WONDERFUL VARIETIES: A GEOMETRICAL REALIZATION

S. CUPIT-FOUTOU

ABSTRACT. We give a geometrical realization of wonderful varieties by means of a suitable class of invariant Hilbert schemes. Consequently, we prove Luna's conjecture asserting that wonderful varieties can be classified by some triples of combinatorial invariants: the spherical systems.

CONTENTS

INTRODUCTION

Wonderful varieties are complex algebraic varieties which generalize De Concini-Procesi compactifications of symmetric spaces studied in [DP]: they are equipped with an action of a connected reductive algebraic group G, they are smooth, toroidal and spherical (i.e. they contain a dense orbit for a Borel subgroup B of G). The (unique) closed orbit, the B-weights of the function field of a wonderful variety as well as its B-stable but not G-stable prime divisors are invariants of special interest. They are/yield combinatorial invariants in the sense that they can be expressed in terms of the root system of the acting group

G. After Wasserman completed the classification of rank 2 wonderful varieties ([W]), Luna highlighted in [Lu3] some properties enjoyed by such triples and took them as axioms to set up the definition of *spherical systems*. In case the group G is of type A, Luna proved in loc. cit. that there corresponds a unique wonderful variety to any spherical system. Luna's conjecture asserts that this holds in general. Thanks to Losev's work ([Lo]), the uniqueness part of this problem is known to be true. The existence part remained an open problem; its has been proved only in a few additional cases ([BP, Bra, BC2]). The approach followed there is Lie theoretical: one provides case by case a subgroup H of G such that G/H admits a wonderful compactification.

Wonderful varieties play an important role in invariant deformation theory ([J, BC1, Bri3]). By means of the so-called invariant Hilbert schemes introduced by Alexeev and Brion in [AB], we were able to construct geometrically in [C] some peculiar wonderful varieties. This allowed us to answer positively Luna's conjecture in the setting of the spherical systems which have as third combinatorial datum the emptyset. We generalize here this approach to a wider class of invariant Hilbert schemes in order to prove Luna's conjecture in full generality.

Many geometrical properties of wonderful varieties can be read off spherical systems and vice versa ([Lu3]). This dictionary allows many reductions to prove Luna's conjecture. In particular, it suffices to consider the so-called spherically closed spherical systems.

After having recalled, in the first section, Luna's definition of such spherical systems, we attach to any such object (for a simply connected reductive algebraic group G) a set of characters (ω_D, χ_D) indexed by a finite set $\mathcal{D}$. The ω_D's are dominant weights of G defined after [F] and the χ_D's are characters of some well-determined diagonalizable group C. The characters ω_D, (resp. χ_D) encapsulate the first, (resp. the third), datum of the spherical system under consideration.

We end up the first section with some brief recalls on wonderful varieties and their associated spherical systems. Further, we shall give the geometrical interpretation of the characters (ω_D, χ_D).

In the second section, given a spherically closed spherical system $\mathcal{S}$, we consider the $G \times C$-module

$$V = \oplus_{\mathcal{D}} V\left((\omega_D, \chi_D)\right)$$

where $V((\omega_D, \chi_D))$ stands for the irreducible $G \times C$-module associated to the weight (ω_D, χ_D). We thus study the invariant Hilbert scheme $\mathrm{Hilb}(\mathcal{S})$ which parameterizes the non-degenerate subvarieties of V whose coordinate ring is isomorphic as a $G \times C$-module to $\oplus_\lambda V(\lambda)$, λ being in the monoid spanned by the dominant weights (λ_D, χ_D).

This invariant Hilbert scheme contains in particular as closed point the $G \times C$-orbitclosure X_0 within V of $\sum_{\mathcal{D}} v_{(\omega_D,\chi_D)}$; $v_{(\omega_D,\chi_D)}$ is a highest weight vector of $V((\omega_D, \chi_D))$.

One of the novelties here compared to [C] is that the algebraic group on which the invariant Hilbert scheme depends is no longer the given group G.

We thus describe the tangent space at X_0 of $\mathrm{Hilb}(\mathcal{S})$; it has a structure of T_{ad}-module ([AB]) where T_{ad} denotes the adjoint torus of the given group G. We shall show that the first datum of the given spherical system $\mathcal{S}$ (the so-called spherical roots of $\mathcal{S}$) is encoded in this tangent space. More specifically, generalizing the arguments developed in [BC1], we obtain

Theorem 1. *(Theorem 12) The tangent space at X_0 of* $\mathrm{Hilb}(\mathcal{S})$ *is a multiplicity free T_{ad}-module; its T_{ad}-weights are the spherical roots of $\mathcal{S}$.*

Afterwards, we show that the corresponding obstruction space is trivial (Theorem 23). As a consequence, we obtain the following statement in the third section.

Theorem 2. *(Theorem 24) The invariant Hilbert scheme* $\mathrm{Hilb}(\mathcal{S})$ *is a toric T_{ad}-variety; it is the affine space $\mathbb{A}^r$ with r being the number of spherical roots of Σ.*

In case of spherical systems with emptyset as third datum, the G-module V is defined up to pairwise distinct dominant weights satisfying an additional property no longer shared when dealing with an arbitrary spherical system. This setting was considered in [J, BC2]. The first assertion of the two previous theorems were obtained therein by means of the already known classification of wonderful varieties of rank 1 (resp. of strict wonderful varieties) ([A], resp. [BC1]); the dimension of the invariant Hilbert scheme is upper bounded by the number of spherical roots.

