Quantized mixed tensor space and Schur–Weyl duality II

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Abstract

In this paper, we show the second part of Schur-Weyl duality for mixed tensor space. The quantum group $\mathbf{U} = U(\mathfrak{gl}_\mathfrak{n})$ of the general linear group and a q-deformation $\mathfrak{B}^n_{r,s}(q)$ of the walled Brauer algebra act on $V^{\otimes r} \otimes V^{*\otimes s}$ where $V = R^n$ is the natural $\mathbf{U}$-module. We show that $\mathrm{End}_{\mathfrak{B}^n_{r,s}(q)}(V^{\otimes r} \otimes V^{*\otimes s})$ is the image of the representation of $\mathbf{U}$, which we call the rational q-Schur algebra. As a byproduct, we obtain a basis for the rational q-Schur algebra. This result holds whenever the base ring R is a commutative ring with one and q an invertible element of R.

Key words: Schur-Weyl duality, walled Brauer algebra, mixed tensor space, rational q-Schur algebra
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Introduction

Schur-Weyl duality plays an important role in representation theory since it relates the representations of the general linear group with the representations of the symmetric group. The classical Schur-Weyl duality due to Schur ([14]) states that the actions of the general linear group $G = GL_n$ and the symmetric group $\mathfrak{S}_m$ on tensor space $V^{\otimes m}$ with $V = \mathbb{C}^n$ satisfy the bicentralizer property, that is $\mathrm{End}_{\mathfrak{S}_m}(V^{\otimes m})$ is generated by the action of G and correpondingly, $\mathrm{End}_G(V^{\otimes m})$ is generated by the action of $\mathfrak{S}_m$. This duality

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has been generalized to subgroups and groups related with G (e. g. orthogonal, symplectic groups, Levi subgroups) and corresponding algebras related with the group algebra of the symmetric group (e. g. Brauer algebras, Ariki-Koike algebras), as well as deformations of these algebras. In general, the phrase 'Schur-Weyl duality' has come to indicate such a bicentralizer property for two algebras acting on some module.

One such generalization is the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ where V is the natural G- or equivalently $\mathbb{C}G$-module and V^* its dual, thus $V^{\otimes r} \otimes V^{*\otimes s}$ has a $\mathbb{C}G$-module structure as the tensor product of $\mathbb{C}G$-modules. The centralizer algebra is known to be the walled Brauer algebra $\mathfrak{B}^n_{r,s}$ ([1]), and $\mathbb{C}G$ and $\mathfrak{B}^n_{r,s}$ satisfy the bicentralizer property on mixed tensor space. For $s = 0$, one recovers the classical Schur-Weyl duality. In that case, the walled Brauer algebra $\mathfrak{B}^n_{r,0}$ coincides with the group algebra of the symmetric group $\mathfrak{S}_r$. A q-deformation of these results also exist ([11, 12]), but only for cases when the centralizer algebra is semisimple.

In this paper, we generalize the results of [1, 11, 12] for a very general setting. Let R be a commutative ring with 1 and $q \in R$ invertible. Let $\mathbf{U}$ be (an integral version of) the quantum group over R, which replaces the general linear group in the quantized case. Let $\mathfrak{B}^n_{r,s}(q)$ be a certain q-deformation of the walled Brauer algebra defined by Leduc [12]. In [4], we gave an alternative, combinatorial desription of this algebra in terms of knot diagrams. This algebra acts on mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ where $V = R^n$ is the natural $\mathbf{U}$-module. Let $S_q(n; r, s) = \mathrm{End}_{\mathfrak{B}^n_{r,s}(q)}(V^{\otimes r} \otimes V^{*\otimes s})$ be the centralizer algebra of the action of $\mathfrak{B}^n_{r,s}(q)$. We call $S_q(n; r, s)$ the *rational q-Schur algebra*. The main result of this paper states that $S_q(n; r, s)$ is the image of the representation of $\mathbf{U}$ on $V^{\otimes r} \otimes V^{*\otimes s}$. So far, this was only known for $R = \mathbb{C}$, $q = 1$ and $R = \mathbb{C}(q)$. In these cases, the algebras act semisimply on the mixed tensor space, so it suffices to decompose this module into irreducible modules. We show that the rational q-Schur algebra is free over R and determine an explicit combinatorial basis. This generalizes the result of [3], which gives a basis for infinite fields with $q = 1$.

The other part of Schur-Weyl duality for mixed tensor space is also true, and was shown in [4]: the centralizer algebra of the $\mathbf{U}$-action on mixed tensor space is generated by $\mathfrak{B}^n_{r,s}(q)$. To show that result, we used the well-known duality between the Hecke algebra $\mathcal{H}_{r+s}(q) \cong \mathfrak{B}^n_{r+s,0}(q)$ and the quantum group $\mathbf{U}$ on tensor space $V^{\otimes r+s}$. This paper now completes the proof of Schur-Weyl duality for mixed tensors.

The main problem to show this part of Schur-Weyl duality is that it is a priori not clear that the centralizer algebra of $\mathfrak{B}^n_{r,s}(q)$ is R-free of rank independent of the ground ring R.

In order to prove Schur-Weyl duality in the general case we make use of the following fact: the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ can be embedded into a tensor space $V^{\otimes r+(n-1)s}$. Although this embedding κ is not a homomorphism of $\mathbf{U}$-modules, it is a homomorphism of $\mathbf{U}'$-modules where $\mathbf{U}'$ is the subalgebra of $\mathbf{U}$ corresponding to the special linear group. We will see that replacing $\mathbf{U}$ by $\mathbf{U}'$ is not significant. Associated with this embedding is an algebra homomorphism $\pi : S_q(n, r + (n-1)s) \to S_q(n; r, s)$ where $S_q(n, r+(n-1)s)$ is the image of the representation of $\mathbf{U}'$ on $V^{\otimes r+(n-1)s}$, the *(ordinary) q-Schur algebra*. This homomorphism was motivated by [3] and is given by restriction to the $\mathbf{U}'$-submodule $V^{\otimes r} \otimes V^{*\otimes s}$ of $V^{\otimes r+(n-1)s}$.

Let $\rho_{\text{ord}} : \mathbf{U}' \to S_q(n, r + (n-1)s)$ be the representation of $\mathbf{U}'$ on $V^{\otimes r+(n-1)s}$. Similarly, let $\rho_{\text{mxd}} : \mathbf{U}' \to S_q(n; r, s)$ be the representation of $\mathbf{U}'$ on $V^{\otimes r} \otimes V^{*\otimes s}$. Since $\rho_{\text{mxd}} = \pi \circ \rho_{\text{ord}}$ and ρ_{ord} is surjective, ρ_{mxd} is surjective (i.e. Schur-Weyl duality for the mixed tensor space holds) if π is surjective. It remains to show that π is surjective or equivalently, that π has a right inverse. We show a stronger statement, namely that the right inverse can be chosen to be a homomorphism of R-modules.

At this point, we switch over to the coefficient spaces: $S_q(n, r + (n-1)s)$ and $S_q(n; r, s)$ are dual algebras of coalgebras $A_q(n, r+(n-1)s)$ and $A_q(n; r, s)$ respectively. We define a map $\iota : A_q(n; r, s) \to A_q(n, r + (n-1)s)$ such that $\pi = \iota^*$. Thus π has a right inverse if ι has a left inverse, namely the dual of this left inverse. So the problem is reduced to find a left inverse of ι.

For this purpose, we give suitable bases for $A_q(n, r + (n-1)s)$ and $A_q(n; r, s)$, such that the matrix of ι with respect to this bases has a nice form. The description of $A_q(n, r + (n-1)s)$ and a basis thereof is well known (see [2, 9]). The basis is indexed by standard bitableaux. We develop a basis for $A_q(n; r, s)$ and thus for $S_q(n; r, s)$ which is indexed by pairs of so-called rational standard bitableaux. To show that the basis elements generate $A_q(n; r, s)$, we develop the *Rational Straightening Algorithm*, which is applied together with the well known straightening algorithm for bideterminants. Note that the resulting basis of $S_q(n; r, s)$ does not coincide with the basis for $q = 1$ in [3] which arises by mapping a cellular basis with the surjection π.

Using q-versions of determinantal identities such as the Laplace Expansion and Jacobi's Ratio Theorem, we show that a basis element of $A_q(n; r, s)$ maps under ι to a multiple of a basis element of $A_q(n, r + (n-1)s)$, thus a left inverse is easy to find, which proves the main results.

1. Preliminaries

Let n be a given positive integer. In this section, we introduce the quantized enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n$ over a commutative ring R with parameter q and summarize some well known results; see for example [8, 10, 13]. We will start by recalling the definition of the quantized enveloping algebra over $\mathbb{Q}(q)$ where q is an indeterminate.

Let $P^\vee$ be the free $\mathbb{Z}$-module with basis $h_1, \ldots, h_n$ and let $\varepsilon_1, \ldots, \varepsilon_n \in P^{\vee *}$ be the corresponding dual basis: ε_i is given by $\varepsilon_i(h_j) := \delta_{i,j}$ for $j = 1, \ldots, n$, where δ is the usual Kronecker symbol. For $i = 1, \ldots, n-1$ let $\alpha_i \in P^{\vee *}$ be defined by $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$.

Definition 1.1. The quantum general linear algebra $U_q(\mathfrak{gl}_n)$ is the associative $\mathbb{Q}(q)$-algebra with 1 generated by the elements e_i, f_i $(i = 1, \ldots, n-1)$ and q^h $(h \in P^\vee)$ with the defining relations

$$
\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \\
&e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \qquad \text{where } K_i := q^{h_i - h_{i+1}}, \\
&e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \qquad \text{for } |i-j| = 1, \\
&f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \qquad \text{for } |i-j| = 1, \\
&e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \qquad \text{for } |i-j| > 1.
\end{aligned}
$$

We note that the subalgebra generated by the K_i, e_i, f_i $(i = 1, \ldots, n-1)$ is isomorphic with $U_q(\mathfrak{sl}_n)$. $U_q(\mathfrak{gl}_n)$ is a Hopf algebra with comultiplication Δ, counit ε and antipode S defined by

$$
\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h, \\
&\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \\
&\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
&S(q^h) = q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i.
\end{aligned}
$$

Note that Δ and ε are homomorphisms of algebras and S is an invertible anti-homomorphism of algebras. Let $V_{\mathbb{Q}(q)}$ be a free $\mathbb{Q}(q)$-vector space with basis $\{v_1, \ldots, v_n\}$. We make $V_{\mathbb{Q}(q)}$ into a $U_q(\mathfrak{gl}_n)$-module via

$$\begin{aligned} q^h v_j &= q^{\varepsilon_j(h)} v_j \text{ for } h \in P^\vee,\ j = 1, \ldots, n \\ e_i v_j &= \begin{cases} v_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \qquad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We call $V_{\mathbb{Q}(q)}$ the *vector representation* of $U_q(\mathfrak{gl}_n)$. This is also a $U_q(\mathfrak{sl}_n)$-module, by restriction of the action.

