

# INVARIANT HILBERT SCHEMES AND WONDERFUL VARIETIES

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ABSTRACT. The invariant Hilbert schemes considered in [J07] and [BCF08] were proved to be smooth. The proof relied on the classification of strict wonderful varieties. We obtain in the present article a classification-free proof of the smoothness of these invariant Hilbert schemes by means of deformation theoretical arguments. As a consequence we recover in a shorter way (case-by-case-free considerations) the classification of strict wonderful varieties. This provides an alternative approach to Luna's conjecture.

## 1. INTRODUCTION

Let  $G$  be a connected reductive complex algebraic group. Subschemes of a given finite dimensional  $G$ -module, whose coordinate rings are isomorphic as  $G$ -modules are parameterized by a quasiprojective scheme, the so-called *invariant Hilbert scheme* introduced by Alexeev and Brion in [AB05].

Let us consider affine multi-cones over some flag  $G$ -varieties which are  $G$ -orbit closures. Invariant Hilbert schemes which contain such a multi-cone as closed point are proved to be smooth in [J07] and [BCF08]. Further, their universal family can be realized as the  $G$ -orbit map of the normalization of a certain multi-cone over a *strict wonderful variety*.

Wonderful  $G$ -varieties are projective  $G$ -varieties which enjoy nice properties like being smooth and having a dense orbit for a Borel subgroup of  $G$  (e.g. flag varieties, De Concini-Procesi compactifications of symmetric spaces). Luna's conjecture asserts that wonderful varieties are classified by combinatorial objects called spherical systems. In [BCF09], we answered this conjecture positively in case of wonderful varieties whose points have selfnormalising stabilizer: the so-called *strict wonderful varieties*; like all (positive) answers obtained so far ([L01, BP05, B07]), we followed Luna's Lie theoretical approach introduced in [L01] which involves case-by-case considerations.

The classification of strict wonderful varieties is one of the main tools which allowed to describe the invariant Hilbert scheme in [J07] and [BCF08].

In this work, we obtain a classification-free proof of the smoothness of the invariant Hilbert scheme under study by means of deformation theoretical arguments. As a consequence we recover in a shorter way (case-by-case-free considerations) the classification of strict wonderful varieties in terms of spherical systems. More specifically, we provide a geometrical construction of these wonderful varieties; this yields an alternative approach to Luna's conjecture.

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**Main notation.** The ground field is the field of complex numbers. Throughout this paper,  $G$  is a connected reductive algebraic group whose Lie algebra is denoted by  $\mathfrak{g}$ . Let  $B$  be a Borel subgroup of  $G$ ,  $U$  its unipotent radical and  $T \subset B$  a maximal torus of  $G$ . The set of dominant weights  $\Lambda^+$  (relatively to  $B$  and  $T$ ) parameterizes the irreducible  $G$ -modules; for a given  $\lambda \in \Lambda^+$ , we shall denote by  $V(\lambda)$  the corresponding  $G$ -module. The  $\mu$ -weight space ( $\mu$  being any character of  $T$ ) of a given  $G$ -module  $V$  is denoted by  $V_\mu$ ; in particular  $V(\lambda)_\lambda$  is one-dimensional.

## 2. INVARIANT HILBERT SCHEMES

**2.1. Definitions and setting.** We recall from [AB05] notions and results concerning invariant Hilbert schemes.

Given a finite dimensional  $G$ -module  $V$  and a scheme  $S$  endowed with the trivial action of  $G$ , a closed  $G$ -subscheme  $\mathfrak{X}$  of  $V \times S$  is called a *family of affine  $G$ -subschemes of  $V$  over  $S$* .

The projection of  $V \times S$  onto  $S$  induces a morphism  $\pi : \mathfrak{X} \rightarrow S$  which is affine, of finite type and  $G$ -equivariant. We thus have a  $G$ -equivariant morphism of  $\mathcal{O}_S$  –  $G$ -modules

$$(1) \quad \pi_* \mathcal{O}_{\mathfrak{X}} \cong \bigoplus_{\lambda \in \Gamma} \mathcal{R}_\lambda \otimes V(\lambda)^*$$

where  $\Gamma$  denotes a subset of  $\Lambda^+$  and  $\mathcal{R}_\lambda$  is the  $U$ -fixed point set  $(\pi_* \mathcal{O}_{\mathfrak{X}})_\lambda^U$ .

Each  $\mathcal{R}_\lambda$  is a coherent sheaf of  $(\pi_* \mathcal{O}_{\mathfrak{X}})^G$ -modules. When each  $\mathcal{R}_\lambda$  is an invertible sheaf of  $\mathcal{O}_S$ -modules, the morphism  $\pi$  is flat and the family  $\mathfrak{X}$  is said to be *of type  $\Gamma$* .

**Theorem and Definition 1.** ([AB05, Theorem 1.7]) *The contravariant functor that associates to any scheme  $S$  the set of families of affine  $G$ -subschemes of  $V$  over  $S$  of type  $\Gamma$  is represented by a quasi-projective scheme, the invariant Hilbert scheme  $\text{Hilb}_\Gamma^G$ .*

We shall be concerned throughout this article by the case where

$$V = V(\lambda_1) \oplus \dots \oplus V(\lambda_s)$$

with  $\lambda_1, \dots, \lambda_s$  linearly independent dominant weights spanning a free and saturated monoid  $\Gamma$ , i.e.  $\Gamma$  is such that

$$\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma.$$

In the remaining, we shall denote

$$\underline{\lambda} = (\lambda_1, \dots, \lambda_s).$$

Let  $X_0$  be the  $G$ -orbit closure within  $V$  of

$$v_{\underline{\lambda}} = v_{\lambda_1} + \dots + v_{\lambda_s}$$

where  $v_{\lambda_i}$  denotes a highest weight vector of  $V(\lambda_i)$  for  $i = 1, \dots, s$ .

Under the saturation assumption, the variety  $X_0$  is spherical under the action of  $G$ , that is  $X_0$  contains a dense orbit for a Borel subgroup of  $G$ . Further,  $X_0$  is normal and the boundary  $X_0 \setminus G.v_{\underline{\lambda}}$  is of codimension greater than 2.

Let  $G_{v_{\underline{\lambda}}}$  be the stabilizer of  $v_{\underline{\lambda}}$  in  $G$  and  $P_{\underline{\lambda}}$  be the normalizer of  $G_{v_{\underline{\lambda}}}$  in  $G$ . The variety  $X_0$  thus coincides with the affine multi-cone

$$\text{Spec } \bigoplus_{\underline{\nu} \in \Gamma} H^0(G/P_{\underline{\lambda}}, \mathcal{L}_{\underline{\nu}})$$

where  $\mathcal{L}_{\underline{\nu}}$  refers to the  $G$ -linearized bundle  $\otimes_i \mathcal{L}_{\lambda_i}^{m_i}$  with  $\underline{\nu} = \sum_{i=1}^s m_i \lambda_i$  a dominant weight and

$$\mathcal{L}_{\lambda} = G \times_B \mathbb{C}_{-\lambda}$$

where  $\mathbb{C}_{-\lambda}$  stands for the one-dimensional  $B$ -module associated to the character  $-\lambda$ .

The subvariety  $X_0$  of  $V$  can thus be regarded as a closed point of  $\text{Hilb}_\Gamma^G$ . More generally, any closed point of  $\text{Hilb}_\Gamma^G$  having a multiplicity-free coordinate ring, is a spherical affine  $G$ -variety; see[B86, V86]. Such an affine  $G$ -variety is said to be non-degenerate if its projection onto any  $V(\lambda_i)$ ,  $i = 1, \dots, s$ , is not trivial.

**Theorem 2.1.** ([AB05, Corollary 1.17 and Theorem 2.7]) *The non-degenerate  $G$ -subvarieties of  $V$  which can be seen as closed points of  $\text{Hilb}_\Gamma^G$  are parameterised by a connected and open subscheme  $\text{Hilb}_{\underline{\lambda}}^G$  of  $\text{Hilb}_\Gamma^G$ .*

**2.2. Tangent space of the invariant Hilbert scheme.** Let  $T_{\text{ad}}$  be the adjoint torus of  $G$ , that is  $T_{\text{ad}} = T/Z(G)$  where  $Z(G)$  is the center of  $G$ . Any invariant Hilbert scheme is endowed with an action of the adjoint torus (see [AB05] for details).