Let X_1 be a closed point of $\mathrm{Hilb}(\mathcal{S})$ whose T_{ad}-orbit is dense in $\mathrm{Hilb}(\mathcal{S})$. Regarding X_1 as a subvariety in V, we consider its coordinate ring $k[X_1]$ and we set

$$R(\mathcal{S}) = \oplus_{\lambda \in \Gamma(\mathcal{S})} k[X_1]_\lambda e^\lambda \otimes k[\mathbb{A}^\Sigma]$$

where $\Gamma(\mathcal{S})$ stands for the monoid spanned by the weights (ω_D, χ_D) and $k[X_1]_\lambda$ for the isotypical component associated to the highest weight λ.

The algebraic torus $\mathbb{G}_m^{\mathcal{D}}$ whose charactergroup is spanned by the (ω_D, χ_D)'s acts naturally on $R(\mathcal{S})$.

Theorem 3. *(Theorem 25) Let $\tilde{X}$ be the affine variety whose spectrum is the ring $R(\mathcal{S})$ and X^{reg} consist of the points of $\tilde{X}$ whose $G\times C$-orbit is of maximal dimension. Then the variety*

$$X(\mathcal{S}) = \tilde{X}^{reg}/\mathbb{G}_m^{\mathcal{D}}$$

is a wonderful G-variety whose spherical system is the given $\mathcal{S}$.

In combination, with Luna's dictionary set up, we can conclude by proving that Luna's conjecture is true (Corollary 26).

Acknowledgments. I am grateful to Michel Brion for his advice and suggestions on and around the subject. I announced the results contained in this paper during the workshop "Enveloping Algebras and Geometric Representation Theory", held in Oberwolfach in March 2009; I thank the organizers of this meeting for having given me this opportunity.

1. Notation and Basic material

The ground field k is the field of complex numbers. Throughout this paper, G denotes a simply connected reductive algebraic group, B a Borel subgroup and $T\subset B$ a maximal torus of G. Considering the relative set of simple roots S of G, we define as usual the support $\mathrm{Supp}\beta$ of any integral linear combination $\beta=\sum n_\alpha\alpha$ of simple roots α to be the set of simple roots α such that $n_\alpha\neq 0$. We label the simple roots as in Bourbaki ([Bo]).

1.1. **Spherical systems.**

Definition 4 ([W, Lu3])**.** A *spherical root* is one of the following characters of T: α_1 and $2\alpha_1$ of type A_1; $\alpha_1+\alpha_1'$ of type $\mathsf{A}_1\times\mathsf{A}_1$; $\alpha_1+2\alpha_2+\alpha_3$ of type A_3; $\alpha_1+\ldots+\alpha_n$ of type A_n, $n\geq 2$; $\alpha_1+2\alpha_2+3\alpha_3$ of type B_3; $\alpha_1+\ldots+\alpha_n$ of type B_n, $n\geq 2$; $\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n$ of type C_n; $2\alpha_1+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ of type D_n, $n\geq 4$; $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ of type F_4; $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$ and $4\alpha_1+2\alpha_2$ of type G_2.

The set of spherical roots of G is the set of spherical roots whose support is a subset of the set of simple roots S of G. We denote it by $\Sigma(G)$.

Definition 5 ([BL], 1.1.6)**.** Let S^p be a subset of S and σ be a spherical root of G. The couple (S^p,σ) is said to be *compatible* if

$$S^{pp}(\sigma)\subset S^p\subset S^p(\sigma)$$

where $S^p(\sigma)$ is the set of simple roots orthogonal to σ and $S^{pp}(\sigma)$ is one of the following sets

- $S^p(\sigma)\cap\mathrm{Supp}\sigma\setminus\{\alpha_r\}$ if $\sigma=\alpha_1+\ldots+\alpha_r$ with $\mathrm{Supp}\sigma$ of type B_r,

- $S^p(\sigma) \cap \mathrm{Supp}\sigma \setminus \{\alpha_1\}$ if $\mathrm{Supp}\sigma$ is of type C_r,
- $S^p(\sigma) \cap \mathrm{Supp}\sigma$ otherwise.

Definition 6. Let S^p be a set of some simple roots, Σ a set of spherical roots of G and A a multiset of functionals on the lattice spanned by Σ. The triple $(S^p, \Sigma, \mathsf{A})$ is called a *spherical system* if

(A1) $\delta(\gamma) \leq 1$ for any $\delta \in \mathsf{A}$ and any $\gamma \in \Sigma$. Further if $\delta(\gamma) = 1$ then $\gamma \in \Sigma \cap S$.

(A2) $\delta_\alpha^+(\gamma) + \delta_\alpha^-(\gamma) = (\alpha^\vee, \gamma)$ for any $\delta \in \mathsf{A}$ and any $\gamma \in \Sigma$.

(A3) A is the union of all $\mathsf{A}(\alpha)$'s with $\alpha \in \Sigma \cap S$.

(Σ1) $(\alpha^\vee, \sigma) \in 2\mathbb{Z}_{\leq 0}$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$ and all $\alpha \in S$ such that $2\alpha \in \Sigma$.

(Σ2) $(\alpha^\vee, \sigma) = (\beta^\vee, \sigma)$ for all $\sigma \in \Sigma$ and all $\alpha, \beta \in S$ which are mutually orthogonal and such that $\alpha + \beta \in \Sigma$.

(S) The couple $(\{\sigma\}, S^p)$ is compatible for any $\sigma \in \Sigma$.

1.2. **Colors of a spherical system.** The purpose of this subsection is to attach to any spherical system $\mathcal{S} = (S^p, \Sigma, \mathsf{A})$ of G, a finite set of pairwise distinct characters (ω_D, χ_D).

Let ω_α denote the fundamental weight associated to the simple root α.