Let $[l]_q$ (in $\mathbb{Z}[q, q^{-1}]$ resp. in R) be defined by $[l]_q := \sum_{i=0}^{l-1} q^{2i-l+1}$, $[l]_q! := [l]_q[l-1]_q \ldots [1]_q$ and let $e_i^{(l)} := \frac{e_i^l}{[l]_q!}$, $f_i^{(l)} := \frac{f_i^l}{[l]_q!}$. Let $\mathbf{U}_{\mathbb{Z}[q,q^{-1}]}$ (resp., $\mathbf{U}'_{\mathbb{Z}[q,q^{-1}]}$) be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{gl}_n)$ generated by the q^h (resp., K_i) and the divided powers $e_i^{(l)}$ and $f_i^{(l)}$ for $l \geq 0$. $\mathbf{U}_{\mathbb{Z}[q,q^{-1}]}$ is a Hopf algebra and we have

$$\begin{aligned} \Delta(e_i^{(l)}) &= \sum_{k=0}^{l} q^{k(l-k)} e_i^{(l-k)} \otimes K_i^{k-l} e_i^{(k)} \\ \Delta(f_i^{(l)}) &= \sum_{k=0}^{l} q^{-k(l-k)} f_i^{(l-k)} K_i^k \otimes f_i^{(k)} \\ S(e_i^{(l)}) &= (-1)^l q^{l(l-1)} e_i^{(l)} K_i^l \\ S(f_i^{(l)}) &= (-1)^l q^{-l(l-1)} K_i^{-l} f_i^{(l)} \\ \varepsilon(e_i^{(l)}) &= \varepsilon(f_i^{(l)}) = 0. \end{aligned}$$

Furthermore, the $\mathbb{Z}[q, q^{-1}]$-lattice $V_{\mathbb{Z}[q,q^{-1}]}$ in $V_{\mathbb{Q}(q)}$ generated by the v_i is invariant under the action of $\mathbf{U}_{\mathbb{Z}[q,q^{-1}]}$ and of $\mathbf{U}'_{\mathbb{Z}[q,q^{-1}]}$. Now, make the transition from $\mathbb{Z}[q, q^{-1}]$ to an arbitrary commutative ring R with 1: Let $q \in R$ be invertible and consider R as a $\mathbb{Z}[q, q^{-1}]$-module via specializing $q \in \mathbb{Z}[q, q^{-1}] \mapsto q \in R$. Then, let $\mathbf{U}_R := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbf{U}_{\mathbb{Z}[q,q^{-1}]}$ and $\mathbf{U}'_R := R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbf{U}'_{\mathbb{Z}[q,q^{-1}]}$. $\mathbf{U}_R$ inherits a Hopf algebra structure from $\mathbf{U}_{\mathbb{Z}[q,q^{-1}]}$ and $V_R := R \otimes_{\mathbb{Z}[q,q^{-1}]} V_{\mathbb{Z}[q,q^{-1}]}$ is a $\mathbf{U}_R$-module and by restriction also a $\mathbf{U}'_R$-module.

If no ambiguity arises, we will henceforth omit the index R and write $\mathbf{U}$, $\mathbf{U}'$ instead of $\mathbf{U}_R$, $\mathbf{U}'_R$ and V instead of V_R. Furthermore, we will write $e_i^{(l)}$ as shorthand for $1 \otimes e_i^{(l)} \in \mathbf{U}_R$, similarly for the $f_i^{(l)}$, K_i short for $1 \otimes K_i$, and q^h short for $1 \otimes q^h$.

Suppose W, W_1 and W_2 are $\mathbf{U}$-modules, then one can define $\mathbf{U}$-module structures on $W_1 \otimes W_2 = W_1 \otimes_R W_2$ and $W^* = \mathrm{Hom}_R(W, R)$ using the comultiplication and the antipode by setting $x(w_1 \otimes w_2) = \Delta(x)(w_1 \otimes w_2)$ and $(xf)(w) = f(S(x)w)$.

Definition 1.2. Let r, s be nonnegative integers. The $\mathbf{U}$-module $V^{\otimes r} \otimes V^{*\otimes s}$ is *called mixed tensor space.*

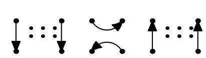
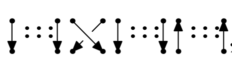
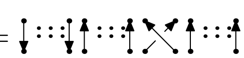
Let $I(n, r)$ be the set of r-tuples with entries in $\{1, \ldots, n\}$ and let $I(n, s)$ be defined similarly. The elements of $I(n, r)$ (and $I(n, s)$) are called *multi indices*. Note that the symmetric groups $\mathfrak{S}_r$ and $\mathfrak{S}_s$ act on $I(n, r)$ and $I(n, s)$ respectively from the right by place permutation, that is if s_j is a Coxeter generator and $\mathbf{i} = (i_1, i_2, \ldots)$ is a multi index, then let $\mathbf{i}.s_j = (i_1, \ldots, i_{j-1}, i_{j+1}, i_j, i_{j+2}, \ldots)$. Then a basis of the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ can be indexed by $I(n, r) \times I(n, s)$. For $\mathbf{i} = (i_1, \ldots, i_r) \in I(n, r)$ and $\mathbf{j} = (j_1, \ldots, j_s) \in I(n, s)$ let

$$v_{\mathbf{i}|\mathbf{j}} = v_{i_1} \otimes \ldots \otimes v_{i_r} \otimes v_{j_1}^* \otimes \ldots \otimes v_{j_s}^* \in V^{\otimes r} \otimes V^{*\otimes s}$$

where $\{v_1^*, \ldots, v_n^*\}$ is the basis of V^* dual to $\{v_1, \ldots, v_n\}$. Then $\{v_{\mathbf{i}|\mathbf{j}} \mid \mathbf{i} \in I(n, r), \mathbf{j} \in I(n, s)\}$ is a basis of $V^{\otimes r} \otimes V^{*\otimes s}$.

We have another algebra acting on $V^{\otimes r} \otimes V^{*\otimes s}$, namely the quantized walled Brauer algebra $\mathfrak{B}_{r,s}^n(q)$ introduced in [4]. This algebra is defined as a diagram algebra, in terms of Kauffman's tangles. A presentation by generators and relations can be found in [4]. Note that this algebra and its action coincides with Leduc's algebra ([12], see the remarks in [4]).

Here, all we need is the action of generators given in the following diagrams. $\mathfrak{B}_{r,s}^n(q)$ is generated by the elements

$E =$ , $S_i =$ , $\hat{S}_j =$ 

where the non-propagating edges in E connect vertices in columns r, $r + 1$ while the crossings in S_i and $\hat{S}_j$ connect vertices in columns i, $i + 1$ and columns $r + j$, $r + j + 1$ respectively. If $v_{\mathbf{i}|\mathbf{j}} = v \otimes v_{i_r} \otimes v_{j_1}^* \otimes v'$, then the action of the generators on $V^{\otimes r} \otimes V^{*\otimes s}$ is given by

$$\begin{aligned} v_{\mathbf{i}|\mathbf{j}}E &= \delta_{i_r,j_1} \sum_{s=1}^{n} q^{2i_r-n-1} v \otimes v_s \otimes v_s^* \otimes v' \\ v_{\mathbf{i}|\mathbf{j}}S_i &= \begin{cases} q^{-1}v_{\mathbf{i}|\mathbf{j}} & \text{if } i_i = i_{i+1} \\ v_{\mathbf{i}.s_i|\mathbf{j}} & \text{if } i_i < i_{i+1} \\ v_{\mathbf{i}.s_i|\mathbf{j}} + (q^{-1}-q)v_{\mathbf{i}|\mathbf{j}} & \text{if } i_i > i_{i+1} \end{cases} \\ v_{\mathbf{i}|\mathbf{j}}\hat{S}_j &= \begin{cases} q^{-1}v_{\mathbf{i}|\mathbf{j}} & \text{if } j_j = j_{j+1} \\ v_{\mathbf{i}|\mathbf{j}.s_j} & \text{if } j_j > j_{j+1} \\ v_{\mathbf{i}|\mathbf{j}.s_j} + (q^{-1}-q)v_{\mathbf{i}|\mathbf{j}} & \text{if } j_j < j_{j+1}. \end{cases} \end{aligned}$$

The action of $\mathfrak{B}_{r,s}^n(q)$ on $V^{\otimes r} \otimes V^{*\otimes s}$ commutes with the action of $\mathbf{U}$.

Theorem 1.3 ([4])**.** *Let $\sigma : \mathfrak{B}_{r,s}^n(q) \to \operatorname{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$ be the representation of the quantized walled Brauer algebra on the mixed tensor space. Then σ is surjective, that is*

$$\operatorname{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathfrak{B}_{r,s}^n(q)/_{\operatorname{ann}_{\mathfrak{B}_{r,s}^n(q)}(V^{\otimes r}\otimes V^{*\otimes s})}.$$

The main result of this paper is the other half of the preceding theorem:

Theorem 1.4. *Let $\rho_{\mathrm{mxd}} : \mathbf{U} \to \operatorname{End}_{\mathfrak{B}_{r,s}^n(q)}(V^{\otimes r} \otimes V^{*\otimes s})$ be the representation of the quantum group. Then ρ_{mxd} is surjective, that is*

$$\operatorname{End}_{\mathfrak{B}_{r,s}^n(q)}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathbf{U}/_{\operatorname{ann}_{\mathbf{U}}(V^{\otimes r}\otimes V^{*\otimes s})}.$$

Theorems 1.3 and 1.4 together state that the mixed tensor space is a $(\mathbf{U}, \mathfrak{B}_{r,s}^n(q))$-bimodule with the double centralizer property. In the literature, this is also called *Schur–Weyl Duality*. Theorem 1.4 will be proved at the end of this paper.

For $s = 0$, this is well known. $\mathfrak{B}_{m,0}^n(q)$ is the Hecke algebra $\mathcal{H}_m$, and $V^{\otimes m}$ is the (ordinary) tensor space.

Definition 1.5. If m is a positive integer, let $\mathcal{H}_m$ be the associative R-algebra with one generated by elements $T_1, \ldots, T_{m-1}$ with respect to the relations

$$\begin{aligned} &(T_i + q)(T_i - q^{-1}) = 0 \text{ for } i = 1, \ldots, m-1 \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \text{ for } i = 1, \ldots, m-2 \\ &T_iT_j = T_jT_i \text{ for } |i-j| \geq 2. \end{aligned}$$

If $w \in \mathfrak{S}_m$ is an element of the symmetric group on m letters, and $w = s_{i_1} s_{i_2} \ldots s_{i_l}$ is a reduced expression as a product of Coxeter generators, let $T_w = T_{i_1} T_{i_2} \ldots T_{i_l}$. Then the set $\{T_w \mid w \in \mathfrak{S}_m\}$ is a basis of $\mathcal{H}_m$.

Note that $\mathcal{H}_m$ acts on $V^{\otimes m}$, since $\mathcal{H}_m \cong \mathfrak{B}^n_{m,0}(q)$, the isomorphism given by $T_i \mapsto S_i$.

Theorem 1.6 ([5, 7])**.** *Let* $\rho_{\mathrm{ord}} : \mathbf{U} \to \mathrm{End}_R(V^{\otimes m})$ *be the representation of* $\mathbf{U}$ *on* $V^{\otimes m}$*. Then* $\operatorname{im} \rho_{\mathrm{ord}} = \mathrm{End}_{\mathcal{H}_m}(V^{\otimes m})$*. This algebra is called the q-Schur algebra and denoted by* $S_q(n, m)$*.*

We will refer to $V^{\otimes m}$ as ordinary tensor space.

2. Mixed tensor space as a submodule

Recall that $\mathbf{U}'$ is the subalgebra of $\mathbf{U}$ corresponding to the Lie algebra $\mathfrak{sl}_n$.

Theorem 2.1. *If* m *is a nonnegative integer, let* $\rho_{\mathrm{ord}} : \mathbf{U} \to \mathrm{End}_R(V^{\otimes m})$ *be the representation of* $\mathbf{U}$ *on* $V^{\otimes m}$*. Then*

$$\rho_{\mathrm{ord}}(\mathbf{U}) = \rho_{\mathrm{ord}}(\mathbf{U}').$$

Proof. Define the *weight* of $\mathbf{i} \in I(n, m)$ to be $\mathrm{wt}(\mathbf{i}) = \lambda = (\lambda_1, \ldots, \lambda_n)$, such that λ_i is the number of entries in $\mathbf{i}$, that are equal to i. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a composition of m into n parts, i. e. $\lambda_1 + \ldots + \lambda_n = m$, let $V^{\otimes m}_\lambda$ be the R-submodule of $V^{\otimes m}$ generated by all $v_{\mathbf{i}}$ with $\mathrm{wt}(\mathbf{i}) = \lambda$. Then $V^{\otimes m}$ is the direct sum of all $V^{\otimes m}_\lambda$, where λ runs through the set of compositions of m into n parts. Let φ_λ be the projection onto $V^{\otimes m}_\lambda$. [7] shows, that the restriction of $\rho_{\mathrm{ord}} : \mathbf{U} \to S_q(n, m)$ to any subalgebra $\mathbf{U}' \subseteq \mathbf{U}$ is surjective, if the subalgebra $\mathbf{U}'$ contains the divided powers $e_i^{(l)}, f_i^{(l)}$ and preimages of the projections φ_λ.