Let us recall how  $T_{\text{ad}}$  acts on the tangent space  $T_{X_0} \text{Hilb}_{\Gamma}^G$  at  $X_0$  of the invariant Hilbert scheme  $\text{Hilb}_{\Gamma}^G$  (see Section 2.1 in [AB05]). Let  $t \in T_{\text{ad}}$  then

$$t.v = (\lambda_i - \mu)(t)v \quad \text{when } v \in V(\lambda_i)_{\mu}.$$

**Theorem 2.2.** ([AB05, Proposition 1.5]) *The tangent space  $T_{X_0} \text{Hilb}_{\Gamma}^G$  at  $X_0$  of  $\text{Hilb}_{\Gamma}^G$  is isomorphic as a  $T_{\text{ad}}$ -module to the  $G_{v_{\lambda}}$ -invariant subspace  $(V/\mathfrak{g}.v_{\lambda})^{G_{v_{\lambda}}}$ .*

**Theorem 2.3.** ([BCF08, Theorem 2.2, Theorem 4.1] and also [J07] if  $s = 1$ ) *The tangent space  $T_{X_0} \text{Hilb}_{\Gamma}^G$  is a multiplicity free  $T_{\text{ad}}$ -module. Further, its  $T_{\text{ad}}$ -weights belong to Table 1.*

TABLE 1.  $T_{\text{ad}}$ -weights in  $(V/\mathfrak{g}.v_{\lambda})^{G_{v_{\lambda}}}$ .

Type of support	Weight
$A_1 \times A_1$	$\alpha + \alpha'$
$A_n$	$\alpha_1 + \dots + \alpha_n, n \geq 2$ $2\alpha, n = 1$ $\alpha_1 + 2\alpha_2 + \alpha_3, n = 3$
$B_n, n \geq 2$	$\alpha_1 + \dots + \alpha_n$ $2\alpha_1 + \dots + 2\alpha_n$ $\alpha_1 + 2\alpha_2 + 3\alpha_3, n = 3$
$C_n, n \geq 3$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$
$D_n, n \geq 4$	$2\alpha_1 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, n = 4$ $\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4, n = 4$
$F_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
$G_2$	$4\alpha_1 + 2\alpha_2$ $\alpha_1 + \alpha_2$

The weights occurring in Table 1 are so-called *spherical roots*; they are closely related to wonderful varieties (see [L97, L01] or Section 4 where we have reported from loc. cit. the results we shall need).

In Proposition 4.16 and Proposition 4.17, we will characterize in a purely combinatorial way the monoids  $\Gamma$  such that  $T_{X_0} \text{Hilb}_{\Gamma}^G$  is not trivial.

**2.3. Obstruction space and smoothness.** Let us recall first from [S73] (see also [S07]) the definition and some properties of the second cotangent module  $T_{X_0}^2$  of  $X_0$ .

Let  $I \subset \text{Sym}(V^*)$  be the ideal of the affine multi-cone  $X_0 \subset V$  and let  $\mathcal{O}(X_0) = \text{Sym}(V^*)/I$ . Take a presentation of  $I$  as  $\mathcal{O}(X_0)$ -modules

$$0 \rightarrow R \rightarrow F \rightarrow I \rightarrow 0$$

where  $F$  is a finitely generated free  $\mathcal{O}(X_0)$ -module.

This induces an exact sequence of  $\mathcal{O}(X_0)$ -modules

$$(2) \quad R/K \rightarrow F \otimes \mathcal{O}(X_0) \rightarrow I/I^2 \rightarrow 0$$

where  $K$  is the module of Koszul relations.

Let us apply  $\text{Hom}(-, \mathcal{O}(X_0)) := \text{Hom}_{\mathcal{O}(X_0)}(-, \mathcal{O}(X_0))$  to the exact sequence (2) then  $T_{X_0}^2$  is defined by the exact sequence

$$\text{Hom}(I/I^2, \mathcal{O}_{X_0}) \rightarrow \text{Hom}(F \otimes \mathcal{O}(X_0), \mathcal{O}_{X_0}) \rightarrow \text{Hom}(R/K, \mathcal{O}_{X_0}) \rightarrow T_{X_0}^2 \rightarrow 0.$$

The  $\mathcal{O}(X_0)$ -module  $T_{X_0}^2$  is independent of the presentation of the ideal  $I$  of  $X_0$ . Since  $X_0$  is normal and  $X_0 \setminus G.v_\lambda$  is of codimension greater or equal to 2, we have:

$$(3) \quad 0 \rightarrow T_{X_0}^2 \rightarrow H^1(G.v_\lambda, \mathcal{N}_\lambda) \rightarrow H^1(G.v_\lambda, F \otimes \mathcal{O}_{X_0}),$$

where  $\mathcal{N}_\lambda$  stands for the normal sheaf of  $G.v_\lambda$  in  $V$ .

By Schlessingers Comparison Theorem (see [S73] and also Section 2.3 and Proposition 3.1.12 in [S07]), we have

**Proposition 2.4.** *If the invariant set  $(T_{X_0}^2)^G$  is trivial then the invariant Hilbert scheme  $\text{Hilb}_\Gamma^G$  is smooth at  $X_0$ .*

**Theorem 2.5.** *The invariant Hilbert scheme  $\text{Hilb}_\lambda^G$  is smooth at  $X_0$ .*

**Corollary 2.6.** *The invariant Hilbert scheme  $\text{Hilb}_\lambda^G$  is an affine space.*

*Proof.* Being smooth and connected,  $\text{Hilb}_\lambda^G$  is irreducible. By Corollary 3.4 in [AB05], the invariant Hilbert scheme is acted on by the adjoint torus  $T_{ad}$  of  $G$  with finitely many  $T_{ad}$ -orbits. Hence it is a toric variety for the adjoint torus of  $G$ . Further, a smooth affine toric variety with a single fixed point being an affine space,  $\text{Hilb}_\lambda^G$  is in turn an affine space.  $\square$

The proof of the above theorem is conducted in details in Section 5. By Proposition 2.4, it amounts to proving that  $(T_{X_0}^2)^G$  is trivial; this will be achieved by means of the two following propositions.

For any  $1 \leq i \neq j \leq s$ , let

$$K_{i,j} = \ker\{V(\lambda_i) \otimes V(\lambda_j) \longrightarrow V(\lambda_i + \lambda_j)\}$$

and

$$K_{i,i} = \ker\{S^2 V(\lambda_i) \longrightarrow V(2\lambda_i)\}.$$

This yields the presentation

$$0 \rightarrow R \rightarrow \bigoplus_{1 \leq i, j \leq s} K_{i,j} \otimes \mathcal{O}(X_0) \rightarrow I/I^2 \rightarrow 0.$$

**Proposition 2.7.** *Let  $\mathfrak{g}_{v_\lambda}$  be the isotropy Lie algebra of  $v_\lambda$  and  $G_{v_\lambda}^\circ$  be the identity component of  $G_{v_\lambda}$ . The invariant set  $(T_{X_0}^2)^G$  is given by the kernel of the map*

$$H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}/G_{v_\lambda}^\circ} \longrightarrow \bigoplus_{1 \leq i, j \leq s} H^1(\mathfrak{g}_{v_\lambda}, K_{i,j})^{G_{v_\lambda}/G_{v_\lambda}^\circ}$$

induced by the map of  $\mathfrak{g}_{v_\lambda}$ -modules

$$v \mapsto \sum_i v \cdot v_{\lambda_i} \quad \text{where } v \in V \text{ and } v \cdot v_{\lambda_i} \in V \cdot V(\lambda_i).$$

*Proof.* The sheaf  $\mathcal{N}_\lambda$  being the  $G$ -linearized sheaf on  $G/G_{v_\lambda}$  associated to the  $G_{v_\lambda}$ -module  $V/\mathfrak{g}.v_\lambda$ , we have:

$$H^1(G.v_\lambda, \mathcal{N}_\lambda)^G = H^1(G.v_\lambda, V/\mathfrak{g}.v_\lambda).$$

We have similarly

$$H^1(G.v_\lambda, \mathcal{O}_{X_0}^{\oplus n})^G = \bigoplus_{1 \leq i, j \leq s} H^1(G.v_\lambda, K_{i,j}).$$

From [H61], we know that

$$H^1(G.v_\lambda, V/\mathfrak{g}.v_\lambda) \simeq H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}/G_{v_\lambda}^\circ}.$$

The proposition follows from the exact sequence 3.  $\square$

Let  $S(\Gamma)$  be the set of simple roots of  $G$  orthogonal to every element of  $\Gamma$  and let  $\Sigma(\Gamma)$  be the set of  $T_{\text{ad}}$ -weights of the tangent space  $T_{X_0} \text{Hilb}_\Gamma^G$ . Take  $\gamma \in \Sigma(\Gamma)$ , we shall denote by  $v_\gamma \in \bigoplus_i V(\lambda_i)_{\lambda_i - \gamma}$  the corresponding weight vector.

**Proposition 2.8.** *The  $T_{\text{ad}}$ -weight vectors of  $H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)$  can be represented by the following cocycles indexed by the simple roots  $\alpha$  in  $S \setminus S(\Gamma)$  and by the  $T_{\text{ad}}$ -weights  $\gamma$  in  $\Sigma(\Gamma)$*

$$\begin{aligned} \varphi_{\alpha, \gamma} : X_\alpha &\longmapsto X_{-\alpha}^r v_\gamma \\ X_\delta &\longmapsto 0 \quad \text{if } \delta \neq \alpha \end{aligned}$$

Here  $r = -(\gamma, \alpha^\vee)$  if  $\alpha \notin \text{Supp } \gamma$  and  $r = 0$  otherwise.

