}}(\psi) = \mathrm{Res}^{\mathbf{T}_J^{F^m}}_{\mathbf{Z}^{F^m}}(\psi').$$

Now, for $h \in H_J^{F^m}$ there is $z_h \in \mathbf{Z}^{F^m}$ with $h_{\mathbf{L}_J}(z_h) = h$ (this is a consequence of the fact that $h_{\mathbf{L}_J}$ is surjective and $H_J^{F^m}$ equals $\{1\}$ or H_J). Hence, there is $t \in \mathbf{T}_J^{\circ F^m}$ satisfying $z_h = th$. Since $\psi(z_h) = \psi'(z_h)$, it follows

$$\psi^\circ(t)\psi_{H_J}(h) = \psi^\circ(t)\psi'_{H_J}(h).$$

Hence, one has $\psi_{H_J}(h) = \psi'_{H_J}(h)$ for all $h \in H_J^{F^m}$ implying $\psi_{H_J} = \psi'_{H_J}$. We then deduce from this discussion that the characters $\psi_\eta = \psi^\circ \otimes \eta$ for $\eta \in \mathrm{Irr}(H_J^{F^m})$ have distinct restrictions on $\mathbf{Z}^{F^m}$.

Let $N'_{i,m}$ be the set of D-orbits on $\mathrm{Irr}(\mathbf{B}_0^{F^m})$ of size i. Note that Remark 4.1 implies $i = |H_J^{F^m}|$, because $|H_J^{F^m}| = |\mathcal{Z}(\mathbf{L}_J^{F^m})| = |H^1(F^m, Z(\mathbf{L}_J))|$. Then $N'_{i,m}$ is the set of $D_{J,\psi,i}$ for $J \subseteq \Delta$ with $|H_J^{F^m}| = i$ and $\psi \in \mathrm{Irr}(\mathbf{T}_J^{F^m})$. Moreover, by Lemma 4.6, $D_{J,\psi,i}$ is F-stable if and only if $F(J) = J$ and $F(\psi) = \psi$. Now, let $\psi \in \mathrm{Irr}(\mathbf{T}_J^{F^m})$ be F-stable. If we write $\psi = \psi^\circ \otimes \psi_{H_J}$ as above, then $F(\psi^\circ) = \psi^\circ$ and $F(\psi_{H_J}) = \psi_{H_J}$, implying

$$N'^F_{i,m} = \{D_{J,\psi^\circ\otimes\eta,i} \in N'_{i,m} \mid J \subseteq \Delta,\, F(\psi^\circ) = \psi^\circ,\, \eta \in \mathrm{Irr}_F(H_J^{F^m})\}.$$

Suppose $i = d$. In particular, this implies $|H_J^{F^m}| = d$. Then for $J \subseteq \Delta$ and $\psi^\circ \in \mathbf{T}_J^{\circ F^m}$ with $F(\psi^\circ) = \psi^\circ$, the F-stable D-orbits $D_{J,\psi^\circ\otimes\eta,d}$ for $\eta \in \mathrm{Irr}_F(H_J^{F^m})$ are over distinct F-stable characters of $\mathbf{Z}^{F^m}$. For all μ, $\mu' \in \mathrm{Irr}_F(\mathbf{Z}^{F^m})$, we deduce

$$|N'_{d,m}(\mu)^F| = |N'_{d,m}(\mu')^F|.$$

Suppose $i = 1$. Then H_J is trivial and $\mathbf{T}_J$ is connected. Using Lemma 2.1, the F-stable characters of $\mathbf{T}_J^{F^m}$ are the character $\psi = \psi_0 \circ N_{F^m/F}$ for any $\psi_0 \in \mathbf{T}_J^F$. If $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is trivial, then $N'_{1,m}(1_{\mathbf{Z}^{F^m}})^F = N'^F_{1,m}$. In particular, for every $\mu \neq 1_{\mathbf{Z}^{F^m}}$ one has $N'_{1,m}(\mu)^F = \emptyset$. If $N_{F^m/F} : \mathbf{Z}^{F^m} \to \mathbf{Z}^F$ is surjective, then Lemma 2.1 implies that every character $\mu \in \mathrm{Irr}_F(\mathbf{Z}^{F^m})$ has the form $\mu_0 \circ N_{F^m/F}$ for some $\mu_0 \in \mathrm{Irr}(\mathbf{Z}^F)$ and

$$\langle \mathrm{Res}^{\mathbf{T}_J^{F^m}}_{\mathbf{Z}^{F^m}}(\psi), \mu \rangle_{\mathbf{Z}^{F^m}} = \langle \mathrm{Res}^{\mathbf{T}_J^{F}}_{\mathbf{Z}^{F}}(\psi_0), \mu_0 \rangle_{\mathbf{Z}^{F}}.$$

Note that ψ_0 lies in $N'_{1,1}$ because H_J is trivial. This implies $|N'_{1,m}(\mu)^F| = |N'_{1,1}(\mu_0)|$. Furthermore, the discussion at the beginning of the proof implies that the numbers