Therefore, we define a partial order on the set of compositions of m into n parts by $\lambda \preceq \mu$ if and only if $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n) \leq (\mu_1 - \mu_2, \mu_2 - \mu_3, \ldots, \mu_{n-1} - \mu_n)$ in the lexicographical order. It suffices to show, that for each composition λ, there exists an element $u \in \mathbf{U}'$ such that $uv_{\mathbf{i}} = 0$ whenever $\mathrm{wt}(\mathbf{i}) \prec \lambda$ (i. e. $\mathrm{wt}(\mathbf{i}) \preceq \lambda$ and $\mathrm{wt}(\mathbf{i}) \neq \lambda$) and $uv_{\mathbf{i}} = v_{\mathbf{i}}$ whenever $\mathrm{wt}(\mathbf{i}) = \lambda$. In Theorem 4.5 of [13], it is shown that certain elements

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} := \prod_{s=1}^{t} \frac{K_i q^{c-s+1} - K_i^{-1} q^{-c+s-1}}{q^s - q^{-s}}$$

are elements of $\mathbf{U}'$ for $i = 1, \dots, n-1, c \in \mathbb{Z}$ and $t \in \mathbb{N}$. Let

$$u := \prod_{i=1}^{n-1} \begin{bmatrix} K_i; m+1 \\ \lambda_i - \lambda_{i+1} + m + 1 \end{bmatrix},$$

which is an element of $\mathbf{U}'$ since $\lambda_i - \lambda_{i+1} + m + 1 > 0$. Then u has the desired properties. $\square$

The next lemma is motivated by [3, §6.3].

Lemma 2.2. *There is a well defined $\mathbf{U}'$-monomorphism $\kappa : V^* \to V^{\otimes n-1}$ given by*

$$\begin{aligned} v_i^* &\mapsto (-q)^i \sum_{w \in \mathfrak{S}_{n-1}} (-q)^{l(w)} v_{(12\dots\hat{i}\dots n).w} \\ &= (-q)^i \sum_{w \in \mathfrak{S}_{n-1}} (-q)^{l(w)} v_{(12\dots\hat{i}\dots n)} T_w = (-q)^i v_{(12\dots\hat{i}\dots n)} \sum_{w \in \mathfrak{S}_{n-1}} (-q)^{l(w)} T_w \end{aligned}$$

where $\hat{i}$ means leaving out i.

Proof. It is clear, that κ is a monomorphism of R-modules. By definition, $K_i v_j^* = q^{\delta_{i+1,j} - \delta_{i,j}} v_j^*$ and $K_i v_{(1\dots\hat{j}\dots n)} = q^{1-\delta_{i,j}} q^{\delta_{i+1,j}-1} v_{(1\dots\hat{j}\dots n)}$. Thus κ commutes with K_i. Now $e_i v_j^* = -\delta_{i,j} q^{-1} v_{j+1}^*$. If $j \neq i, i+1$ then

$$\begin{aligned} e_i \kappa(v_j^*) &= (-q)^j e_i \sum_w (-q)^{l(w)} v_{(1\dots i i+1 \dots \hat{j} \dots n)} T_w \\ &= -(-q)^j \sum_w (-q)^{l(w)} v_{(1\dots i i \dots \hat{j} \dots n)} T_w = 0 = \kappa(e_i v_j^*) \end{aligned}$$

For $j = i$ resp. $i+1$ we get

$$\begin{aligned} e_i \kappa(v_{i+1}^*) &= (-q)^{i+1} \sum_w (-q)^{l(w)} (e_i v_{(1\dots\widehat{i+1}\dots n)}) T_w = 0 \\ e_i \kappa(v_i^*) &= (-q)^i \sum_w (-q)^{l(w)} (e_i v_{(1\dots\hat{i} i+1 \dots n)}) T_w \\ &= (-q)^i \sum_w (-q)^{l(w)} v_{(1\dots i \widehat{i+1} \dots n)} T_w = -q^{-1} \kappa(v_{i+1}^*) \end{aligned}$$

Furthermore, for $l \geq 2$ we clearly have $e_i^{(l)} v_j^* = 0$ and $e_i^{(l)} \kappa(v_j^*) = 0$. The argument for f_i works similarly. $\square$

Lemma 2.2 enables us to consider the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ as a $\mathbf{U}'$-submodule $T^{r,s}$ of $V^{\otimes r+(n-1)s}$ via an embedding which we will also denote by κ. Thus $\mathfrak{B}^n_{r,s}(q)$ acts on $T^{r,s}$.

If we restrict the action of an element of $\mathbf{U}'$ on $V^{\otimes r+(n-1)s}$ or equivalently of the q-Schur algebra $S_q(n, r+(n-1)s)$ to $T^{r,s}$, then we get an element of $\mathrm{End}_R(T^{r,s})$. Since the actions of $\mathbf{U}'$ and $\mathfrak{B}^n_{r,s}(q)$ commute, this is also an element of $\mathrm{End}_{\mathfrak{B}^n_{r,s}(q)}(T^{r,s})$. Let $S_q(n; r, s) := \mathrm{End}_{\mathfrak{B}^n_{r,s}(q)}(V^{\otimes r} \otimes V^{*\otimes s})$, thus we have an algebra homomorphism $\pi : S_q(n, r+(n-1)s) \to S_q(n; r, s)$ by restriction of the action to $T^{r,s} \cong V^{\otimes r} \otimes V^{*\otimes s}$. Our aim is to show that π is surjective, for then each element of $\mathrm{End}_{\mathfrak{B}^n_{r,s}(q)}(V^{\otimes r} \otimes V^{*\otimes s})$ is given by the action of an element of $\mathbf{U}'$.

Lemma 2.3. *Let M be a free R-module with basis $\mathcal{B} = \{b_1, \ldots, b_l\}$ and let U be a submodule of M given by a set of linear equations on the coefficients with respect to the basis $\mathcal{B}$, i. e. there are elements $a_{ij} \in R$ such that $U = \{\sum c_i b_i \in M : \sum_j a_{ij} c_j = 0$ for all $i\}$. Let $\{b_1^*, \ldots, b_l^*\}$ be the basis of $M^* = \mathrm{Hom}_R(M, R)$ dual to $\mathcal{B}$ and let X be the submodule generated by all $\sum_j a_{ij} b_j^*$. Then $U \cong (M^*/X)^*$.*

Proof. $(M^*/X)^*$ is isomorphic to the submodule of M^{**} given by linear forms on M^* that vanish on X. Via the natural isomorphism $M^{**} \cong M$, this is isomorphic to the set of elements of M that are annihilated by X. An element $m = \sum_k c_k b_k$ is annihilated by X if and only if $0 = \sum_{j,k} a_{ij} b_j^*(c_k b_k) = \sum_k a_{ik} c_k$ for all i and this is true if and only if $m \in U$. □

Note that an element $\tilde{\varphi} \in (M^*/X)^*$ corresponds to the element $\varphi = \sum_i \tilde{\varphi}(b_i^* + X) b_i$ of U. In our case $S_q(n, m)$ and $S_q(n; r, s)$ are R-submodules of R-free algebras, namely $\mathrm{End}_R(V^{\otimes m})$ and $\mathrm{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$ resp., given by a set of linear equations, which we will determine more precisely in Sections 3 and 4.

Definition 2.4. Let $M = \mathrm{End}_R(V^{\otimes m})$ and $U = S_q(n, m)$. Then U is defined as the algebra of endomorphisms commuting with a certain set of endomorphisms and thus is given by a system of linear equations on the coefficients. Let $A_q(n, m) = M^*/X$ as in Lemma 2.3. Similarly let $A_q(n; r, s) = M^*/X$ with $M = \mathrm{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$ and $U = S_q(n; r, s)$.

By Lemma 2.3 we have $A_q(n, m)^* = S_q(n, m)$ and $A_q(n; r, s)^* = S_q(n; r, s)$.

We will proceed as follows: We will take $m = r + (n-1)s$ and define an R-homomorphism $\iota : A_q(n; r, s) \to A_q(n, r + (n-1)s)$ such that $\iota^* = \pi : S_q(n, r + (n-1)s) \to S_q(n; r, s)$. Then we will define an R-homomorphism $\phi : A_q(n, r + (n-1)s) \to A_q(n; r, s)$ such that $\phi \circ \iota = \mathrm{id}_{A_q(n;r,s)}$ by giving suitable bases for $A_q(n, r+(n-1)s)$ and $A_q(n; r, s)$. Dualizing this equation, we get $\pi \circ \phi^* = \iota^* \circ \phi^* = \mathrm{id}_{S_q(n;r,s)}$, and this shows that π is surjective. Actually $A_q(n, r+(n-1)s)$ and $A_q(n; r, s)$ are coalgebras and ι is a morphism of coalgebras, but we do not need this for our results.

3. $A_q(n, m)$

The description of $A_q(n, m)$ is well known, see e. g. [2]. Let $A_q(n)$ be the free R-algebra on generators x_{ij} $(1 \leq i, j \leq n)$ subject to the relations

$$\begin{aligned}
x_{ik}x_{jk} &= qx_{jk}x_{ik} \quad \text{if } i < j \\
x_{ki}x_{kj} &= qx_{kj}x_{ki} \quad \text{if } i < j \\
x_{ij}x_{kl} &= x_{kl}x_{ij} \quad \text{if } i < k \text{ and } j > l \\
x_{ij}x_{kl} &= x_{kl}x_{ij} + (q - q^{-1})x_{il}x_{kj} \quad \text{if } i < k \text{ and } j < l.
\end{aligned}$$

Note that these relations define the commutative algebra in n^2 commuting intederminates x_{ij} in case $q = 1$. The free algebra on the generators x_{ij} is obviously graded (with all the generators in degree 1), and since the relations are homogeneous, this induces a grading on $A_q(n)$. Then

Lemma 3.1 ([2])**.** *$A_q(n, m)$ is the R-submodule of $A_q(n)$ of elements of homogeneous degree m.*

Proof. Since our relations of the Hecke algebra differ from those in [2] ($(T_i - q)(T_i + 1) = 0$ is replaced by $(T_i + q)(T_i - q^{-1}) = 0$), and thus $A_q(n, m)$ differs as well, we include a proof here.

Suppose that φ is an endomorphism commuting with the action of a generator S_i. For convenience, we assume that $m = 2$ and $S = S_1$. φ can be written as a linear combination of the basis elements $E_{(ij),(kl)}$ mapping $v_k \otimes v_l$ to $v_i \otimes v_j$, and all other basis elements to 0. For the coefficient of $E_{(ij),(kl)}$, we write $c_{ik}c_{jl}$, such that $\varphi = \sum_{i,j,k,l} c_{ik}c_{jl}E_{(ij),(kl)}$. On the one side we have

$$S(\varphi(v_k \otimes v_l)) = S\left(\sum_{i,j} c_{ik}c_{jl}v_i \otimes v_j\right)$$
$$= \sum_{i<j} c_{ik}c_{jl}v_j \otimes v_i + q^{-1}\sum_i c_{ik}c_{il}v_i \otimes v_i$$
$$+ \sum_{i>j} c_{ik}c_{jl}(v_j \otimes v_i + (q^{-1}-q)v_i \otimes v_j)$$
$$= \sum_{i\neq j} c_{ik}c_{jl}v_j \otimes v_i + q^{-1}\sum_i c_{ik}c_{il}v_i \otimes v_i + (q^{-1}-q)\sum_{i<j} c_{jk}c_{il}v_j \otimes v_i$$

Now, suppose that $k > l$. Then

$$\begin{aligned}\varphi(S(v_k \otimes v_l)) &= \varphi(v_l \otimes v_k + (q^{-1}-q)v_k \otimes v_l)\\ &= \sum_{i,j}\left(c_{jl}c_{ik} + (q^{-1}-q)c_{jk}c_{il}\right)v_j \otimes v_i\end{aligned}$$

Similar formulas hold for $k = l$ and $k < l$. Comparing coefficients leads to the relations given above. □

$A_q(n, m)$ has a basis consisting of monomials, but it will turn out to be more convenient for our purposes to work with a basis of standard bideterminants (see [9]). Note that the supersymmetric quantum letterplace algebra in [9] for $L^- = P^- = \{1, \ldots, n\}, L^+ = P^+ = \emptyset$ is isomorphic to $A_{q^{-1}}(n) \cong A_q(n)^{\mathrm{opp}}$, and we will adjust the results to our situation.