The proof of Proposition 2.8 is postponed to Section 5.

### 3. WONDERFUL VARIETIES

**Definition 3.9.** An algebraic  $G$ -variety  $X$  is said to be *wonderful of rank  $r$*  if it satisfies the following conditions.

- (i)  $X$  is smooth and complete.
- (ii)  $X$  contains an open  $G$ -orbit whose complement is the union of  $r$  smooth prime  $G$ -divisors  $D_1, \dots, D_r$  with normal crossings and such that  $\cap_1^r D_i \neq \emptyset$ .
- (iii) The  $G$ -orbit closures of  $X$  are given by the intersections  $\cap_I D_i$  where  $I$  is a subset of  $\{1, \dots, r\}$ .

As examples of wonderful varieties, one may consider flag varieties or De Concini-Procesi compactifications of symmetric spaces; see [DCP82].

Wonderful varieties are projective and spherical; see [L96].

**Definition 3.10.** A wonderful variety whose points have a selfnormalising stabilizer is called *strict*.

Luna introduced several invariants attached to any wonderful  $G$ -variety  $X$ : *spherical roots, colors*. We shall recall freely results concerning these notions; see [L97, L01] for details.

Let  $Y$  be the (unique) closed  $G$ -orbit of  $X$  and  $z \in Y$  be the unique point fixed by the Borel subgroup  $B^-$  of  $G$  such that  $B \cap B^- = T$ .

The *spherical roots* of  $X$  are the  $T$ -weights of the quotient  $T_z X / T_z Y$  where  $T_z X$  (resp.  $T_z Y$ ) denotes the tangent space at  $z$  of  $X$  (resp. of  $Y$ ). The rank of  $X$  is equal to the number of spherical roots of  $X$ .

Let  $P_X$  be the stabilizer of the point  $z \in Y$ . The subgroup  $P_X$  is a parabolic subgroup hence it corresponds to a subset  $S_X^p$  of the set of simple roots  $S$ .

In case of strict wonderful varieties, the couple  $(S_X^p, \Sigma_X, \emptyset)$  shares nice properties: it is a *spherical system for  $G$* . Spherical systems were introduced by Luna as triples which fulfill certain axiomatic conditions.

Luna's conjecture asserts that there corresponds a unique (non-necessarily strict) wonderful  $G$ -variety to a given spherical system.

Before recalling Luna's definition of spherical systems in case the third datum is the empty set, let us set some further notation. Let  $S$  be the set of simple roots of  $G$  relatively to  $B$  and  $T$ . Given a simple root  $\alpha \in S$ , let  $\alpha^\vee$  be its associated coroot, that is  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ . Given  $\beta = \sum_\alpha n_\alpha \alpha$  where the sum runs over  $S$  with  $n_\alpha \geq 0$  (resp.  $n_\alpha \leq 0$ ) for all  $\alpha$ . The *support* of  $\beta$  is defined as usual as the set of simple roots  $\alpha$  such that  $n_\alpha \neq 0$ ; we shall denote it by  $\text{Supp}\beta$ .

#### 3.1. Spherical systems.

**Definition 3.11.** *The set of spherical roots of  $G$  is the set of characters of Table 1 whose support is a subset of  $S$ . We denote it by  $\Sigma(G)$ .*

**Definition 3.12.** ([BL08, 1.1.6]) Let  $S^p \subset S$  and  $\sigma \in \sigma(G)$ . The couple  $(S^p, \sigma)$  is said to be *compatible* if

$$S^{pp}(\sigma) \subset S^p \subset S^p(\sigma)$$

where  $S^{pp}(\sigma)$  is one of the following sets

- $S^p(\sigma) \cap \text{Supp}\sigma \setminus \{\alpha_r\}$  if  $\sigma = \alpha_1 + \dots + \alpha_r$  with  $\text{Supp}\sigma$  of type  $B_r$ ,
- $S^p(\sigma) \cap \text{Supp}(\sigma) \setminus \{\alpha_1\}$  if  $\text{Supp}\sigma$  is of type  $C_r$ ,
- $S^p(\sigma) \cap \text{Supp}(\sigma)$  otherwise.

**Definition 3.13.** Let  $S^p \subset S$  and  $\Sigma \subset \Sigma(G)$ . The couple  $(S^p, \Sigma)$  is called a *spherical system* if

( $\Sigma 1$ )  $(\alpha^\vee, \sigma) \in 2\mathbb{Z}_{\leq 0}$  for all  $\sigma \in \Sigma \setminus \{2\alpha\}$  and all  $\alpha \in S$  such that  $2\alpha \in \Sigma$ .

( $\Sigma 2$ )  $(\alpha^\vee, \sigma) = (\beta^\vee, \sigma)$  for all  $\sigma \in \Sigma$  and all  $\alpha, \beta \in S$  which are mutually orthogonal and such that  $\alpha + \beta \in \Sigma$ .

( $S$ ) The couple  $(\{\sigma\}, S^p)$  is compatible for any  $\sigma \in \Sigma$ .

( $St$ ) The couple  $(\{2\sigma\}, S^p)$  is not compatible for any  $\sigma \in \Sigma$ .

**3.2. Colors.** Given a wonderful  $G$ -variety  $X$ , the  $B$ -stable but not  $G$ -stable prime divisors in  $X$  are called the *colors of  $X$* . The subgroup  $P_X$  (see the previous paragraph) coincides with the stabilizer of the colors of  $X$ .

**Definition 3.14.** *The set of colors  $\Delta$  of a given spherical system  $(S^p, \Sigma)$  is defined as the set of the following dominant weights:*

- $\omega_\alpha$  (resp.  $2\omega_\alpha$ ) if  $\alpha \in S \setminus S^p$  and  $2\alpha \notin \Sigma$  (resp.  $2\alpha \in \Sigma$ );
- $\omega_\alpha + \omega_\beta$  if  $\alpha, \beta \in S$  and  $\alpha + \beta \in \Sigma$ .

Here  $\omega_\alpha$  stands for the fundamental weight associated to any simple root  $\alpha$ , that is  $\omega_\alpha(\beta^\vee)$  equals 1 if  $\beta = \alpha$  and equals 0 for any other simple root  $\beta$ .

In case the spherical system is given by a wonderful  $G$ -variety  $X$ , the set  $\Delta$  coincides with the set of colors of  $X$  in the following way. Let  $D$  be a color of  $X$ . Consider its inverse image  $\pi^{-1}(D)$  through the canonical quotient  $\pi : G \rightarrow G/H$  where  $G/H$  is isomorphic to the dense  $G$ -orbit of  $X$ . Choose  $H$  such that  $BH$  is open in  $G$  (recall that  $X$  is  $G$ -spherical). Then, whenever the Picard group of  $G$  is trivial,  $\pi^{-1}(D)$  is a  $B \times H$ -stable divisor of  $G$  hence it has an equation  $f_D$  which is a  $B \times H$ -eigenvector. The set  $\Delta$  is thus given by the set of the  $B$ -weights of the  $f_D$ 's.



## 4. INVARIANT HILBERT SCHEMES AND WONDERFUL VARIETIES

**4.1. Tangent spaces and spherical systems.** Given a saturated monoid  $\Gamma$ , recall the definition of the sets  $\Sigma(\Gamma)$  and  $S^p(\Gamma)$ ; see the paragraph right before Proposition 2.8.

**Theorem 4.15.** ([BCF08, Theorem 4.1]) *The couple  $(S^p(\Gamma), \Sigma(\Gamma))$  is a spherical system.*

Denote by  $\Delta(\Gamma)$  the set of colors of  $(S^p(\Gamma), \Sigma(\Gamma))$ . Let  $S(\gamma)$  be the set of simple roots  $\delta$  such that  $\gamma - \delta$  is a root.

**Proposition 4.16.** *Assume that the tangent space  $T_{X_0} \text{Hilb}_\Gamma^G$  is not trivial. Let  $\lambda$  be one of the dominant weights defining the monoid  $\Gamma$ . If the support of the set  $\Sigma(\Gamma)$  coincides with the set of simple roots  $S$  then one of the following possibilities may occur.*

- $\lambda$  belongs to  $\Delta(\Gamma)$  up to a scalar,
- $\lambda$  equals  $\omega_\alpha + \omega_\beta$  if there exists  $\gamma \in \Sigma(\Gamma)$  such that  $(\gamma, \alpha) > 0$  and  $(\gamma, \beta) > 0$ ,
- $\lambda = \omega_\alpha + \sum a_\delta \omega_\delta$  with  $(\gamma, \delta) = 0$  along with  $S(\gamma) \neq \{\alpha, \beta\}$  for all  $\gamma \in \Sigma(\Gamma)$ .