Let us start by recalling Foschi's definition ([F]) of the dominant weights ω_D. Set

$$\begin{array}{rl} \sum_\beta \omega_\beta & \beta \in S \text{ and } \delta_\alpha^+(\beta) = 1 \text{ for a fixed } \alpha \text{ in } S \cap \Sigma \\ \sum_\beta \omega_\beta & \beta \in S \text{ and } \delta_\alpha^-(\beta) = 1 \text{ for a fixed } \alpha \text{ in } S \cap \Sigma \\ 2\omega_\alpha & \text{if } \alpha \in S^{2a} \\ \omega_\alpha + \omega_\beta & \text{if } \alpha + \beta \in \Sigma \text{ with } \alpha \text{ and } \beta \text{ orthogonal simple roots} \\ \omega_\alpha & \text{for the remaining } \alpha \text{ in } S \setminus S^p. \end{array}$$

Note that these weights may not be pairwise distinct: among the first two defined dominant weights, some may occur twice– but no more. These weights define thus a multiset and in the remainder we denote by $\mathcal{D}$ its indexing set. Set

$$S^a = 2S \cap \Sigma \quad \text{ and } S^b = S \setminus (S^p \cup (S \cap \Sigma) \cup S^a)\,.$$

If α and β are orthogonal simple roots whose sum is in Σ, write $\alpha \sim \beta$. Then, regarding A as an abstract set, $\mathcal{D}$ can be written as the following disjoint union

$$\mathcal{D} = \mathsf{A} \cup S^a \cup (S^b/\sim).$$

We now introduce some additional characters χ_D indexed by $\mathcal{D}$ (*see* also Lemma 3.2.1 in [Bri3]).

Given any γ in Σ and any $D \in \mathcal{D}$, define (after [Lu2])

$$a_D^\gamma = \begin{cases} \delta_\alpha^+(\gamma) & \text{if } D = \delta_\alpha^+ \\ \delta_\alpha^-(\gamma) & \text{if } D = \delta_\alpha^- \\ \frac{1}{2}(\gamma, \alpha^\vee) & \text{if } D = D_\alpha \in S^a \\ (\gamma, \alpha^\vee) & \text{if } D = D_\alpha \in S^b \end{cases}.$$

Let $\mathbb{G}_m^r$ be the torus whose charactergroup is spanned by the set Σ of spherical roots and $\mathbb{G}_m^{\mathcal{D}}$ the torus $\mathrm{GL}(V)^G$, V being the G-module $\oplus_{\mathcal{D}} V(\omega_D)$. Consider the epimorhism

$$\mathbb{G}_m^{\mathcal{D}} \to \mathbb{G}_m^r : (t_D)_{D\in\mathcal{D}} \mapsto \left(\prod_{D\in\mathcal{D}} t_D^{a_D^\gamma} \right)_{\gamma\in\Sigma}.$$

Let C be its kernel C; it is a diagonalizable group.

The character χ_D is defined as the image of the D-component character

$$\varepsilon_D : (t_D)_{\mathcal{D}} \mapsto t_D$$

through the dual map of $C \to \mathbb{G}_m^{\mathcal{D}}$.

Lemma 7. *The (ω_D, χ_D)'s are linearly independent and form in turn a basis of the charactergroup of $\mathbb{G}_m^{\mathcal{D}}$. Further they satisfy the following equalities*

$$(\gamma, 0) = \sum_{\mathcal{D}} a_D^\gamma (\omega_D, \chi_D) \quad \textit{for all } \gamma \in \Sigma.$$

The set of characters (ω_D, χ_D) will be referred in the remainder as *the set of colors of* $\mathcal{S}$.

1.3. **Wonderful varieties.**

Definition 8. A smooth complete algebraic variety equipped with an action of G is said to be *wonderful*

(i) if it contains a dense G-orbit whose complementary is a finite union of smooth prime divisors D_i $(i = 1, ..., r)$ with normal crossings;

(ii) its G-orbitclosures are given by the $\cap_{i\in I} D_i$'s, I being a subset of the indexing set $\{1, ..., r\}$.

Wonderful varieties X are spherical (*see* [Lu1]) and as already recalled in the introduction one can attach to any X a spherical system $(S_X^p, \Sigma_X, \mathsf{A}_X)$ as follows; *see* Section 5.1 in [Lu3] for details. The Picard group of X has as basis the B-stable but not G-stable divisors D of X. Let H be the generic stabilizer of X such that BH is open in G and $\pi : G \to G/H$ be the quotient morphism. Then, $\pi^{-1}(D)$ can be represented by an equation which is a $B \times H$-eigenvector. The corresponding $B \times H$-character of $\pi^{-1}(D)$ is given by one of the colors

(ω_D, χ_D) associated to $(S^p_X, \Sigma_X, \mathsf{A}_X)$; *see* for instance Lemma 3.2.1 and its proof in [Bri3].

2. Invariant Hilbert scheme attached to a spherical system

Fix a spherical system $\mathcal{S} = (S^p, \Sigma, \mathsf{A})$ of G.

2.1. **Definition of** $\mathrm{Hilb}(\mathcal{S})$**.** Set

$$\mathbf{G} = G \times C \quad \text{and} \quad V(\mathcal{S}) = \oplus_{\mathcal{D}} V(\lambda_D)$$

where $V(\lambda_D)$ is the irreducible $G \times C$-module associated to the color $\lambda_D = (\omega_D, \chi_D)$ of $\mathcal{S}$.

Denote Γ the monoid spanned by the characters λ_D.

Consider the functor $\mathcal{H}ilb(\mathcal{S})$ introduced (in a more general setting) by Alexeev and Brion which assigns to any scheme S (endowed with the trivial action of $\mathbf{G}$) the following set of families $\pi : \mathcal{X} \to S$ such that

$$\pi_* \mathcal{O}_{\mathcal{X}} \cong \oplus_{\lambda \in \Gamma} \mathcal{F}_\lambda \otimes V(\lambda)^* \quad \text{as } \mathcal{O}_S - \mathbf{G}\text{-modules}$$

where $\mathcal{F}_\lambda$ denotes an invertible sheaf.