A *partition* λ of m is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = m$. Denote the set of partitions of m by $\Lambda^+(m)$. The *Young diagram* $[\lambda]$ of a partition λ is $\{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$. It can be represented by an array of boxes, λ_1 boxes in the first row, λ_2 boxes in the second row, etc.

A λ-*tableau* $\mathfrak{t}$ is a map $f : [\lambda] \to \{1, \ldots, n\}$. A tableau can be represented by writing the entry $f(i, j)$ into the (i, j)-th box. A tableau $\mathfrak{t}$ is called *standard*, if the entries in each row are strictly increasing from left to right, and the entries in each column are nondecreasing downward. In the literature, this property is also called semi-standard, and the role of rows and columns may be interchanged. A pair $[\mathfrak{t}, \mathfrak{t}']$ of λ-tableaux is called a *bitableau*. It is standard if both $\mathfrak{t}$ and $\mathfrak{t}'$ are standard λ-tableaux.

Note that the next definition differs from the definition in [9] by a sign.

Definition 3.2. Let $i_1, \ldots, i_k, j_1, \ldots, j_k$ be elements of $\{1, \ldots, n\}$, For $i_1 < i_2 < \ldots < i_k$ let the *right quantum minor* be defined by

$$(i_1 i_2 \ldots i_k | j_1 j_2 \ldots j_k)_r := \sum_{w \in \mathfrak{S}_k} (-q)^{l(w)} x_{i_{w1} j_1} x_{i_{w2} j_2} \ldots x_{i_{wk} j_k}.$$

For arbitrary $i_1, \ldots, i_k$, the right quantum minor is then defined by the rule

$$(i_1 \ldots i_l i_{l+1} \ldots i_k | j_1 j_2 \ldots j_k)_r := -q^{-1} (i_1 \ldots i_{l-1} i_{l+1} i_l i_{l+2} \ldots i_k | j_1 j_2 \ldots j_k)_r$$

for $i_l > i_{l+1}$. Similarly, let the *left quantum minor* be defined by

$$\begin{aligned}
(i_1 \ldots i_k | j_1 \ldots j_k)_l &:= \sum_{w \in \mathfrak{S}_k} (-q)^{l(w)} x_{i_1, j_{w1}} x_{i_2 j_{w2}} \ldots x_{i_k j_{wk}} \text{ if } j_1 < \ldots < j_k, \\
(i_1 \ldots i_k | j_1 \ldots j_k)_l &:= -q^{-1} (i_1 \ldots i_k | j_1 \ldots j_{l+1} j_l \ldots j_k)_l \text{ if } j_l > j_{l+1}.
\end{aligned}$$

Finally let the *quantum determinant* be defined by

$$\det_q := (12 \ldots n | 12 \ldots n)_r = (12 \ldots n | 12 \ldots n)_l.$$

If $[\mathfrak{t}, \mathfrak{t}']$ is a bitableau, and $\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_k$ resp. $\mathfrak{t}'_1, \mathfrak{t}'_2, \ldots, \mathfrak{t}'_k$ are the rows of $\mathfrak{t}$ resp. $\mathfrak{t}'$, then let

$$(\mathfrak{t} | \mathfrak{t}') := (\mathfrak{t}_k | \mathfrak{t}'_k)_r \ldots (\mathfrak{t}_2 | \mathfrak{t}'_2)_r (\mathfrak{t}_1 | \mathfrak{t}'_1)_r.$$

$(\mathfrak{t} | \mathfrak{t}')$ is called a *bideterminant*.

Remark 3.3. We note the following properties of quantum minors:

1.

$$\begin{aligned}
(i_1 \ldots i_k | j_1 \ldots j_k)_r &= -q (i_1 \ldots i_k | j_1 \ldots j_{l+1} j_l \ldots j_k)_r \text{ for } j_l > j_{l+1} \\
(i_1 \ldots i_k | j_1 \ldots j_k)_l &= -q (i_1 \ldots i_{l+1} i_l \ldots i_k | j_1 \ldots j_k)_l \text{ for } i_l > i_{l+1}.
\end{aligned}$$

2. If $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$, then right and left quantum minors coincide, and we simply write $(i_1 \ldots i_k | j_1 \ldots j_k)$. This notation thus indicates that the sequences of numbers are increasing. In general, right and left quantum minors differ by a power of $-q$.
3. If two i_l's or j_l's coincide, then the quantum minors vanish.
4. The quantum determinant $\det_q$ is an element of the center of $A_q(n)$.

Definition 3.4. Let the *content* of a monomial $x_{i_1j_1}\dots x_{i_mj_m}$ be defined as the tuple $(\alpha,\beta)=((\alpha_1,\dots,\alpha_n),(\beta_1,\dots,\beta_n))$ where α_i is the number of indices i_t such that $i_t=i$, and β_j is the number of indices j_t such that $j_t=j$. Note that $\sum\alpha_i=\sum\beta_j=m$ for each monomial of homogeneous degree m. For such a tuple (α,β), let $P(\alpha,\beta)$ be the subspace of $A_q(n,m)$ generated by the monomials of content (α,β). Furthermore, let the *content* of a bitableau $[\mathfrak{t},\mathfrak{t}']$ be defined similarly as the tuple (α,β), such that α_i is the number of entries in $\mathfrak{t}$ equal to i and β_j is the number of entries in $\mathfrak{t}'$ equal to j.

Theorem 3.5 ([9])**.** *The bideterminants $(\mathfrak{t}|\mathfrak{t}')$ of the standard λ-tableaux with λ a partition of m form a basis of $A_q(n,m)$, such that the bideterminants of standard λ-tableaux of content (α,β) form a basis of $P(\alpha,\beta)$.*

The proof in [9] works over a field, but the arguments are valid if the field is replaced by a commutative ring with 1. The reversed order of the minors is due to the opposite algebra. Note that for $i_1<i_2<\dots<i_k$ and $j_1<j_2<\dots<j_k$ we have

$$q^{\frac{k(k-1)}{2}}(i_1i_2\dots i_k|j_1j_2\dots j_k)_r=\sum_{w\in\mathfrak{S}_k}(-q)^{-l(w)}x_{i_{wk}j_1}x_{i_{w(k-1)}j_2}\dots x_{i_{w1}j_k},$$

which is a quantum minor of $A_{q^{-1}}(n)^{\mathrm{opp}}$.

Lemma 3.6 (Laplace's expansion [9])**.**

1. *For $j_1<j_2<\dots<j_l<j_{l+1}<\dots<j_k$ we have*

$$\begin{aligned}&(i_1i_2\dots i_k|j_1j_2\dots j_k)_l\\&=\sum_w(-q)^{l(w)}(i_1\dots i_l|j_{w1}\dots j_{wl})_l(i_{l+1}\dots i_k|j_{w(l+1)}\dots j_{wk})_l\end{aligned}$$

where the summation is over all $w\in\mathfrak{S}_k$, such that $w1<w2<\dots<wl$ and $w(l+1)<w(l+2)<\dots<wk$.

2. *For $i_1<i_2<\dots<i_k$ we have*

$$\begin{aligned}&(i_1i_2\dots i_k|j_1j_2\dots j_k)_r\\&=\sum_w(-q)^{l(w)}(i_{w1}\dots i_{wl}|j_1\dots j_l)_r(i_{w(l+1)}\dots i_{wk}|j_{l+1}\dots j_k)_r\end{aligned}$$

the summation again over all $w\in\mathfrak{S}_k$, such that $w1<w2<\dots<wl$ and $w(l+1)<w(l+2)<\dots<wk$.

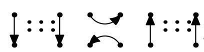
4. $A_q(n; r, s)$

A basis of $\mathrm{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$ is given by matrix units $E_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}}$ such that $E_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}} v_{\mathbf{s}|\mathbf{t}} = \delta_{\mathbf{k}|\mathbf{l},\mathbf{s}|\mathbf{t}} v_{\mathbf{i}|\mathbf{j}}$. Suppose $\varphi = \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} c_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}} E_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}} \in \mathrm{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$ commutes with the action of $\mathfrak{B}^n_{r,s}(q)$ or equivalently with a set of generators of $\mathfrak{B}^n_{r,s}(q)$. Since coefficient spaces are multiplicative, we can write

$$c_{i_1k_1} c_{i_2k_2} \ldots c_{i_rk_r} c^*_{j_1l_1} c^*_{j_2l_2} \ldots c^*_{j_sl_s}$$

for the coefficient $c_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}}$. It is easy to see from the description of $A_q(n,m)$ that φ commutes with the generators without non-propagating edges if and only if the c_{ij} satisfy the relations of $A_q(n)$ and the c^*_{ij} satisfy the relations of $A_{q^{-1}}(n)$.

Now suppose that φ in addition commutes with the action of the generator

$e =$ .

We assume that $r = s = 1$ (the general case being similar) and $\varphi = \sum_{i,j,k,l=1}^{n} c_{ik} c^*_{jl} E_{i|j\ k|l}$. Let $v = v_i \otimes v^*_j$ be a basis element of $V \otimes V^*$. We have (the indices in the sums always run from 1 to n)

$$\begin{aligned}
\varphi(v)e &= \sum_{s,t} c_{si} c^*_{tj} (v_s \otimes v^*_t) e = \sum_{s,k} q^{2s-n-1} c_{si} c^*_{sj} (v_k \otimes v^*_k) \\
\varphi(ve) &= \delta_{ij} q^{2i-n-1} \sum_{k} \varphi(v_k \otimes v^*_k) = \delta_{ij} q^{2i-n-1} \sum_{k,s,t} c_{sk} c^*_{tk} v_s \otimes v^*_t
\end{aligned}$$

Comparing coefficients, we get the following conditions:

$$\begin{gathered}
\sum_{k=1}^{n} c_{ik} c^*_{jk} = 0 \text{ for } i \neq j \\
\sum_{k=1}^{n} q^{2k} c_{ki} c^*_{kj} = 0 \text{ for } i \neq j \\
\sum_{k=1}^{n} q^{2k-2i} c_{ki} c^*_{ki} = \sum_{k=1}^{n} c_{jk} c^*_{jk}.
\end{gathered}$$

This, combined with Lemma 2.3 shows that

Lemma 4.1.

$$A_q(n;r,s) \cong (F(n,r) \otimes_R F_*(n,s))/Y$$

where $F(n,r)$ resp. $F_*(n,s)$ is the R-submodule of the free algebra on generators x_{ij} resp. x^*_{ij} generated by monomials of degree r resp. s and Y is the R-submodule of $F(n,r) \otimes_R F_*(n,s)$ generated by elements of the form $h_1h_2h_3$ where h_2 is one of the elements

$$x_{ik}x_{jk} - qx_{jk}x_{ik} \text{ for } i<j \tag{4.1.1}$$

$$x_{ki}x_{kj} - qx_{kj}x_{ki} \text{ for } i<j \tag{4.1.2}$$

$$x_{ij}x_{kl} - x_{kl}x_{ij} \text{ for } i<k, j>l \tag{4.1.3}$$

$$x_{ij}x_{kl} - x_{kl}x_{ij} - (q-q^{-1})x_{il}x_{kj} \text{ for } i<k, j<l \tag{4.1.4}$$

$$x^*_{ik}x^*_{jk} - q^{-1}x^*_{jk}x^*_{ik} \text{ for } i<j \tag{4.1.5}$$

$$x^*_{ki}x^*_{kj} - q^{-1}x^*_{kj}x^*_{ki} \text{ for } i<j \tag{4.1.6}$$

$$x^*_{ij}x^*_{kl} - x^*_{kl}x^*_{ij} \text{ for } i<k, j>l \tag{4.1.7}$$

$$x^*_{ij}x^*_{kl} - x^*_{kl}x^*_{ij} + (q-q^{-1})x^*_{il}x^*_{kj} \text{ for } i<k, j<l \tag{4.1.8}$$

$$\sum_{k=1}^{n} x_{ik}x^*_{jk} \text{ for } i \neq j \tag{4.1.9}$$

$$\sum_{k=1}^{n} q^{2k}x_{ki}x^*_{kj} \text{ for } i \neq j \tag{4.1.10}$$

$$\sum_{k=1}^{n} q^{2k-2i}x_{ki}x^*_{ki} - \sum_{k=1}^{n} x_{jk}x^*_{jk} \tag{4.1.11}$$

and h_1, h_3 are monomials of appropriate degree.