*In case  $\lambda = a\lambda_D$  with  $a > 1$  and  $\lambda_D \in \Delta(\Gamma)$ , the set  $\Sigma(\Gamma)$  has to be a singleton. Further, if the second possibility occurs for some  $\lambda$  then  $(\lambda', \alpha + \beta) = 0$  for all  $\lambda' \neq \lambda$ .*

As a converse, we have

**Proposition 4.17.** *Given a spherical system  $(S^p, \Sigma)$  of  $G$ , let  $\Delta$  be its set of colors. Let  $\Gamma$  be a saturated monoid spanned by linearly independent dominant weights which satisfy the conditions stated in the previous proposition (with  $\Delta(\Gamma) = \Delta$  and  $\Sigma(\Gamma) = \Sigma$ ). Then the tangent space of the corresponding invariant Hilbert scheme is not trivial. Further the elements of  $\Sigma$  are  $T_{\text{ad}}$ -weights of this tangent space.*

**Remark 4.18.** In Section 6 of [BCF08], one can find analogous versions of the two above statements whose proofs are quite indirect since requiring the use of wonderful varieties. We will provide a purely combinatorial proof; see Section 5.

**Definition 4.19.** A spherical system  $(S^p, \Sigma)$  of  $G$  is *primitive* if the support of  $\Sigma$  coincides with the whole set  $S$  of simple roots of  $G$ , and if there is no spherical system  $(S^p, \Sigma')$  such that  $\Sigma'$  strictly contains  $\Sigma$ .

**4.2. Classification of wonderful varieties.** Given the decomposition of  $V$  into irreducible  $G$ -modules,

$$V = V(\lambda_1) \oplus \dots \oplus V(\lambda_s)$$

the corresponding  $s$ -multi-cone  $\mathcal{C}(X)$  generated by any  $X \subset V$  is (after [BK79])

$$\cup_{\underline{t} \in \mathbb{C}^s} \underline{t}.X$$

where

$$\underline{t}.X = \{(t_1 x_1, \dots, t_s x_s) : \underline{t} = (t_i)_i \in \mathbb{C}^s, (x_i)_i \in X\}.$$

**Theorem 4.20.** *Take a primitive spherical system  $(S^p, \Sigma)$  and let  $\Gamma$  be the monoid spanned by its set of colors  $\lambda_i$  ( $i = 1, \dots, s$ ). Consider the invariant Hilbert scheme  $\text{Hilb}_{\underline{\lambda}}^G$ . Let  $X_1$  be a closed point such that its  $T_{\text{ad}}$ -orbit is dense within  $\text{Hilb}_{\underline{\lambda}}^G$ . Regarding  $X_1$  as a subvariety of  $V = \oplus_i V(\lambda_i)$ , let  $\mathcal{C}(X_1)$  be the corresponding  $s$ -multi-cone.*

*Then the multihomogeneous spectrum of the regular ring  $\mathcal{R}(\mathcal{C}(X_1))$  of  $\mathcal{C}(X_1)$*

$$X_{\Gamma} = \text{Proj} \mathcal{R}(\mathcal{C}(X_1))$$

*is a wonderful  $G$ -variety with spherical system  $(S^p, \Sigma)$ .*

*Proof.* Recall that by Proposition 4.17 and Theorem 2.5, the invariant Hilbert scheme  $\text{Hilb}_{\underline{\lambda}}^G$  is not trivial whence the existence of  $X_1$  by the proof of Corollary 2.6. Further the dimension of  $\text{Hilb}_{\underline{\lambda}}^G$  equals the cardinality of  $\Sigma$ .

Let  $v \in V$  be such that  $X_1$  is the  $G$ -orbit closure of  $v$  within  $V$ . Since the dominant weights  $\lambda_i$  defining the monoid  $\Gamma$  are linearly independent, the  $G \times T_{\text{ad}}$ -orbit closure of  $v$  within  $V$  coincides with the multicone  $\mathcal{C}(X_1)$  generated by  $X_1$ .

Write  $v = v_1 + \dots + v_s$  with  $v_i \in V(\lambda_i)$ . Since the saturation assumption is fulfilled by the set of colors  $\lambda_i$ , the  $G$ -variety  $X_{\Gamma}$  coincides with the  $G$ -orbit closure of  $([v_1], \dots, [v_s])$  within the multiprojective space  $\mathbb{P}(V(\lambda_1)) \times \dots \times \mathbb{P}(V(\lambda_s))$ .

The projective variety  $X_{\Gamma}$  is thus smooth and spherical for the action of  $G$  with a single closed  $G$ -orbit given by the multihomogeneous spectrum of the regular ring of  $\overline{G.v_{\underline{\lambda}}} \subset V$ . Being also toroidal, the variety  $X_{\Gamma}$  is thus a wonderful  $G$ -variety (see [L97]). Further its rank equals the cardinality of the given set  $\Sigma$ . We shall prove that the spherical system  $(S_X^p, \Sigma_X)$  of  $X_{\Gamma}$  is indeed  $(S^p, \Sigma)$ . This follows readily from the characterization of the closed  $G$ -orbit of  $X_{\Gamma}$  and from Corollary 2.6 along with the definition of spherical roots recalled in the previous section.  $\square$

As already noticed and proved in [BCF08] (see Corollary 2.5 in [loc.cit.]; see also [B07]), we get the following.

**Corollary 4.21.** *Given a connected reductive algebraic group  $G$  and  $\Gamma$  a saturated monoid, consider the set  $\tilde{\Gamma}$  of colors of the spherical system*

$(S^p(\Gamma), \Sigma(\Gamma))$ . Let  $X_{\tilde{\Gamma}}$  be the wonderful  $G$ -variety obtained in the above theorem. The universal family of  $\text{Hilb}_{\underline{\lambda}}^G$  ( $\underline{\lambda}$  defining the monoid  $\Gamma$ ) is given by the quotient map

$$\tilde{X}_{\tilde{\Gamma}} \longrightarrow \tilde{X}_{\tilde{\Gamma}}//G$$

where  $\tilde{X}_{\tilde{\Gamma}}$  denotes the normalization of the affine multi-cone

$$\text{Spec } \bigoplus_{\underline{\nu} \in \Gamma} H^0(X_{\tilde{\Gamma}}, \mathcal{L}_{\underline{\nu}}).$$

Here  $\mathcal{L}_{\chi}$  denotes the  $G$ -linearized invertible sheaf associated to the character  $\chi$ .

As a consequence of the two above statements, we can answer positively to Luna's conjecture in the context of strict wonderful varieties.

**Corollary 4.22.** *Given a spherical system  $(S^p, \Sigma)$  of some reductive algebraic group  $G$ , there exists a unique wonderful  $G$ -variety whose spherical system is  $(S^p, \Sigma)$ .*

*Proof.* By [L01], it suffices to consider primitive spherical systems. The existence part thus follows from Theorem 2.6; the uniqueness is a consequence of the corollary above as already noticed in Section 6.2 of [BCF09].  $\square$

**Remark 4.23.** The existence part of the above theorem is proved in [BCF09] by different methods; the proof follows Luna's approach introduced in [L01]. This proof is Lie theoretical. More specifically, for a given spherical system of  $G$ , a subgroup of  $G$  is exhibited. One thus proves that this subgroup is wonderful (it has a compactification which is a wonderful variety) and by ad-hoc arguments that it has indeed the spherical system under consideration. See [BP05] and [B07] for other partial positive answers to Luna's conjecture.

The uniqueness part (in full generality) of this theorem follows from [L09]. The approach developed here uses the so-called colored fans introduced in [LV83].

## 5. PROOFS

**5.1. Auxiliary lemmas.** For convenience, we shall recall the following statements from [BCF08].

**Lemma 5.24.** ([BCF08, Proposition 3.4]) *Let  $\gamma$  be a  $T_{\text{ad}}$ -weight vector of  $(V/\mathfrak{g}\cdot v_{\underline{\lambda}})^{G_{v_{\underline{\lambda}}}}$ . If  $\delta$  is a simple root in the support of  $\gamma$  such that  $\gamma - \delta$  is not a root then  $(\gamma, \delta) \geq 0$ . Further if  $\gamma$  and  $\delta$  are orthogonal then  $\delta$  is orthogonal to all the  $\lambda_i$ 's.*

**Remark 5.25.** We will generalize the above statement in Proposition 5.33.

**Lemma 5.26.** ([BCF08, Proof of Theorem 3.10]) *If  $[v]$  is a  $T_{\text{ad}}$ -weight vector of  $(V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$ , then one of its representatives  $v \in V$  can be taken as follows*

$$[v] \in (V(\lambda)/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}} \quad \text{or} \quad v = X_{-\gamma}v_\lambda$$

where  $\lambda$  is one of the given dominant weights  $\lambda_i$ . The second case occurs only when  $(V(\lambda)/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  is trivial.

**Lemma 5.27.** ([BCF08, Lemma 3.13]) *If  $G$  is of type  $F_4$ ,  $\lambda_1 = \omega_4 + a\omega_3$  and  $\lambda_i = a_i\omega_3$  for  $a_i > 0$  for some  $i$ , then the space  $(V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  is trivial.*

**Remark 5.28.** The above lemma is basically the same as Lemma 3.13 in [BCF08]. Here we have not required that  $a$  be non-zero. The proof conducted there is however still valid.