This functor is representable by a quasiprojective scheme, the invariant Hilbert scheme $\mathrm{Hilb}^{\mathbf{G}}_\Gamma(V)$ (Theorem 1.7 in [AB]).

In particular, $\mathrm{Hilb}(\mathcal{S})$ contains as closed point the horospherical $\mathbf{G}$-variety $X_0(\mathcal{S})$ given by the $\mathbf{G}$-orbitclosure within V of

$$v_{\mathcal{D}} = \sum_{D \in \mathcal{D}} v_{\lambda_D}.$$

A G-subvariety of V is said to be *non-degenerate* if its projections onto the irreducible modules $V(\lambda_D)$ are all non-trivial. The non-degenerate subvarieties of V whose coordinate ring is isomorphic as a G-module to that of $X_0(\mathcal{S})$ are parameterized by an open scheme, $\mathrm{Hilb}(\mathcal{S})$, of $\mathrm{Hilb}^{\mathbf{G}}_\Gamma(V)$ (Corollary 1.17 in [AB]).

By abuse of terminology, we shall refer to $\mathrm{Hilb}(\mathcal{S})$ as the *invariant Hilbert scheme associated to the spherical system* $\mathcal{S}$.

There exists an action of the adjoint torus T_{ad} of $\mathbf{G}$ on the invariant Hilbert scheme $\mathrm{Hilb}(\mathcal{S})$ and $X_0(\mathcal{S})$ is the unique T_{ad}-fixed point of $\mathrm{Hilb}(\mathcal{S})$; *see* Section 2 in [AB].

Note that whenever the dominant weights ω_D are pairwise distinct the invariant Hilbert scheme associated to the group G itself and to V as a G-module maps to $\mathrm{Hilb}(\mathcal{S})$. As stated in the introduction, this case falls in the setting of [J, BC1]. Indeed, the invariant Hilbert schemes studied therein are those associated to a finite dimensional G-module $V = \oplus_i V(\lambda_i)$ such that the monoid Γ spanned by the dominant weights

λ_i is saturated. Namely, the dominant weights in the integral span of Γ are themselves elements of Γ. This saturation property is fulfilled by the colors of a spherical system only if the latter has the emptyset as a third datum.

2.2. **Tangent space of** Hilb($\mathcal{S}$)**.** Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$ and $\mathbf{G}_{v_\mathcal{D}}$ the stabilizer of $v_\mathcal{D}$ in $\mathbf{G}$. By Proposition 1.15 in [AB], the tangent space at X_0 of Hilb($\mathcal{S}$) is included in the $T_{\rm ad}$-module $(V/\mathfrak{g}.v_{\underline{\lambda}})^{\mathbf{G}_{v_\mathcal{D}}}$. Further, both of these $T_{\rm ad}$-modules are isomorphic whenever the codimension of the boundary $X_0 \setminus \mathbf{G}.v_\mathcal{D}$ is greater or equal to 2. This condition occurs exactly when the third datum of the spherical system $\mathcal{S}$ is the emptyset.

Let us recall the action of the adjoint torus $T_{\rm ad}$ on $(V/\mathfrak{g}.v_{\underline{\lambda}})^{\mathbf{G}_{v_\mathcal{D}}}$ defined in [AB]. For any element t of $T_{\rm ad}$, set

$$t.v = (\lambda_D - \mu)(t)v \quad \text{for any weightvector } v \text{ in } V(\lambda_D)_\mu.$$

Let denote $\Sigma(\mathcal{D})$ the set of $T_{\rm ad}$-weights of $(V/\mathfrak{g}.v_{\underline{\lambda}})^{\mathbf{G}_{v_\mathcal{D}}}$.

Note that the elements of $\Sigma(\mathcal{D})$ are in the integral span of the weights λ_D; they are in particular characters of $T \times C$ of shape $(\gamma, 0)$ where γ is in the integral span of the dominant weights ω_D. However, we shall refer to them below just as characters γ of T.

In the following, two simple roots which are not orthogonal are called adjacent.

Definition 9. A spherical system is called *spherically closed* if none of its spherical roots $\gamma \in \Sigma \setminus S$ is such that $(S^p, 2\gamma)$ is compatible. $(S^p, \{\})$ is compatible.

Proposition 10. *Let $\mathcal{S}$ be a spherically closed spherical system. Then*

$$\Sigma(\mathcal{D}) = \Sigma \cup \{\alpha + \alpha' : \alpha, \alpha' \textit{ adjacent simple roots in } \Sigma\}.$$

Proof. Take γ a character of T whose support does not contain any simple root in Σ. If γ belongs to Σ then it is a weight in $\Sigma(\mathcal{D})$; further the triple $(S^p, \Sigma(\mathcal{D}), \emptyset)$ is a spherical system of G (*see* [BC1]). Note that the dominant weights ω_D of this spherical systems coincide with those of $\mathcal{S}$ - which is no longer true for the respective characters χ_D's. Let γ be in $\Sigma(\mathcal{D})$. If γ does not belong to Σ then it is not in the integral span of the colors (ω_D, χ_D) of $\mathcal{S}$: a contradiction.

Observe now that any $\alpha + \alpha'$ as stated in the proposition is clearly in $\Sigma(\mathcal{D})$. Moreover, any spherical root whose support contains a spherical root α is either α itself or of shape $\alpha + \alpha'$ where α is orthogonal to $\alpha + \alpha'$ and α' is a simple root adjacent to α but not in Σ. Both α and $\alpha + \alpha'$ are obviously in $\Sigma(\mathcal{D})$ when they are spherical roots of $\mathcal{S}$.