$|N'_{d,1}(\mu_0)|$ do not depend on $\mu_0 \in \mathrm{Irr}(\mathbf{Z}^F)$. In particular, for every $\mu_0 \in \mathrm{Irr}(\mathbf{Z}^F)$ one has

$$|N'_{d,1}(\mu_0)| = \frac{1}{d}|N'_{d,1}|.$$

Moreover, one has $|N'_{1,1}(\mu_0)| = |\mathrm{Irr}_{p'}(\mathbf{B}_0^F|\mu_0)| - d|N'_{d,1}(\mu_0)|$. Using Lemma 4.7, we deduce that $|N'_{1,1}(\mu_0)|$ does not depend on μ_0, proving, for all μ, $\mu' \in \mathrm{Irr}_F(\mathbf{Z}^{F^m})$

$$|N'_{1,m}(\mu)^F| = |N'_{1,m}(\mu')^F|.$$

This yields the claim. □

5.5. **An A-equivariant bijection respecting central characters.** We keep the above notation and suppose that Hypothesis 5.1 holds. In particular, let d be the order of D. For $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, we write A_ν for the subgroup of A defined in equation (21). In this subsection, we will give conditions to find an A_ν-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$.

As before, let $\mathcal{O}_{D,\nu}$ be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathcal{O}'_{D,\nu}$ be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. Furthermore, for $i \in \{1, d\}$, we define $N_i(\nu)$ (resp. $N'_i(\nu)$) to be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ (resp. $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$) of size i.

Proposition 5.10. *Suppose that $(\mathbf{G}, F)$ satisfies Hypothesis 5.1. Let $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$. If there are K_ν-equivariant bijections $f_i : N_i(\nu) \to N'_i(\nu)$ for $i \in \{1, d\}$, then there is an A_ν-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$.*

Proof. We define $\eta : \mathcal{O}_{D,\nu} \to \mathcal{O}'_{D,\nu}$ by $\eta(\omega) = f_1(\omega)$ if $\omega \in N_1(\nu)$, and $\eta(\omega) = f_d(\omega)$ if $\omega \in N_d(\nu)$. Since $\mathcal{O}_{D,\nu} = N_1(\nu) \sqcup N_d(\nu)$ and $\mathcal{O}'_{D,\nu} = N'_1(\nu) \sqcup N'_d(\nu)$ and $f_i : N_i(\nu) \to N'_i(\nu)$ are K_ν-equivariant bijections, we deduce that η is a K_ν-equivariant bijection between $\mathcal{O}_{D,\nu}$ and $\mathcal{O}'_{D,\nu}$. Write $\Omega = \{\Omega_i \mid i \in I\}$ (resp. Ω') for the set of K_ν-orbits on $\mathcal{O}_{D,\nu}$ (resp. $\mathcal{O}'_{D,\nu}$). For every $\Omega_i \in \Omega$, we fix $\omega_i \in \Omega_i$. Then $\eta(\omega_i)$ is a set of representative for Ω', because η is an K_ν-equivariant bijection. If $|\Omega_i| > 1$, we choose any character $\chi_{\omega_i} \in \omega_i$ and any character $\chi'_{\omega_i} \in \eta(\omega_i)$.

Let γ be a generator of the cyclic group K_ν. Note that γ is the restriction of a Frobenius map $F' : \mathbf{G} \to \mathbf{G}$. Now, suppose that $|\Omega_i| = 1$ (resp. $|\Omega'_i| = 1$), i.e. ω_i is fixed by γ. Recall that there is a semisimple element s of $\mathbf{G}^{*F^*}$ such that ω_i is the set of constituents of ρ_s. Since $F'(\omega_i) = \omega_i$, we deduce that s and $F'^*(s)$ are $\mathbf{G}^{*F^*}$-conjugate. In particular, Theorem 3.6 implies that the constituent $\rho_{s,1}$ of ρ_s is F'-stable. We put $\rho_{s,1} = \chi_{\omega_i}$. Moreover, $\eta(\omega_i)$ is fixed by γ. Furthermore, following the notation of Lemma 4.5, one has $\eta(\omega_i) = D_{J,\psi,k}$ for some $J \subseteq \Delta$, $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$ and $k \in \{1, d\}$. Using Lemma 4.6, $F'(D_{J,\psi,k}) = D_{J,\psi,k}$ implies $F'(J) = J$ and $F'(\psi) = \psi$. Thus, the character $\chi_{J,1,\psi} \in \eta(\omega_i)$ is F'-stable and we put $\chi'_{\omega_i} = \chi_{J,1,\psi}$.