**5.2. Proof of Proposition 2.8.** Let

$$V = V(\lambda_1) \oplus \dots \oplus V(\lambda_s)$$

where the dominant weights  $\lambda_i$  satisfy the properties of Proposition 4.16.

Let  $\dot{\varphi}$  be a  $T_{\text{ad}}$ -weight vector of  $H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}/G_{v_\lambda}^\circ}$ . We shall prove that  $\varphi$  can be represented by some cocycle  $\varphi = \varphi_{\alpha,\gamma}$  as follows. Let  $\alpha$  be a simple root of the isotropy Lie algebra  $\mathfrak{g}_{v_\lambda}$  and  $\gamma$  be either a  $T_{\text{ad}}$ -weight in  $\Sigma(\Gamma)$  or one of the given dominant weights  $\lambda_i$ . If  $\gamma \in \Sigma(\Gamma)$ , let  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  be the corresponding  $T_{\text{ad}}$ -weight vector otherwise take  $v_\gamma$  to be the highest weight vector  $\lambda_i$ . Then define

$$\begin{aligned} \varphi_{\alpha,\gamma} : X_\alpha &\longmapsto [X_{-\alpha}^r v_\gamma] \\ X_\delta &\longmapsto 0 \quad \text{if } \delta \neq \alpha, \delta \in S \end{aligned}$$

where

$$r = \max\{i \geq 0 : X_{-\alpha}^i v_\gamma \neq 0 \text{ in } V\} \quad \text{and} \quad v_\gamma \in V(\lambda_j)_{\lambda_j - \gamma} \text{ for some } \lambda_j.$$

By Kostant's theorem (see [K61] and also [K02], Chap. III-2), whenever  $\delta \in S^p(\Gamma)$ , namely  $\delta$  is orthogonal to every dominant weight  $\lambda_i$  then  $\dot{\varphi}(X_\alpha) = 0$ . Hence in the definition of  $\varphi_{\alpha,\gamma}$  above, the simple root  $\alpha$  does not belong to  $S^p(\Gamma)$ .

**Lemma 5.29.** *Let  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  be a  $T_{\text{ad}}$ -weight vector and take  $v_\gamma \in V(\lambda_j)_{\lambda_j - \gamma}$  for some  $\lambda_j$  to be one of its representatives. Suppose that  $\varphi_{\alpha,\gamma}$  defines a cocycle for some simple root  $\alpha$ . Then the vector*

$$[X_{-\alpha}^r v_\gamma] \quad \text{with} \quad r = \max\{i \geq 0 : X_{-\alpha}^i v_\gamma \neq 0 \text{ in } V\}$$

does not depend upon the choice of the dominant weight  $\lambda_j$ .

*Proof.* As recalled in Lemma 5.26, the choice of the dominant weight  $\lambda_j$  is not unique whenever  $v_\gamma = X_{-\gamma}v_{\lambda_j}$ . More specifically, we can choose also the representative  $v_\gamma$  to be  $X_{-\gamma}v_{\lambda_i}$  for some other  $\lambda_i \neq \lambda_j$ . If  $\lambda_i$  and  $\lambda_j$  are both orthogonal to the given simple root  $\alpha$  then by Proposition 4.16 then so is  $\gamma$ . Further  $\alpha$  can not belong to the support of  $\gamma$ ; the lemma follows. Suppose that  $\lambda_i$  is not orthogonal to  $\alpha$  and that  $\alpha$  does not belong to the support of  $\gamma$ . Then one checks that  $\varphi_{\alpha,\gamma}$  can not be a cocycle. We are thus left with  $\lambda_i$  non-orthogonal to  $\alpha$  and  $\alpha$  lying in the support of  $\gamma$ . By Table 1 and Proposition 5.24,  $\gamma - \alpha$  is a root. In case  $\gamma + \alpha$  is a root, the lemma is obvious otherwise one has that  $(\lambda, \alpha^\vee) = 1$  and  $(\lambda_j, \alpha^\vee) = 0$ . The lemma follows.  $\square$

**Lemma 5.30.** *Let  $\alpha$  be a simple root. Suppose there exists a unique dominant weight  $\lambda_i$  which is not orthogonal to  $\alpha$ . Let  $[v] \in V/\mathfrak{g}.v_\lambda$  be non-trivial. Then either there exists a positive root  $\beta \neq \alpha$  such that  $X_\beta.v \neq 0$  in  $V$  or  $v = X_{-\alpha}^r v_{\lambda_i}$  in  $V$  with  $r = 2$  or  $4$ .*

*Proof.* If  $X_\beta v = 0$  for all positive roots  $\beta$  distinct to  $\alpha$  then  $v$  will be equal either to  $v_{\lambda_i}$ ,  $X_{-\alpha}v_{\lambda_i}$  or to  $X_{-\alpha}^r v_{\lambda_i}$  with  $r > 1$  for some  $\lambda_i$ . Under the assumptions of the lemma, the class of  $v$  in  $V/\mathfrak{g}.v_\lambda$  will be trivial unless it is equal to the latter possibility. By Proposition 4.16, the integer  $r$  equals 2 or 4.  $\square$

**Remark 5.31.** The condition of the previous lemma is fulfilled whenever the  $\lambda_i$ 's are the colors of a spherical system.

**Lemma 5.32.** *We have  $X_\beta\varphi(X_\alpha) = 0$  in  $V/\mathfrak{g}.v_\lambda$  for every root  $\beta \neq \alpha$  of the isotropy Lie algebra  $\mathfrak{g}_{v_\lambda}$ .*

*Proof.* Let  $\mu$  be the  $T_{\text{ad}}$ -weight of  $\varphi(X_\alpha)$  then  $\varphi(X_\beta)$  should be of  $T_{\text{ad}}$ -weight  $\mu - \alpha + \beta$ , for any root  $\beta \neq \alpha$  of the isotropy Lie algebra  $\mathfrak{g}_{v_\lambda}$ .

Let  $\beta$  be a simple root. Suppose that  $X_\beta\varphi(X_\alpha) \notin \mathfrak{g}.v_\lambda$ . Since  $X_\beta\varphi(X_\alpha) - X_\alpha\varphi(X_\beta) \in \mathfrak{g}.v_\lambda$ , it follows that  $X_\alpha\varphi(X_\beta) \notin \mathfrak{g}.v_\lambda$ . And in particular  $\varphi(X_\beta)$  is then a weight vector of weight  $\mu - \beta + \alpha$ , which is possible only in case  $\beta = \alpha$ . The lemma follows.  $\square$

The following proposition is the announced generalization of Lemma 5.24.

**Proposition 5.33.** *Take a  $T_{\text{ad}}$ -weightvector  $[v_\gamma] \in V/\mathfrak{g}.v_\lambda$  of weight  $\gamma$ . Let  $\alpha, \delta$  be orthogonal simple roots with  $\delta$  in the support of  $\gamma$ . Suppose that  $\gamma - \delta$  is not a root and that*

$$[X_\beta v_\gamma] = 0 \quad \text{for all } \beta \neq \alpha.$$

*Then every  $\lambda_i$  is orthogonal to  $\delta$ .*

*Proof.* We start similarly as for the proof of Proposition 3.4 in [BCF08]. Suppose there exists a dominant weight among the given  $\lambda_i$ 's, say  $\lambda_k$  which is not orthogonal to  $\delta$  then in particular such a  $\lambda_k$  is not orthogonal to  $\gamma$  neither since  $\delta$  lies in the newort of  $\gamma$ .

Note that if  $\gamma$  is not equal to  $\alpha$  (up to a scalar) then there exists a positive root  $\beta \neq \alpha$  whose support is contained in that of  $\gamma$  and such that

$$(\lambda_k - \gamma, \beta) < 0.$$

We claim that the component  $v_\gamma^k$  of  $v_\gamma$  in  $V(\lambda_k)$  can be assumed to be trivial. Indeed if  $v_\gamma^k \neq 0$  then  $X_\beta v_\gamma \neq 0$  in  $V$  because of the inequality stated right above. And in turn, we will get that  $X_\beta v_\gamma = X_{-\gamma+\beta} v_\lambda$  by Lemma 5.32. Considering instead the representative  $v_\gamma - X_{-\gamma+\beta} v_\lambda$  whose component in  $V(\lambda_k)$  is trivial, we shall obtain the claim.

Assume thus for the remaining of the proof that  $v_\gamma^k = 0$ . It follows that if  $\lambda_k$  is not orthogonal to  $\delta$  then  $\delta$  should lie in the support of  $\beta'$  where  $\beta'$  is a positive root distinct to  $\alpha$  such that

$$0 \neq X_{\beta'} v_\gamma \in \mathfrak{g}.v_\lambda.$$

If such a root  $\beta'$  can be taken to be simple then  $\beta' = \delta$  whence a contradiction since  $\gamma - \delta$  is not a root.

