NONVANISHING OF KRONECKER COEFFICIENTS FOR RECTANGULAR SHAPES

PETER BÜRGISSER, MATTHIAS CHRISTANDL, AND CHRISTIAN IKENMEYER

Abstract. We prove that for any partition $(\lambda_1, \dots, \lambda_{d^2})$ of size dm there exists $k \geq 1$ such that the tensor square of the irreducible representation of the symmetric group S_{kdm} with respect to the rectangular partition $(km, \dots, km)$ contains the irreducible representation corresponding to the stretched partition $(k\lambda_1, \dots, k\lambda_{d^2})$. We also prove a related approximate version of this statement in which the stretching factor k is effectively bounded in terms of d. This investigation is motivated by questions of geometric complexity theory.

1. Introduction

Kronecker coefficients are the multiplicities occuring in tensor product decompositions of irreducible representations of the symmetric groups. These coefficients play a crucial role in geometric complexity theory of [12, 13], which is an approach to arithmetic versions of the famous P versus NP problem and related questions in computational complexity via geometric representation theory. As pointed out in [1], for implementing this approach, one needs to identify certain partitions $\lambda \vdash_{d^2} dm$ with the property that the Kronecker coefficient associated with $\lambda, \square, \square$ vanishes, where $\square := (m, \dots, m)$ stands for the rectangle partition of length d. Computer experiments show that such λ occur rarely. Our main result confirms this experimental finding. We prove that for any $\lambda \vdash_{d^2} dm$ there exists a stretching factor k such that the Kronecker coefficient of $k\lambda, k\square, k\square$ is nonzero. (Here, $k\lambda$ stands for the partition arising by multiplying all components of λ by k.) We also prove a related approximate version of this statement (Theorem 2) that suggests that the stretching factor k may be chosen not too large.

Our proof relies on a recently discovered connection between Kronecker coefficients and the spectra of composite quantum states [2, 8]. Let ρ_{AB} be the density operator of a bipartite quantum system and ρ_A, ρ_B denote the

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density operators corresponding to the systems A and B, respectively. It turns out that the set of possible triples of spectra $(\mathrm{spec}\rho_{AB}, \mathrm{spec}\rho_A, \mathrm{spec}\rho_B)$ is obtained as the closure of the set of triples $(\overline{\lambda}, \overline{\mu}, \overline{\nu})$ of normalized partitions λ, μ, ν with nonvanishing Kronecker coefficient, where we set $\overline{\lambda} := \frac{1}{|\lambda|}\lambda$. For proving the main theorem it is therefore sufficient to construct, for any prescribed spectrum $\overline{\lambda}$, a density matrix ρ_{AB} having this spectrum and such that the spectra of ρ_A and ρ_B are uniform distributions.

In [8] the set of possible triples of spectra $(\mathrm{spec}\rho_{AB}, \mathrm{spec}\rho_A, \mathrm{spec}\rho_B)$ is interpreted as the moment polytope of a complex algebraic group variety, thus linking the problem to geometric invariant theory. We do not not use this connection in our paper. Instead we argue as in [2] using the estimation theorem of [7]. The exponential decrease rate in this estimation allows us to derive the bound on the stretching factor in Theorem 2.

2. Preliminaries

2.1. Kronecker coefficients and its moment polytopes.

A *partition* λ of $n \in \mathbb{N}$ is a monotonically decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that $\lambda_i = 0$ for all i but finitely many i. The length $\ell(\lambda)$ of λ is defined as the number of its nonzero parts and its size as $|\lambda| := \sum_i \lambda_i$. One writes $\lambda \vdash_\ell n$ to express that λ is a partition of n with $\ell(\lambda) \le d$. Note that $\bar{\lambda} := \lambda/n = (\lambda_1/n, \lambda_2/n, \ldots)$ defines a probability distribution on $\mathbb{N}$.

It is well known [6] that the complex irreducible representations of the symmetric group S_n can be labeled by partitions $\lambda \vdash n$ of n. We shall denote by $[\lambda]$ the irreducible representation of S_n associated with λ. The *Kronecker coefficient* $g_{\lambda,\mu,\nu}$ associated with three partitions λ, μ, ν of n is defined as the dimension of the space of S_n-invariants in the tensor product $[\lambda] \otimes [\mu] \otimes [\nu]$. Note that $g_{\lambda,\mu,\nu}$ is invariant with respect to a permutation of the partitions. It is known that $g_{\lambda,\mu,\nu} = 0$ vanishes if $\ell(\lambda) > \ell(\mu)\ell(\nu)$. Equivalently, $g_{\lambda,\mu,\nu}$ may also be defined as the multiplicity of $[\lambda]$ in the tensor product $[\mu] \otimes [\nu]$.