Let $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$. We denote by ω its D-orbit. Then there is a unique $i \in I$ such that $\omega \in \Omega_i$ implying there exists a unique $l \in \mathbb{N}$ such that $\omega = \gamma^l(\omega_i)$. Hence for every $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F)$, there are unique $i \in I$, $k, l \in \mathbb{N}$ such that

$$\chi = \delta^k \gamma^l(\chi_{\omega_i}).$$

We define $\Psi : \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu) \to \mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ by setting

$$\Psi(\chi) = \delta^k F'^l(\chi'_{\omega_i}),$$

where i, j, k are as above. Then Ψ is an A_ν-equivariant bijection. □

Recall that the characteristic $p > 0$ of $\mathbb{F}_q$ is a good prime for $\mathbf{G}$ if p does not divide the coefficients of the highest root of the root system associated to $\mathbf{G}$ (see [8, p. 125]).

Theorem 5.11. *Suppose that p is a good prime for $\mathbf{G}$ and that $(\mathbf{G}, F)$ satisfies Hypothesis 5.1. We suppose the subgroup of field automorphisms of $\mathbf{G}^F$ acts trivially on $\mathbf{Z}^F$. Then for $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, the sets $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ are A-equivalent.*

Proof. Write $q = p^n$. Recall that $K = \langle F_0 \rangle$ and $F = F_0^n$. For any divisor j of n, we put $F_j = F_0^j$ and for $i \in \{1, d\}$, let $N_{i,j}$ and $N'_{i,j}$ be the set of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^{F_j})$ and $\mathrm{Irr}_{p'}(\mathbf{B}^{F_j})$, respectively.

For $i \in \{1, d\}$ and j a divisor of n, let $\mathcal{T}_i^{[j]}$ be a set of representatives s for the F_j^*-stable geometric semisimple classes of $\mathbf{G}^*$, such that $|A_{\mathbf{G}^*}(s)^{F_j^*}| = i$. By the Lang-Steinberg theorem, we can suppose that the elements of $\mathcal{T}_i^{[j]}$ are F_j^*-stable. Let

$$\mathcal{T}_{i,j} = \{s \in \mathcal{T}_i^{[n]} \,|\, F_j^*([s]_{\mathbf{G}^*}) = [s]_{\mathbf{G}^*}\}.$$

Remark 5.6 implies

$$|N_{i,n}^{F_j}| = \sum_{s \in \mathcal{T}_{i,j}} |H^1(F^*, A_{\mathbf{G}^*}(s))^{F_j^*}|.$$

Let s be in $\mathcal{T}_{i,j}$. Again, by the Lang-Steinberg theorem, there is $t \in [s]_{\mathbf{G}^*}$ satisfying $F_j^*(t) = t$. Furthermore, j divides n implies $F^*(t) = t$. We can then suppose that every $s \in \mathcal{T}_{i,j}$ is F_j^*-stable and F^*-stable. By assumption, F_0 acts trivially on $\mathbf{Z}^F$. For every F_j^*-stable semisimple element $s \in \mathbf{G}^*$, one has $A_{\mathbf{G}^*}(s) = A_{\mathbf{G}^*}(s)^{F_j^*}$. In particular, one has $\mathcal{T}_{i,j} = \mathcal{T}_i^{[j]}$ and $|H^1(F^*, A_{\mathbf{G}^*}(s))^{F_j^*}| = i$ for every $s \in \mathcal{T}_{i,j}$. Thus, we have

$$|N_{i,j}| = \sum_{s \in \mathcal{T}_i^{[j]}} |H^1(F_j^*, A_{\mathbf{G}^*}(s))| = \sum_{s \in \mathcal{T}_i^{[j]}} i = \sum_{s \in \mathcal{T}_{i,j}} i = |N_{i,n}^{F_j}|. \tag{23}$$

By [3, 1.1, 6.5] and [18, Theorem 1], the relative McKay conjecture holds for $\mathbf{G}^{F_j}$ and $\widetilde{\mathbf{G}}^{F_j}$ (here we need that p is a good prime for $\mathbf{G}$). So, by Lemma 5.4, for every $\mu \in \mathrm{Irr}(\mathbf{Z}^{F^n})$, there is a D-equivariant bijection between $\mathrm{Irr}_{p'}(\mathbf{G}^{F_j}|\mu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^{F_j}|\mu)$. So, in particular, we get

$$|N_{i,j}| = |N'_{i,j}|. \tag{24}$$

Moreover, with the notation of Lemma 4.5, for $i \in \{1, d\}$, one has

$$N_{i,n}^{\prime F_j} = \{D_{J,\psi,i} \,|\, J \subseteq \Delta,\, F_j(\psi) = \psi\}.$$