Remark 4.2. The map given by $x_{ik} \mapsto q^{2k-2i}x_{ki}$ and $x^*_{ik} \mapsto x^*_{ki}$ induces an R-linear automorphism of $A_q(n;r,s)$.

Bideterminants can also be formed using the variables x^*_{ij}. In this case let

$$(\mathfrak{t}|\mathfrak{t}')^* := (\mathfrak{t}_1|\mathfrak{t}'_1)^*_r(\mathfrak{t}_2|\mathfrak{t}'_2)^*_r \dots (\mathfrak{t}_k|\mathfrak{t}'_k)^*_r$$

where the quantum minors $(i_1 \dots i_k|j_1 \dots j_k)^*_{r/l}$ are defined as above with q replaced by q^{-1}.

5. The map $\iota : A_q(n;r,s) \to A_q(n, r+(n-1)s)$

For any $1 \le i,j \le n$ let $\iota(x_{ij}) = x_{ij}$ and

$$\iota(x^*_{ij}) = (-q)^{j-i}(12\ldots\hat{\imath}\ldots n|12\ldots\hat{\jmath}\ldots n) \in A_q(n,n-1),$$

then there is a unique R-linear map

$$\iota : F(n,r) \otimes_R F_*(n,s) \to A_q(n, r+(n-1)s)$$

such that $\iota(x_{i_1j_1}\ldots x_{i_rj_r}x^*_{k_1l_1}\ldots x^*_{k_sl_s}) = \iota(x_{i_1j_1})\ldots\iota(x_{i_rj_r})\iota(x^*_{k_1l_1})\ldots\iota(x^*_{k_sl_s})$.

Lemma 5.1. *ι factors through Y and thus induces an R-linear map*

$$A_q(n;r,s) \to A_q(n, r+(n-1)s)$$

which we will then also denote by ι.

Proof. We have to show that the generators of Y lie in the kernel of ι. Generators of Y involving the elements (4.1.1) up to (4.1.4) are obviously in the kernel of ι. [6, Theorem 7.3] shows that generators involving elements (4.1.5) up to (4.1.8) are also in the kernel. Laplace's Expansion shows that

$$\begin{aligned}
\iota\left(\sum_{k=1}^n x_{ik}x^*_{jk}\right) &= \sum_{k=1}^n (-q)^{(k-1)-(j-1)} x_{ik} \cdot (1\ldots\hat{\jmath}\ldots n|1\ldots\hat{k}\ldots n)_l \\
&= (-q)^{1-j}(i1\ldots\hat{\jmath}\ldots n|1\ldots n)_l = \delta_{i,j}\cdot \det{}_q \text{ and} \\
\iota\left(\sum_{k=1}^n q^{2k-2i} x_{ki}x^*_{kj}\right) &= q^{-2i+j+1}\sum_{k=1}^n (-q)^{k-1} x_{ki} \cdot (1\ldots\hat{k}\ldots n|1\ldots\hat{\jmath}\ldots n)_r \\
&= (-q)^{j-2i+1}(1\ldots n|i1\ldots\hat{\jmath}\ldots n)_r = \delta_{i,j}\cdot \det{}_q,
\end{aligned}$$

thus the generators involving the elements (4.1.9) up to (4.1.11) are in the kernel of ι. □

Now, we have maps

$$\iota^* : A_q(n, r+(n-1)s)^* \to A_q(n;r,s)^* \text{ and } \pi : S_q(n, r+(n-1)s) \to S_q(n;r,s).$$

By definition $A_q(n, r+(n-1)s)^* \cong S_q(n, r+(n-1)s)$ and $A_q(n;r,s)^* \cong S_q(n;r,s)$. Under these identifications we have

Lemma 5.2. $\iota^* = \pi$.

Proof. We will write

$$\begin{aligned} x_{i_1\ldots i_l\, j_1\ldots j_l} &= x_{i_1,j_1}\ldots x_{i_l,j_l} \text{ and} \\ x_{i_l\ldots i_1|l_1\ldots l_m\, j_l\ldots j_1|k_1\ldots k_m} &= x_{i_l,j_l}\ldots x_{i_1,j_1}x^*_{l_1,k_1}\ldots x^*_{l_m,k_m}. \end{aligned}$$

Suppose that $\tilde{\varphi} \in A_q(n, r+(n-1)s)^*$. The corresponding element of $S_q(n, r+(n-1)s)$ is $\varphi = \sum_{\mathbf{i},\mathbf{j}\in \mathrm{I}(n,r+(n-1)s)} \tilde{\varphi}(x_{\mathbf{ij}})E_{\mathbf{ij}}$. Since $\iota^*(\tilde{\varphi}) = \tilde{\varphi}\circ\iota$, we have

$$\iota^*(\varphi) = \sum_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} \tilde{\varphi}\circ\iota(x_{\mathbf{i}|\mathbf{j}\,\mathbf{k}|\mathbf{l}})E_{\mathbf{i}|\mathbf{j}\,\mathbf{k}|\mathbf{l}}$$

In other words: The coefficient of $E_{\mathbf{i}|\mathbf{j}\,\mathbf{k}|\mathbf{l}}$ in $\iota^*(\varphi)$ can be computed by substituting each $x_{\mathbf{st}}$ in $\iota(x_{\mathbf{i}|\mathbf{j}\,\mathbf{k}|\mathbf{l}})$ by $\tilde{\varphi}(x_{\mathbf{st}})$. On the other hand, to compute the coefficient of $E_{\mathbf{i}|\mathbf{j}\,\mathbf{k}|\mathbf{l}}$ in $\pi(\varphi)$, one has to consider the action of φ on a basis element $v = \kappa(v_{\mathbf{k}|\mathbf{l}})$ of $T^{r,s}$. For a multi index $\mathbf{l} \in \mathrm{I}(n,s)$ let $\mathbf{l}^* \in \mathrm{I}(n,(n-1)s)$ be defined by

$$\mathbf{l}^* = (1\ldots\hat{l_1}\ldots n1\ldots\hat{l_2}\ldots n\ldots 1\ldots\hat{l_s}\ldots n).$$

Then

$$v = \kappa(v_{\mathbf{k}|\mathbf{l}}) = (-q)^{l_1+l_2+\ldots+l_s}\sum_{w\in\mathfrak{S}_{n-1}^{\times s}} (-q)^{l(w)} v_{\mathbf{k}}\otimes(v_{\mathbf{l}^*}T_w)$$

and thus we have

$$\begin{aligned} \varphi(v) &= (-q)^{\sum l_k}\sum_{\mathbf{s},\mathbf{t},w}(-q)^{l(w)}\tilde{\varphi}(x_{\mathbf{st}})E_{\mathbf{st}}\,(v_{\mathbf{k}}\otimes(v_{\mathbf{l}^*}T_w)) \\ &= \sum_{\mathbf{s},w}(-q)^{l(w)+\sum l_k}\tilde{\varphi}(x_{\mathbf{s}\,\mathbf{k}\mathbf{l}^*.w})v_{\mathbf{s}}. \end{aligned}$$

Since φ leaves $T^{r,s}$ invariant, $\varphi(v)$ is a linear combination of the basis elements $\kappa(v_{\mathbf{i}|\mathbf{j}})$ of $T^{r,s}$. Distinct $\kappa(v_{\mathbf{i}|\mathbf{j}})$ involve distinct basis vectors of $V^{\otimes r+(n-1)s}$. Thus if $\varphi(v) = \sum_{\mathbf{i}|\mathbf{j}}\lambda_{\mathbf{i}|\mathbf{j}}\kappa(v_{\mathbf{i}|\mathbf{j}}) = \sum_{\mathbf{i}|\mathbf{j},w}\lambda_{\mathbf{i}|\mathbf{j}}(-q)^{l(w)+j_1+\ldots+j_s}v_{\mathbf{i}\mathbf{j}^*.w}$ then $(-q)^{\sum j_k}\lambda_{\mathbf{i}|\mathbf{j}}$ is equal to the coefficient of $v_{\mathbf{i}\mathbf{j}^*}$ when $\varphi(v)$ is written as a linear combination of basis vectors of $V^{\otimes r+(n-1)s}$. The coefficient of $v_{\mathbf{i}\mathbf{j}^*}$ in $\varphi(v)$ is, by the formula above,

$$(-q)^{\sum l_k}\sum_{w}(-q)^{l(w)}\tilde{\varphi}(x_{\mathbf{i}\mathbf{j}^*\,\mathbf{k}\mathbf{l}^*.w}).$$

Thus

$$\lambda_{\mathbf{i}|\mathbf{j}} = (-q)^{\sum l_k - j_k} \sum_w (-q)^{l(w)} \tilde{\varphi}(x_{\mathbf{i}\mathbf{j}^*\ \mathbf{k}\mathbf{l}^* w}) = \tilde{\varphi} \circ \iota(x_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}}).$$

But $\lambda_{\mathbf{i}|\mathbf{j}}$ is also the coefficient of $E_{\mathbf{i}|\mathbf{j}\ \mathbf{k}|\mathbf{l}}$ in $\pi(\varphi)$ which shows the result. □

Theorem 5.3 (Jacobi's Ratio Theorem)**.** *Suppose* $n \geq l \geq 0$*, and* $i_1 < i_2 < \ldots < i_l$ *and* $j_1 < j_2 < \ldots < j_l$*. Let* $i'_1 < i'_2 < \ldots < i'_{n-l}$ *and* $j'_1 < j'_2 < \ldots < j'_{n-l}$ *be the unique numbers such that* $\{1, \ldots, n\} = \{i_1, \ldots, i_l, i'_1, \ldots, i'_{n-l}\} = \{j_1, \ldots, j_l, j'_1, \ldots, j'_{n-l}\}$*. Then*

$$\iota\left((i_1 \ldots i_l | j_1 \ldots j_l)^*\right) = (-q)^{\sum_{t=1}^{l}(j_t - i_t)} \det_q^{l-1} (i'_1 \ldots i'_{n-l} | j'_1 \ldots j'_{n-l}).$$

Proof. We argue by induction on l. For $l = 0$ we have

$$\iota(1) = 1 = \det_q^{-1}(1 \ldots n | 1 \ldots n).$$

For $l = 1$ the theorem is true by the definition of $\iota(x^*_{ij})$. Now assume the theorem is true for $l-1$. Apply Laplace's expansion and use induction to get

$$\begin{aligned}
\iota\left((i_1 \ldots i_l | j_1 \ldots j_l)^*\right) &= \iota\left(\sum_{k=1}^{l} (-q)^{-(k-1)} x^*_{i_k j_1} (i_1 \ldots \hat{i_k} \ldots i_l | j_2 \ldots\ldots j_l)^*\right) \\
&= \sum_{k=1}^{l} (-q)^{1-k} (-q)^{j_1 - i_k} (1 \ldots \hat{i_k} \ldots n | 1 \ldots \hat{j_1} \ldots n) \cdot (-q)^{\sum\limits_{t \neq 1} j_t - \sum\limits_{t \neq k} i_t} \det_q^{l-2} \\
&\cdot (1 \ldots \hat{i_1} \ldots \hat{i_2} \ldots\ldots \hat{i_{k-1}} \ldots \hat{i_{k+1}} \ldots\ldots \hat{i_l} \ldots n | 1 \ldots \hat{j_2} \ldots \hat{j_3} \ldots\ldots \hat{j_l} \ldots n)
\end{aligned}$$