The Kronecker coefficients also appear when studying representations of the general linear groups $\mathrm{GL}(d)$ over $\mathbb{C}$. We recall that irreducible $\mathrm{GL}(d)$-modules are labeled by their highest weight, a monotonically decreasing list of d integers, cf. [6]. We will only be concerned with highest weights consisting of nonnegative numbers, which are therefore of the form $\lambda \vdash_d k$ for modules of degree k. We shall denote by V_λ the irreducible $\mathrm{GL}(d)$-module with highest weight λ.

Suppose now that $\lambda \vdash_{d_1 d_2} k$. When restricting with respect to the morphism $\mathrm{GL}(d_1) \times \mathrm{GL}(d_2) \to \mathrm{GL}(d_1 d_2), (\alpha, \beta) \mapsto \alpha \otimes \beta$, then the module V_λ splits as follows:

$$V_\lambda = \bigoplus_{\mu,\nu} g_{\lambda,\mu,\nu} V_\mu \otimes V_\nu. \tag{1}$$

Even though being studied for more than fifty years, Kronecker coefficients are only understood in some special cases. For instance, giving a combinatorial interpretation of the numbers $g_{\lambda,\mu,\nu}$ is a major open problem, cf. [14, 15] for more information.

We are mainly interested in whether $g_{\lambda,\mu,\nu}$ vanishes or not. For studying this in an asymptotic way one may consider, for fixed $d = (d_1, d_2, d_3) \in \mathbb{N}^3$ with $d_1 \le d_2 \le d_3 \le d_1 d_2$, the set

$$\mathsf{Kron}(d) := \Big\{ \frac{1}{n}(\lambda_1, \lambda_2, \lambda_3) \mid \exists n\ \lambda_i \vdash_{d_i} n,\ g_{\lambda_1,\lambda_2,\lambda_3} \neq 0 \Big\}.$$

It turns out that $\mathsf{Kron}(d)$ is a rational polytope in $\mathbb{Q}^{d_1+d_2+d_3}$. This follows from general principles from geometric invariant theory, namely $\mathsf{Kron}(d)$ equals the *moment polytope* of the projective variety $\mathbb{P}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})$ with respect to the standard action of the group $\mathrm{GL}(d_1) \times \mathrm{GL}(d_2) \times \mathrm{GL}(d_3)$, cf. [11, 5, 8]. For an elementary proof that $\mathsf{Kron}(d)$ is a polytope see [4].

2.2. **Spectra of density operators.** Let $\mathcal{H}$ be a d-dimensional complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear operators mapping $\mathcal{H}$ into itself. For $\rho \in \mathcal{L}(\mathcal{H})$ we write $\rho \ge 0$ to denote that ρ is positive semidefinite. The set of *density operators* on $\mathcal{H}$ is defined as

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho \ge 0, \mathrm{tr}\rho = 1\}.$$

Density operators are the mathematical formalism to describe the states of quantum objects. The spectrum $\mathrm{spec}\rho$ of ρ is a probability distribution on $[d] := \{1, \dots, d\}$.

The state of a system composed of particles A and B is described by a density operator on a tensor product of two Hilbert spaces, $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The partial trace $\rho_A = \mathrm{tr}_B(\rho_{AB}) \in \mathcal{L}(\mathcal{H}_A)$ of ρ_{AB} obtained by tracing over B then defines the state of particle A. We recall that the *partial trace* tr_B is the linear map $\mathrm{tr}_B \colon \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ uniquely characterized by the property $\mathrm{tr}_B(\rho_A \otimes \rho_B) = \mathrm{tr}(\rho_B)\,\rho_A$ for all $\rho_A \in \mathcal{L}(\mathcal{H}_A)$ and $\rho_B \in \mathcal{L}(\mathcal{H}_B)$.