Consequently, to obtain the proposition, we are left to prove the inclusion of $\Sigma(\mathcal{D})$ in the right-hand side set in case of weight whose support intersects Σ. This is achieved in Proposition 13 below. □

Remark 11. Considering the case of pairwise distinct dominant weights ω_D, this proposition does not hold in general while forgetting the C-module structure. Firstly, if γ is a loose spherical root of $\mathcal{S}$ then 2γ belongs to $\Sigma(\mathcal{D})$ but γ itself does not. Secondly, $\Sigma(\mathcal{D})$ is contained in Σ only for peculiar spherical systems (among those containing no loose spherical root); *see* [C] and also [BC1].

Essentially from the characterization of the **G**-orbit closures of codimension 1 in X_0 along with the preceding proposition, we obtain.

Corollary 12. *The tangent space at X_0 of the invariant Hilbert scheme* $\mathrm{Hilb}(\mathcal{S})$ *is a multiplicity free T_{ad}-module; its T_{ad}-weights are the spherical roots of $\mathcal{S}$.*

Proposition 13. *Suppose $\mathcal{S}$ is a spherically closed spherical system of G. Let γ be a T_{ad}-weight in $\Sigma(\mathcal{S})$ whose support contains a simple root α in Σ. Then γ is a root and it is equal to α or $\alpha+\alpha'$ with α' a simple root. In the latter case, either γ or α' belongs to Σ.*

Proof. Note that $(\alpha+\alpha', 0)$ is in the integral span of the λ_D's if and only if either α and α' are both in Σ or $\alpha+\alpha'$ is in Σ. The latter assertion of the proposition follows.

In order to prove the first assertion of the proposition, we shall proceed as follows. Note first that α is clearly in $\Sigma(\mathcal{D})$. When γ is not equal to α, we shall prove, in the following lemmas, that $\gamma-\alpha$, $\gamma-\alpha'$ (α' being one of the simple roots adjacent to α) and γ are roots. □

Lemma 14. *[Proposition 3.4 in* [BC1]*] Let γ be a weight in $\Sigma(\mathcal{D})$. Suppose there exists a simple root δ in the support of γ such that $\gamma-\delta$ is not a root. Then (γ,δ) is positive. Moreover, if γ is orthogonal to δ then so are the dominant weights ω_D.*

Proof. The proof is conducted in loc. cit. in case the ω_D's are pairwise distinct. Nevertheless, it is still valid thanks mainly to the following property of spherical systems: If α and β are arbitrary simple roots and spherical roots of a same spherical system then there exists at most one dominant weight among the ω_D's which is neither orthogonal to α nor to β. This property is due to Axiom (A2) of spherical systems. □

Lemma 15. *Let α be a simple root in Σ and α' a simple root adjacent to α. One of the dominant weights ω_D is orthogonal to α but not to α'.*

Proof. Being a spherical root of $\mathcal{S}$, the simple root α lies in the integral span of the λ_D's hence at least one of the ω_D's is not orthogonal to α'. The lemma follows from Axiom (A2) of spherical systems. □

Let us fix some further notation and convention for the remainder. In the following lemmas, γ and α are distinct weights in $\Sigma(\mathcal{D})$ with α lying in the support of γ. A weightvector of T_{ad}-weight γ is represented by a vector v_γ in the weightspace $\oplus V_{\lambda_D-\gamma}$. The root vector associated to a root β of G is denoted by X_β. As element of the Lie algebra of G, they act on the G-module V; the corresponding action is simply denoted by $X_\beta v$ for some v in V.

Lemma 16. *The character $\gamma-\alpha$ is a root.*

Proof. Let us proceed by contradiction: suppose $\gamma-\alpha$ is not a root. Since $X_\alpha v_\gamma$ has to lie in $\mathfrak{g}.v_{\underline{\lambda}}$, the vector $X_\alpha v_\gamma$ is trivial in V. Moreover, by Lemma 14, (γ,α) is strictly positive. Consequently, the representative v_γ thus lies in $V(\lambda_\alpha^+)\oplus V(\lambda_\alpha^-)$ where λ_α^+ and λ_α^- are the dominant weights among the ω_D's which are not orthogonal to α. Since the vector v_γ can not be dominant, there exists a simple root δ in the support of γ such that the vector $X_\delta v$ is not trivial in V. It follows that the weight $\gamma-\delta$ is a root. Thanks to Lemma 15, γ is thus equal to $\alpha+\delta$ hence γ has to be a root: a contradiction. □

Lemma 17. *If the weight γ is not a root then the vector $X_\alpha v$ is not trivial in V.*

Proof. Thanks to Lemma 16, $(\gamma-\alpha,\alpha^\vee)$ is positive hence (γ,α) is strictly positive. By the same arguments as those used in the proof of Lemma 16, we get a contradiction whenever $X_\alpha v$ is trivial in V. □

Lemma 18. *The supports of α and $\gamma-\alpha$ are not orthogonal.*

Proof. Let us proceed by contradiction. Then the weightvector v_γ can be written as $X_{-\alpha}v_{\gamma-\alpha}$ where $v_{\gamma-\alpha}$ is a weightvector of T_{ad}-weight $\gamma-\alpha$. In particular, $X_\alpha v$ is not trivial in V. By means of Lemma 15, we get a contradiction. □

Lemma 19. *There exists a simple root α' adjacent to α such that the character $\gamma-\alpha'$ is a root. In particular, α' lies in the support of γ.*

Proof. Note first that by the previous lemma, the support of γ contains a simple root α' adjacent to α. One of the dominant weights ω_D is thus non-orthogonal to α' since so is α. Let us proceed by contradiction: suppose $\gamma-\alpha'$ is not a root. Then $X_{\alpha'}v$ is trivial in V and (γ,α') is strictly positive by Lemma 14. It follows that γ is not a root.