We claim that this is equal to

$$\begin{aligned}
(-q)^{\sum_{t=1}^{l}(j_t - i_t)} \det_q^{l-2} \sum_w (-q)^{l(w)+1-n} (w1w2 \ldots w(n-1) | 1 \ldots \hat{j_1} \ldots n) \\
\cdot (wn\, 1 \ldots \hat{i_1} \ldots\ldots \hat{i_l} \ldots n | 1 \ldots \hat{j_2} \ldots\ldots \hat{j_l} \ldots n)_l \qquad (5.3.1)
\end{aligned}$$

where the summation is over all $w \in \mathfrak{S}_n$ such that $w1 < w2 < \ldots < w(n-1)$. If wn is not one of the i_k's, then the summand in (5.3.1) vanishes, since wn appears twice in the row on the left side of the second minor. Thus the summation is over all w as above with $wn = i_k$ for some k. Note that $l(w) = n - i_k$ and

$$(i_k 1 \ldots \hat{i_1} \ldots\ldots \hat{i_l} \ldots n | \mathbf{t})_l = (-q)^{i_k - k} (1 \ldots \hat{i_1} \ldots \hat{i_{k-1}} \ldots \hat{i_{k+1}} \ldots \hat{i_l} \ldots n | \mathbf{t}),$$

the claim follows. Again apply Laplace's expansion to the second minor in (5.3.1) to get

$$\begin{aligned}&(wn\,1\ldots\hat{i_1}\ldots\ldots\hat{i_l}\ldots n|1\ldots\hat{j_2}\ldots\ldots\hat{j_l}\ldots n)_l\\ &=\sum_v(-q)^{l(v)}x_{wn\,v1}(1\ldots\hat{i_1}\ldots\ldots\hat{i_l}\ldots n|v2v3\ldots v\hat{j_2}\ldots\ldots v\hat{j_l}\ldots vn),\end{aligned}$$

the summation being over all $v\in\mathfrak{S}_{\{1,\ldots,\hat{j_2},\ldots,\hat{j_l},\ldots,n\}}$ with $v2<v3<\ldots<vn$. After substituting this term in (5.3.1), one can again apply Laplace's expansion, to get that (5.3.1) is equal to

$$\begin{aligned}&(-q)^{\sum(j_t-i_t)}\det\nolimits_q^{l-2}\sum_v(-q)^{l(v)+1-n}(12\ldots n|1\ldots\hat{j_1}\ldots n\,v1)_r\\ &\quad\cdot(1\ldots\hat{i_1}\ldots\ldots\hat{i_l}\ldots n|v2v3\ldots v\hat{j_2}\ldots\ldots v\hat{j_l}\ldots vn)\end{aligned}\qquad(5.3.2)$$

The only summand in (5.3.2) that does not vanish, is the term for $v1=j_1$ with $l(v)=j_1-1$. Thus (5.3.2) is equal to

$$\begin{aligned}&(-q)^{\sum(j_t-i_t)}\det\nolimits_q^{l-2}(-q)^{j_1-n}(12\ldots n|1\ldots\hat{j_1}\ldots nj_1)_r\cdot(i'_1\ldots i'_{n-l}|j'_1\ldots j'_{n-l})\\ &\quad=(-q)^{\sum_{t=1}^{l}(j_t-i_t)}\det\nolimits_q^{l-1}(i'_1\ldots i'_{n-l}|j'_1\ldots j'_{n-l}).\end{aligned}$$

□

6. A basis for $A_q(n;r,s)$

Theorem 5.3 enables us to construct elements of $A_q(n;r,s)$ that are mapped to standard bideterminants under ι. First, we will introduce the notion of rational tableaux, although we will slightly differ from the definition of rational tableaux in [15]. Recall that $\Lambda^+(k)$ is the set of partitions of k.

Definition 6.1. A *rational* (ρ,σ)*-tableau* is a pair $(\mathfrak{r},\mathfrak{s})$ such that $\mathfrak{r}$ is a ρ-tableau and $\mathfrak{s}$ is a σ-tableau for some $k\geq 0$, $\rho\in\Lambda^+(r-k)$ and $\sigma\in\Lambda^+(s-k)$ with $\rho_1+\sigma_1\leq n$.

Let $\mathrm{first}_i(\mathfrak{r},\mathfrak{s})$ be the number of entries of the first row of $\mathfrak{r}$ which are $\leq i$ plus the number of entries of the first row of $\mathfrak{s}$ which are $\leq i$. A rational tableau is called *standard* if $\mathfrak{r}$ and $\mathfrak{s}$ are standard tableaux and the following condition holds:

$$\mathrm{first}_i(\mathfrak{r},\mathfrak{s})\leq i \text{ for all } i=1,\ldots,n \qquad(6.1.1)$$

A pair $[(\mathfrak{r}, \mathfrak{s}), (\mathfrak{r}', \mathfrak{s}')]$ of rational (ρ, σ)-tableaux is called a *rational bitableau*, and it is called a standard rational bitableau if both $(\mathfrak{r}, \mathfrak{s})$ and $(\mathfrak{r}', \mathfrak{s}')$ are standard rational tableaux.

Remark 6.2. In [15], condition (6.1.1) is already part of the definition of rational tableaux. The condition $\rho_1 + \sigma_1 \leq n$ is equivalent to condition (6.1.1) for $i = n$. The reason for the difference will be apparent in the proof of the next lemma.

Lemma 6.3. *There is a bijection between the set consisting of all standard rational (ρ, σ)-tableaux for $\rho \in \Lambda^+(r-k)$, $\sigma \in \Lambda^+(s-k)$, as k runs from 0 to* $\min(r, s)$ *and the set of all standard λ-tableaux for $\lambda \in \Lambda^+(r + (n-1)s)$ satisfying $\sum_{i=1}^{s} \lambda_i \geq (n-1)s$.*

Proof. Given a rational (ρ, σ)-tableau $(\mathfrak{r}, \mathfrak{s})$ we construct a λ-tableau $\mathfrak{t}$ as follows: Draw a rectangular diagram with s rows and n columns. Rotate the tableau $\mathfrak{s}$ by 180 degrees and place it in the bottom right corner of the rectangle. Place the tableau $\mathfrak{r}$ on the left side below the rectangle. Fill the empty boxes of the rectangle with numbers, such that in each row, the entries that do not appear in $\mathfrak{t}$ appear in the empty boxes in increasing order. Let $\mathfrak{t}$ be the tableau consisting of the formerly empty boxes and the boxes of $\mathfrak{r}$. We illustrate this procedure with an example. Let $n = 5, r = 4, s = 5, k = 1$ and let

$$(\mathfrak{r}, \mathfrak{s}) = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 3 & 5 \\ \hline \end{array} \right).$$

Then

$$(\mathfrak{r}, \mathfrak{s}) \rightsquigarrow \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & 5 & 3 \\ \hline \ & \ & \ & 4 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 4 & 5 & 3 \\ \hline 1 & 2 & 5 & 4 & 3 \\ \hline 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \rightsquigarrow \mathfrak{t} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 1 & 2 & 5 \\ \cline{1-3} 1 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}$$

It is now easy to give an inverse: Just draw the rectangle into the tableau $\mathfrak{t}$, fill the empty boxes of the rectangle in a similar way as before, rotate these back to obtain $\mathfrak{s}$. $\mathfrak{r}$ is the part of the tableau $\mathfrak{t}$, that lies outside the rectangle. We have to show, that these bijections provide standard tableaux of the right shape.

Suppose $(\mathfrak{r}, \mathfrak{s})$ is a rational (ρ, σ)-tableau, then $\mathfrak{t}$ is a λ-tableau, with $\lambda_i = n - \sigma_{s+1-i}$ for $i \leq s$ and $\lambda_i = \rho_{i-s}$ for $i > s$. Thus $\lambda_i \geq \lambda_{i+1}$ for $i < s$ is equivalent to $\sigma_{s+1-i} \leq \sigma_{s-i}$, and for $i > s$ it is equivalent to $\rho_{i-s} \geq \rho_{i+1-s}$. Now $\rho_1 + \sigma_1 = \lambda_{s+1} - (\lambda_s - n)$. This shows that λ is a partition if and only if ρ and σ are partitions with $\rho_1 + \sigma_1 \leq n$. We still have to show that $(\mathfrak{r}, \mathfrak{s})$ is standard if and only if $\mathfrak{t}$ is standard.

By definition, all standard tableaux have increasing rows. A tableau has nondecreasing columns if and only if for all $i = 1, \ldots, n$ and all rows (except for the last row) the number of entries $\leq i$ in this row is greater or equal than the number of entries $\leq i$ in the next row. Now, it follows from the construction that $\mathfrak{t}$ has nondecreasing columns inside the rectangle if and only if $\mathfrak{s}$ has nondecreasing columns, $\mathfrak{t}$ has nondecreasing columns outside the rectangle if and only if $\mathfrak{r}$ has nondecreasing columns, and the columns in $\mathfrak{t}$ do not decrease from row s to row $s+1$ if and only if condition (6.1.1) holds. □

Definition 6.4. For $k \geq 1$ let $\mathfrak{det}_q^{(k)} \in A_q(n; k, k)$ be recursively defined by $\mathfrak{det}_q^{(1)} := \sum_{l=1}^n x_{1l} x_{1l}^*$ and $\mathfrak{det}_q^{(k)} := \sum_{l=1}^n x_{1l} \mathfrak{det}_q^{(k-1)} x_{1l}^*$ for $k > 1$.

Let a (rational) bideterminant $((\mathfrak{r}, \mathfrak{s})|(\mathfrak{r}', \mathfrak{s}')) \in A_q(n; r, s)$ be defined by

$$((\mathfrak{r}, \mathfrak{s})|(\mathfrak{r}', \mathfrak{s}')) := (\mathfrak{r}|\mathfrak{r}')\ \mathfrak{det}_q^{(k)}\ (\mathfrak{s}|\mathfrak{s}')^*$$

whenever $[(\mathfrak{r}, \mathfrak{s}), (\mathfrak{r}', \mathfrak{s}')]$ is a rational (ρ, σ)-bitableau such that $\rho \in \Lambda^+(r-k)$, $\sigma \in \Lambda^+(s-k)$, for some $k = 0, 1, \ldots, \min(r, s)$.

Note that the proof of Lemma 5.1 shows that $\iota(\mathfrak{det}_q^{(k)}) = \det_q^k$. Furthermore, if ρ_1 or $\sigma_1 > n$, then the bideterminant of a (ρ, σ)-bitableau vanishes. As a direct consequence of Theorem 5.3 we get

Lemma 6.5. *Let $(\mathfrak{r}, \mathfrak{s})$ and $(\mathfrak{r}', \mathfrak{s}')$ be two standard rational tableaux, and let $\mathfrak{t}$ and $\mathfrak{t}'$ be the (standard) tableaux obtained from the correspondence of Lemma 6.3. Then we have*

$$\iota((\mathfrak{r}, \mathfrak{s})|(\mathfrak{r}', \mathfrak{s}')) = (-q)^{c(\mathfrak{t}, \mathfrak{t}')} (\mathfrak{t}|\mathfrak{t}')$$

for some integer $c(\mathfrak{t}, \mathfrak{t}')$. In particular, the bideterminants of standard rational bitableaux are linearly independent.