2.3. **Admissible spectra and Kronecker coefficients.** The *quantum marginal problem* asks for a description of the set of possible triples of spectra $(\mathrm{spec}\rho_{AB}, \mathrm{spec}\rho_A, \mathrm{spec}\rho_B)$ for fixed $d_A = \dim \mathcal{H}_A$ and $d_B = \dim \mathcal{H}_B$. In [2, 8, 4] it was shown that this set equals the closure of the moment polytope for Kronecker coefficients, so

$$\overline{\mathsf{Kron}(d_A, d_B, d_A d_B)} = \Big\{ (\mathrm{spec}\rho_{AB}, \mathrm{spec}\rho_A, \mathrm{spec}\rho_B) \mid \rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \Big\}.$$

We remark that this result is related to *Horn's problem* that asks for the compatibility conditions of the spectra of Hermitian operators A, B, and $A+B$ on finite dimensional Hilbert spaces. In [9] a similar characterization of these triple of spectra in terms of the Littlewood Richardson coefficients was given. The latter are the multiplicities occuring in tensor products of irreducible representations of the general linear groups. For Littlewood Richardson coefficients one can actually avoid the asymptotic description since the so called saturation conjecture is true [10].

2.4. **Estimation theorem.** We will need a consequence of the estimation theorem of [7]. The group $S_k \times \mathrm{GL}(d)$ naturally acts on the tensor power $(\mathbb{C}^d)^{\otimes k}$. Schur-Weyl duality describes the isotypical decomposition of this module as

$$(\mathbb{C}^d)^{\otimes k} = \bigoplus_{\lambda \vdash_d k} [\lambda] \otimes V_\lambda. \tag{2}$$

We note that this is an orthogonal decomposition with respect to the standard inner product on $(\mathbb{C}^d)^{\otimes k}$. Let P_λ denote the orthogonal projection of $(\mathbb{C}^d)^{\otimes k}$ onto $[\lambda] \otimes V_\lambda$. The estimation theorem [7] states that for any density operator $\rho \in \mathcal{L}(\mathbb{C}^d)$ with spectrum r we have

$$\mathrm{tr}(P_\lambda\, \rho^{\otimes k}) \leq (k+1)^{d(d-1)/2} \exp\big(-\frac{k}{2}\|\overline{\lambda} - r\|_1^2\big) \tag{3}$$

(see [2] for a simple proof). This shows that the probability distribution $\overline{\lambda} \mapsto \mathrm{tr}(P_\lambda\, \rho^{\otimes k})$ is concentrated around r with exponential decay in the distance $\|\overline{\lambda} - r\|_1$.

3. Main results

Theorem 1. **(1)** *For all probability distributions r on $[d^2]$, the triple (r, u, u) is contained in $\overline{\mathsf{Kron}(d, d, d^2)}$, where $u = (\frac{1}{d}, \dots, \frac{1}{d})$ denotes the uniform distribution on $[d]$.*

(2) *Let $\lambda \vdash dm$ be a partition into at most d^2 parts for $m, d \geq 1$ and let $\square := (m, \dots, m)$ denote the rectangular partition of dm into d parts. Then there exists a stretching factor $k \geq 1$ such that $g_{k\lambda, k\square, k\square} \neq 0$.*

This result shows that finding partitions λ with $g_{\lambda,\square,\square} = 0$, as required for the purposes of geometric complexity theory, requires a careful search.

The next result indicates that the stretching factor k may be chosen not too large.

Theorem 2. *Let $\lambda \vdash_{d^2} dm$ and $\epsilon > 0$. Then there exists a stretching factor $k = O(\frac{d^4}{\epsilon^2} \log \frac{d}{\epsilon})$ and there exist partitions $\Lambda \vdash_{d^2} kdm$ and $R_1, R_2 \vdash_d kdm$ of kdm such that $g_{k\lambda, R_1, R_2} \neq 0$ and*

$$\|\Lambda - k\lambda\|_1 \leq \epsilon|\Lambda| \quad \|R_i - k\square\|_1 \leq \epsilon|R_i| \quad \textit{for } i = 1, 2.$$

3.1. **Proof of Theorem 1.** We know that $\mathsf{Kron}(d, d, d^2)$ is a rational polytope, i.e., defined by finitely many affine linear inequalities with rational coefficients. This easily implies that a rational point in $\overline{\mathsf{Kron}(d, d, d^2)}$ actually lies in $\mathsf{Kron}(d, d, d^2)$. Hence the second part of Theorem 1 follows from the first part.