Thanks to the previous lemmas, α' is not a weight in $\Sigma(\mathcal{D})$ hence it is not a spherical root in Σ. Recalling that γ is in the integral span of the weights ω_D, we get that $(\omega_D - \gamma, \alpha')$ is negative for every D. But since $X_{\alpha'}v$ is trivial in V, the representative of v_γ can be taken in the module $V(\lambda_D)$ associated to the (single) dominant weight which is not orthogonal to α'. Since $X_\alpha v$ is not trivial in V by Lemma 17, the support of $\gamma - \alpha$ does not contain the root α. Together with the fact that α' belongs to the support of γ, we get that$(\gamma - \alpha, \alpha)$ is strictly negative. It follows that γ is a root since so is $\gamma - \alpha$ by Lemma 16: a contradiction. □

Lemma 20. *The weight γ is a root.*

Proof. We first claim: Let α and α' be non-orthogonal pairwise simple roots and δ an arbitrary root. If $\delta + \alpha$ is not a root then neither is $\delta + \alpha - \alpha'$. Apply the claim to $\delta := \gamma - \alpha$ which is a root as previously proved. We get that if γ is not a root then neither is $\gamma - \alpha'$ for any simple root α' adjacent to α. This yields a contradiction with Lemma 19. □

2.3. **Obstruction space.** Given D, D' in $\mathcal{D}$, consider the morphism of G-modules

$$m_{D,D'} : V(\lambda_D) \oplus V(\lambda_{D'}) \longrightarrow V(\lambda_D + \lambda_{D'})$$

such that

$$v_{\lambda_D} \otimes v_{\lambda_{D'}} \longmapsto v_{\lambda_D + \lambda_{D'}}.$$

We shall regard $m_{D,D}$ on the symmetric product $V(\lambda_D) \cdot V(\lambda_D)$. Set

$$K_{D,D'} = \ker m_{D,D'} \qquad \text{for any } D, D' \text{ in } \mathcal{D}.$$

As a slight generalization of Proposition 2.7 in [C], we have

Proposition 21. *The $\mathbf{G}$-invariants $\left(T^2_{X_0}\right)^{\mathbf{G}}$ of the second cotangent module $T^2_{X_0}$ at X_0 is an obstruction space for the functor $\mathcal{H}ilb(\mathcal{S})$. Further it is contained in the kernel of*

$$H^1(\mathbf{G}_{v_{\underline{\lambda}}}, V/\mathfrak{g}.v_{\underline{\lambda}}) \longrightarrow \oplus_{D,D'} H^1(\mathbf{G}_{v_{\underline{\lambda}}}, K_{D,D'})$$

induced by $v \mapsto \sum_D v \cdot v_{\lambda_D}$ for $v \in V$ and $v \cdot v_{\lambda_D} \in V \cdot V(\lambda_D)$.

Recall the definition of $\Sigma(\mathcal{D})$ stated in the previous section.

Proposition 22. *Any T_{ad}-weight vector of $H^1(\mathfrak{g}_{v_{\underline{\lambda}}}, V/\mathfrak{g}.v_{\underline{\lambda}})$ can be represented by a cocycle $\varphi_{\alpha,\gamma}$ indexed by a simple root α in $S \setminus S^p$ and a T_{ad}-weight γ in $\Sigma(\mathcal{D})$ and defined as follows*

$$\varphi_{\alpha,\gamma} : \begin{cases} X_\alpha \mapsto [X^r_{-\alpha} v_\gamma] & \\ X_\delta \mapsto 0 & \text{if } \delta \neq \alpha \text{ with } \delta \in S \text{ or } -\delta \in S^p \end{cases}$$

The vector $[v_\gamma]$ *stands for the* γ*-weightvector of* $(V/\mathfrak{g}.v_{\underline{\lambda}})^{\mathbf{G}_{v_{\underline{\lambda}}}}$ *with* v_γ *chosen in some suitable* $\oplus_D V(\lambda_D)_{\lambda_D-\gamma}$*;* $r = \max\{i \geq 0 : X^i_{-\alpha}v_\gamma \neq 0$ *in* $V\}$.

Proof. In case the support of $\gamma + r\alpha$ does not contain any spherical root, this proposition is already proved in [C]. Suppose thus we are not in this situation. Let φ be a T_{ad}-weight of $H^1(\mathfrak{g}_{v_{\underline{\lambda}}}, V/\mathfrak{g}.v_{\underline{\lambda}})$. As in Lemma 5.31 in [C], φ satisfies the following property: given a simple root α, $X_\beta\varphi(X_\alpha) \in \mathfrak{g}.v_{\underline{\lambda}}$ for every β distinct to α. The proposition follows. □

Corollary 23. *The obstruction space for the functor* $\mathcal{H}ilb(\mathcal{S})$ *of invariant deformations is trivial.*

Proof. Keep the notation of the two propositions right above. We will show that no $\varphi_{\alpha,\gamma}$ maps trivially through the map of Proposition 21. As before, we are left to consider the weights $\gamma+r\alpha$ whose support does not contain any spherical root of $\mathcal{S}$. We follow the lines of the proof of Proposition 5.33 in [C]: we provide a dominant weight λ_D such that $v_\gamma.v_{\lambda_D} \in \oplus_{D,D'}K_{D,D'}$ is not trivial and annihilated by the root operator $X_{-\alpha}$.

Note that for the weights γ under consideration, there are at least two dominant weights among the ω_D's which are not orthogonal to γ.

