COUNTING p'-CHARACTERS IN FINITE REDUCTIVE GROUPS

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Abstract. This article is concerned with the relative McKay conjecture for finite reductive groups. Let $\mathbf{G}$ be a connected reductive group defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$ with corresponding Frobenius map F. We prove that if the F-coinvariants of the component group of the center of $\mathbf{G}$ has prime order and if p is a good prime for $\mathbf{G}$, then the relative McKay conjecture holds for $\mathbf{G}^F$ at the prime p. In particular, this conjecture is true for $\mathbf{G}^F$ in defining characteristic for a simple and simply-connected group $\mathbf{G}$ of type B_n, C_n, E_6 and E_7. Our main tools are the theory of Gelfand-Graev characters for connected reductive groups with disconnected center developed by Digne-Lehrer-Michel and the theory of cuspidal Levi subgroups. We also explicitly compute the number of semisimple classes of $\mathbf{G}^F$ for any simple algebraic group $\mathbf{G}$.

1. Introduction

Let G be a finite group and p be a prime divisor of $|G|$. As usually, we denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G and by $\mathrm{Irr}_{p'}(G)$ the subset of irreducible characters with degree prime to p. For any fixed p-Sylow subgroup P of G, John McKay has conjectured that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|$. This is actually proved for some groups but remains open in general. Recently, Isaacs, Malle and Navarro reduced this conjecture to a new question, the so-called inductive McKay condition, which concerns properties of perfect central extensions of finite simple groups; see [8].

In this article, we are interested in the relative McKay conjecture, asserting that for every linear character ν of the center Z of G, if $\mathrm{Irr}_{p'}(G|\nu)$ denotes the subset of characters $\chi \in \mathrm{Irr}_{p'}(G)$ lying over ν (*i.e.* satisfying $\langle \chi, \mathrm{Ind}_Z^G(\nu) \rangle_G \neq 0$), then one has the equality $|\mathrm{Irr}_{p'}(G|\nu)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P)|\nu)|$. In order to prove the inductive McKay condition, we in particular have to show that the relative McKay conjecture holds for some perfect central extensions of finite simple groups. This is one of the motivations to consider this question in this work.

Let $\mathbf{G}$ be a connected reductive group defined over a finite field with q elements $\mathbb{F}_q$ of characteristic $p > 0$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$. Throughout this paper, we will always assume that p is a good prime for $\mathbf{G}$, that is p does not divide the coefficients of the highest root of the root system associated to $\mathbf{G}$ (see [4, 1.14]). Let $\mathbf{T}$ be a maximal F-stable torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and let $\mathbf{U}$ denote the unipotent radical of $\mathbf{B}$ (which is F-stable). Note that, if $\mathbf{U}$ is not trivial, then the prime p divides the order of the finite fixed-point subgroup $\mathbf{G}^F$ and the subgroup $\mathbf{U}^F \subseteq \mathbf{G}^F$ is a p-Sylow subgroup of $\mathbf{G}^F$. Moreover, one has $\mathrm{N}_{\mathbf{G}^F}(\mathbf{U}^F) = \mathbf{B}^F$. If the center of $\mathbf{G}$ is connected, then the McKay conjecture is true for the group $\mathbf{G}^F$ at the prime p. We will see in the

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following that the relative McKay conjecture holds in this case (see Proposition 6.5). This question is more difficult when the center of $\mathbf{G}$ is disconnected. In this article, we will solve it in a special situation. Denote by $\mathcal{Z}(\mathbf{G}) = \mathrm{Z}(\mathbf{G})/\mathrm{Z}(\mathbf{G})^\circ$ the group of components of the center of $\mathbf{G}$ and by $H^1(F, \mathcal{Z}(\mathbf{G}))$ the set of the F-classes of $\mathcal{Z}(\mathbf{G})$. Then our main result is the following.

Theorem 1.1. *Let $\mathbf{G}$ be a connected reductive group defined over the finite field $\mathbb{F}_q$ of characteristic $p > 0$ and let $F : \mathbf{G} \to \mathbf{G}$ denote the corresponding Frobenius map. Let $\mathbf{T}$ be a maximal F-stable torus contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. If p is a good prime for $\mathbf{G}$ and if the group $H^1(F, \mathcal{Z}(\mathbf{G}))$ is trivial or has prime order, then for every linear character ν of $\mathrm{Z}(\mathbf{G}^F)$, one has*

$$|\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu)| = |\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)|.$$

As consequence, this proves the relative McKay conjecture in defining characteristic for $\mathbf{G}^F$ with $\mathbf{G}$ a simple group given in Table 4.

This paper is organized as follows. In Section 2, we recall some results from Bonnafé [1] on the cuspidal Levi subgroups of connected reductive groups. We will need this theory first, in order to associate to every linear character of $\mathcal{Z}(\mathbf{G})$ a cuspidal Levi subgroup of $\mathbf{G}$ (corresponding to a cuspidal local system in Lusztig theory), and secondly to control the disconnected part of the inertial subgroup of linear characters of $\mathbf{U}^F$. In Section 3, we apply the theory of Gelfand-Graev characters of $\mathbf{G}^F$ for connected reductive group $\mathbf{G}$ with disconnected center, developed by Digne-Lehrer-Michel in [5]. Note that we need here that p is a good prime for $\mathbf{G}$. In particular, we give a formula to compute the scalar product of two Gelfand-Graev characters; see Proposition 3.2. As consequence, we obtain an explicit formula for the number of semisimple classes of $\mathbf{G}^F$ (see Theorem 3.5) and compute this number for $\mathbf{G}^F$ with $\mathbf{G}$ any simple algebraic group; see Corollary 3.6. Recall that the constituents of the duals of Gelfand-Graev characters (for the Alvis-Curtis duality functor) are the so-called semisimple characters of $\mathbf{G}^F$. When p is a good prime for $\mathbf{G}$, the semisimple characters are the p'-characters of $\mathbf{G}^F$ (that is, the elements of $\operatorname{Irr}_{p'}(\mathbf{G}^F)$). In Section 4, using the results of Section 3, we compute the number of semisimple characters of $\mathbf{G}^F$ when $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order; see Proposition 4.2. In Section 5, we give a formula for the number of p'-characters of $\mathbf{B}^F$ depending on the cuspidal Levi subgroups of $\mathbf{G}$; see Proposition 5.6. Finally, in Section 6, we show that if the center of $\mathbf{G}$ is connected or if $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order, then for a linear character ν of $\mathrm{Z}(\mathbf{G}^F)$ the number of semisimple characters of $\mathbf{G}^F$ lying over ν does not depend on ν; see Proposition 6.5 and Proposition 6.6. We can then prove Theorem 1.1; see Remark 6.7.

2. Cuspidal Levi subgroups and central characters

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$. As above, we denote by $\mathbf{T}$ a maximal F-stable torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Write Φ for the root system of $\mathbf{G}$ and Φ^+ for the set of positive roots with respect to $\mathbf{B}$. Denote by Δ the set of corresponding simple roots and by W the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$, identified with the quotient $\mathrm{N}(\mathbf{T})/\mathbf{T}$. Moreover, we associate to every $\alpha \in \Phi$ a reflection $w_\alpha \in W$ and for any subset I of Δ, we denote by W_I the subgroup of W generated by w_α for $\alpha \in I$. The subgroup $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$ is a standard parabolic subgroup of $\mathbf{G}$ (relative to $\mathbf{B}$). We denote by $\mathbf{L}_I$ the Levi subgroup of $\mathbf{P}_I$ containing $\mathbf{T}$. Note that every Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is conjugate in $\mathbf{G}$ to a Levi $\mathbf{L}_I$ for some subset I of Δ.

Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. Then the inclusion $\mathrm{Z}(\mathbf{G}) \subseteq \mathrm{Z}(\mathbf{L})$ induces a surjective map

$$h_{\mathbf{L}} : \mathcal{Z}(\mathbf{G}) \to \mathcal{Z}(\mathbf{L}),$$

where $\mathcal{Z}(\mathbf{G}) = \mathrm{Z}(\mathbf{G})/\mathrm{Z}(\mathbf{G})^\circ$. We recall that $\mathbf{G}$ is cuspidal if $\ker(h_{\mathbf{L}}) \neq \{1\}$ for every proper Levi $\mathbf{L}$ of $\mathbf{G}$. Moreover, a linear character ζ of $\mathcal{Z}(\mathbf{G})$ is cuspidal if, for every Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, the subgroup $\ker(h_{\mathbf{L}})$ is not contained in $\ker(\zeta)$.

Let ζ be a linear character of $\mathcal{Z}(\mathbf{G})$. Then there is a Levi subgroup $\mathbf{L}$ (which is cuspidal) and a cuspidal character $\zeta_{\mathbf{L}}$ of $\mathcal{Z}(\mathbf{L})$ such that

$$\zeta = \zeta_{\mathbf{L}} \circ h_{\mathbf{L}}.$$

More precisely, for a subgroup K of $\mathcal{Z}(\mathbf{G})$, denote by $\mathcal{L}_0(K)$ the set of Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ such that $\ker(h_{\mathbf{L}}) \subseteq K$ and by $\mathcal{L}_{\min}(K)$ the subset of minimal elements of $\mathcal{L}_0(K)$. In [1, 2.16], Bonnafé proves that the Levi subgroups of $\mathcal{L}_{\min}(K)$ are cuspidal and $\mathbf{G}$-conjugate. Therefore, we associate to the linear character ζ of $\mathcal{Z}(\mathbf{G})$ a standard Levi $\mathbf{L}_I$ of $\mathcal{L}_{\min}(\ker(\zeta))$. Note that all Levi subgroups in $\mathcal{L}_{\min}(K)$ have the same semisimple rank.

Let $H^1(F, \mathcal{Z}(\mathbf{G}))$ be the set of F-classes of $\mathcal{Z}(\mathbf{G})$. Since $\mathcal{Z}(\mathbf{G})$ is abelian, the Lang map $\mathcal{L} : \mathcal{Z}(\mathbf{G}) \to \mathcal{Z}(\mathbf{G}) : g \mapsto g^{-1}F(g)$ is a morphism of groups and we have $H^1(F, \mathcal{Z}(\mathbf{G})) = \mathcal{Z}(\mathbf{G})/\mathcal{L}(\mathcal{Z}(\mathbf{G}))$. In particular, a character ζ of $H^1(F, \mathcal{Z}(\mathbf{G}))$ can be seen as a character of $\mathcal{Z}(\mathbf{G})$ with $\mathcal{L}(\mathcal{Z}(\mathbf{G}))$ in its kernel. Hence, we can associate to every character ζ of $H^1(F, \mathcal{Z}(\mathbf{G}))$ a cuspidal Levi $\mathbf{L}$ of $\mathbf{G}$ and a cuspidal $\zeta_{\mathbf{L}}$ of $\mathcal{Z}(\mathbf{L})$. Note that $\mathbf{L}$ can be chosen F-stable and with this choice, $\zeta_{\mathbf{L}}$ is F-stable.