We are thus left with the situation where  $X_\delta v_\gamma = 0$  in  $V$  for all simple roots  $\delta \neq \alpha$ . Since  $X_{\beta'} v_\gamma \neq 0$  in  $V$  for some positive root  $\beta'$  as above, it follows that  $X_{\alpha'+\alpha} \neq 0$  in  $V$  whenever  $\alpha'$  is a simple root in the support of  $\gamma$  and adjacent to  $\alpha$ . Therefore  $\delta$  has to be adjacent to  $\alpha$ .  $\square$

*Proof of Proposition 2.8.* Let  $\gamma$  be the  $T_{\text{ad}}$ -weight of  $[\varphi(X_\alpha)]$  and denote by  $v_\gamma$  a representative of  $\varphi(X_\alpha)$  in  $V/\mathfrak{g}.v_\lambda$ . Note first that whenever  $\alpha$  does not belong to the support then  $X_\alpha v_\gamma = 0$  in  $V$  hence  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  by Lemma 5.32.

We shall assume in the remaining of the proof that  $\alpha$  does belong to the support of the  $T_{\text{ad}}$ -weight  $\gamma$ . We shall proceed along the type of the support of  $\gamma$ . Let us work out a few cases in details. The main ingredients are Proposition 5.33 and Proposition 4.16. Let  $v_\gamma$  be non-equal to  $X_{-\alpha}^r v_{\lambda_i}$  otherwise the proposition is already proved. As a consequence of Lemma 5.32, the weight  $\gamma$  can be written as a sum of two positive roots, say  $\beta_1$  and  $\beta_2$ .

Consider first the case where the supports of the roots  $\beta_1$  and  $\beta_2$  are orthogonal. Then by Proposition 5.33 and its proof, the roots  $\beta_1$  and  $\beta_2$  have to be simple. Applying now Proposition 4.16, we obtain that there is a single dominant weight, say  $\lambda$ , which is neither orthogonal to  $\beta_1$  nor to  $\beta_2$ . Further if this dominant weight  $\lambda$  is a color then  $\gamma \in \Sigma(\Gamma)$

otherwise there may exist a second dominant weight non-orthogonal to one of the roots  $\beta_i$ , say  $\beta_1$ . The latter possibility is ruled out. First note that in that case,  $G_{v_\lambda}$  is not connected (see Proposition 4.16). Further  $\alpha$  has to equal  $\beta_2$  and  $\varphi$  is of weight  $\beta_1$ . But  $\beta_1$  can not be written as integral combination of the  $\lambda_i$ 's (see Proposition 4.16) hence  $\varphi$  is not fixed by  $G_{v_\lambda}/G_{v_\lambda}^\circ$ .

Suppose now that the support of  $\gamma$  is of type  $A_n$ . From Proposition 5.33, we obtain that if  $\gamma$  is not a root then  $\gamma = \alpha_{i-1} + 2\alpha_i + \alpha_{i+1}$  with  $\alpha = \alpha_i$ . Thanks to Proposition 4.16, all the dominant weights  $\lambda_k$  have to be orthogonal to both  $\alpha_{i-1}$  and  $\alpha_{i+1}$ . One thus has clearly that  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  by Lemma 5.32. If  $\gamma$  is now a root then one gets:  $\gamma = \alpha_i + \dots + \alpha_j$ . Then Proposition 5.33 (and its proof) along with Lemma 5.32 yield:  $\alpha = \alpha_i$  or  $\alpha = \alpha_j$ . Therefore if the dominant weights are orthogonal to all the simple roots in the support of  $\gamma$  except  $\alpha_i$  and  $\alpha_j$  then  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$ . If the dominant weights are not so then obviously  $\gamma - \alpha \in \Sigma(\Gamma)$ . Note that the same arguments can be applied to  $\gamma = \alpha_i + \dots + \alpha_j$  with support of type  $B_n$ .

In case of type  $B_n$ , the weight vector  $v_\gamma$  may be equal to  $X_{-\alpha}^r.v_\lambda$  whenever  $\alpha = \alpha_n$ ,  $\alpha_1 + \dots + \alpha_n \in \Sigma(\Gamma)$  and there are at least two dominant weights among the given  $\lambda_i$ 's which are not orthogonal to  $\alpha_n$ . Suppose  $\gamma$  is a root. Similarly as before, one obtains that  $\gamma = \alpha_i + \dots + \alpha_n$  or  $\gamma = 2(\alpha_i + \dots + \alpha_n)$  if  $\alpha$  lies in the support of  $\gamma$ . Let  $\gamma = 2(\alpha_i + \dots + \alpha_n)$ . Then since  $\gamma - \delta$  is not a root for each simple root distinct to  $\alpha_i$ ,  $X_{\alpha_i}.v_\gamma \neq 0$  in  $V$  and Proposition 5.33 gives:  $\alpha = \alpha_i$  otherwise  $\alpha$  will be in  $S^p$ - a contradiction. Finally, one gets that  $[v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$  whenever  $\alpha_{i+1} \in S^p$  otherwise  $\gamma - 2\alpha \in \Sigma(\Gamma)$ .

The other types can be conducted similarly. □

**5.3. Proof of Theorem 2.5.** To show the smoothness of the invariant Hilbert scheme (Theorem 2.5), we shall use the characterization of the second cotangent module given in Proposition 2.7 and thus prove the following statement by means of Proposition 2.8.

Let

$$V = V(\lambda_1) \oplus \dots \oplus V(\lambda_s)$$

where the dominant weights  $\lambda_i$  satisfy the properties of Proposition 4.16.

For short, set

$$S^2V/V(2\lambda) = \bigoplus_{1 \leq i, j \leq s} V(\lambda_i) \cdot V(\lambda_j)/V(\lambda_i + \lambda_j).$$

Take  $\dot{\varphi}$  in  $H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)$  and denote by  $\varphi$  a cocycle representing  $\dot{\varphi}$ . Recall the definition of the map  $f$ : let  $X_\alpha$  be any root vector of the

isotropy Lie algebra  $\mathfrak{g}_{v_\lambda}$ , we have

$$f\varphi(X_\alpha) = \sum_i \varphi(X_\alpha) \cdot v_{\lambda_i}.$$

**Proposition 5.34.** *The map*

$$f : H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda) \rightarrow H^1(\mathfrak{g}_{v_\lambda}, S^2V/V(2\lambda))$$

*is injective.*

*Proof.* To prove the injectivity of  $f$ , we can assume without loss of generality that  $\varphi$  is a  $T_{\text{ad}}$ -weight vector. By Proposition 2.8,  $\varphi = \varphi_{\alpha, \gamma}$  for some  $\alpha$  simple and  $\gamma \in S(\Gamma)$  or  $\gamma = \lambda_i$ . Set for convenience  $\varphi(X_\alpha) = [v_{s_\alpha * \gamma}]$ .

It is thus enough to prove that there exists  $v_{s_\alpha * \gamma} \cdot v_{\lambda_i}$  non-trivial in  $S^2V/V(2\lambda)$  for which there is no  $v \in S^2V/V(2\lambda)$  such that  $v_{s_\alpha * \gamma} \cdot v_{\lambda_i} = X_\alpha v$  in  $S^2V/V(2\lambda)$ .

Note that this assertion holds whenever

$$(4) \quad X_{-\alpha}(v_{s_\alpha * \gamma} \cdot v_{\lambda_i}) = 0 \quad \text{in } S^2V/V(2\lambda)$$

or

$$(5) \quad X_\alpha^a v_{s_\alpha * \gamma} \neq 0 \text{ in } V \quad \text{for } a = (\lambda_i, \alpha^\vee).$$

Let us consider first  $X_{-\alpha}(v_{s_\alpha * \gamma} \cdot v_{\lambda_i})$ . Note that by definition of  $r$ , we have:  $X_{-\alpha} v_{s_\alpha * \gamma} = 0$  in  $V$ . We thus have

$$X_{-\alpha}(v_{s_\alpha * \gamma} \cdot v_{\lambda_i}) = v_{s_\alpha * \gamma} \cdot X_{-\alpha} v_{\lambda_i}.$$

When  $\lambda_i$  is orthogonal to  $\alpha$ , assertion (4) to be proved is thus clear.