Proof. This follows directly from Theorem 5.3, the construction of the bijection and $\iota(\mathfrak{det}_q^{(k)}) = \det_q^k$. The second statement follows from the fact that the $(\mathfrak{t}|\mathfrak{t}')$'s are linearly independent. □

Lemma 6.6. *We have*

$$\sum_{l=1}^{n} x_{il}\mathfrak{det}_q^{(k)} x_{jl}^* = 0 \text{ for } i \neq j \tag{6.6.1}$$

$$\sum_{l=1}^{n} q^{2l} x_{li}\mathfrak{det}_q^{(k)} x_{lj}^* = 0 \text{ for } i \neq j \tag{6.6.2}$$

$$\sum_{l=1}^{n} q^{2l-2i} x_{li}\mathfrak{det}_q^{(k)} x_{li}^* = \sum_{l=1}^{n} x_{jl}\mathfrak{det}_q^{(k)} x_{jl}^* \tag{6.6.3}$$

Proof. Without loss of generality, we may assume that $k=1$. Suppose that $i, j \neq 1$. Then

$$\begin{aligned}
\sum_{l=1}^{n} x_{il}\mathfrak{det}_q^{(1)} x_{jl}^* &= \sum_{k,l=1}^{n} x_{ik}x_{1l}x_{1l}^*x_{jk}^* = \sum_{k<l} x_{1l}x_{ik}x_{jk}^*x_{1l}^* + q^{-2}\sum_{k} x_{1k}x_{ik}x_{jk}^*x_{1k}^* \\
&+ \sum_{k>l}\left(x_{1l}x_{ik}x_{jk}^*x_{1l}^* + (q^{-1}-q)(x_{1k}x_{il}x_{1l}^*x_{jk}^* + x_{1l}x_{ik}x_{1k}^*x_{jl}^*)\right) \\
&= \sum_{k,l} x_{1l}x_{ik}x_{jk}^*x_{1l}^* + (q^{-2}-1)\sum_{k} q x_{1k}x_{ik}x_{1k}^*x_{jk}^* \\
&+ (q^{-1}-q)\sum_{k>l}(x_{1k}x_{il}x_{1l}^*x_{jk}^* + x_{1l}x_{ik}x_{1k}^*x_{jl}^*) \\
&= \delta_{ij}\mathfrak{det}_q^{(2)} + (q^{-1}-q)\sum_{k,l} x_{1k}x_{il}x_{1l}^*x_{jk}^* = \delta_{ij}\mathfrak{det}_q^{(2)}.
\end{aligned}$$

For $j \neq 1$ we have

$$\begin{aligned}
\sum_{l=1}^{n} x_{1l}\mathfrak{det}_q^{(1)} x_{jl}^* &= \sum_{k,l=1}^{n} x_{1k}x_{1l}x_{1l}^*x_{jk}^* = \sum_{k<l} q x_{1l}x_{1k}x_{jk}^*x_{1l}^* + q^{-1}\sum_{k} x_{1k}x_{1k}x_{jk}^*x_{1k}^* \\
&+ \sum_{k>l}\left(q^{-1}x_{1l}x_{1k}x_{jk}^*x_{1l}^* + (q^{-1}-q)x_{1k}x_{1l}x_{jl}^*x_{1k}^*\right) \\
&= \sum_{k,l} q^{-1}x_{1l}x_{1k}x_{jk}^*x_{1l}^* = 0.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
\sum_{l=1}^{n} x_{il}\mathfrak{det}_q^{(1)}x_{1l}^* &= 0 \text{ for } i\neq 1\\
\sum_{l=1}^{n} q^{2l-2i}x_{li}\mathfrak{det}_q^{(1)}x_{lj}^* &= \delta_{ij}\sum_{l=1}^{n} q^{2l-2}x_{l1}\mathfrak{det}_q^{(1)}x_{l1}^* \text{ for } i,j\neq 1\\
\sum_{l=1}^{n} q^{2l-2}x_{l1}\mathfrak{det}_q^{(1)}x_{lj}^* &= 0 \text{ for } j\neq 1\\
\sum_{l=1}^{n} q^{2l-2i}x_{li}\mathfrak{det}_q^{(1)}x_{l1}^* &= 0 \text{ for } i\neq 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_{l=1}^{n} q^{2l-2}x_{l1}\mathfrak{det}_q^{(1)}x_{l1}^* = \sum_{l,k} q^{2l-2}x_{l1}x_{1k}x_{1k}^*x_{l1}^* = \sum_{l,k\neq 1} q^{2l-2}x_{1k}x_{l1}x_{l1}^*x_{1k}^*\\
&\quad + \sum_{l\neq 1} q^{2l-4}x_{11}x_{l1}x_{l1}^*x_{11}^* + \sum_{k\neq 1} q^2 x_{1k}x_{11}x_{11}^*x_{1k}^* + x_{11}x_{11}x_{11}^*x_{11}^*\\
&\quad = \mathfrak{det}_q^{(2)} + \sum_{l\neq 1} q^{2l-4}(1-q^2)x_{11}x_{l1}x_{l1}^*x_{11}^* + \sum_{k\neq 1}(q^2-1)x_{1k}x_{11}x_{11}^*x_{1k}^*\\
&\quad = \mathfrak{det}_q^{(2)} + (1-q^2)\left(\sum_{l\neq 1} q^{2l-4}x_{11}x_{l1}x_{l1}^*x_{11}^* - q^{-2}\sum_{k\neq 1} x_{11}x_{1k}x_{1k}^*x_{11}^*\right)\\
&\quad = \mathfrak{det}_q^{(2)}.
\end{aligned}$$

The proof is complete. □

Lemma 6.7. *Suppose* $\mathbf{r}=(r_1,\ldots,r_k),\mathbf{s}=(s_1,\ldots,s_k)\in I(n,k)$ *are fixed. Let* $j\in\{1,\ldots,n\}$ *and* $k\geq 1$*. Then we have, modulo* $\mathfrak{det}_q^{(1)}$*,*

$$\begin{aligned}
&\sum_{j<j_1<j_2<\ldots<j_k} (\mathbf{r}|j_k\ldots j_2 j_1)_r(\mathbf{s}|j_1 j_2\ldots j_k)_r^*\\
&\qquad \equiv (-1)^k q^{2\sum_{i=0}^{k-1} i} \sum_{j_1<j_2<\ldots<j_k\leq j} (\mathbf{r}|j_k\ldots j_2 j_1)_r(\mathbf{s}|j_1 j_2\ldots j_k)_r^*
\end{aligned}$$

Proof. $(\mathbf{s}|j_1 j_2\ldots j_k)_r^*$ and $(\mathbf{s}|j_1 j_2\ldots j_k)_l^*$ differ only on a power of $-q$ not depending on $j_1,j_2,\ldots,j_k$. Thus we can show the lemma with $(\_,\_)_r^*$ replaced by $(\_,\_)^*_l$. Similarly, we can assume that $r_1 < r_2 < \ldots < r_k$ and $s_1 > s_2 > \ldots > s_k$. Note that modulo $\mathfrak{det}_q^{(1)}$ we have the relations $\sum_{k=1}^n x_{ik} x^*_{jk} \equiv 0$. It follows that the lemma is true for $k = 1$. Assume that the lemma holds for $k-1$. If M is an ordered set, let $M^{k,<}$ be the set of k-tuples in M with increasing entries. For a subset $M \subset \{1, \ldots, n\}$ we have

$$\begin{aligned}
&\sum_{\mathbf{j} \in M^{k,<}} (\mathbf{r}|j_k \ldots j_2 j_1)_r (\mathbf{s}|j_1 j_2 \ldots j_k)^*_l \\
&\qquad = \sum_{\mathbf{j} \in M^{k,<}, w} (-q)^{-l(w)} (\mathbf{r}|j_k \ldots j_2 j_1)_r x^*_{s_1 j_{w1}} \ldots x^*_{s_k j_{wk}} \\
&\qquad = \sum_{\mathbf{j} \in M^{k,<}, w} (\mathbf{r}|j_{wk} \ldots j_{w1})_r x^*_{s_1 j_{w1}} \ldots x^*_{s_k j_{wk}} \\
&\qquad = \sum_{\mathbf{j} \in M^{k}} (\mathbf{r}|j_k \ldots j_1)_r x^*_{s_1 j_1} \ldots x^*_{s_k j_k}
\end{aligned}$$

Applying Laplace's Expansion, we can write a quantum minor $(\mathbf{r}|\mathbf{j}_1\mathbf{j}_2)_r$ as a linear combination of products of quantum minors, say

$$(\mathbf{r}|\mathbf{j}_1\mathbf{j}_2)_r = \sum_l c_l (\mathbf{r}'_l|\mathbf{j}_1)_r (\mathbf{r}''_l|\mathbf{j}_2)_r.$$

Then with $\epsilon_k := (-1)^k q^{2\sum_{i=0}^{k-1} i}$, $\mathbf{j} = (j_1, \ldots, j_k)$ and $\mathbf{j}' = (j_1, \ldots, j_{k-1})$, $D = \{j+1 \ldots n\}$ and $C = \{1 \ldots j\}$, we have

$$\sum_{\mathbf{j}\in D^{k,<}} (\mathbf{r}|j_k\dots j_2j_1)_r(\mathbf{s}|j_1j_2\dots j_k)^*_l = \sum_{\mathbf{j}\in D^k} (\mathbf{r}|j_k\dots j_1)_r x^*_{s_1j_1}\dots x^*_{s_kj_k}$$

$$= \sum_{\mathbf{j}\in D^k,l} c_l(\mathbf{r}'_l|j_k)_r(\mathbf{r}''_l|j_{k-1}\dots j_1)_r x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}x^*_{s_kj_k}$$

$$\equiv \epsilon_{k-1} \sum_{\substack{\mathbf{j}'\in C^{k-1},l\\ j_k>j}} c_l(\mathbf{r}'_l|j_k)_r(\mathbf{r}''_l|j_{k-1}\dots j_1)_r x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}x^*_{s_kj_k}$$

$$= \epsilon_{k-1} \sum_{\substack{\mathbf{j}'\in C^{k-1}\\ j_k>j}} (\mathbf{r}|j_kj_{k-1}\dots j_1)_r x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}x^*_{s_kj_k}$$

$$= \epsilon_{k-1} \sum_{\substack{\mathbf{j}'\in C^{k-1}\\ j_k>j}} (-q)^{k-1}(\mathbf{r}|j_{k-1}\dots j_1j_k)_r x^*_{s_kj_k}x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}$$

$$= \epsilon_{k-1} \sum_{\substack{\mathbf{j}'\in C^{k-1},l\\ j_k>j}} (-q)^{k-1}c_l(\mathbf{r}'_l|j_{k-1}\dots j_1)_r x_{\mathbf{r}''_l j_k} x^*_{s_kj_k}x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}$$

$$\equiv -\epsilon_{k-1} \sum_{\mathbf{j}\in C^k,l} (-q)^{k-1}c_l(\mathbf{r}'_l|j_{k-1}\dots j_1)_r x_{\mathbf{r}''_l j_k} x^*_{s_kj_k}x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}$$

$$= -\epsilon_{k-1} \sum_{\mathbf{j}\in C^k} (-q)^{k-1}(\mathbf{r}|j_{k-1}\dots j_1j_k)_r x^*_{s_kj_k}x^*_{s_1j_1}\dots x^*_{s_{k-1}j_{k-1}}$$

$$= -\epsilon_{k-1} \sum_{\mathbf{j}\in C^{k,<}} (-q)^{k-1}(\mathbf{r}|j_k\dots j_1)_r(s_ks_1\dots s_{k-1}|j_1\dots j_k)^*_l$$

$$= -\epsilon_{k-1} \sum_{\mathbf{j}\in C^{k,<}} (-q)^{2(k-1)}(\mathbf{r}|j_k\dots j_1)_r(s_1\dots s_k|j_1\dots j_k)^*_l$$

$$= \epsilon_k \sum_{\mathbf{j}\in C^{k,<}} (\mathbf{r}|j_k\dots j_2j_1)_r(\mathbf{s}|j_1j_2\dots j_k)^*_l$$

and the proof is complete. □

Lemma 6.8. *Let* $\mathbf{r}'$ *and* $\mathbf{s}'$ *be strictly increasing multi indices, considered as tableaux with one row. Let* i *be the maximal entry appearing and suppose that* i *is minimal such that* i *violates condition* (6.1.1). *Let* I *be the set of entries appearing in both* $\mathbf{r}'$ *and* $\mathbf{s}'$, *then we have* $i \in I$. *Let* $L_1 = \{k_1, \dots, k_{l_1}\}$ *(resp.* $L_2 = \{k'_1, \dots, k'_{l_2}\}$*) be the set of entries of* $\mathbf{r}'$ *(resp.* $\mathbf{s}'$*) not appearing in* $\mathbf{s}'$*(resp.* $\mathbf{r}'$*), and let* $i_1 < i_2 < \dots < i_k = i$ *be the entries of* I. *Let*