If $\gamma = \alpha$ then clearly there exists λ_D such that $v_\gamma.v_{\lambda_D}$ is not trivial and annihilated by $X_{-\alpha}$ since $X_{-\alpha}.v_\gamma = 0$ in V and $(\lambda_D, \alpha^\vee) = 1$.

Thanks to the axioms $(\mathsf{A}1)-(\mathsf{A}3)$ (*see* also Lemma 15), the remaining weights γ share the following property. There exists a dominant weight, say λ, which is not orthogonal to γ, orthogonal to α and such that the representative of the T_{ad}-weightvector of weight γ can be chosen in the weightspace $\oplus_{\mathcal{D}}V(\lambda_D)_{\lambda_D-\gamma}$ and such that the vector $v_\gamma.v_\lambda$ is not trivial in $\oplus_{D,D'}K_{D,D'}$. The dominant weight λ is thus the required weight. □

3. The scheme $\mathrm{Hilb}(\mathcal{S})$ and the wonderful variety $X(\mathcal{S})$

Many ideas in the two last subsections are in the spirit of those developed in [Bri3].

Take a spherical system $\mathcal{S} = (S^p, \Sigma, \mathsf{A})$ of G.

3.1. Geometrical description of $\mathrm{Hilb}(\mathcal{S})$.

Theorem 24. *The invariant Hilbert scheme* $\mathrm{Hilb}(\mathcal{S})$ *is an affine toric variety for the adjoint torus of* G*; its weights are the spherical roots of* $\mathcal{S}$*. More specifically, it is an affine space.*

Proof. We argue as in [C]. Schlessinger's criterion along with Theorem 23 imply that Hilb($\mathcal{S}$) is smooth. The characters (ω_D, χ_D) defining the monoid Γ being linearly independent, Hilb$(\mathcal{S})^\circ$ is affine, connected and acted on by the adjoint torus with finitely many orbits and a single fixed point; *see* Corollary1.17, Theorem 2.7 and Corollary 4.3 in [AB]. Therefore Hilb$(\mathcal{S})^\circ$ is an affine space and its dimension is as stated by Theorem 12. □

3.2. **Geometrical realization of** $X(\mathcal{S})$**.** Let X_1 be a closed point of the invariant Hilbert scheme Hilb($\mathcal{S}$) whose T_{ad}-orbit is dense. Regarding X_1 as a subvariety of V, consider the $\mathbf{G} \times \mathbf{T}$-algebra

$$R(\mathcal{S}) = \oplus_{\lambda\in\Gamma(\mathcal{S})} k[X_1]_\lambda e^\lambda \otimes k[e^{(\gamma,0)} : \gamma \in \Sigma].$$

Define

$$\tilde{X} = \mathrm{Spec} R(\mathcal{S}).$$

Then $\tilde{X}$ is an affine spherical $\mathbf{G} \times T$-variety and $\tilde{X} \to \mathrm{Spec}\left(R(\mathcal{S})^{\mathbf{G}}\right)$ is the universal family of Hilb($\mathcal{S}$).

The algebraic torus $\mathbb{G}_m^{\mathcal{D}}$ acts naturally on $\tilde{X}$; $e^{(\omega_D,\chi_D)}$ is of weight ε_D hence e^γ is of weight $\sum a_D^\gamma \varepsilon_D$ since $(\gamma, 0) = \sum_{\mathcal{D}} a_D^\gamma(\omega_D, \chi_D)$. Note that the $G \times \mathbb{G}_M^{\mathcal{D}}$-variety $\tilde{X}$ is spherical.

Let $\tilde{X}^{reg} \subset \tilde{X}$ consist of the points of $\tilde{X}$ whose $\mathbf{G}$-orbit is of maximal dimension.

Theorem 25. *The G-variety*

$$X(\mathcal{S}) = \tilde{X}^{reg}/\mathbb{G}_m^{\mathcal{D}}$$

is wonderful. Further its spherical system is the given $\mathcal{S}$ in case the latter is spherically closed.

Proof. By construction, the G-variety $X(\mathcal{S})$ is complete, smooth, toroidal and spherical hence wonderful. Let $(S_X^p, \Sigma_X, \mathsf{A}_X)$ be its spherical system. Its closed G-orbit is given by $X_0/\mathbb{G}_m^{\mathcal{D}}$ whence $S_X^p = S^p$ where X_0 is the T_{ad}-fixed closed point of Hilb($\mathcal{S}$). The spherical roots of X coincides with Σ by Theorem 12. Let us now determine the matrix $\mathsf{A}(X)$. First note that the B-weights of the colors of X are given by the ω_D's. Let $f_{\lambda_D} \in k[X_1]_{\lambda_D}$ where $\lambda_D = (\omega_D, \chi_D)$. The invariant algebra R^U of the unipotent radical U of the Borel subgroup B is generated by the $f_{\lambda_D} e^{\lambda_D}$'s and the e^γ's. The canonical section of any prime divisor of X being G-invariant, it can be identified to some e^γ with $\gamma \in \Sigma$. It follows that the canonical section of a color X of B-weight ω_D can be identified to $f_{\lambda_D} e^{\lambda_D}$ and in turn that $\mathsf{A}(X)$ is the matrix with a_D^γ as coefficients since $\prod f_{\lambda_D}^{a_D^\gamma}$ is a $\mathbb{G}_m^{\mathcal{D}}$-invariant function of $\tilde{X}$ for any $\gamma \in \Sigma$. □

Corollary 26. *Luna's conjecture is true: to any spherical system, there corresponds a unique wonderful variety.*

Proof. It suffices to consider spherical systems which are spherically closed; *see* Section 6 in [Lu3]. The existence part is given by the previous theorem. The proof of the uniqueness is conducted as follows.