Suppose that  $\lambda_i$  is not orthogonal to  $\alpha$ . If  $(\lambda_i, \gamma) \neq 0$ , we have  $(\gamma, \alpha) \geq 0$  by Proposition 4.16. We shall prove assertion (5) considering the cases where  $(\lambda_i, \gamma) = 0$  and  $(\lambda_i, \gamma) \neq 0$  separately; this is done in the next lemmas.  $\square$

**Lemma 5.35.** *Let  $\varphi(X_\alpha) = X_{-\alpha}^r v_{\lambda_j}$  for some  $\lambda_j$ . Then Assertion (5) holds with  $\lambda_i = \lambda_j$ .*

*Proof.* Note that  $r = (\lambda_j, \alpha^\vee)$ . Hence if  $r > 1$  then  $X_{-\alpha}^r v_{\lambda_j} \cdot v_{\lambda_j} \neq 0$  and Assertion (5) is clear. If  $r = 1$  then there exists  $\lambda_i \neq \lambda_j$  which is not orthogonal to  $\alpha$ . But by Proposition 4.16 this implies that such a  $\lambda_i$  can be chosen to be fundamental. Assertion (5) follows with that chosen  $\lambda_i$ .  $\square$

**Lemma 5.36.** *Let  $v_{s_\alpha * \gamma} \cdot v_{\lambda_i} \neq 0$  in  $S^2V/V(2\lambda)$  with  $\gamma \in \Sigma(\Gamma)$ . If  $\lambda_i$  is orthogonal to  $\gamma$  then Assertion (5) holds.*



*Proof.* Note first that the support of  $\gamma$  does not contain  $\alpha$ . Indeed  $\lambda_i$  being orthogonal to  $\gamma$  it can not be orthogonal to  $\alpha$  otherwise  $v_{s_\alpha * \gamma} \cdot v_{\lambda_i}$  will be 0. Hence  $(\gamma, \alpha^\vee) \leq 0$  and further  $X_\alpha v_\gamma = 0$  in  $V$ . It follows in turn that  $v_{s_\alpha * \gamma} = X_{-\alpha}^r v_\gamma$  with  $r = (\lambda - \gamma, \alpha^\vee)$  and  $v_\gamma \in V(\lambda)$ . Further, since  $v_{s_\alpha * \gamma} \cdot v_{\lambda_i} \neq 0$  and  $(\lambda_i, \gamma) = 0$ , we have:  $v_{s_\alpha * \gamma} \neq v_\gamma$ .

The weight  $\lambda$  being non-orthogonal to  $\gamma$ , it is different from  $\lambda_i$ .

Assume that  $(\gamma, \alpha) = 0$  then since  $\alpha$  does not belong to the support of  $\gamma$ , it has to be orthogonal to every simple root lying in the support of  $\gamma$ . Let  $\delta$  be a simple root  $\delta$  in the support of  $\gamma$  such that  $X_\delta v_\gamma \neq 0$  in  $V$ . Considering  $X_\delta \varphi(X_\alpha)$  along with  $(\lambda, \alpha)$ , one ends up with a contradiction.

We deduce that  $(\gamma, \alpha) < 0$ . Thanks to Proposition 4.16,  $\lambda_i$  is the single weight among the  $\lambda_j$ 's which is not orthogonal to  $\alpha$ . Recall that  $\gamma$  belongs to  $\mathbb{Z}\Gamma$ . It follows that  $r = (\lambda - \gamma, \alpha^\vee) \geq (\lambda_i, \alpha^\vee)$  whence the lemma.  $\square$

**Lemma 5.37.** *Suppose  $(\gamma, \alpha^\vee) > 1$  with  $\gamma \in \Sigma(\Gamma)$ . Then Assertion (5) holds with  $\lambda_i$  such that  $v_\gamma \in V(\lambda_i)$ .*

*Proof.* If  $\gamma = 2\alpha$ , we fall in the case of Lemma 5.35.

By Table 1, the weight  $\gamma$  under consideration is such that  $(\gamma, \alpha^\vee) = 2$ . Further  $\gamma - \delta$  is a root with  $\delta$  a simple root if and only if  $\delta$  equals  $\alpha$ . Hence  $X_\delta v_\gamma = 0$  in  $V$  for all simple roots  $\delta$  distinct to  $\alpha$  and a fortiori  $X_\alpha v_\gamma \neq 0$  in  $V$ . Moreover  $\gamma - 2\alpha$  not being a root, we have:  $X_\alpha^2 v_\gamma = 0$  in  $V$ . Since  $(\lambda_i, \alpha^\vee)$  equals 1 or 2, we have respectively  $v_{s_\alpha * \gamma}$  equals  $v_\gamma$  or  $X_{-\alpha} v_\gamma$ . The lemma follows readily.  $\square$

**Lemma 5.38.** *Let  $\gamma \in \Sigma(\Gamma)$  and  $(\gamma, \alpha^\vee) = 1$  for some simple root  $\alpha$ . Then Assertion (5) holds.*

*Proof.* Note that  $\gamma - \alpha$  is a root; see Table 1. By Proposition 4.16, there exists a unique  $\lambda$  non-orthogonal to  $\alpha$  and  $(\lambda, \alpha^\vee) = 1$ . Further  $v_\gamma$  can be chosen in  $V(\lambda)$ . If  $X_\alpha v_\gamma = 0$  in  $V$ , it follows from [J07] and Lemma 5.26 that  $[v_\gamma] = [X_{-\gamma} v_\lambda] = [X_{-\gamma} v_{\lambda_j}]$  for some  $\lambda_j \neq \lambda$  and such that  $(\lambda_j, \gamma - \alpha) \neq 0$ . In particular  $X_\alpha v_\gamma \neq 0$  in  $V$  for  $v_\gamma = X_{-\gamma} v_{\lambda_j}$ . Then  $v_\gamma$  can be chosen such that  $v_\gamma \in V(\lambda_k)$  with  $X_\alpha v_\gamma \neq 0$  in  $V$  and  $\lambda_k = \lambda$  or  $\lambda_j$  as above. Assertion (5) thus holds with  $\lambda_i = \lambda$ .  $\square$

**Lemma 5.39.** *Let  $\alpha$  be a simple root non-orthogonal to the monoid  $\Gamma$  ( $\alpha \notin S^p(\Gamma)$ ). Suppose  $(\gamma, \alpha) = 0$  then Assertion (4) or Assertion (5) holds.*

*Proof.* First assume that  $\alpha$  does not belong to the support of  $\gamma$ . Then  $\alpha$  is orthogonal to every simple root in the support of  $\gamma$ . Let  $v_\gamma \in V(\lambda)$ . It follows that  $v_{s_\alpha * \gamma} = v_\gamma$  in  $V$  if and only if  $(\lambda, \alpha) = 0$ . If  $v_\gamma \cdot v_\lambda \neq 0$  then

Assertion (4) holds whenever  $(\lambda, \alpha) = 0$ . If  $v_\gamma \cdot v_\lambda = 0$  then  $v_\gamma = X_{-\gamma}v_\lambda$  and there exists  $\lambda_j \neq \lambda$  such that  $0 \neq v_\gamma \cdot v_{\lambda_j} = v_\lambda \cdot X_{-\gamma}v_{\lambda_j}$  whence Assertion (4) whenever  $(\lambda, \alpha) = 0$ . Let now  $(\lambda, \alpha) \neq 0$ . Note that  $X_\alpha v_\gamma = 0$  in  $V$  since  $\alpha$  does not belong to the support of  $\gamma$ . Then  $v_{s_\alpha \gamma} = X_{-\alpha}^r v_\gamma$  with  $r = (\lambda - \gamma, \alpha^\vee) = (\lambda, \alpha^\vee)$ . Further  $v_{s_\alpha \gamma} \notin \mathfrak{g} \cdot v_\lambda$ ; Assertion (5) thus holds with  $\lambda_i = \lambda$ .

Assume now that  $\alpha$  lies in the support of  $\gamma$ . Then by Lemma 5.24,  $\gamma - \alpha$  has to be a root; the type  $F_4$  is ruled out by Lemma 5.27. More precisely  $\gamma$  is a root of type  $B_n$  or  $C_n$ . Further in type  $B_n$ , we can choose  $v_\gamma = X_{-\gamma}v_\lambda$  whereas  $v_\gamma \in V(\lambda) \setminus \mathfrak{g} \cdot v_\lambda$  in type  $C_n$  along with  $(\lambda, \alpha) = 0$  in both cases. In the first situation,  $v_{s_\alpha \gamma} = X_{-\gamma-\alpha}v_\lambda$  and there exists  $\lambda_i \neq \lambda$  non-orthogonal to  $\gamma$ . In type  $B_n$ , we then have  $0 \neq v_{s_\alpha \gamma} \cdot v_{\lambda_i} = X_{-\gamma-\alpha}v_{\lambda_i} \cdot v_\lambda$  whence Assertion (4). In type  $C_n$ , Assertion (4) holds with  $\lambda_i = \lambda$ .  $\square$

**5.4. Proof of Proposition 4.16.** Given  $\gamma$  a spherical root, recall the definition of  $S(\gamma)$ ; see Proposition 4.16. Note that  $S(\gamma)$  is of cardinality at most 2; see Table 1.

In this section, we are given a saturated monoid  $\Gamma$  spanned by linearly independent dominant weights  $\lambda_i, i = 1, \dots, s$ . Recall the definitions of  $S^p(\Gamma)$  and  $\Sigma(\Gamma)$ ; see the paragraph before Proposition 2.8. We assume also that the support of  $\Sigma(\Gamma)$  consists of the whole set of simple roots of the group  $G$ .

Proposition 4.16 follows from the lemmas stated below.

**Lemma 5.40.** *Let  $\gamma \in \Sigma(\Gamma)$  be such that  $S(\gamma)$  consists of two distinct simple roots  $\alpha$  and  $\beta$ . If one of the dominant weights  $\lambda_i$  is neither orthogonal to  $\alpha$  nor to  $\beta$  then all the others are orthogonal to both  $\alpha$  and  $\beta$ .*

*Proof.* The case of  $\gamma$  being of type  $F_4$  is solved in Lemma 5.27.