Recall that $\mathbf{T}_J = \mathbf{T}_J^\circ \times H_J$ and F_j acts trivially on H_J^F (because the groups H_J and $\mathcal{Z}(\mathbf{L}_J)$ are F_0-equivalent, and if $H_J^{F_0}$ is non trivial, then $H_J^{F_0} = H_J$). For $\psi \in \mathrm{Irr}(\mathbf{T}_J^{F_j})$, we write $\psi = \psi^\circ \otimes \eta$ for the decomposition of ψ with respect to the direct product $\mathbf{T}_J^{F_j} \times H_J^{F_j}$. Then $F_j(\psi) = \psi$ if and only if $F_j(\psi^\circ) = \psi^\circ$. Put

$$\varphi_i : N_{i,j} \to N_{i,n}^{F_j},\, D_{J,\psi^\circ \otimes \eta, i} \mapsto D_{J, \psi^\circ \circ N_{F/F_j} \otimes \eta, i}.$$

Note that on the right hand side, the character η is considered as an F_j-stable linear character of H_J^F. This is possible because $H_J^F = H_J^{F_j}$. Since $N_{F/F_j} : \mathbf{T}_J^{\circ F} \to \mathbf{T}_J^{\circ F_j}$

induces a bijection between the linear characters of $\mathbf{T}_J^{\circ F_j}$ and the F_j-stable linear characters of $\mathbf{T}_J^F$ (see Lemma 2.1), we conclude that φ_i is bijective for $i \in \{1, d\}$. In particular, one has

$$|N_{i,n}'^{F_j}| = |N_{i,j}'|. \tag{25}$$

From relations (23), (24) and (25), we deduce

$$|N_{i,n}^{F_j}| = |N_{i,n}'^{F_j}| \quad \text{for all divisors } j \text{ of } n.$$

Let $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$. Suppose that $N_{F/F_j} : \mathbf{Z}^F \to \mathbf{Z}^{F_j}$ is surjective. Then by Lemma 5.8 and 5.9, the numbers $|N_{i,n}(\nu)^F|$ and $|N_{i,n}'(\nu)^F|$ do not depend on ν. In particular, $|N_{i,n}^{F_j}| = |N_{i,n}'^{F_j}|$ implies

$$|N_{i,n}(\nu)^{F_j}| = N_{i,n}'(\nu)^{F_j}|, \tag{26}$$

where $N_{i,j}(\nu)$ and $N_{i,j}'(\nu)$ are the sets of D-orbits on $\mathrm{Irr}_{p'}(\mathbf{G}^{F_j}|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^{F_j}|\nu)$, respectively. Suppose that $N_{F/F_j} : \mathbf{Z}^F \to \mathbf{Z}^{F_j}$ is trivial. Using the same argument, Lemma 5.8 and 5.9 imply $|N_{d,n}(\nu)^{F_j}| = |N_{d,n}'(\nu)^{F_j}|$. Moreover, for $\nu \neq 1_{\mathbf{Z}^F}$, we have

$$N_{1,n}(\nu)^{F_j} = \emptyset = N_{1,n}(\nu)^{F_j}. \tag{27}$$

In particular, equation (27) implies

$$|N_{1,n}(1_{\mathbf{Z}^F})^{F_j}| = |N_{1,n}^{F_j}| \quad \text{and} \quad |N_{1,n}'(1_{\mathbf{Z}^F})^{F_j}| = |N_{1,n}'^{F_j}|.$$

Since $|N_{1,n}^{F_j}| = |N_{1,n}'^{F_j}|$, we have

$$|N_{1,n}(1_{\mathbf{Z}^F})^{F_j}| = |N_{1,n}'(1_{\mathbf{Z}^F})^{F_j}| \tag{28}$$

Thus, equations (26), (27) and (28) imply that, for all divisors j of n and all $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, one has

$$|N_{i,n}(\nu)^{F_j}| = |N_{i,n}'(\nu)^{F_j}|.$$

Using [15, 13.23], we deduce that the sets $N_{i,n}(\nu)$ and $N_{i,n}'(\nu)$ are K-equivalent for $i \in \{1, d\}$. We conclude with Proposition 5.10. □

Remark 5.12. *Note that, in the proof of Theorem 5.11, we need the assumption that p is a good prime for $\mathbf{G}$ only in order to apply the results of* [3].

Corollary 5.13. *Suppose $(\mathbf{G}, F)$ satisfies Hypothesis 5.1 and $d = 2$. If p is a good prime for $\mathbf{G}$, then for $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$, the sets $\mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ and $\mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ are A-equivalent.*

Proof. If $d = 2$, then every automorphisms of $\mathbf{G}^F$ acts trivially on $\mathbf{Z}^F$. The result is then as direct consequence of Theorem 5.11. □

6. Inductive McKay condition

In this section, we prove our main result, Theorem 1.1. Our main ingredient will be the equivariant bijection constructed in Corollary 5.13. The largest part of this section will be devoted to the verification of the cohomology condition.