The first part of Theorem 1 follows from the spectral characterization of $\overline{\mathsf{Kron}(d, d, d^2)}$ described in §2.3 and the following result.

Proposition 1. *For any probability distribution r on $[d^2]$ there exists a density operator $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with spectrum r such that $\mathrm{tr}_A(\rho_{AB}) = \frac{1}{d}\mathrm{Id}$ and $\mathrm{tr}_B(\rho_{AB}) = \frac{1}{d}\mathrm{Id}$, where $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathbb{C}^d$.*

The proof of Proposition 1 proceeds by different lemmas. It will be convenient to use the bra and ket notation of quantum mechanics. Suppose that $\mathcal{H}_A$ and $\mathcal{H}_B$ are d-dimensional Hilbert spaces. We recall first the *Schmidt decomposition*: for any $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal bases $\{|u_i\rangle\}$ of $\mathcal{H}_A$ and $\{|v_i\rangle\}$ of $\mathcal{H}_B$ as well as nonnegative real numbers α_i, called *Schmidt coefficients*, such that $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$. Indeed, the α_i are just the singular values of $|\psi\rangle$ when we interpret it as a linear operator in $\mathcal{L}(\mathcal{H}_A^*, \mathcal{H}_B) \simeq \mathcal{H}_A \otimes \mathcal{H}_B$.

Lemma 3. *Suppose that $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has the Schmidt coefficients α_i and consider $\rho := |\psi\rangle\langle\psi| \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then $\mathrm{tr}_B(\rho) \in \mathcal{L}(\mathcal{H}_A)$, obtained by tracing over the B-spaces, has eigenvalues α_i^2.*

Proof. We have $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$ for some orthonormal bases $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. This implies

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j} \alpha_i \alpha_j |u_i\rangle\langle u_j| \otimes |v_i\rangle\langle v_j|$$

and tracing over the B-spaces yields $\mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_i \alpha_i^2 |u_i\rangle\langle u_i|$. $\square$

Let $|0\rangle, \ldots, |d-1\rangle$ denote the standard orthonormal basis of $\mathbb{C}^d$. We consider the discrete Weyl operators $X, Z \in \mathcal{L}(\mathbb{C}^d)$ from [3] defined by

$$X|i\rangle = |i+1\rangle, \quad Z|i\rangle = \omega^i\, |i\rangle,$$

where ω denotes a primitive dth root of unity and the addition is modulo d. We note that X and Z are unitary matrices and $X^{-1}ZX = \omega Z$.

We consider now two copies $\mathcal{H}_A$ and $\mathcal{H}_B$ of $\mathbb{C}^d$ and define the "maximal entangled state" $|\psi_{00}\rangle := \frac{1}{\sqrt{d}} \sum_\ell |\ell\rangle|\ell\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$. By definition, $|\psi_{00}\rangle$ has the Schmidt coefficients $\frac{1}{\sqrt{d}}$. Hence the vectors

$$|\psi_{ij}\rangle := (\mathrm{id} \otimes X^i Z^j)|\psi_{00}\rangle,$$

obtained from $|\psi_{00}\rangle$ by applying a tensor product of unitary matrices, have the Schmidt coefficients $\frac{1}{\sqrt{d}}$ as well.

Lemma 4. *The vectors $|\psi_{ij}\rangle$, for $0 \le i, j < d$, form an orthonormal bases of $\mathcal{H}_A \otimes \mathcal{H}_B$.*

Proof. We have, for some dth root of unity θ,

$$\begin{aligned}
\langle\psi_{ij}|\psi_{k\ell}\rangle &= \langle\psi_{00}|(\mathrm{id} \otimes Z^{-j}X^{-i})(\mathrm{id} \otimes X^k Z^\ell)\ |\psi_{00}\rangle \\
&= \theta\, \langle\psi_{00}|\mathrm{id} \otimes X^{k-i}Z^{\ell-j}|\psi_{00}\rangle \\
&= \theta \sum_{m,m'} \langle mm|\mathrm{id} \otimes X^{k-i}Z^{\ell-j}|m'm'\rangle \\
&= \theta \sum_m \langle m|X^{k-i}Z^{\ell-j}|m\rangle = \theta\, \mathrm{tr}\big(X^{k-i}Z^{\ell-j}\big).
\end{aligned}$$