In the following, we write $H^1(F, \mathcal{Z}(\mathbf{G}))^\wedge$ for the set of irreducible characters of $H^1(F, \mathcal{Z}(\mathbf{G}))$.

3. Number of semisimple classes

3.1. **Gelfand-Graev characters.** Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ with Frobenius map $F : \mathbf{G} \to \mathbf{G}$. We denote by $\mathbf{T}$ a maximal F-stable torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We write $\mathbf{U}$ for the unipotent radical of $\mathbf{B}$. We recall that p is supposed to be a good prime for $\mathbf{G}$.

As above, we denote by Φ the root system of $\mathbf{G}$, by Φ^+ the set of positive roots with respect to $\mathbf{B}$ and by Δ the set of corresponding simple roots. We write $\mathbf{X}_\alpha$ for the non-trivial minimal closed unipotent subgroup of $\mathbf{U}$ normalized by $\mathbf{T}$ and corresponding to the root $\alpha \in \Phi^+$. Recall that the Frobenius map F induces a permutation on Φ such that $F(\Phi^+) = \Phi^+$ and $F(\Delta) = \Delta$. Put

$$\mathbf{U}_0 = \prod_{\alpha \in \Phi^+ \setminus \Delta} \mathbf{X}_\alpha.$$

Denote by $\mathbf{U}_1$ the quotient $\mathbf{U}/\mathbf{U}_0$ and write $\pi_{\mathbf{U}_0} : \mathbf{U} \to \mathbf{U}_1$ for the canonical projection map. Then we have $\mathbf{U}_1 \simeq \prod_{\alpha \in \Delta} \mathbf{X}_\alpha$ and

$$\mathbf{U}_1^F = \prod_{\omega \in \mathcal{O}} \mathbf{X}_\omega^F, \tag{1}$$

where $\mathcal{O}$ is the set of F-orbits on Δ and $\mathbf{X}_\omega = \prod_{\alpha \in \omega} \mathbf{X}_\alpha$. Recall that an element of $\mathbf{G}$ is regular if its centralizer has a minimal possible dimension. By [6, 14.14] the regular unipotent elements of $\mathbf{U}$ are the elements $u \in \mathbf{U}$ such that for every $\alpha \in \Delta$,

$\pi_{\mathbf{U}_0}(u)_\alpha \neq 1$. Moreover by [6, 14.25], the set of regular unipotent classes of $\mathbf{G}^F$ are parametrized by $H^1(F, \mathcal{Z}(\mathbf{G}))$. For $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$, denote by $\mathcal{U}_z$ the conjugacy class of unipotent elements corresponding to z and put

$$\gamma_z : \mathbf{G}^F \to \mathbb{C}, g \mapsto \begin{cases} |\mathcal{U}_z|/|\mathbf{G}^F| & \text{if } g \in \mathcal{U}_z \\ 0 & \text{otherwise} \end{cases}$$

Recall that a linear character ψ of $\mathbf{U}^F$ is a regular character if it has $\mathbf{U}_0^F$ in its kernel and if the induced linear character on $\mathbf{U}_1^F$ (always denoted by ψ) satisfies $\mathrm{Res}^{\mathbf{U}_1^F}_{\mathbf{X}_\omega^F}(\psi) \neq 1_{\mathbf{X}_\omega^F}$ for every $\omega \in \mathcal{O}$. By [6, 14.28], the set of $\mathbf{T}^F$-orbits of regular characters of $\mathbf{U}^F$ is parametrized by $H^1(F, \mathcal{Z}(\mathbf{G}))$ as follows.

Fix ψ_1 a regular linear character of $\mathbf{U}^F$ and $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$. Choose $t_z \in \mathbf{T}$ such that $t_z^{-1}F(t_z)\,\mathrm{Z}(\mathbf{G}^F) = z$. Then the $\mathbf{T}^F$-orbit of the regular characters of $\mathbf{U}^F$ corresponding to z has $\psi_z =^{t_z} \psi_1$ for representative.

We now can define the Gelfand-Graev characters of $\mathbf{G}^F$ by setting for every $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$

$$\Gamma_z = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\phi_z).$$

Denote by $D_{\mathbf{G}}$ the Alvis-Curtis duality map. For $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$, there is a virtual character φ_z of $\mathbf{U}^F$ (see the proof of [6, 14.33]) with $\mathbf{U}_0^F$ in its kernel, which is zero outside regular unipotent elements and satisfying

$$D_{\mathbf{G}}(\Gamma_z) = \mathrm{Ind}_{\mathbf{U}^F}^{\mathbf{G}^F}(\varphi_z).$$

In particular, $D_{\mathbf{G}}(\Gamma_z)$ is constant on $\mathcal{U}_z$ and there are complex numbers $c_{z,z'}$ (for $z' \in H^1(F, \mathcal{Z}(\mathbf{G})))$ with

$$D_{\mathbf{G}}(\Gamma_z) = \sum_{z' \in H^1(F,\mathcal{Z}(\mathbf{G}))} c_{z,z'}\gamma_{z'}. \tag{2}$$

Following [5], we now recall how to compute the coefficients $c_{z,z'}$. For this, we need some notations. For $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$, put

$$\sigma_z = \sum_{\psi \in \Psi_{z^{-1}}} \psi(u),$$

where $u \in \mathcal{U}_1$ and Ψ_z denotes the $\mathbf{T}^F$-orbit of ψ_z. Moreover, for any character ζ of $H^1(F, \mathcal{Z}(\mathbf{G}))$, we define

$$\sigma_\zeta = \sum_{z \in H^1(F,\mathcal{Z}(\mathbf{G}))} \zeta(z)\sigma_z.$$

In [5, 2.3, 2.5], the following result is proven.

Proposition 3.1. *With the above notation, if p is a good prime for $\mathbf{G}$, then the matrix $(c_{z,z'})_{z,\, z' \in H^1(F,\mathcal{Z}(\mathbf{G}))}$ is invertible and its inverse is $(\eta_{\mathbf{G}}\sigma_{z(z')^{-1}})_{z,z' \in H^1(F,\mathcal{Z}(\mathbf{G}))}$, where $\eta_{\mathbf{G}} = (-1)^{\mathbb{F}_q\text{-rk}(\mathbf{G})}$. Moreover, we have $c_{z,z'} = c_{z(z')^{-1},1}$ and if we put $c_\zeta = \sum_{z \in H^1(F,\mathcal{Z}(\mathbf{G}))} \zeta(z)c_{z,1}$ for any character ζ of $H^1(F, \mathcal{Z}(\mathbf{G}))$, then there is a fourth root of unity ξ_ζ such that*

$$c_\zeta = \eta_{\mathbf{G}}\eta_{\mathbf{L}}q^{-\frac{1}{2}(\text{ss-rk}(\mathbf{L}_\zeta))}\xi_\zeta,$$

where $\mathbf{L}_\zeta$ is the cuspidal Levi of $\mathbf{G}$ associated to the character ζ as explained in Section 2.

Proposition 3.2. *With the notation as above, if p is a good prime for $\mathbf{G}$, then for z_1, $z_2 \in H^1(F, \mathcal{Z}(\mathbf{G}))$, one has*

$$\langle \Gamma_{z_1}, \Gamma_{z_2} \rangle_{\mathbf{G}^F} = |\operatorname{Z}(\mathbf{G})^{\circ F}| \sum_{\zeta \in H^1(F,\mathcal{Z}(\mathbf{G}))^\wedge} \overline{\zeta(z_1)}\zeta(z_2) q^{l-(\text{ss-rk}(\mathbf{L}_\zeta))},$$

where $\mathbf{L}_\zeta$ is the cuspidal Levi of $\mathbf{G}$ associated to the character ζ of $H^1(F, \mathcal{Z}(\mathbf{G}))$ and l is the semisimple rank of $\mathbf{G}$.

Proof. Fix z_1 and z_2 in $H^1(F, \mathcal{Z}(\mathbf{G}))$ and put $I = \langle \Gamma_{z_1}, \Gamma_{z_2} \rangle_{\mathbf{G}^F}$. Since the duality functor $D_{\mathbf{G}}$ is an isometry, one has $I = \langle D_{\mathbf{G}}(\Gamma_{z_1}), D_{\mathbf{G}}(\Gamma_{z_2}) \rangle_{\mathbf{G}^F}$. Furthermore, thanks to Equation (2), we deduce

$$\langle D_{\mathbf{G}}(\Gamma_{z_1}), D_{\mathbf{G}}(\Gamma_{z_2}) \rangle_{\mathbf{G}^F} = \sum_{z,z' \in H^1(F,\mathcal{Z}(\mathbf{G}))} c_{z_1,z}\overline{c_{z_2,z'}} \langle \gamma_z, \gamma_{z'} \rangle_{\mathbf{G}^F}.$$

Note that, if $z' \neq z$, then $\langle \gamma_z, \gamma_{z'} \rangle_{\mathbf{G}^F} = 0$. Moreover, $\langle \gamma_z, \gamma_z \rangle_{\mathbf{G}^F} = |\operatorname{C}_{\mathbf{G}^F}(u_z)|$ for $u_z \in \mathcal{U}_z$. We deduce

$$I = \sum_{z \in H^1(F,\mathcal{Z}(\mathbf{G}))} c_{z_1,z}\overline{c_{z_2,z}} |\operatorname{C}_{\mathbf{G}^F}(u_z)|, \tag{3}$$

However, the group $\operatorname{C}_{\mathbf{G}}(u_1)$ is abelian (because the characteristic is good for $\mathbf{G}$). It then follows that $|\operatorname{C}_{\mathbf{G}^F}(u_z)| = |\operatorname{C}_{\mathbf{G}^F}(u_1)|$ for every $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$; see [6, 14.22]. Moreover, [6, 14.23] implies

$$|H^1(F, \mathcal{Z}(\mathbf{G}))| \frac{|\mathbf{G}^F|}{|\operatorname{C}_{\mathbf{G}^F}(u_z)|} = \frac{|\mathbf{G}^F|}{|\operatorname{Z}(\mathbf{G})^{\circ F}| q^l}.$$

Since $|H^1(F, \mathcal{Z}(\mathbf{G}))| = |\operatorname{Z}(\mathbf{G})^F| / |\operatorname{Z}(\mathbf{G})^{\circ F}|$, we deduce

$$|\operatorname{C}_{\mathbf{G}^F}(u_z)| = |\operatorname{Z}(\mathbf{G})^F| q^l. \tag{4}$$