6.1. **Setup.** Throughout this whole Section 6, let $\mathbf{G}$ be a simple simply-connected algebraic group defined over $\mathbb{F}_q$ of characteristic $p > 0$ with corresponding Frobenius map F such that the quotient

$$X = \mathbf{G}^F / Z(\mathbf{G}^F)$$

is a finite simple group and $\mathbf{G}^F$ is the universal cover of X. Let $\mathbf{Z} = Z(\mathbf{G})$ be the center of the algebraic group $\mathbf{G}$ and suppose that $|\mathbf{Z}| = 2$. As in Section 2.1, we embed $\mathbf{G}$ in a connected reductive group $\widetilde{\mathbf{G}}$ with an $\mathbb{F}_q$-rational structure obtained by extending F to $\widetilde{\mathbf{G}}$, such that the center of $\widetilde{\mathbf{G}}$ is connected and the groups $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ have the same derived subgroup. We assume that p is a good prime for $\mathbf{G}$. Fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\widetilde{\mathbf{T}}$ for the unique F-stable torus of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$. Fix an F-stable Borel subgroup $\widetilde{\mathbf{B}}$ of $\widetilde{\mathbf{G}}$ containing $\widetilde{\mathbf{T}}$ and $\mathbf{B}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. As in Section 5, we put $K = \langle F_0 \rangle$. Let D be the group of diagonal automorphisms of $\mathbf{G}^F$ induced by the action of $\widetilde{\mathbf{G}}^F/\mathbf{G}^F$ on $\mathbf{G}^F$. We set

$$\mathcal{A} = \widetilde{\mathbf{G}}^F \rtimes K.$$

We assume that F acts trivially on the Weyl group of $\mathbf{G}$. Note that our assumptions imply that $|D| = 2$ and that $\mathbf{G}^F$ has no graph automorphisms.

6.2. **Extensions of characters.** We assume the setup from Subsection 6.1. Fix $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$ and let $\Phi_\nu : \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu) \to \mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ be an $\mathcal{A}$-equivariant bijection from Corollary 5.13. For a D-stable character $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$, we define

$$G_\chi = \widetilde{\mathbf{G}}^F \rtimes \mathrm{Stab}_K(\chi) \quad \text{and} \quad G'_\chi = \widetilde{\mathbf{B}}^F \rtimes \mathrm{Stab}_K(\Phi_\nu(\chi)).$$

Lemma 6.1. *Let $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ be D-stable. Then, there is an extension of χ to G_χ and an extension of $\Phi_\nu(\chi)$ to G'_χ.*

Proof. There is an integer $i > 0$ dividing $|K|$ such that $\mathrm{Stab}_K(\chi) = \langle F_0^i \rangle$. Let $F' = F_0^i$ and $m = |K|/i$. Then $F'^m = F$.

First, we show that χ extends to G_χ. Since χ is semisimple, there is a semisimple element $s \in \mathbf{G}^{*F^*}$ satisfying $\chi = \rho_s$, where ρ_s is defined in equation (4). So, one has $F'(\rho_s) = \rho_s$ if and only if $F'(s)$ and s are conjugate in $\mathbf{G}^{*F^*}$ (see Theorem 3.6). Since $\widetilde{\mathbf{G}}^F$ stabilizes ρ_s, we have $A_{\mathbf{G}^*}(s)^{F^*} = \{1\}$. In particular, we can suppose that s is F'-stable. Choose an F'-stable element $\widetilde{s} \in \widetilde{\mathbf{G}}^{*F^*}$ such that $i^*(\widetilde{s}) = s$. Then $\rho_{\widetilde{s}}$ is an extension of ρ_s and $F'(\rho_{\widetilde{s}}) = \rho_{\widetilde{s}}$ because $F'(\widetilde{s}) = \widetilde{s}$ by Theorem 3.6. Moreover, since $\mathrm{Stab}_K(\rho_s)$ is cyclic, $\rho_{\widetilde{s}}$ extends to G_χ and so $\chi = \rho_s$ extends to G_χ, as required.

Next, we prove that $\Phi_\nu(\chi)$ extends to G'_χ. In the notation of Lemma 4.2, we have $\Phi_\nu(\chi) = \chi_{J,j,\psi}$ for some $J \subseteq \Delta$, some positive integer j and some $\psi \in \mathrm{Irr}(\mathbf{T}_J^F)$, where $\mathbf{T}_J$ is the stabilizer of $x \in \mathbf{U}_J^*$ in $\mathbf{T}$ as in equation (20). Since $\chi_{J,j,\psi}$ is $\widetilde{\mathbf{B}}_0^F$-stable, it follows that $i(J) = 1$ and $j = 1$. Let $\widetilde{\mathbf{T}}_J$ be the stabilizer of $x \in \mathbf{U}_J^*$ in $\widetilde{\mathbf{T}}$ and write ϕ_J for the irreducible character of $\mathbf{U}_J^F$ chosen for the construction of $\chi_{J,1,\psi}$ in Lemma 4.2. As in equation (18), we extend ϕ_J to the characters $\hat{\phi}_J$ and $\check{\phi}_J$ of $\mathbf{U}_1^F \rtimes \mathbf{T}_J^F$ and $\mathbf{U}_1^F \rtimes \widetilde{\mathbf{T}}_J^F$, respectively. The groups $\mathbf{T}_J$ and $\widetilde{\mathbf{T}}_J$ are connected. Then by [8, 0.5], there is an F-stable torus $\mathbf{H}$ such that $\widetilde{\mathbf{T}}_J = \mathbf{T}_J \times \mathbf{H}$. In particular,