Let  $[v_\gamma]$  be the  $T_{\text{ad}}$ -weight vector in  $(V/\mathfrak{g} \cdot v_\lambda)^{G_{v_\lambda}}$  of weight the given  $\gamma$ . Choose  $v_\gamma \in \oplus V(\lambda_i)_{\lambda_i - \gamma}$ . Let  $\lambda$  be one of the given  $\lambda_i$ 's. Suppose it is neither orthogonal to  $\alpha$  nor to  $\beta$ .

First note that because of saturation, all the  $\lambda_i$ 's except  $\lambda$  have to be orthogonal to (for instance)  $\beta$ . If  $\gamma$  is orthogonal to one of the simple roots,  $\alpha$  and  $\beta$ , then this simple root has to be  $\alpha$ .

Take a representative  $v_\gamma$  of  $[v_\gamma]$  as in Lemma 5.26.

Recall that  $\gamma - \delta$  is not a root for all simple roots  $\delta$  distinct to  $\alpha$  and  $\beta$ . Since  $X_\delta \cdot v_\gamma \in \mathfrak{g} \cdot v_\lambda$ , we have:  $X_\delta \cdot v_\gamma = 0$  in  $V$  for such  $\delta$ 's. It follows that  $X_\alpha \cdot v_\gamma \neq 0$  or  $X_\beta \cdot v_\gamma \neq 0$  in  $V$ ,  $v_\gamma$  not being a highest weight vector in  $V$ .

If  $(V(\lambda)/\mathfrak{g} \cdot v_\lambda)^{G_{v_\lambda}}$  is not trivial then  $(\gamma, \alpha)$  and  $(\gamma, \beta)$  are both strictly positive; see [J07]. Further one can thus choose the representative  $v_\gamma$

in  $V(\lambda)$  and such that  $X_{\beta}.v_{\gamma} \neq 0$  in  $V$ ; in particular  $X_{\beta}.v_{\gamma} \in \mathfrak{g}.v_{\lambda}$ . For  $X_{\beta}.v_{\gamma} \in \mathfrak{g}.v_{\lambda}$ , we have  $X_{-\gamma+\beta}.v_{\lambda_i} = 0$  for every  $\lambda_i \neq \lambda$  hence the  $\lambda_i$ 's except  $\lambda$  have to be orthogonal to  $\alpha$ .

If  $(V(\lambda)/\mathfrak{g}.v_{\lambda})^{G_{v_{\lambda}}}$  is trivial then  $v_{\gamma} = X_{-\gamma}.v_{\lambda}$ . Since  $\gamma - \beta$  is a root, we have:  $X_{\alpha}.v_{\gamma} = X_{-\gamma+\beta}.v_{\lambda} \neq 0$  in  $V$ . We conclude as before that all the  $\lambda_i$ 's but  $\lambda$  are orthogonal to  $\alpha$ .  $\square$

As a straightforward consequence of Lemma 5.40, we obtain the following statement.

**Lemma 5.41.** *Consider a spherical root system with a spherical root  $\gamma = \alpha_1 + \dots + \alpha_n$  of type  $A_n$ ,  $n > 1$ . Then we may choose one of the dominant weight to be  $\omega_1 + \omega_n$  if and only if one of the following occurs*

- (i) *this spherical system is of rank 1 or*
- (ii) *all the spherical roots  $\gamma'$  such that  $(\gamma', \gamma) \neq 0$  are of type  $A_1 \times A_1$ .*

**Lemma 5.42.** *Let  $\lambda$  be one of the given dominant weights  $\lambda_i$ . Let  $\alpha$  and  $\beta$  be two distinct simple roots. Assume that  $(\lambda_i, \alpha) = (\lambda_i, \beta) = 0$  for all  $\lambda_i = \lambda$  and that  $(\lambda, \alpha).(\lambda, \beta) \neq 0$ . Then one of the following occurs:*

- (i)  $(\gamma, \alpha) = 0 = (\gamma, \beta)$  for all  $\gamma \in \Sigma(\Gamma)$ .
- (ii)  $(\lambda, \delta) = 0$  for every simple root  $\delta$  distinct to  $\alpha$  and  $\beta$ .

*Proof.* Since every  $T_{\text{ad}}$ -weight  $\gamma$  in  $\Sigma(\Gamma)$  is an integral sum of the dominant weights, we have:  $(\gamma, \alpha) > 0$  if and only if  $(\gamma, \beta) > 0$  for all  $\gamma \in \Sigma(\Gamma)$ . Suppose there exists  $\gamma \in \Sigma(\Gamma)$  such that  $(\gamma, \alpha) > 0$  then  $(\gamma, \delta) \leq 0$  for every simple root  $\delta$  distinct to  $\alpha$  and  $\beta$  by Lemma 5.24. The second item of the lemma thus follows from the definition of spherical systems and Lemma 5.41.  $\square$

**Lemma 5.43.** *Let  $\alpha$  be a simple root. Suppose there exist two distinct dominant weights  $\lambda$  and  $\lambda'$  among the given  $\lambda_i$ 's which are not orthogonal to  $\alpha$ . Then  $(\gamma, \alpha) = 0$  for all  $\gamma \in \Sigma(\Gamma)$ .*

*Proof.* Take  $\gamma \in \Sigma(\Gamma)$  such that  $\alpha$  belongs to the support of  $\gamma$ . Note that such a  $\gamma$  does exist by assumption of the support of  $\Sigma(\Gamma)$ . Further  $(\gamma, \alpha) \geq 0$  (see Table 1) and by Proposition 5.24,  $\gamma - \alpha$  has to be a root. By Lemma 5.40 we should have:  $(\gamma, \alpha) = 0$ . But this happens only in type  $B_n$  or  $C_n$  (see Table 1). In type  $B_n$ ,  $\alpha$  being extremal, the proposition follows readily. Consider now the type  $C_n$  and suppose there exists  $\gamma' \in \Sigma(\Gamma)$  such that  $(\gamma', \alpha) < 0$ . Then there is another simple root, say  $\alpha'$ , such that  $(\gamma', \alpha') < 0$  because of saturation. Such a root  $\alpha'$  lies in the support of some  $\gamma'' \in \Sigma(\Gamma)$ . Further note that  $(\gamma'', \alpha) = 0$  and  $(\gamma'', \alpha') > 0$ . Hence there should exist at least two

dominant weights among the  $\lambda_i$ 's which are not orthogonal to  $\alpha'$ . The previous arguments applied to  $\gamma''$  and  $\alpha'$  lead to a contradiction. Hence  $\gamma'$  has to be orthogonal to  $\alpha$ .  $\square$

**Lemma 5.44.** *If  $\lambda_i$  is (up to a scalar) a fundamental weight. Then  $\lambda_i$  is a (up to a scalar) a color in  $\Delta(\Gamma)$ .*

*Proof.* Let  $\alpha$  be the simple root which is not orthogonal to the given weight  $\lambda_i$ . Take  $\gamma \in \Sigma(\Gamma)$  whose support contains  $\alpha$ . Then  $\gamma - \alpha$  has to be a root by Lemma 5.24. If  $\gamma$  is orthogonal to  $\alpha$  then  $\gamma$  is of type  $B_n$  or  $C_n$  whence the lemma. If  $(\gamma, \alpha) > 0$  then  $\lambda_i$  is a color except if  $\gamma$  is of type  $A_1 \times A_1$ . In the latter case, observe that  $\gamma \in \Sigma(\Gamma)$  whenever one of the  $\lambda_j$ 's is neither orthogonal to  $\alpha$  nor  $\alpha'$  for  $\gamma = \alpha + \alpha'$ . But because of saturation this is not possible.  $\square$

*Proof of Proposition 4.17.* Let  $(S^p, \Sigma)$  be a spherical system of  $G$  and  $\Delta$  be its set of colors. Recall the definition of  $\Gamma$  stated in Proposition 4.17 and let  $V$  be the corresponding  $G$ -module. Take  $\gamma \in \Sigma$ . We shall prove that  $\gamma \in \Sigma(\Gamma)$ , namely that  $\gamma$  is a  $T_{\text{ad}}$ -weight of  $(V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$ .

By the characterization of the monoid  $\Gamma$  along with Lemma 5.26 (see also Lemma 5.27), one gets that in most cases, there exists a unique dominant weight among the given  $\lambda_i$ 's which is not orthogonal to the given  $\gamma$ . Proposition 4.17 thus follows from [J07].

Suppose now that there are more than one dominant weight which is not orthogonal to  $\gamma$ . Then by definition, this occurs whenever  $S(\gamma)$  is a 2-set. By Table 1, the weight  $\gamma$  has to be a root. One thus checks easily that  $X_{-\gamma}v_\lambda$  is indeed a  $T_{\text{ad}}$ -weightvector of  $(V(\underline{\lambda})/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}}$ .  $\square$

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