For every $\zeta \in H^1(F, \mathcal{Z}(\mathbf{G}))^\wedge$, we have $c_\zeta = \sum_{z \in H^1(F,\mathcal{Z}(\mathbf{G}))} \zeta(z) c_{z,1}$. Denote by T the character table of $H^1(F, \mathcal{Z}(\mathbf{G}))$ (identified with the quotient group $\mathcal{Z}(\mathbf{G})/\mathcal{L}(\mathcal{Z}(\mathbf{G}))$ as above). Write $m = |H^1(F, \mathcal{Z}(\mathbf{G}))|$. Since T is the character table of a finite abelian group, it follows that T is invertible and $T^{-1} = \frac{1}{m}{}^t\overline{T}$. We then deduce that, for every $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$

$$c_z = \frac{1}{m} \sum_{\zeta \in H^1(F,\mathcal{Z}(\mathbf{G}))^\wedge} \overline{\zeta(z)} c_\zeta. \tag{5}$$

Furthermore, by Proposition 3.1 one has $c_{z_i,z} = c_{z_i(z)^{-1},1}$. Then Equations (3), (4) and (5) imply

$$\begin{aligned}
I &= \sum_{z \in H^1(F,\mathcal{Z}(\mathbf{G}))} \frac{1}{m^2} \sum_{\zeta,\zeta' \in H^1(F,\mathcal{Z}(\mathbf{G}))^\wedge} \overline{\zeta(z_1 z^{-1})\zeta'(z_2 z^{-1})} |\operatorname{C}_{\mathbf{G}^F}(u_z)| c_\zeta \overline{c_{\zeta'}} \\
&= \frac{|\operatorname{Z}(\mathbf{G})^F| q^l}{m} \sum_{\zeta,\zeta' \in H^1(F,\mathcal{Z}(\mathbf{G}))^\wedge} \overline{\zeta(z_1)}\zeta'(z_2) \langle \zeta, \zeta' \rangle_{H^1(F,\mathcal{Z}(\mathbf{G}))} c_\zeta \overline{c_{\zeta'}} \\
&= \frac{|\operatorname{Z}(\mathbf{G})^F| q^l}{m} \sum_{\zeta \in H^1(F,\mathcal{Z}(\mathbf{G}))^\wedge} \overline{\zeta(z_1)}\zeta(z_2) |c_\zeta|^2.
\end{aligned}$$

Now, Proposition 3.1 implies $c_\zeta = \eta_{\mathbf{G}}\eta_{\mathbf{L}} q^{-\frac{1}{2}(\text{ss-rk}(\mathbf{L}_\zeta))}\xi_\zeta$. Thus

$$|c_\zeta|^2 = q^{-(\text{ss-rk}(\mathbf{L}_\zeta))}|\xi_\zeta|^2 = q^{-(\text{ss-rk}(\mathbf{L}_\zeta))}.$$

Moreover,

$$\frac{|\,\mathrm{Z}(\mathbf{G})^F|}{m} = |\,\mathrm{Z}(\mathbf{G})^{\circ F}|.$$

This proves the claim. □

Remark 3.3. Note that $\langle\, \Gamma_z, \Gamma_{z'}\,\rangle_{\mathbf{G}^F}$ does not depend on the fourth roots of unity ξ_ζ associated to $\zeta \in H^1(F, \mathcal{Z}(\mathbf{G}))^\wedge$ as in Proposition 3.1.

Remark 3.4. If the center of $\mathbf{G}$ is connected, there is only one Gelfand-Graev character Γ_1 and the cuspidal Levi subgroup associated to the trivial character of $H^1(F, \mathcal{Z}(\mathbf{G}))$ is a maximal torus, which has semisimple rank equal to zero. Thus, we obtain

$$\langle\, \Gamma_1, \Gamma_1\,\rangle_{\mathbf{G}^F} = |\,\mathrm{Z}(\mathbf{G})^F|q^l,$$

which is a well-known result [4, 8.3.1].

3.2. **Number of semisimple classes.**

Theorem 3.5. *Let $\mathbf{G}$ be a connected reductive group defined over a finite field of characteristic $p > 0$ with q elements $\mathbb{F}_q$ and let $F : \mathbf{G} \to \mathbf{G}$ denote the corresponding Frobenius map. Write $\mathcal{S}$ for a set of representatives of semisimple classes of $\mathbf{G}^F$. Denote by $(\mathbf{G}^*, F^*)$ a dual pair of $(\mathbf{G}, F)$. With the above notation, if p is a good prime for $\mathbf{G}$, then we have*

$$|\mathcal{S}| = |\,\mathrm{Z}(\mathbf{G})^{\circ F}| \sum_{\zeta \in H^1(F^*, \mathcal{Z}(\mathbf{G}^*))^\wedge} q^{l-(\text{ss-rk}(\mathbf{L}^*_\zeta))},$$

where l is the semisimple rank of $\mathbf{G}$ and $\mathbf{L}^*_\zeta$ is a cuspidal Levi subgroup of $\mathbf{G}^*$ associated to $\zeta \in H^1(F^*, \mathcal{Z}(\mathbf{G}^*))^\wedge$ as explained in Section 2.

Proof. Denote by $(\mathbf{G}^*,\, F^*)$ a pair dual to $(\mathbf{G},\, F)$. As explained in Section 3.1, we can associate to every $z \in H^1(F^*, \mathcal{Z}(\mathbf{G}^*))$ a Gelfand-Graev character Γ_z of $\mathbf{G}^{*F^*}$. Recall that Γ_z is multiplicity free. We can describe more precisely the constituents of Γ_z as follows. Fix $s \in \mathcal{S}$. Using Deligne-Lusztig characters, Digne-Michel defined in [6, 14.40] a class function χ_s and proved that for every $z \in H^1(F^*, \mathcal{Z}(\mathbf{G}^*))$, there is exactly one irreducible character of $\mathbf{G}^F$, denoted by $\chi_{s,z}$, which is a common constituent of χ_s and Γ_z and satisfying (see [6, 14.49]):

$$\Gamma_z = \sum_{s \in \mathcal{S}} \chi_{s,z}. \tag{6}$$

Equation (6) implies $|\mathcal{S}| = \langle\, \Gamma_1, \Gamma_1\,\rangle_{\mathbf{G}^{*F^*}}$. Now, thanks to Proposition 3.2, the result follows. □

We now will precise some notations. For a simple algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$, if the corresponding Frobenius map is split, then we denote it by F^+. Otherwise, if the $\mathbb{F}_q$-structure is given by a non-split Frobenius, we denote it by F^-. Moreover, if $\mathbf{G}$ is of type X and has split and non-split Frobenius map F^+ and F^-, then we put ${}^\epsilon X(q) = \mathbf{G}^{F^\epsilon}$ for $\epsilon \in \{-1, 1\}$.

Fix some positive integer n and denote by $\mathbf{G}_{\mathrm{sc}}$ a simple simply-connected algebraic group of type A_n. For any divisor r of $n+1$, there is a simple algebraic group $\mathbf{G}_r$ of type A_n and a surjective morphism $\pi_r : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_r$ satisfying $\ker(\pi_r)$ equals

Type			$\lvert\mathcal{S}\rvert$
${}^{\epsilon}A_n^r(q)$	$r \mid (n+1)$	$m = \gcd(r, q-\epsilon)$	$\sum_{d/m} \phi(d) q^{\frac{n+1}{d}-1}$
$B_n(q)$	adjoint	$q = 0 \mod 2$	q^n
		$q = 1 \mod 2$	$q^n + q^{n-1}$
$C_n(q)$	adjoint	$q = 0 \mod 2$	q^n
		$q = 1 \mod 2$	$q^n + q^{\lfloor n/2 \rfloor}$
${}^{\epsilon}D_{2n+1}(q)$	adjoint	$q = 0, 2 \mod 4$	q^{2n+1}
		$q = \epsilon \mod 4$	$q^{2n+1} + 2q^{n-1} + q^{2n-1}$
		$q = -\epsilon \mod 4$	$q^{2n+1} + q^{2n-1}$
$\mathrm{SO}_{4n+2}^{\epsilon}(q)$		$q = 0 \mod 2$	q^{2n+1}
		$q = 1 \mod 2$	$q^{2n+1} + q^{2n-1}$
${}^{\epsilon}D_{2n}(q)$	adjoint	$q = 0 \mod 2$	q^{2n}
		$q = 1 \mod 2$	$q^{2n} + 2q^n + q^{2n-2}$
$\mathrm{SO}_{4n}^{\epsilon}(q)$		$q = 0 \mod 2$	q^{2n}
		$q = 1 \mod 2$	$q^{2n} + q^{2n-2}$
$\mathrm{HS}_{4n}(q)$		$q = 0 \mod 2$	q^{2n}
		$q = 1 \mod 2$	$q^{2n} + q^n$
${}^{\epsilon}E_6(q)$	adjoint, $p \neq 2$	$q = 0, -\epsilon \mod 3$	q^6
		$q = \epsilon \mod 3$	$q^6 + 2q^2$
E_7	adjoint, $p \neq 3$	$q = 0 \mod 2$	q^7
		$q = 1 \mod 2$	$q^7 + q^4$

TABLE 1. Number of semisimple classes for simple algebraic groups.

the subgroup of $\mathrm{Z}(\mathbf{G}_{\mathrm{sc}})$ of order r. If $\mathbf{G}_r$ is defined over $\mathbb{F}_q$ with Frobenius map F^{ϵ}, then put ${}^{\epsilon}A_n^r(q) = \mathbf{G}_r^{F^{\epsilon}}$.

Corollary 3.6. *Let $\mathbf{G}$ be a simple algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius map F. If $\mathbf{G}^F$ is isomorphic to $\mathbf{G}_{\mathrm{sc}}^F$, then the number of semisimple classes of $\mathbf{G}^F$ is q^n, where n is the semisimple rank of $\mathbf{G}$. Otherwise, the number of semisimple classes of $\mathbf{G}^F$ is given in Table 1. As usually, we denote by ϕ the Euler function.*

Proof. Let $\mathbf{G}$ be a simple algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius F. Denote by $(\mathbf{G}^*, F^*)$ a pair dual to $(\mathbf{G}, F)$. In table 2, we recall simple algebraic groups in duality.