$\widetilde{\mathbf{T}}_J^F = \mathbf{T}_J^F \times \mathbf{H}^F$ and by [15, 6.17]

$$\mathrm{Ind}_{\mathbf{U}_1^F \rtimes \mathbf{T}_J^F}^{\mathbf{U}_1^F \rtimes \widetilde{\mathbf{T}}_J^F}(\hat{\phi}_J \otimes \psi) = \sum_{\eta \in \mathrm{Irr}(\mathbf{H}^F)} \check{\phi}_J \otimes \psi \otimes \eta.$$

Then we deduce

$$\mathrm{Ind}_{\mathbf{B}_0^F}^{\widetilde{\mathbf{B}}_0^F}(\chi_{J,1,\psi}) = \sum_{\eta \in \mathrm{Irr}(\mathbf{H}^F)} \widetilde{\chi}_{J,1,\psi\otimes\eta},$$

where $\widetilde{\chi}_{J,1,\psi\otimes\eta}$ is the irreducible character of $\widetilde{\mathbf{B}}_0^F$ constructed in Lemma 4.2. In particular, the characters $\widetilde{\chi}_{J,1,\psi\otimes\eta}$ are the extensions of $\chi_{J,1,\psi}$ to $\widetilde{\mathbf{B}}_0^F$. Because Φ_ν is $\mathcal{A}$-equivariant, we have $\mathrm{Stab}_K(\chi_{J,1,\psi}) = \mathrm{Stab}_K(\Phi_\nu(\chi)) = \mathrm{Stab}_K(\chi) = \langle F' \rangle$ where $F'^m = F$ as at the beginning of the proof. Since $F'(\chi_{J,1,\psi}) = \chi_{J,1,\psi}$, Lemma 4.6 implies $F'(\psi) = \psi$. Thus, $\widetilde{\psi} = \psi \otimes 1_{\mathbf{H}^F}$ is F'-stable. Hence, the character $\widetilde{\chi}_{J,1,\widetilde{\psi}}$ is an F'-stable extension of $\chi_{J,1,\psi}$ to $\widetilde{\mathbf{B}}_0^F$. So, there is an F'-stable extension of $\Phi_\nu(\chi)$ to $\widetilde{\mathbf{B}}^F$. Since $G'_\chi/\widetilde{\mathbf{B}}^F$ is cyclic, we get an extension of $\Phi_\nu(\chi)$ to G'_χ, and the proof is complete. □

6.3. **Proof of Theorem 1.1.** We assume the setup from Subsection 6.1. As described in [16, Section 10], for each irreducible character χ of a covering group of X, we have to construct a group G_χ satisfying certain conditions. Because $\mathbf{Z}^F$ has order 2, there is only one non-trivial covering group of X, namely $S = \mathbf{G}^F$. Instead of treating the faithful irreducible characters of S and the irreducible characters of X separately, we consider the latter as (non-faithful) characters of S and deal with both cases simultaneously; see also the proof of [16, (15.3)]. In case χ is not faithful, all of the groups constructed in the following have to be considered "modulo $\ker(\chi)$".

Since $\mathbf{G}^F$ has no graph automorphisms, the conjugation action of $\mathcal{A}$ induces the group of automorphisms of $\mathbf{G}^F$ stabilizing $\mathbf{U}^F$. Fix a linear character $\nu \in \mathrm{Irr}(\mathbf{Z}^F)$. Then $\mathcal{A}$ fixes ν, since $\mathbf{Z}^F$ has order 2. By Corollary 5.13, there is an $\mathcal{A}$-equivariant bijection $\Phi_\nu : \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu) \to \mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$. In particular, we see that the conditions (1),(2),(3),(4) of [16, Section 10] are satisfied. The dictionary between our notation and the one in [16, Section 10] is: $\mathbf{U}^F \leftrightarrow Q$, $\mathbf{B}^F \leftrightarrow T$, $\Phi_\nu \leftrightarrow ()^*$ and $\mathcal{A} \leftrightarrow A$ (more precisely: A corresponds to the set of automorphisms induced by the conjugation action of $\mathcal{A}$ on $\mathbf{G}^F$).