Let X be a wonderful G-variety with spherical system $\mathcal{S}$, $\mathcal{D}$ its set of colors and ω_D the B-weight of a color $D \in \mathcal{D}$. The variety X coincides with the normalization (in the function field of X) of some G-orbitclosure X' within $\prod_{\mathcal{D}} \mathbb{P}\left(V(\omega_D)\right)$. This fact slightly generalizes Proposition 2 in [Bri2]. Further, $H^0(X, \mathcal{O}_X(D))$ is either isomorphic as a G-module to $V(\omega_D)$ or $V(\omega_D) \oplus V(0)$ (*see* [Bri1]). We thus have a finite morphism

$$X \to \prod_{\mathcal{D}} \mathbb{P}\left(H^0(X, \mathcal{O}_X(D))\right).$$

Consider the affine multicone $\hat{X}$ (resp. $\hat{X}'$) over X (resp. $\hat{X}$) with respect to the above morphism. We have in particular

$$\hat{X} = \operatorname{Spec} \oplus_{(n_D)_D \in \mathbb{Z}^{\mathcal{D}}} H^0\left(X, \mathcal{O}_X\left(\sum_{\mathcal{D}} n_D D\right)\right).$$

From Proposition 3.1.1 in [Bri3], we know that $\hat{X}//G = \operatorname{Spec}(k[\hat{X}]^G)$ is isomorphic to $\mathbb{A}^r$ where r is the rank of X.

Let H be the generic stabilizer of X and K the largest subgroup of H of X such that H/K is diagonalizable. Then H/K is isomorphic to the diagonalizable group C defined in Subsection 1.2.

The principal fiber of the quotient map $\pi : \hat{X} \to \hat{X}//G$ is isomorphic to the spherical $G \times H/K$-variety $CE(G/K) = \operatorname{Spec} k[G/K]$ and π realizes a degeneration of $CE(G/K)$ into $X_0(\mathcal{S})$; *see* Section 3.2 in [Bri3]. Together with Theorem 24, it follows that $\pi' : \hat{X}' \to \hat{X}'//G$ is the universal family of the invariant Hilbert scheme $\operatorname{Hilb}(\mathcal{S})$.

This implies that if X and X' are wonderful G-varieties which have the same spherical system then K and K' are isomorphic. And since H/K and H'/K' are isomorphic so are the subgroups H and H'. In turn, X and X' are isomorphic thanks to the uniqueness of the wonderful compactification. □

References

[A] Ahiezer, D., *Equivariant completions of homogeneous algebraic varieties by homogeneous divisors*, Ann. Global Anal. Geom. **1** (1983), no. 1, 49–78.

[AB] Alexeev, V. and Brion, M., *Moduli of affine schemes with reductive group action*, J. Algebraic Geom., **14** (2005), no. 1, 83–117.

[Bo] Bourbaki, N., Éléments de mathématique. Groupes et Algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337 *Hermann, Paris,* 1968.
[Bra] Bravi, P., *Wonderful varieties of type E*, Represent. Theory **11** (2007), 174–191.
[BC1] Bravi, P. and Cupit-Foutou S., *Equivariant deformations of the affine multicone over a flag variety*, Adv. Math. **217** (2008), 2800–2821.
[BC2] Bravi, P. and Cupit-Foutou S., *Classification of strict wonderful varieties*, preprint `arXiv:math.AG/08062263`.
[BL] Bravi, P. and Luna D., *An introduction to wonderful varieties with many examples of type* F_4, preprint `arXiv:math.0812.2340`.
[BP] Bravi, P. and Pezzini, G., *Wonderful varieties of type D*, Represent. Theory **9** (2005), 578–637.
[Bri1] Brion, M., *Groupe de Picard*, Duke Math. Journal, Volume **58**, Number 2 (1989), 397-424.
[Bri2] Brion, M., *Variétés sphériques*, Notes de la session de la S. M. F. "Opérations hamiltoniennes et opérations de groupes algébriques", Grenoble, 1997, 1–60.
[Bri3] Brion, M., *The total coordinate ring of* , J. Algebra **313** (2007), 61–99.
[C] Cupit-Foutou, S., *Invariant Hilbert schemes and wonderful varieties*, preprint `arXiv:math.AG/0811.1567`.
[DP] De Concini, C. and Procesi, C., *Complete symmetric varieties*, Invariant theory (Montecatini, 1982), Lecture Notes in Math., 996, Springer, Berlin, 1983, 1–44.
[F] Foschi, A., *Variétés maginifiques et polytopes moment*, PhD thesis, Institut Fourier, Universit J. Fourier, Grenoble, 1998.
[J] Jansou, S., *Déformations invariantes des cônes de vecteurs primitifs*, Math. Ann. **338** (2007), 627–647.
[K] Knop, F., The Luna-Vust theory of spherical embeddings. *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989),* 225–249, *Manoj Prakashan, Madras,* 1991.
[Lo] Losev, I.V., Uniqueness property for spherical homogeneous spaces, Duke Math. Journal, Volume **147**, Number 2 (2009), 315–343.
[Lu1] Luna, D., *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), 249–258.
[Lu2] Luna, D., *Grosses cellules pour les variétés magnifiques*, in Algebraic Groups and Lie Groups, ed. by G. I. Lehrer, Australian Math. Soc. Lecture, Series 9 (1997), 267-280.
[Lu3] Luna, D., *Variétés sphériques de type A*, Inst. Hautes Études Sci. Publ. Math. **94** (2001), 161–226.
[W] Wasserman, B., *Wonderful varieties of rank two*, Transform. Groups **1** (1996), 375–403.

Mathematisches Institut, Universität zu Köln, Weyertal Str. 86–90, 50931 Köln, Germany
E-mail address: `scupit@mi.uni-koeln.de`