Fix a linear character ζ of $\mathcal{Z}(\mathbf{G}^*)$ and denote by $\mathbf{L}_{\zeta}^*$ a cuspidal Levi subgroup of $\mathcal{L}_{\min}(\ker(\zeta))$. Write $\mathbf{G}_{\mathrm{sc}}^*$ for a simple simply-connected group of the same version as $\mathbf{G}^*$ and by $\pi : \mathbf{G}_{\mathrm{sc}}^* \to \mathbf{G}^*$ the universal cover of $\mathbf{G}^*$. The endomorphism F^* of

	$\mathbf{G}$	$\mathbf{G}^*$
A_n	$\mathbf{G}_r$	$\mathbf{G}_{(n+1)/r}$
B_n	simply-connected	C_n of type adjoint
	adjoint	C_n of type simply-connected
D_{2n+1}	simply-connected	adjoint
	SO_{4n+2}	SO_{4n+2}
D_{2n}	simply-connected	adjoint
	SO_{4n}	SO_{4n}
	HS_{4n}	HS_{4n}
E_6	simply-connected	adjoint
E_7	simply-connected	adjoint

TABLE 2. Groups in duality

$\mathbf{G}^*$ is induced by a unique Frobenius map (also denoted by F^*) of $\mathbf{G}^*_{\mathrm{sc}}$. Now, put $\widehat{\mathbf{L}}^*_\zeta = \pi^{-1}(\mathbf{L}^*_\zeta)$. Note that $\widehat{\mathbf{L}}^*_\zeta$ is a Levi subgroup of $\mathbf{G}^*_{\mathrm{sc}}$ with the same semisimple rank as $\mathbf{L}^*_\zeta$. Moreover, following [1, 2.10], we deduce that $\widehat{\mathbf{L}}^*_\zeta \in \mathcal{L}_{\min}(\pi^{-1}(\ker(\zeta)))$. Note that, since $\mathbf{G}^*$ is simple, one has $\ker(h_{\widehat{\mathbf{L}}^*_\zeta}) = \pi^{-1}(\ker(\zeta))$; see [1, 2.9].

Suppose now that $\mathcal{Z}(\mathbf{G}^*_{\mathrm{sc}})$ is cyclic of order N. Then $\mathcal{Z}(\mathbf{G}^*)$ is cyclic of order $N' = N/|\ker(\pi)|$. Since $\mathrm{Im}(\zeta)$ is a subgroup of $\mathbb{C}^\times$ of order $\mathrm{o}(\zeta)$ (we consider here $\mathrm{Irr}(\mathcal{Z}(\mathbf{G}^*))$ as a group with product the tensor product of characters). it in particular follows that $\ker(\zeta)$ has order $N'/\mathrm{o}(\zeta)$. But there is only one subgroup K of $\mathcal{Z}(\mathbf{G}^*)$ of order $N'/\mathrm{o}(\zeta)$ and $\mathbf{L}^*_\zeta$ is then a standard Levi of $\mathcal{L}_{\min}(K)$ only depending on $\mathrm{o}(\zeta)$. Furthermore, one has

$$|\pi^{-1}(K)| = |K||\ker(\pi)| = \mathrm{N}/\mathrm{o}(\zeta).$$

Since $\mathcal{Z}(\mathbf{G}^*_{sc})$ is cyclic, $\pi^{-1}(K)$ is then the unique subgroup of order $N/\mathrm{o}(\zeta)$. Then $\widehat{\mathbf{L}}^*_\zeta$ is a Levi subgroup of $\mathbf{G}^*_{\mathrm{sc}}$ satisfying $|\ker(h_{\widehat{\mathbf{L}}^*_\zeta})| = N/\mathrm{o}(\zeta)$.

In [1, Table 2.17], Bonnafé explicitly computes $\mathcal{L}_{\min}(K)$ for any subgroup K of $\mathcal{Z}(\mathbf{G}^*_{\mathrm{sc}})$. In Table 3, we recall some information that we need. For more details, we refer to [1]. For the notation in Table 3, we put $\mu_n = \{z \in \overline{\mathbb{F}}_p^\times \,|\, z^n = 1\}$.

Hence, using Table 3 we then can find the cuspidal Levi subgroup (and its semisimple rank) associated to every linear character of $\mathcal{Z}(\mathbf{G}^*)$ for $\mathbf{G}^*$ of type A_n, B_n, C_n, E_6 and E_7 and D_{2n+1}. For example, suppose $\mathbf{G}$ is of type A_n. Then using the notation preceding Corollary 3.6 , there is an integer r such that $\mathbf{G} = \mathbf{G}_r$. Moreover, one has $\mathbf{G}^*_r = \mathbf{G}_{r'}$ with $r' = (n+1)/r$. Note that $|\mathcal{Z}(\mathbf{G}_{r'})| = r$. Let d be a divisor of r and let ζ be a linear character of $\mathcal{Z}(\mathbf{G}_{r'})$ of order d. Then $\widehat{\mathbf{L}}^*_\zeta$ has semisimple rank equal to $\frac{n+1}{d}(d-1)$.

Suppose $\mathbf{G}$ is of type D_{2n} and denote by $\pi : \mathbf{G}^*_{\mathrm{sc}} \to \mathbf{G}^*$ the universal cover of $\mathbf{G}^*$ as above. The group $\mathcal{Z}(\mathbf{G}^*_{\mathrm{sc}})$ has order 4 and exponent 2. Moreover, the three non-trivial characters of $\mathcal{Z}(\mathbf{G}^*_{\mathrm{sc}})$ have distinct kernel. These kernels are the subgroups of order 2 of $\mathcal{Z}(\mathbf{G}^*_{\mathrm{sc}})$ denoted by c_1, c_2 and c_3 in Table 3. Note that if

Type of $\mathbf{G}$	$\mathcal{Z}(\mathbf{G})$	K	ss-rk($\mathbf{L}$) for $\mathbf{L} \in \mathcal{L}_{\min}(K)$	$\mathcal{Z}(\mathbf{L})$
A_n	μ_{n+1}	$\mu_{(n+1)/d}$ $d \mid (n+1)$ $p \nmid d$	$\frac{n+1}{d}(d-1)$	μ_d
B_n $p \neq 2$	μ_2	1	$\lfloor \frac{n+1}{2} \rfloor$	μ_2
C_n $p \neq 2$	μ_2	1	1	μ_2
D_{2n+1} $p \neq 2$	μ_4	1	$n+2$	μ_4
		μ_2	2	μ_2
D_{2n} $p \neq 2$	$\mu_2 \times \mu_2$	1	$n+1$	$\mu_2 \times \mu_2$
		c_1	n	μ_2
		c_2	n	μ_2
		c_3	2	μ_2
E_6 $p \neq 3$	μ_3	1	4	μ_3
E_7 $p \neq 2$	μ_2	1	3	μ_2

TABLE 3. $\mathcal{L}_{\min}(K)$ for simple simply-connected groups

$\ker(\pi) = c_3$ then $\mathbf{G}^* = \mathrm{SO}_{4n}$ and if $\ker(\pi) \in \{c_1, c_2\}$, then $\mathbf{G}^* = \mathrm{HS}_{4n}$. Let ζ be a non-trivial linear character of $\mathcal{Z}(\mathbf{G}^*)$. Suppose first that $\mathbf{G}^* = \mathbf{G}^*_{\mathrm{sc}}$. Then, the corresponding cuspidal Levi $\mathbf{L}^*_\zeta$ is a cuspidal standard Levi subgroup $\mathbf{G}^*_{\mathrm{sc}}$ such that ζ and $h_{\mathbf{L}_\zeta}$ have the same kernel. If $\mathbf{G}^* = \mathrm{SO}_{4n}$ or $\mathbf{G}^* = \mathrm{HS}_{4n}$, then $\mathcal{Z}(\mathbf{G}^*)$ has order 2 and the semisimple rank of the cuspidal Levi associated to the non-trivial character of $\mathcal{Z}(\mathbf{G}^*)$ equals the semisimple rank of any elements of $\mathcal{L}_{\min}(\ker(\pi))$ (in the group $\mathbf{G}^*_{\mathrm{sc}}$).

We now discuss the conditions on q given in the second column of Table 1. Suppose that $\mathcal{Z}(\mathbf{G}^*)$ is cyclic of order N. Then, using [7, Table 1.12.6, 1.15.2], we show that the order of $H^1(F^{\epsilon*}, \mathcal{Z}(\mathbf{G}^*))$ is the gcd of N and $q - \epsilon$. If $\mathcal{Z}(\mathbf{G}^*)$ is not cyclic (i.e. $\mathbf{G}$ is of type D_{2n}) and if $p \neq 2$, then $H^1(F^{\epsilon*}, \mathcal{Z}(\mathbf{G}^*)) = \mathcal{Z}(\mathbf{G}^*)$; see [7, Table 1.12.6, 1.15.2].

The result then follows from Theorem 3.5.

□

4. Results on semisimple characters

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ (with Frobenius map F) as above and let $(\mathbf{G}^*, F^*)$ denote a dual pair of $(\mathbf{G}, F)$. Write $\mathcal{S}$ (resp. $\mathcal{T}$) for a set of representatives of semisimple classes of $\mathbf{G}^{*F^*}$ (resp. a set of representatives of F^*-stable semisimple classes of $\mathbf{G}^*$). Moreover, we suppose that the elements of $\mathcal{T}$ are F^*-stable (which is possible because by Lang-Steinberg Theorem, we can choose an F^*-stable representative in every F^*-stable geometric class of $\mathbf{G}^*$). Put $A_{\mathbf{G}^*}(s) = \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}_{\mathbf{G}^*}(s)^\circ$. Recall that the classes of $\mathbf{G}^{*F^*}$ with representative $t \in \mathbf{G}^{*F^*}$ conjugate to s in $\mathbf{G}^*$ are parametrized by the set of F^*-classes of $A_{\mathbf{G}^*}(s)$. Moreover, $A_{\mathbf{G}^*}(s)$ is abelian, implying $|H^1(F^*, A_{\mathbf{G}^*}(s))| = |A_{\mathbf{G}^*}(s)^{F^*}|$. Note that there is an injective morphism between $A_{\mathbf{G}^*}(s)^{F^*}$ and $H^1(F, \mathcal{Z}(\mathbf{G}))^\wedge$. Hence $|A_{\mathbf{G}^*}(s)^{F^*}|$ divides $|H^1(F, \mathcal{Z}(\mathbf{G}))|$ and for every divisor d of $|H^1(F, \mathcal{Z}(\mathbf{G}))|$, we put

$$\mathcal{T}_d = \left\{ s \in \mathcal{T} \mid d = |A_{\mathbf{G}^*}(s)^{F^*}| \right\}. \tag{7}$$

For $s \in \mathcal{S}$ and $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$, we set $\rho_{s,z} = D_{\mathbf{G}}(\chi_{s,z})$, where the character $\chi_{s,z}$ is the constituent of the Gelfand-Graev character Γ_z defined in Equation (6). Put

$$\mathrm{Irr}_s(\mathbf{G}^F) = \{\rho_{s,z} \mid s \in \mathcal{S},\ z \in H^1(F, \mathcal{Z}(\mathbf{G}))\}.$$

The irreducible characters $\rho_{s,z}$ are the so-called semisimple characters of $\mathbf{G}^F$.