Next, we construct the group G_χ. Fix $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$. First, suppose that χ is not D-stable. In this case, we set

$$G_\chi = \mathbf{G}^F \rtimes \mathrm{Stab}_{KD}(\chi) \quad \text{and} \quad G'_\chi = \mathbf{B}^F \rtimes \mathrm{Stab}_{KD}(\Phi_\nu(\chi)).$$

We have to verify conditions (5)-(8) of [16, Section 10]. We have $\mathbf{Z}^F \subseteq Z(G_\chi)$ and the stabilizer of χ in $\mathcal{A}$ is induced by the conjugation action of $\mathrm{N}_{G_\chi}(\mathbf{U}^F)$. Note that $\mathrm{N}_{G_\chi}(\mathbf{B}^F) = G'_\chi$. Moreover, the group $C = \mathrm{C}_{G_\chi}(\mathbf{G}^F) = \mathrm{Z}(\mathbf{G}^F)$ is abelian and the set $\mathrm{Irr}(C|\nu)$ contains the G_χ-invariant character $\gamma = \nu$. Thus, conditions (5)-(7) are true in this case. Note that the factor groups $G_\chi/\mathbf{G}^F$ and $G'_\chi/\mathbf{B}^F$ are isomorphic to a subgroup of K and hence are cyclic. So, the cohomology groups $H^2(G_\chi/\mathbf{G}^F, \mathbb{C}^\times)$ and $H^2(G'_\chi/\mathbf{B}^F, \mathbb{C}^\times)$ are trivial, and therefore condition (8) holds in this case, too.

Now suppose that $\chi \in \mathrm{Irr}_{p'}(\mathbf{G}^F|\nu)$ is D-stable. In this case, we set

$$G_\chi = \widetilde{\mathbf{G}}^F \rtimes \mathrm{Stab}_K(\chi) \quad \text{and} \quad G'_\chi = \widetilde{\mathbf{B}}^F \rtimes \mathrm{Stab}_K(\Phi_\nu(\chi)).$$

Again, we have to verify conditions (5)-(8) of [16, Section 10]. We have $\mathbf{Z}^F \subseteq Z(G_\chi)$ and the stabilizer of χ in $\mathcal{A}$ is induced by the conjugation action of $\mathrm{N}_{G_\chi}(\mathbf{U}^F)$. Note that $\mathrm{N}_{G_\chi}(\mathbf{B}^F) = G'_\chi$. Moreover, the group $C = \mathrm{C}_{G_\chi}(\mathbf{G}^F) = \mathrm{Z}(\widetilde{\mathbf{G}}^F)$ is abelian. So, conditions (5) and (6) of [16, Section 10] are satisfied.

By Lemma 6.1, there exists an extension $\widetilde{\chi}'$ of $\chi' = \Phi_\nu(\chi) \in \mathrm{Irr}_{p'}(\mathbf{B}^F|\nu)$ to G'_χ. For abbreviation, we set $M = \mathbf{B}^F$. Since $M \subseteq MC \subseteq G'_\chi$, the character $\mathrm{Res}^{G'_\chi}_{MC}(\widetilde{\chi}')$ is irreducible. Since MC is a central product, there are $\chi'_M \in \mathrm{Irr}(M)$ and $\chi'_C \in \mathrm{Irr}(C)$ satisfying

$$\mathrm{Res}^{G'_\chi}_{MC}(\widetilde{\chi}') = \chi'_M \cdot \chi'_C, \tag{29}$$

see Subsection 2.6. Furthermore, for $g \in M$, one has $\mathrm{Res}^{G'_\chi}_{MC}(\widetilde{\chi}')(g) = \chi'(g)$, so $\chi'_M = \chi'$. Moreover, since χ_M and $\widetilde{\chi}'$ are G'_χ-stable, it follows from equation (29) that χ'_C is G'_χ-stable. Note that

$$\mathrm{Res}^{G'_\chi}_{\mathrm{Z}(M)}(\widetilde{\chi}') = \mathrm{Res}^{M}_{\mathrm{Z}(M)}(\chi') = \chi'(1) \cdot \nu.$$

Hence, equation (29) implies $\mathrm{Res}^{C}_{\mathrm{Z}(S)}(\chi'_C) = \nu$. We put

$$\gamma = \chi'_C. \tag{30}$$

It follows that γ is a G'_χ-stable and hence G_χ-stable element of $\mathrm{Irr}(C|\nu)$. So, condition (7) of [16, Section 10] holds in this situation.

We choose a positive integer i dividing $|K|$ such that $\mathrm{Stab}_K(\chi) = \langle F_0^i \rangle$. Let $F' = F_0^i$ and $m = |K|/i$. Then $F'^m = F$. Put

$$E(\chi) = \{\vartheta \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F) \mid \mathrm{Res}^{\widetilde{\mathbf{G}}^F}_{\mathbf{G}^F}(\vartheta) = \chi\}.$$

Since the semisimple character χ is F'-stable, there is an F'-stable semisimple element $s \in \mathbf{G}^{*F^*}$ satisfying $\chi = \rho_s$. Choose an F'-stable semisimple element $\widetilde{s} \in \widetilde{\mathbf{G}}^{*F^*}$ such that $i^*(\widetilde{s}) = s$. Then,

$$E(\chi) = \{\rho_{\widetilde{s}t} \mid t \in \mathrm{Z}(\widetilde{\mathbf{G}}^*)^{F^*}\}.$$