It is easy to check that $\theta\, \mathrm{tr}\big(X^{k-i}Z^{\ell-j}\big) = 0$ if $\ell \ne j$ or $k \ne i$. $\square$

Proof of Proposition 1. Let r_{ij} be the given probability distribution. According to Lemma 4, the density operator $\rho_{AB} := \sum_{ij} r_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|$ has the eigenvalues r_{ij}. Lemma 3 tells us that $\mathrm{tr}_B(|\psi_{ij}\rangle\langle\psi_{ij}|)$ has the eigenvalues $1/d$, hence $\mathrm{tr}_B(|\psi_{ij}\rangle\langle\psi_{ij}|) = \frac{1}{d}\,\mathrm{Id}$. It follows that $\mathrm{tr}_B(\rho_{AB}) = \frac{1}{d}\,\mathrm{Id}$. Analogously, we get $\mathrm{tr}_A(\rho_{AB}) = \frac{1}{d}\,\mathrm{Id}$. □

3.2. **Proof of Theorem 2.** The proof is essentially the one of Theorem 2 in [2]. Suppose that $\lambda \vdash_{d^2} dm$. By Proposition 1 there is a density operator ρ_{AB} having the spectrum $r := \overline{\lambda}$. Let P_X denote the orthogonal projection of $(\mathcal{H}_A)^{\otimes k}$ onto the sum of its isotypical components $[\mu] \otimes V_\mu$ satisfying $\|\overline{\mu} - u\|_1 \le \epsilon$, where u denotes the uniform distribution on $[d]$. Then $P_{\overline{X}}$ equals the orthogonal projection of $(\mathcal{H}_A)^{\otimes k}$ onto the sum of its isotypical components $[\mu] \otimes V_\mu$ satisfying $\|\overline{\mu} - u\|_1 > \epsilon$. The estimation theorem (3) implies that

$$\mathrm{tr}(P_{\overline{X}}\,(\rho_A)^{\otimes k}) \le (k+1)^d (k+1)^{d(d-1)/2}\, e^{-\frac{k}{2}\epsilon^2} \le (k+1)^{d(d+1)/2}\, e^{-\frac{k}{2}\epsilon^2},$$

since there at most $(k+1)^d$ partitions of k of length at most d.

Let P_Y denote the orthogonal projection of $(\mathcal{H}_B)^{\otimes k}$ onto the sum of its isotypical components $[\nu] \otimes V_\nu$ satisfying $\|\overline{\nu} - u\|_1 \le \epsilon$, and let P_Z denote the orthogonal projection of $(\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes k}$ onto the sum of its isotypical components $[\Lambda] \otimes V_\Lambda$ satisfying $\|\overline{\Lambda} - r\|_1 \le \epsilon$. Then we have, similarly as for P_X,

$$\begin{aligned} \mathrm{tr}(P_{\overline{Y}}\,(\rho_B)^{\otimes k}) &\le (k+1)^{d(d+1)/2}\, e^{-\frac{k}{2}\epsilon^2}, \\ \mathrm{tr}(P_{\overline{Z}}\,(\rho_{AB})^{\otimes k}) &\le (k+1)^{d^2(d^2+1)/2}\, e^{-\frac{k}{2}\epsilon^2}. \end{aligned}$$

By choosing $k = O(\frac{d^4}{\epsilon^2}\log\frac{d}{\epsilon})$ we can achieve that

$$\mathrm{tr}(P_{\overline{X}}\,(\rho_A)^{\otimes k}) < \frac{1}{3}, \quad \mathrm{tr}(P_{\overline{Y}}\,(\rho_B)^{\otimes k}) < \frac{1}{3}, \quad \mathrm{tr}(P_{\overline{Z}}\,(\rho_{AB})^{\otimes k}) < \frac{1}{3}.$$

We put $\sigma := (\rho_{AB})^{\otimes k}$ in order to simplify notation and claim that

$$\mathrm{tr}((P_X \otimes P_Y)\sigma P_Z) > 0. \tag{4}$$

In order to see this, we decompose $\mathrm{id} = P_X \otimes P_Y + P_{\overline{X}} \otimes \mathrm{id} + P_X \otimes P_{\overline{Y}}$. From the definition of the partial trace we have

$$\mathrm{tr}\big((P_{\overline{X}} \otimes \mathrm{id})\sigma\big) = \mathrm{tr}\big(P_{\overline{X}}(\rho_A)^{\otimes k}\big) < \frac{1}{3}.$$