Proposition 4.1. *With the above notation, we have*

$$|\mathrm{Irr}_s(\mathbf{G}^F)| = \sum_{d/|H^1(F,\mathcal{Z}(\mathbf{G}))|} d^2\, |\mathcal{T}_d|.$$

Proof. As explained in [6, p. 139], we embed $\mathbf{G}$ in a connected reductive group with connected center $\widetilde{\mathbf{G}}$ with the same derived subgroup and such that $\mathbf{G}$ is normal in $\widetilde{\mathbf{G}}$. We extend F to $\widetilde{\mathbf{G}}$ (denoted by the same symbol). The inclusion $\mathbf{G} \subseteq \widetilde{\mathbf{G}}$ induces a surjective F^*-equivariant morphism $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$. For $s \in \mathcal{S}$, there is an F^*-stable semisimple $\widetilde{s}$ of $\widetilde{\mathbf{G}}^*$ such that $i^*(\widetilde{s}) = s$. Write $\rho_{\widetilde{s}}$ for the semisimple character of $\widetilde{\mathbf{G}}^F$ corresponding to s (this character is unique because $H^1(F, \mathcal{Z}(\widetilde{\mathbf{G}}))$ is trivial). Then by [6, 14.49], the character $\rho_{s,1}$ is a constituent of $\mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}})$. Moreover, the inertial group $\widetilde{\mathbf{G}}^F(s)$ of $\rho_{s,1}$ in $\widetilde{\mathbf{G}}^F$ is such that $\widetilde{\mathbf{G}}^F/\widetilde{\mathbf{G}}^F(s) \simeq A_{\mathbf{G}^*}(s)^{F^*}$. Thus by Clifford theory, since $\mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}})$ is multiplicity free (see [9]), we deduce that $\mathrm{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}})$ has $|A_{\mathbf{G}^*}(s)^{F^*}|$ constituents. It follows that

$$|\mathrm{Irr}_s(\mathbf{G}^F)| = \sum_{s\in\mathcal{S}} |A_{\mathbf{G}^*}(s)^{F^*}| = \sum_{t\in\mathcal{T}} \sum_{s\in\mathcal{S}\cap[t]_{\mathbf{G}^*}} |A_{\mathbf{G}^*}(s)^{F^*}| = \sum_{t\in\mathcal{T}} |A_{\mathbf{G}^*}(t)^{F^*}|^2.$$

The result follows. □

Proposition 4.2. *We keep the same notation as above and we suppose that p is a good prime for $\mathbf{G}$. Suppose that $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order ℓ. Let ζ be a non trivial linear character of $H^1(F, \mathcal{Z}(\mathbf{G}))$. Write $\mathbf{L}$ for its associated cuspidal Levi subgroup. Then we have*

$$|\mathrm{Irr}_s(\mathbf{G}^F)| = |\mathrm{Z}(\mathbf{G})^{\circ F}| \left(q^l + (\ell^2 - 1) q^{l - (\text{ss-rk}(\mathbf{L}))} \right),$$

where l denotes the semisimple rank of $\mathbf{G}$. In particular, in Table 4, we give the number of semisimple characters of $\mathbf{G}^F$ for simple groups $\mathbf{G}$ with $\mathrm{Z}(\mathbf{G}))^F$ of prime order. For the notation of Table 4, we put $m = \gcd(r, q - \epsilon)$.

$\mathbf{G}_{\mathrm{sc}}^F$		$\lvert \operatorname{Irr}_s(\mathbf{G}_{\mathrm{sc}}^F)\rvert$
${}^\epsilon A_n^r(q)$	m *prime*	$q^n + (m^2-1)q^{\frac{n+1}{m}-1}$
$B_n(q)$	$q = 1 \mod 2$	$q^n + 3q^{\lfloor n/2 \rfloor}$
$C_n(q)$	$q = 1 \mod 2$	$q^n + 3q^{n-1}$
${}^\epsilon D_{2n+1}(q)$	$q = -\epsilon \mod 4$	$q^{2n+1} + 3q^{2n-1}$
$\mathrm{SO}_{2n}^\epsilon(q)$	$q = 1 \mod 2$	$q^n + 3q^{n-2}$
$\mathrm{HS}_{4n}(q)$	$q = 1 \mod 2$	$q^{2n} + 3q^n$
${}^\epsilon E_6(q)$, $p \neq 2$	$q = \epsilon \mod 3$	$q^6 + 8q^2$
$E_7(q)$, $p \neq 3$	$q = 1 \mod 2$	$q^7 + 3q^4$

TABLE 4. Number of semisimple characters.

Proof. We denote by $\mathcal{T}_1$ and $\mathcal{T}_\ell$ the sets as defined in Equation (7). We have $|\mathcal{T}| = |\mathcal{T}_1| + |\mathcal{T}_\ell|$ and $|\mathcal{S}| = |\mathcal{T}_1| + \ell|\mathcal{T}_\ell|$ implying

$$|\mathcal{T}_1| = \frac{1}{\ell-1}(\ell|\mathcal{T}| - |\mathcal{S}|) \quad \text{and} \quad |\mathcal{T}_\ell| = \frac{1}{\ell-1}(|\mathcal{S}| - |\mathcal{T}|).$$

Furthermore, from [6, 14.42] we deduce that $|\mathcal{T}| = |\operatorname{Z}(\mathbf{G})^{\circ F}|q^l$. Moreover, since ℓ is prime, all non trivial linear characters of $H^1(F, \mathcal{Z}(\mathbf{G}))$ are faithful on $H^1(F, \mathcal{Z}(\mathbf{G}))$. Their corresponding characters of $\mathcal{Z}(\mathbf{G})$ then have the same kernel (equal to $\mathcal{L}(\mathcal{Z}(\mathbf{G}))$). Thus, they are associated to a same cuspidal Levi subgroup $\mathbf{L}$, which is the standard Levi of $\mathcal{L}_{\min}(\mathcal{L}(\mathcal{Z}(\mathbf{G})))$. Thanks to Theorem 3.5 we deduce that

$$|\mathcal{S}| = |\operatorname{Z}(\mathbf{G})^{\circ F}| \left(q^l + (\ell-1)q^{l-(\text{ss-rk}(\mathbf{L}))}\right).$$

Now, using Proposition 4.1, we obtain

$$\begin{aligned} |\operatorname{Irr}_s(\mathbf{G}^F)| &= |\mathcal{T}_1| + \ell^2|\mathcal{T}_\ell| \\ &= (\ell+1)|\mathcal{S}| - \ell|\mathcal{T}| \\ &= |\operatorname{Z}(\mathbf{G})^{\circ F}| \left((\ell+1)q^l + (\ell^2-1)q^{l-(\text{ss-rk}(\mathbf{L}))} - \ell q^l\right) \\ &= |\operatorname{Z}(\mathbf{G})^{\circ F}| \left(q^l + (\ell^2-1)q^{l-(\text{ss-rk}(\mathbf{L}))}\right). \end{aligned}$$

Now, Table 4 follows from Table 3. However, note that for $\mathbf{G} = \mathrm{SO}_{2n}$, we have to distinguish whether n is even or not. If $n = 2k+1$, then the number of semisimple characters of $\mathrm{SO}_{4k+2}^\epsilon(q)$ is $q^{2k+1} + 3q^{2k-1} = q^n + 3q^{n-2}$. If $n = 2k$, then the number of semisimple characters of $\mathrm{SO}_{4k}^\epsilon(q)$ is $q^{2k} + 3q^{2k-2} = q^n + 3q^{n-2}$. □

5. CHARACTERS OF p'-ORDER IN BOREL SUBGROUPS

5.1. **Formula for the number of p'-characters.** In this section, we keep the same notation as above. In particular, $\mathbf{T}$ denotes a maximal F-stable torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We consider the group

$$\mathbf{B}_0 = \mathbf{U}_1 \rtimes \mathbf{T},$$

where $\mathbf{U}_1 = \mathbf{B}/\mathbf{U}_0$ (see §3.1 for the notation). Note that $\mathbf{B}_0$ is F-stable and $\mathbf{B}_0^F = \mathbf{U}_1^F \rtimes \mathbf{T}^F$. Moreover, the set $\operatorname{Irr}_{p'}(\mathbf{B}^F)$ is in bijection with the set $\operatorname{Irr}(\mathbf{B}_0^F)$; see [2, Lemma 4]. As in the proof of Proposition 4.1, we consider $\widetilde{\mathbf{G}}$ a connected reductive group with connected center containing $\mathbf{G}$ and such that they have the same derived subgroup. We denote by $\widetilde{\mathbf{T}}$ the unique F-stable maximal torus of $\widetilde{\mathbf{G}}$ containing $\mathbf{T}$. We denote by Ω and $\widetilde{\Omega}$ the sets of $\mathbf{T}^F$-orbits and $\widetilde{\mathbf{T}}^F$-orbits on $\mathrm{Irr}(\mathbf{U}_1^F)$, respectively. As in Equation (1), we denote by $\mathcal{O}$ the set of F-orbits on Δ. Moreover, for every $\omega \in \mathcal{O}$, we fix a non-trivial character ϕ_ω of $\mathbf{X}_\omega^F$ (for the notation, see Equation (1)). For $J \subseteq \mathcal{O}$, we set

$$\phi_J = 1_{\overline{J}} \otimes \prod_{\omega\in J} \phi_\omega,$$

where $1_{\overline{J}} = \prod_{\omega\notin J} 1_{\mathbf{X}_\omega^F}$. Then by [4, 2.9, 8.1.2], the set $\{\phi_J \mid J \subseteq \mathcal{O}\}$ is a set of representatives of $\widetilde{\Omega}$.

Proposition 5.1. *We keep the notation as above. For every $J \subseteq \mathcal{O}$, we denote by Ω_J (resp. $\Omega_{J,1}$) the element of $\widetilde{\Omega}$ (resp. Ω) containing ϕ_J. Moreover, we set $n_J = |\Omega_J|/|\Omega_{J,1}|$. Then*

$$|\mathrm{Irr}_{p'}(\mathbf{B}^F)| = \sum_{J\subseteq\mathcal{O}} n_J |\,\mathrm{C}_{\mathbf{T}^F}(\phi_J)|.$$

Proof. First remark that n_J is an integer. Indeed, since $\mathbf{T}^F \subseteq \widetilde{\mathbf{T}}^F$, we deduce that Ω_J is a disjoint union of $\mathbf{T}^F$-orbits. In particular, there is k such that

$$\Omega_J = \bigsqcup_{i=1}^{k} \Omega_{J,i}, \tag{8}$$

where $\Omega_{J,i} \in \Omega$ (the notation is chosen such that $\phi_J = \phi_{J,1} \in \Omega_{J,1}$). Moreover, for every $1 \leq i \leq k$, $|\Omega_{J,i}| = |\Omega_{J,1}|$ because $\Omega_{J,i}$ and $\Omega_{J,1}$ are conjugate by an element of $\widetilde{\mathbf{T}}^F$. Then $|\Omega_{J,1}|$ divides $|\Omega_J|$ and $n_J = k$. For $1 \leq i \leq n_J$, fix $t_i \in \widetilde{\mathbf{T}}^F$ such that $\phi_{J,i} = {}^{t_i}\phi_{J,1} \in \Omega_{J,i}$ and denote by $\mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})$ the stabilizer of $\phi_{J,i}$ in $\mathbf{T}^F$. Then the inertial subgroup $I_{J,i}$ of $\phi_{J,i}$ in $\mathbf{B}_0^F$ is $\mathbf{U}_1^F \rtimes \mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})$. Moreover, since $\mathbf{U}_1^F$ is abelian, we can extend $\phi_{J,i}$ to $I_{J,i}$ setting $\widetilde{\phi}_{J,i}(ut) = \phi_{J,i}(u)$ for $u \in \mathbf{U}_1^F$ and $t \in \mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})$. Then, by Clifford theory, the characters of $\mathbf{B}_0^F$ such that $\phi_{J,i}$ is a constituent of their restrictions to $\mathbf{U}_1^F$ are exactly the irreducible characters $\mathrm{Ind}_{I_{J,i}}^{\mathbf{B}_0^F}(\widetilde{\phi}_{J,i} \otimes \psi)$ with $\psi \in \mathrm{Irr}(\mathrm{C}_{\mathbf{T}^F}(\phi_{J,i}))$. There are $|\,\mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})|$ such characters. Hence, we deduce

$$|\mathrm{Irr}(\mathbf{B}_0^F)| = \sum_{J\subseteq\mathcal{O}} \sum_{i=1}^{n_J} |\,\mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})|.$$

Furthermore, we have $|\,\mathrm{C}_{\mathbf{T}^F}(\phi_{J,i})| = |{}^{t_i}\,\mathrm{C}_{\mathbf{T}^F}(\phi_{J,1})|$. The result follows. □

For $J \subseteq \mathcal{O}$, we define

$$m(J) = \bigsqcup_{\omega\in J} \omega. \tag{9}$$

Note that $m(J) \subseteq \Delta$ and $F(m(J)) = m(J)$.