In particular, $|E(\chi)| = |\,\mathrm{Z}(\widetilde{\mathbf{G}}^F)|$. Moreover, note that $\rho_{\widetilde{s}t}$ is F'-stable if and only if $F'^*(t) = t$. In particular, one has

$$|E(\chi)^{F'}| = |\,\mathrm{Z}(\widetilde{\mathbf{G}}^*)^{F'^*}| = p^i - 1.$$

Put

$$L(\chi) = \{\varphi \in \mathrm{Irr}(\mathbf{G}^F C) \mid \mathrm{Res}^{\mathbf{G}^F C}_{\mathbf{G}^F}(\varphi) = \chi\}.$$

Since $\mathbf{G}^F C$ is a central product, every character of $\mathbf{G}^F$ extends to $\mathbf{G}^F C$. Now, using [15, 6.17], one has $L(\chi) = \{\varphi \otimes \xi \mid \xi \in \mathbf{G}^F C/\mathbf{G}^F\}$. In particular, $|L(\chi)| = |C|/2$. Since χ is F'-stable, we have $\varphi = \chi \otimes \xi \in L(\chi)^{F'}$ if and only if ξ is F'-stable. Because $C = \mathrm{Z}(\widetilde{\mathbf{G}}^F)$, it follows that

$$|L(\chi)^{F'}| = \frac{1}{2} \cdot |\,\mathrm{Z}(\widetilde{\mathbf{G}})^{F'}| = \frac{1}{2}(p^i - 1) = \frac{1}{2}|E(\chi)^{F'}|. \tag{31}$$

Consider the map

$$\Theta : E(\chi)^{F'} \to L(\chi)^{F'},\ \vartheta \mapsto \mathrm{Res}^{\widetilde{\mathbf{G}}^F}_{\mathbf{G}^F C}(\vartheta).$$

Note that Θ is well-defined because

$$F'(\Theta(\vartheta)) = \Theta(F'(\vartheta)) = \Theta(\vartheta).$$

By [1, 6.B], we know that $|\widetilde{\mathbf{G}}^F/\mathbf{G}^F C| = 2$. So Clifford theory (see [15, 6.19]) and equation (31) imply

$$|\Theta(E(\chi)^{F'})| \geq \frac{1}{2}|E(\chi)^{F'}| = |L(\chi)^{F'}|. \tag{32}$$

So, Θ is surjective. In particular, for the irreducible F'-stable character $\gamma \in \mathrm{Irr}(C|\nu)$ defined in equation (30), the character $\chi \centerdot \gamma$ (which lies in $L(\chi)^{F'}$) extends to an F'-stable character $\widetilde{\chi}$ of $\widetilde{\mathbf{G}}^F$. So, by [15, 11.22], the character $\widetilde{\chi}$ extends to G_χ, because $\mathrm{Stab}_K(\chi)$ is cyclic and stabilizes $\widetilde{\chi}$. In particular, $\chi \centerdot \gamma$ extends to G_χ.

So, we have shown that there is a G_χ-stable irreducible character γ of C over ν such that $\chi \centerdot \gamma$ and $\Phi_\nu(\chi) \centerdot \gamma$ extend to extend to G_χ and $\mathrm{N}_{G_\chi}(M)$, respectively. Then, by [15, 11.7] we have

$$[\chi \centerdot \gamma]_{G_\chi/\mathbf{G}^F C} = [\Phi_\nu(\chi) \centerdot \gamma]_{\mathrm{N}_{G_\chi}(M)/MC}.$$

Thus, condition (8) of [16, Section 10] is satisfied, too. Hence, X is "good" for p and the proof of Theorem 1.1 is complete. □

Remark 6.2. *The following can be said about those finite simple groups of type B_m, C_m or E_7 which are not included in Corollary 1.2:*

(a) *By* [4, Remark 2] *and* [5], *all simple groups $B_m(2^n)$ (isomorphic to $C_m(2^n)$) are "good" for the defining characteristic $p = 2$.*
(b) *Theorem 5 in* [4] *implies that all simple groups $E_7(2^n)$ are "good" for $p = 2$.*
(c) *Our methods were not sufficient to show that the simple groups $E_7(3^n)$ are "good" for $p = 3$. By Remark 5.12, it is enough to prove that the following properties hold. First, the group $\mathbf{G}^F$ (here, $\mathbf{G}$ denotes a simple simply-connected group of type E_7 defined over a finite field of characteristic* 3 *and F denotes the corresponding Frobenius map) satisfies the relative McKay conjecture at the prime $p = 3$. Second, the number of semisimple characters ρ_s of $\mathbf{G}^F$ lying over $\nu \in \mathrm{Irr}(\mathrm{Z}(\mathbf{G}^F))$ and such that the group $\mathrm{C}_{\mathbf{G}^{*F^*}}(s)$ is connected, does not depend on ν.*

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