Similarly,

$$\mathrm{tr}\big((P_X \otimes P_{\overline{Y}})\sigma\big) \le \mathrm{tr}\big((\mathrm{id} \otimes P_{\overline{Y}})\sigma\big) = \mathrm{tr}\big(P_{\overline{Y}}(\rho_B)^{\otimes k}\big) < \frac{1}{3}.$$

Hence $\mathrm{tr}\big((P_X \otimes P_Y)\sigma\big) > \frac{1}{3}$. Using $\mathrm{tr}\big((P_X \otimes P_Y)\sigma P_{\overline{Z}}\big) \le \mathrm{tr}(\sigma P_{\overline{Z}}) < \frac{1}{3}$, we get

$$\mathrm{tr}\big((P_X \otimes P_Y)\sigma P_Z\big) = \mathrm{tr}\big((P_X \otimes P_Y)\sigma\big) - \mathrm{tr}\big((P_X \otimes P_Y)\sigma P_{\overline{Z}}\big)\frac{1}{3} - \frac{1}{3} = 0,$$

which proves Claim (4).

Claim (4) implies that there exist partitions μ, ν, Λ with normalizations ϵ-close to u, u, r, respectively, such that $(P_\mu \otimes P_\nu)P_\Lambda \neq 0$. Recalling the isotypical decomposition (2), we infer that

$$([\Lambda] \otimes V_\Lambda) \cap ([\mu] \otimes V_\mu) \otimes ([\nu] \otimes V_\nu) \neq \emptyset.$$

Statement (1) implies that $g_{\mu,\nu,\Lambda} \neq 0$ and hence the assertion follows for $R_1 = \mu, R_2 = \nu$. □

References

[1] Peter Bürgisser, J.M. Landsberg, Laurent Manivel, and Jerzy Weyman. An overview of mathematial issues arising in the geometric complexity theory approach to VP≠VNP. arXiv:0907.2850v1.

[2] Matthias Christandl and Graeme Mitchison. The spectra of density operators and the Kronecker coefficients of the symmetric group. *Comm. Math. Phys.*, 261(3):789–797, February 2006.

[3] Matthias Christandl and Andreas Winter. Uncertainty, monogamy, and locking of quantum correlations. *IEEE Trans.Inf. Theory*, 51:3159, 2005.

[4] Matthias Christiandl, Aram Harrow, and Graeme Mitchison. On nonzero Kronecker coefficients and what they tell us about spectra. *Comm. Math. Phys.*, 270(3):575–585, 2007.

[5] Matthias Franz. Moment polytopes of projective G-varieties and tensor products of symmetric group representations. *J. Lie Theory*, 12(2):539–549, 2002.

[6] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[7] Michael Keyl and Reinhard Werner. Estimating the spectrum of a density operator. *Phys. Rev. A (3)*, 64(5):052311, 5, 2001.

[8] Alexander Klyachko. Quantum marginal problem and representations of the symmetric group. 2004. quant-ph/0409113.

[9] Alexander A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.

[10] Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[11] Laurent Manivel. Applications de Gauss et pléthysme. *Ann. Inst. Fourier (Grenoble)*, 47(3):715–773, 1997.

[12] Ketan D. Mulmuley and Milind Sohoni. Geometric complexity theory. I. An approach to the P vs. NP and related problems. *SIAM J. Comput.*, 31(2):496–526 (electronic), 2001.

[13] Ketan D. Mulmuley and Milind Sohoni. Geometric complexity theory. II. Towards explicit obstructions for embeddings among class varieties. *SIAM J. Comput.*, 38(3):1175–1206, 2008.

[14] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[15] Richard P. Stanley. Positivity problems and conjectures in algebraic combinatorics. In *Mathematics: frontiers and perspectives*, pages 295–319. Amer. Math. Soc., Providence, RI, 2000.

Institute of Mathematics, University of Paderborn, D-33098 Paderborn, Germany
E-mail address: pbuerg@upb.de

Faculty of Physics, Ludwig-Maximilians-Universität München, Theresienstr. 37, D-80333 Munich, Germany
E-mail address: christandl@lmu.de

Institute of Mathematics, University of Paderborn, D-33098 Paderborn, Germany
E-mail address: ciken@math.upb.de