Lemma 5.2. *We keep the notation as above. For $J \subseteq \mathcal{O}$, we associate to ϕ_J the F-stable standard Levi subgroup $\mathbf{L}_{m(J)}$ where $m(J)$ is the subset of Δ defined in Relation (9). Then we have*

$$n_J = |H^1(F, \mathcal{Z}(\mathbf{L}_{m(J)}))| \quad \text{and} \quad |\,\mathrm{C}_{\mathbf{T}^F}(\phi_J)| = n_J |\,\mathrm{Z}(\mathbf{G})^{\circ F}| \prod_{\omega\in\mathcal{O}\setminus J} (q^{|\omega|} - 1),$$

where n_J is the integer defined in Proposition 5.1.

Proof. Recall that Ω_J (resp. $\Omega_{J,1}$) is the $\widetilde{\mathbf{T}}^F$-orbit (resp. $\mathbf{T}^F$-orbit) of ϕ_J. By Equation (8), one has

$$|\Omega_J| = n_J|\Omega_{J,1}|.$$

Moreover, as explained in the proof of [4, 8.1.2], we have $|\Omega_J| = \prod_{\omega\in J}(q^{\omega}-1)$. It then follows that

$$|\operatorname{C}_{\mathbf{T}^F}(\phi_J)| = n_J\frac{|\mathbf{T}^F|}{\prod_{\omega\in J}(q^{|\omega|}-1)}.$$

Furthermore, by [4, 2.9], we have $|\mathbf{T}^F| = |\operatorname{Z}(\mathbf{G})^{\circ F}|\prod_{\omega\in\mathcal{O}}(q^{|\omega|}-1)$. Hence we deduce

$$|\operatorname{C}_{\mathbf{T}^F}(\phi_J)| = n_J|\operatorname{Z}(\mathbf{G})^{\circ F}|\prod_{\omega\in\mathcal{O}\backslash J}(q^{|\omega|}-1).$$

Let $\mathbf{L}_{m(J)}$ be the standard F-stable Levi subgroup of $\mathbf{G}$ corresponding to the subset of simple roots $m(J)$. Denote by $\mathbf{B}_{m(J)}\subseteq\mathbf{B}$ an F-stable Borel subgroup of $\mathbf{L}_{m(J)}$ and by $\mathbf{U}_{m(J)}$ the unipotent radical of $\mathbf{B}_{m(J)}$. The set $m(J)$ is the set of simple roots of $\mathbf{L}_{m(J)}$ associated to $\mathbf{B}_{m(J)}$. In particular, Equation (1) applied to the connected reductive group $\mathbf{L}_{m(J)}$ gives

$$\mathbf{U}_{1,m(J)}^F = \prod_{\omega\in J}\mathbf{X}_\omega^F.$$

We denote by ϕ'_J the restriction of ϕ_J to $\mathbf{U}^F_{1,m(J)}$. Then $\phi'_J\in\operatorname{Irr}(\mathbf{U}^F_{1,m(J)})$ and the map $\operatorname{Irr}(\mathbf{U}^F_{1,m(J)})\to\operatorname{Irr}(\mathbf{U}^F_1)$, $\vartheta\mapsto 1_{\overline{J}}\otimes\vartheta$, is $\mathbf{T}^F$-equivariant. Moreover, note that ϕ'_J is a regular character of $\mathbf{U}^F_{1,m(J)}$. Hence, using [6, 14.28], we deduce that $n_J = |H^1(F,\mathcal{Z}(\mathbf{L}_{m(J)}))|$ as required. □

Corollary 5.3. *With the above notation, one has*

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)| = |\operatorname{Z}(\mathbf{G})^{\circ F}|\sum_{J\subseteq\mathcal{O}}|\mathcal{Z}(\mathbf{L}_{m(J)})^F|^2\prod_{\omega\in\mathcal{O}\backslash J}(q^{|\omega|}-1),$$

where $m(J)$ is the subset of Δ associated to J as in Equation (9).

Proof. It is a direct consequence of Proposition 5.1 and Lemma 5.2 and the equality $|H^1(F,\mathcal{Z}(\mathbf{L}_{m(J)}))| = |\mathcal{Z}(\mathbf{L}_{m(J)})^F|$. □

In the following, we will need the following result.

Lemma 5.4. *Fix $I\in\mathcal{O}$ and put $\overline{I} = \mathcal{O}\backslash I$. Then we have*

$$\sum_{I\subseteq J\subseteq\mathcal{O}}\prod_{\omega\notin J}(q^{|\omega|}-1) = q^{|m(\overline{I})|},$$

where m is the map defined in Equation (9).

Proof. First remark that

$$\sum_{I\subseteq J\subseteq\mathcal{O}}\prod_{\omega\notin J}(q^{|\omega|}-1) = \sum_{J\subseteq\overline{I}}\prod_{\omega\in J}(q^{|\omega|}-1).$$

Furthermore, for every finite set A and $f:A\to\mathbb{R}$, one has

$$\prod_{a\in A}(f(a)+1) = \sum_{J\subseteq A}\prod_{a\in J}f(a). \tag{10}$$

We apply Equation (10) with $A=\overline{I}$ and $f:\overline{I}\to\mathbb{R}$, $\omega\mapsto q^{|\omega|}-1$ and we deduce

$$\begin{aligned}\sum_{J\subseteq\overline{I}}\prod_{\omega\in J}(q^{|\omega|}-1) &= \prod_{\omega\in\overline{I}}q^{|\omega|}\\ &= q^{\sum_{\omega\in\overline{I}}|\omega|}\end{aligned}$$

Moreover, Equation (9) implies $|m(\overline{I})|=\sum_{\omega\in\overline{I}}|\omega|$ and the result follows. □

Remark 5.5. If the center of $\mathbf{G}$ is connected, then the center of every Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is connected (because the map $h_{\mathbf{L}}$ is surjective). In particular, Corollary 5.3 and Lemma 5.4 (applied with $I=\emptyset$) give

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)|=|\operatorname{Z}(\mathbf{G})^F|q^{|m(\mathcal{O})|}=|\operatorname{Z}(\mathbf{G})^F|q^{|\Delta|},$$

which is a well-known result; see [2, Remark 1].

5.2. **The case of quasi-simple groups.** In this section, we suppose that $\mathbf{G}$ is a quasi-simple algebraic group. We keep the notation as above. Recall that for $I\subseteq\Delta$, the map $h_{L_I}:\mathcal{Z}(\mathbf{G})\to\mathcal{Z}(\mathbf{L}_I)$ denotes the surjective map induced by the inclusion $\operatorname{Z}(\mathbf{G})\subseteq\operatorname{Z}(\mathbf{L}_I)$. Moreover, recall that for every subgroup K of $\mathcal{Z}(\mathbf{G})$, there is $I\subseteq\Delta$ such that $K=\ker(h_{\mathbf{L}_I})$ (we use here the fact that $\mathbf{G}$ is quasi-simple; see [1, 2.9]). Then we denote by I_K a subset of Δ such that $K=\ker(h_{\mathbf{L}_{I_K}})$ and I_K is minimal (for the inclusion). In particular, $\mathbf{L}_{I_K}\in\mathcal{L}_{\min}(K)$.

Proposition 5.6. *With the above notation, if p is good for $\mathbf{G}$, we have*

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)|=|\operatorname{Z}(\mathbf{G})^{\circ F}|\sum_{K\leq\mathcal{Z}(\mathbf{G})^F}\frac{|\mathcal{Z}(\mathbf{G})^F|^2}{|K|^2}\left(q^{|\overline{I}_K|}-\sum_{K'\in\max(K)}q^{|\overline{I}_{K'}|}\right),$$

where $\max(K)$ *denotes the set of maximal proper subgroups of* K.

Proof. For a subgroup K of $\mathcal{Z}(\mathbf{G})$, we define

$$A_K=\{I\in\Delta\mid I_K\subseteq I\}\quad\text{and}\quad B_K=\{J\in\mathcal{O}\mid\ker(h_{\mathbf{L}_{m(J)}})=K\}.$$

where $m(J)$ is the subset of Δ associated to J defined in Equation (9). Then Corollary 5.3 implies

$$\begin{aligned}|\operatorname{Irr}_{p'}(\mathbf{B}^F)| &= |\operatorname{Z}(\mathbf{G})^{\circ F}|\sum_{K\leq\mathcal{Z}(\mathbf{G})^F}\sum_{J\in B_K}|\mathcal{Z}(\mathbf{L}_{m(J)})^F|^2\prod_{\omega\notin J}(q^{|\omega|}-1)\\ &= |\operatorname{Z}(\mathbf{G})^{\circ F}|\sum_{K\leq\mathcal{Z}(\mathbf{G})^F}|\mathcal{Z}(\mathbf{L}_{m(J)})^F|^2\sum_{J\in B_K}\prod_{\omega\notin J}(q^{|\omega|}-1),\end{aligned}$$

because for $J\in B_K$, the numbers $|\mathcal{Z}(\mathbf{L}_{m(J)})^F|$ are constant. Furthermore, one has

$$B_K=\{J\in\mathcal{O}\mid\ker(h_{\mathbf{L}_{m(J)}})\subseteq K\}\backslash\{J\in\mathcal{O}\mid\ker(h_{\mathbf{L}_{m(J)}})\subsetneq K\}.$$

Note that $\mathbf{L}_{I_K}$ is F-stable. Then I_K is a union of some F-orbits lying in a subset $\widetilde{I}_K$ of $\mathcal{O}$, such that $m(\widetilde{I}_K)=I_K$. Since $\mathbf{L}_{I_K}\in\mathcal{L}_{\min}(K)$, it follows

$$\{J\in\mathcal{O}\mid\ker(h_{\mathbf{L}_{m(J)}})\subseteq K\}=\{J\in\mathcal{O}\mid\widetilde{I}_K\subseteq J\}.$$

Moreover, one has

$$\begin{aligned}\{J\in\mathcal{O}\mid\ \ker(h_{\mathbf{L}_{m(J)}})\subsetneq K\} &= \bigsqcup_{K'\in\max(K)}\{J\in\mathcal{O}\mid\ker(h_{\mathbf{L}_{m(J)}})\subseteq K'\}\\ &= \bigsqcup_{K'\in\max(K)}\{J\in\mathcal{O}\mid\widetilde{I}_{K'}\subseteq J\}.\end{aligned}$$

Thus, if we put $C_K = \{J \in \mathcal{O} \mid \widetilde{I}_K \subseteq J\}$, then it follows

$$\begin{aligned}\sum_{J\in B_K}\prod_{\omega\notin J}(q^{|\omega|}-1) &= \sum_{J\in C_K}\prod_{w\notin J}(q^{|w|}-1) - \sum_{K'\in\max(K)}\sum_{J\in C_{K'}}\prod_{\omega\notin J}(q^{|\omega|}-1)\\ &= q^{|\Delta\setminus m(\widetilde{I}_K)|} - \sum_{K'\in\max K} q^{|\Delta\setminus m(\widetilde{I}_{K'})|}.\end{aligned}$$

The last equality comes from Lemma 5.4. Moreover, we have $h_{\mathbf{L}_{m(J)}}(\mathcal{Z}(\mathbf{G})^F) = \mathcal{Z}(\mathbf{L}_{m(J)})^F$ implying $|\mathcal{Z}(\mathbf{L}_{m(J)})^F| = |\mathcal{Z}(\mathbf{G})^F/K|$. The result follows. □

Proposition 5.7. *Let* $\mathbf{G}$ *be a connected reductive group defined over* $\mathbb{F}_q$ *with corresponding Frobenius* F. *Suppose* p *is a good prime for* $\mathbf{G}$ *and* $\mathcal{Z}(\mathbf{G})^F$ *has prime order* ℓ. *Put* $r = |I|$ *for* $\mathbf{L}_I$ *in* $\mathcal{L}_{\min}(\{1\})$. *Then we have*

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)| = |\operatorname{Z}(\mathbf{G})^{\circ F}|\left(q^l + (\ell^2-1)q^{l-r}\right),$$

where l *is the semisimple rank of* $\mathbf{G}$.

Proof. First remark that we do not suppose that $\mathbf{G}$ is quasi-simple. Indeed, the set $\mathcal{L}_{\min}(\{1\})$ is non-empty. If we denote by $\mathbf{L}_I$ a standard Levi lying in $\mathcal{L}_{\min}(\{1\})$, then we have $\ker(h_{\mathbf{L}_I}) = \{1\}$. Hence $I = I_{\{1\}}$ (see the beginning of §5.2 for the notation). Moreover, we always have $I_{\mathcal{Z}(\mathbf{G})^F} = \emptyset$. We can then apply the proof of Proposition 5.6. We obtain

$$\begin{aligned}|\operatorname{Irr}_{p'}(\mathbf{B}^F)| &= |\operatorname{Z}(\mathbf{G})^{\circ F}|\left(|\mathcal{Z}(\mathbf{G})^F|^2 q^{|\Delta|-r} + q^{|\Delta|} - q^{|\Delta|-r}\right)\\ &= |\operatorname{Z}(\mathbf{G})^{\circ F}|\left(q^{|\Delta|} + (|\mathcal{Z}(\mathbf{G})^F|^2-1)q^{|\Delta|-r}\right).\end{aligned}$$

Since $|\Delta|$ is the semisimple rank of $\mathbf{G}$, the result follows. □

Remark 5.8. For a group $\mathbf{G}$ as in Proposition 5.7, if ζ denotes a non-trivial character of $H^1(F,\mathcal{Z}(\mathbf{G}))$ and $\mathbf{L}_\zeta$ its associated cuspidal Levi of $\mathbf{G}$, then $\mathbf{L}_\zeta$ is F-stable and $\ker(h_{\mathbf{L}_\zeta})$ is trivial. Then $\mathbf{L}_\zeta \in \mathcal{L}_{\min}(\{1\})$. In particular, the number r of Proposition 5.7 is equal to the semisimple rank of $\mathbf{L}_\zeta$, implying

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)| = |\operatorname{Z}(\mathbf{G})^{\circ F}|\left(q^l + (\ell^2-1)q^{l-(\text{ss-rk}(\mathbf{L}_\zeta))}\right).$$

Comparing with Proposition 4.2, we deduce

$$|\operatorname{Irr}_{p'}(\mathbf{G}^F)| = |\operatorname{Irr}_{p'}(\mathbf{B}^F)|.$$

Hence, this then proves that, if p is a good prime for $\mathbf{G}$ and $H^1(F,\mathcal{Z}(\mathbf{G})$ has prime order, then the McKay conjecture holds for $\mathbf{G}$ in defining characteristic.

Proposition 5.9. *If* $\mathbf{G}$ *is a simple and simply-connected algebraic group of type* D_{2n}, *then*

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)| = q^{2n} + 3q^{2n-2} + 6q^n + 4q^{n-1}.$$

If $\mathbf{G}$ *is a simple and simply-connected algebraic group of type* D_{2n+1} *with* $H^1(F,\mathcal{Z}(\mathbf{G}))$ *of order* 4, *then*

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F)| = q^{2n+1} + 3q^{2n-1} + 12q^{n-1}.$$

Proof. If $\mathbf{G}$ is simple and simply-connected group of type D_{2n}, then $\mathcal{Z}(\mathbf{G})^F$ is a finite group of order 4 with exponent 2. Denote by c_1, c_2 and c_3 its subgroups of order 2. Moreover, using Table 3, we deduce

K	$\lvert\overline{I}_K\rvert$
$\{1\}$	$n-1$
c_1	n
c_2	n
c_3	$2n-2$
$\mathcal{Z}(\mathbf{G})^F$	$2n$

Type D_{2n}

K	$\lvert\overline{I}_K\rvert$
$\{1\}$	$n-1$
$\mathbb{Z}_2$	$2n-1$
$\mathcal{Z}(\mathbf{G})^F$	$2n+1$

Type D_{2n+1}

The result then follows from Proposition 5.6. □

6. Restriction of semisimple characters to the center

In this section, we keep the notation as above. To simplify the notation, we set $G = \mathbf{G}^F$, $Z = \mathrm{Z}(\mathbf{G})^F$ and $U = \mathbf{U}^F$. For $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$ and $\nu \in \mathrm{Irr}(Z)$, we put

$$\Gamma_{z,\nu} = \mathrm{Ind}_{ZU}^G(\nu \otimes \phi_z),$$

where ϕ_z is the regular character of U corresponding to z. Note that by Clifford theory, one has

$$\mathrm{Ind}_U^{ZU}(\phi_z) = \sum_{\nu\in\mathrm{Irr}(Z)} \nu \otimes \phi_z.$$

We then deduce that

$$\Gamma_z = \sum_{\nu\in\mathrm{Irr}(Z)} \Gamma_{z,\nu}.$$

Lemma 6.1. *Denote by E_z and $E_{z,\nu}$ the set of constituents of Γ_z and $\Gamma_{z,\nu}$, respectively. Then*

$$E_{z,\nu} = \{\chi \in E_z \mid \langle \mathrm{Res}_Z^G(\chi), \nu \rangle_Z \neq 0\}.$$

Proof. We denote by R a set of representatives of the double cosets $ZU\backslash G/Z$. Therefore, for $\varphi \in \mathrm{Irr}(ZU)$, Mackey's theorem implies

$$\begin{aligned} \mathrm{Res}_Z^G(\mathrm{Ind}_{ZU}^G(\varphi)) &= \sum_{r\in R} \mathrm{Ind}_{^r(ZU)\cap Z}^Z \left(\mathrm{Res}_{^r(ZU)\cap Z}(^r\varphi)\right) \\ &= \sum_{r\in R} \mathrm{Res}_Z\left(^r\varphi\right). \qquad (11) \end{aligned}$$

Fix now ν, $\nu' \in \mathrm{Irr}(Z)$. Then Equation (11) applied with $\varphi = \nu \otimes \phi_z$ implies

$$\begin{aligned} \langle \Gamma_{z,\nu}, \mathrm{Ind}_Z^G(\nu') \rangle_G &= \langle \mathrm{Res}_Z^G\left(\Gamma_{z,\nu}\right), \nu' \rangle_Z \\ &= \langle \sum_{r\in R} \nu, \nu' \rangle_Z \\ &= |R| \langle \nu, \nu' \rangle_Z \\ &= |R|\delta_{\nu,\nu'}. \end{aligned}$$

The result then follows. □

Remark 6.2. Note that if we denote by F_z and $F_{z,\nu}$ the set of constituents of $D_{\mathbf{G}}(\Gamma_z)$ and $D_{\mathbf{G}}(\Gamma_{z,\nu})$, respectively, then $F_{z,\nu} = \{\chi \in E_z \,|\, \langle \operatorname{Res}_Z^G(\chi, \nu \rangle_Z \neq 0\}$. Indeed, by [6, 12.8] and [10, 2.2], $D_{\mathbf{G}}(\operatorname{Ind}_Z^G(\nu)) = \operatorname{Ind}_Z^G(\nu)$. In particular, D_G induces a bijection between $E_{z,\nu}$ and $F_{z,\nu}$.

Lemma 6.3. *With the above notation, for z, $z' \in H^1(F, \mathcal{Z}(\mathbf{G}))$ and ν, $\nu' \in \operatorname{Irr}(Z)$, one has*

$$\langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_G = \langle \Gamma_{z,\nu'}, \Gamma_{z',\nu'} \rangle_G.$$

Proof. We have to show that the scalar product $\langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_G$ does not depend on ν. First remark that it follows from Lemma 6.1 that

$$\langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_G = \langle \Gamma_{z,\nu}, \Gamma_{z'} \rangle_G.$$

Denote by R a set of representatives of the double cosets $UZ\backslash G/U$. Then Mackey's theorem implies

$$\begin{aligned} \langle \Gamma_{z,\nu}, \Gamma_{z'} \rangle_G &= \langle \operatorname{Res}_U^G \left(\operatorname{Ind}_{ZU}^G(\nu \otimes \phi_z)\right), \phi_{z'} \rangle_U \\ &= \sum_{r\in R} \langle \operatorname{Ind}^U_{{}^r(UZ)\cap U} \left(\operatorname{Res}_{{}^r(UZ)\cap U}({}^r(\nu \otimes \phi_z)\right), \phi_{z'} \rangle_U \\ &= \sum_{r\in R} \langle \operatorname{Ind}^U_{{}^rU\cap U}({}^r\phi_z), \phi_{z'} \rangle_U. \end{aligned}$$

Note that the scalar product in the last equality does not depend on ν. This proves the claim. $\square$

Corollary 6.4. *With the above notation, for z, $z' \in H^1(F, \mathcal{Z}(\mathbf{G}))$ and $\nu \in \operatorname{Irr}(Z)$, we have*

$$\langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_G = \frac{1}{|Z|} \langle \Gamma_z, \Gamma_{z'} \rangle_G.$$

Proof. We have

$$\langle \Gamma_z, \Gamma_{z'} \rangle_G = \sum_{\nu,\, \nu' \in \operatorname{Irr}(Z)} \langle \Gamma_{z,\nu}, \Gamma_{z',\nu'} \rangle_G.$$

If $\nu \neq \nu'$, we have $\langle \Gamma_{z,\nu}, \Gamma_{z',\nu'} \rangle_G = 0$ because by Lemma 6.1, the constituents of $\Gamma_{z,\nu}$ (resp. of $\Gamma_{z',\nu'}$) are constituents of $\operatorname{Ind}_Z^G(\nu)$ (resp. $\operatorname{Ind}_Z^G(\nu')$) and the characters $\operatorname{Ind}_Z^G(\nu)$ and $\operatorname{Ind}_Z^G(\nu')$ have no constituents in common. Then

$$\langle \Gamma_z, \Gamma_{z'} \rangle_G = \sum_{\nu \in \operatorname{Irr}(Z)} \langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_G.$$

The result is now a consequence of Lemma 6.3 $\square$

Proposition 6.5. *With the above notation, if p is a good prime for $\mathbf{G}$ and the center of $\mathbf{G}$ is connected, then for every linear character ν of $\mathrm{Z}(\mathbf{G}^F)$, one has*

$$|\operatorname{Irr}_s(\mathbf{G}^F|\nu)| = \frac{1}{|\mathrm{Z}(\mathbf{G}^F)|} |\operatorname{Irr}_s(\mathbf{G}^F)|.$$

Proof. Since the center of $\mathbf{G}$ is connected, there is only one Gelfand-Graev character Γ_1. Moreover, Remark 6.2 implies

$$|\operatorname{Irr}_s(\mathbf{G}^F|\nu)| = \langle \Gamma_{1,\nu}, \Gamma_{1,\nu} \rangle_{\mathbf{G}^F}.$$

Furthermore, one has $|\operatorname{Irr}_s(\mathbf{G}^F)| = \langle \Gamma_1, \Gamma_1 \rangle_{\mathbf{G}^F}$. The result follows from Lemma 6.4 □

Proposition 6.6. *With the above notation, if p is a good prime for $\mathbf{G}$ and the group $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order ℓ, then for every linear character ν of $\mathrm{Z}(\mathbf{G}^F)$, one has*

$$|\operatorname{Irr}_s(\mathbf{G}^F|\nu)| = \frac{1}{|\mathrm{Z}(\mathbf{G}^F)|}|\operatorname{Irr}_s(\mathbf{G}^F)|.$$

Proof. We consider $\widetilde{\mathbf{G}}$ a connected reductive group with connected center as in the proof of Proposition 4.1. Fix s a semisimple element of $\mathbf{G}^{*F^*}$ and $\widetilde{s}$ a semisimple element of $\widetilde{\mathbf{G}}^{*F^*}$ such that $i^*(\widetilde{s}) = s$. In the proof Proposition 4.1, we have seen that $\operatorname{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}})$ has $|A_{\mathbf{G}^*}(s)^{F^*}|$ constituents. In fact, the constituents of $\operatorname{Res}_{\mathbf{G}^F}^{\widetilde{\mathbf{G}}^F}(\rho_{\widetilde{s}})$ are in bijection with $\operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$. We denote by $\rho_{s,\vartheta}$ the constituent corresponding to $\vartheta \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$. Moreover, this bijection could be chosen such that there is a surjective morphism $\omega_s : H^1(F, \mathcal{Z}(\mathbf{G})) \to \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$ satisfying $\rho_{s,\vartheta}$ (for $\vartheta \in \operatorname{Irr}(A_{\mathbf{G}^*}(s)^{F^*})$) is a constituent of $D_{\mathbf{G}}(\Gamma_z)$ for $z \in H^1(F, \mathcal{Z}(\mathbf{G}))$ if and only if $\omega_s(z) = \vartheta$. In particular, the character $\rho_{s,\vartheta}$ lies in $|H^1(F, \mathcal{Z}(\mathbf{G}))|/|A_{\mathbf{G}^*}(s)^{F^*}|$ different duals of Gelfand-Graev characters of $\mathbf{G}^F$. Furthermore, $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order ℓ. It follows that a semisimple character of $\mathbf{G}^F$ is either a constituent of only one $D(\Gamma_z)$ or of all. We keep the notation of Remark 6.2 and put, for $\nu \in \operatorname{Irr}(\mathrm{Z}(\mathbf{G}^F))$

$$F_\nu = \bigcap_{z \in H^1(F, \mathcal{Z}(\mathbf{G}))} F_{z,\nu}.$$

The above discussion implies that if $z \neq z'$, then

$$F_{z,\nu} \cap F_{z',\nu} = F_\nu. \tag{12}$$

Moreover, one has

$$\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu) = \bigcup_{z \in H^1(F, \mathcal{Z}(\mathbf{G}))} F_{z,\nu}.$$

Therefore,

$$\begin{aligned}
|\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu)| &= |\bigcup_{z \in H^1(F, \mathcal{Z}(\mathbf{G}))} F_{z,\nu}| \\
&= \sum_{k=1}^{\ell} (-1)^{k+1} \sum_{I \subseteq H^1(F, \mathcal{Z}(\mathbf{G})), |I|=k} |\bigcap_{z \in I} F_{z,\nu}| \\
&= \sum_z |F_{z,\nu}| + |F_\nu| \sum_{k=2}^{\ell} (-1)^{k+1} \sum_{I \subseteq H^1(F, \mathcal{Z}(\mathbf{G})), |I|=k} 1 \\
&= \sum_z |F_{z,\nu}| + |F_\nu| \sum_{k=2}^{\ell} (-1)^{k+1} \binom{\ell}{k} \\
&= \sum_z |F_{z,\nu}| + |F_\nu|(1-\ell).
\end{aligned}$$

Note that, since the characters $\Gamma_{z,\nu}$ are multiplicity free, one has $|F_{z,\nu}| = \langle \Gamma_{z,\nu}, \Gamma_{z,\nu} \rangle_{\mathbf{G}^F}$ and $|F_\nu| = \langle \Gamma_{z,\nu}, \Gamma_{z',\nu} \rangle_{\mathbf{G}^F}$ where z and z' are two fixed distinct elements of

$H^1(F, \mathcal{Z}(\mathbf{G}))$. Fix two such elements z and z'. Then Corollary 6.4 implies

$$|F_{z,\nu}| = \frac{1}{|\operatorname{Z}(\mathbf{G}^F)|}\langle \Gamma_z, \Gamma_z \rangle_{\mathbf{G}^F} \quad \text{and} \quad |F_\nu| = \frac{1}{|\operatorname{Z}(\mathbf{G}^F)|}\langle \Gamma_z, \Gamma_{z'} \rangle_{\mathbf{G}^F}.$$

Denote by $\mathbf{L}$ the cuspidal Levi subgroup associated to every non-trivial character of $H^1(F, \mathcal{Z}(\mathbf{G}))$ and by l the semisimple rank of $\mathbf{G}$. Proposition 3.2 gives

$$\langle \Gamma_z, \Gamma_z \rangle = |Z^\circ|\left(q^l - (\ell-1)q^{l-(\text{ss-rk}(\mathbf{L}))}\right) \text{ and } \langle \Gamma_z, \Gamma_{z'} \rangle = |Z^\circ|\left(q^l - q^{l-(\text{ss-rk}(\mathbf{L}))}\right),$$

with $Z^\circ = \operatorname{Z}(\mathbf{G})^{\circ F}$. It follows

$$\begin{aligned} |\operatorname{Irr}_s(\mathbf{G}^F|\nu)| &= \frac{1}{|\operatorname{Z}(\mathbf{G}^F)|}|Z^\circ|\left(q^l - (\ell^2-1)q^{l-(\text{ss-rk}(\mathbf{L}))}\right) \\ &= \frac{1}{|\operatorname{Z}(\mathbf{G}^F)|}|\operatorname{Irr}_s(\mathbf{G}^F)|. \end{aligned}$$

The last equality comes from Proposition 4.2. □

Remark 6.7. As we remark in [3], the number $|\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)|$ does not depend on ν for all $\nu \in \operatorname{Z}(\mathbf{G}^F)$ and

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)| = \frac{1}{|\operatorname{Z}(\mathbf{G}^F)|}|\operatorname{Irr}_{p'}(\mathbf{B}^F)|.$$

Suppose now that p is a good prime for $\mathbf{G}$ and $H^1(F, \mathcal{Z}(\mathbf{G}))$ has prime order. Then, thanks to Remark 5.8 and Proposition 6.6, we deduce

$$|\operatorname{Irr}_{p'}(\mathbf{B}^F|\nu)| = |\operatorname{Irr}_{p'}(\mathbf{G}^F|\nu)|,$$

for every $\nu \in \operatorname{Irr}(\operatorname{Z}(\mathbf{G}^F))$. This proves Theorem 1.1.

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References

[1] C. Bonnafé. Éléments unipotents réguliers des sous-groupes de Levi. *Canad. J. Math.*, 56(2):246–276, 2004.

[2] O. Brunat. On the inductive McKay condition in the defining characteristic. *to appear in Math. Z.*

[3] O. Brunat and F. Himstedt. On equivariant bijections of characters in finite reductive groups. *In preparation.*

[4] R.W. Carter. *Finite groups of Lie type.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[5] F. Digne, G. I. Lehrer, and J. Michel. On Gel′fand-Graev characters of reductive groups with disconnected centre. *J. Reine Angew. Math.*, 491:131–147, 1997.

[6] F. Digne and J. Michel. *Representations of finite groups of Lie type*, volume 21 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1991.

[7] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups, Number 3*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Cambridge, 1991.

[8] I.M. Isaacs, G. Malle, and G. Navarro. A reduction theorem for McKay conjecture. *Invent. Math.*, 170:33–101, 2007.

[9] G. Lusztig. On the representations of reductive groups with disconnected centre. *Astérique*, 168:157–166, 1988.
[10] G. Malle. Height 0 characters of finite groups of Lie type. *Represent. Theory*, 11:192–220, 2007.

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