$D = \{i_1, \dots, i_k, i_k+1, i_k+2, \dots, n\}$ *and* $C = \{1, \dots, n\} \setminus (D \cup L_1 \cup L_2)$. *Furthermore, for* $j_1, \dots, j_t \in \{1, \dots, n\}$ *let*

$$m(j_1, \dots, j_t) = |\{(l, c) \in \{1, \dots, t\} \times C : j_l < c\}|.$$

Let $\mathbf{k} = (k_1, \dots, k_{l_1}), \mathbf{k}' = (k'_1, \dots, k'_{l_2})$ *and let* $\mathbf{r}$ *and* $\mathbf{s}$ *be multi indices of the same length as* $\mathbf{r}'$ *resp.* $\mathbf{s}'$*, then we have*

$$\sum_{\mathbf{j} \in D^{k,<}} q^{2m(\mathbf{j})} (\mathbf{r}|\mathbf{k} j_k \dots j_1)_r (\mathbf{s}|j_1 \dots j_k \mathbf{k}')_r^* \equiv 0 \text{ modulo } \mathfrak{det}_q^{(1)}.$$

Proof. Note that $i \in I$ and $i = 2k + l_1 + l_2 - 1$, otherwise $i - 1$ would violate condition (6.1.1). Thus $|C| = k - 1$. Let c_{max} be the maximal element of C, $\tilde{D} = \{c_{max}+1, c_{max}+2, \dots, n\} \subset D \cup L_1 \cup L_2$, $\tilde{C} = \{1, \dots, c_{max}\}$, $D_- = \{d \in D : d < c_{max}\}$ and $D_+ = \{d \in D : d > c_{max}\}$. With $\tilde{\mathbf{j}} = (j_1, \dots, j_l)$ and $\hat{\mathbf{j}} = (j_{l+1}, \dots, j_k)$ we have

$$\sum_{\mathbf{j} \in D^{k,<}} q^{2m(\mathbf{j})} (\mathbf{r}|\mathbf{k} j_k \dots j_1)_r (\mathbf{s}|j_1 \dots j_k \mathbf{k}')_r^*$$
$$= \sum_{l=0}^{k} \sum_{\tilde{\mathbf{j}} \in D_-^{l,<}} q^{2m(\tilde{\mathbf{j}})} \sum_{\hat{\mathbf{j}} \in D_+^{k-l,<}} (\mathbf{r}|\mathbf{k} j_k \dots j_1)_r (\mathbf{s}|j_1 \dots j_k \mathbf{k}')_r^*. \tag{6.8.1}$$

Without loss of generality we may assume that the entries in $\mathbf{s}$ are increasing. We apply Laplace's Expansion and Lemma 6.7 to get for fixed l and $\tilde{\mathbf{j}}$

$$\sum_{\hat{\mathbf{j}} \in D_+^{k-l,<}} (\mathbf{r}|\mathbf{k} j_k \dots j_1)_r (\mathbf{s}|j_1 \dots j_k \mathbf{k}')_r^* = \sum_{\hat{\mathbf{j}} \in \tilde{D}^{k-l,<}} (\mathbf{r}|\mathbf{k} j_k \dots j_1)_r (\mathbf{s}|j_1 \dots j_k \mathbf{k}')_r^*$$
$$= q^{2l(k-l)} \sum_{\hat{\mathbf{j}} \in \tilde{D}^{k-l,<}} (\mathbf{r}|\mathbf{k} j_l \dots j_1 j_k \dots j_{l+1})_r (\mathbf{s}|j_{l+1} \dots j_k j_1 \dots j_l \mathbf{k}')_r^*$$
$$\equiv \epsilon_{k-l} q^{2l(k-l)} \sum_{\hat{\mathbf{j}} \in \tilde{C}^{k-l,<}} (\mathbf{r}|\mathbf{k} j_l \dots j_1 j_k \dots j_{l+1})_r (\mathbf{s}|j_{l+1} \dots j_k j_1 \dots j_l \mathbf{k}')_r^*$$
$$= \epsilon_{k-l} q^{2l(k-l)} \sum_{\hat{\mathbf{j}} \in (C \cup D_-)^{k-l,<}} (\mathbf{r}|\mathbf{k} j_l \dots j_1 j_k \dots j_{l+1})_r (\mathbf{s}|j_{l+1} \dots j_k j_1 \dots j_l \mathbf{k}')_r^*.$$

This expression can be substituted into (6.8.1). Each nonzero summand belongs to a disjoint union $S_1 \dot{\cup} S_2 = S \subset C \cup D_-$ such that $|S| = k$,

$S_1 = \{j_1, \ldots, j_l\}$ and $S_2 = \{j_{l+1}, \ldots, j_k\}$. We will show that the summands belonging to some fixed set S cancel out.

Therefore, we claim that for each subset $S \subset C \cup D_-$ with k elements there exists some $d \in D \cap S$ such that $m(d) = |\{s \in S : s > d\}|$. Suppose not. S contains at least one element of D since $|C| = k-1$. Let $s_1 < s_2 < \ldots < s_m$ be the elements of $D \cap S$. We show by downward induction that $m(s_l) > |\{s \in S : s > s_l\}|$ for $1 \leq l \leq m$: $m(s_m)$ is the cardinality of $\{s_m+1, \ldots, c_{max}\} \cap C$. Since all $s \in S$ with $s > s_m$ are elements of C we have $\{s_m+1, \ldots, c_{max}\} \cap S \subset \{s_m+1, \ldots, c_{max}\} \cap C$, and thus $m(s_m) \geq |\{s \in S : s > s_m\}|$. By assumption we have $>$ instead of $\geq$. Suppose now, that $m(s_l) > |\{s \in S : s > s_l\}|$. We have $\{s \in S : s_{l-1} < s \leq s_l\} = \{s \in S \cap C : s_{l-1} < s < s_l\} \cup \{s_l\}$, thus S contains at most $m(s_{l-1}) - m(s_l)$ elements between s_{l-1} and s_l, and thus at most $m(s_{l-1}) - m(s_l) + 1 + m(s_l) - 1 = m(s_{l-1})$ elements $> s_{l-1}$. By assumption we have $m(s_{l-1}) > |\{s \in S : s > s_{l-1}\}|$. We have shown that S contains less than $m(s_1)$ elements greater than s_1, thus S contains less than $|C| + 1 = k$ elements which is a contradiction. This shows the claim.

Let $S \subset C \cup D_-$ be fixed subset of cardinality k. By the previous consideration there is an element $d \in D \cap S$ with $m(d) = |\{s \in S : s > d\}|$. We claim that the summand for S_1, S_2 with $d \in S_1$ cancels the summand for $S_1 \backslash \{d\}, S_2 \cup \{d\}$. Note that

$$\begin{aligned}
&(\mathbf{r}|\mathbf{k}j_l \ldots \hat{d} \ldots j_1 j_k \ldots d \ldots j_{l+1})_r (\mathbf{s}|j_{l+1} \ldots d \ldots j_k j_1 \ldots \hat{d} \ldots j_l \mathbf{k}')^*_r \\
&\quad = q^{2|\{s \in S : s > d\}| - 2(l-1)} (\mathbf{r}|\mathbf{k}j_l \ldots j_1 j_k \ldots j_{l+1})_r (\mathbf{s}|j_{l+1} \ldots j_k j_1 \ldots j_l \mathbf{k}')^*_r.
\end{aligned}$$

Comparing coefficients, we see that both summands cancel. □

Theorem 6.9 (Rational Straightening Algorithm)**.** *The set of bideterminants of standard rational bitableaux forms an R-basis of $A_q(n; r, s)$.*

Proof. We have to show that the bideterminants of standard rational bitableaux generate $A_q(n; r, s)$. Clearly, the bideterminants $((\mathfrak{r}, \mathfrak{s})|(\mathfrak{r}', \mathfrak{s}'))$ with $\mathfrak{r}, \mathfrak{r}', \mathfrak{s}, \mathfrak{s}'$ standard tableaux generate $A_q(n; r, s)$. Let $\mathrm{cont}(\mathfrak{r})$ resp. $\mathrm{cont}(\mathfrak{s})$ be the content of $\mathfrak{r}$ resp. $\mathfrak{s}$ defined in Definition 3.4.

Let $\mathfrak{r}, \mathfrak{r}', \mathfrak{s}, \mathfrak{s}'$ be standard tableaux and suppose that the rational bitableau $[(\mathfrak{r}, \mathfrak{s}), (\mathfrak{r}', \mathfrak{s}')]$ is not standard. It suffices to show that the bideterminant $((\mathfrak{r}, \mathfrak{s})|(\mathfrak{r}', \mathfrak{s}'))$ is a linear combination of bideterminants $((\hat{\mathfrak{r}}, \hat{\mathfrak{s}})|(\hat{\mathfrak{r}}', \hat{\mathfrak{s}}'))$ such that $\hat{\mathfrak{r}}$ has fewer boxes than $\mathfrak{r}$ or $\mathrm{cont}(\mathfrak{r}) > \mathrm{cont}(\hat{\mathfrak{r}})$ or $\mathrm{cont}(\mathfrak{s}) > \mathrm{cont}(\hat{\mathfrak{s}})$ in the lexicographical order. Without loss of generality we make the following assumptions:

- In the nonstandard rational bitableau $[(\mathfrak{r},\mathfrak{s}),(\mathfrak{r}',\mathfrak{s}')]$ the rational tableau $(\mathfrak{r}',\mathfrak{s}')$ is nonstandard. Note that the automorphism of Remark 4.2 maps a bideterminant $((\mathfrak{r},\mathfrak{s})|(\mathfrak{r}',\mathfrak{s}'))$ to the bideterminant $((\mathfrak{r}',\mathfrak{s}')|(\mathfrak{r},\mathfrak{s}))$.

- Suppose that $(\mathfrak{r},\mathfrak{s})$ and $(\mathfrak{r}',\mathfrak{s}')$ are (ρ,σ)-tableaux. In view of Lemma 6.6 we can assume that $\rho \in \Lambda^+(r)$ and $\sigma \in \Lambda^+(s)$.

- $\mathfrak{r},\mathfrak{r}',\mathfrak{s},\mathfrak{s}'$ are tableaux with only one row (each bideterminant has a factor of this type, and we can use Theorem 3.5 to write nonstandard bideterminants as a linear combination of standard ones of the same content.

- Let i be minimal such that condition (6.1.1) of Definition 6.1 is violated for i. Applying Laplace's Expansion, we may assume that there is no greater entry than i in $\mathfrak{r}'$ and in $\mathfrak{s}'$.

Note that all elements of $A_q(n;r,s)$ having a factor $\mathfrak{Det}_q^{(1)}$ can be written as a linear combination of bideterminants of rational (ρ,σ)-bitableaux with $\rho \in \Lambda^+(r-k), k > 0$. Thus, it suffices to show that $((\mathfrak{r},\mathfrak{s})|(\mathfrak{r}',\mathfrak{s}'))$ is , modulo $\mathfrak{Det}_q^{(1)}$, a linear combination of bideterminants of 'lower content'. The summand of highest content in Lemma 6.8 is that one for $\mathbf{j} = (i_1, i_2, \ldots, i_k)$, and this summand is a scalar multiple (a power of $-q$, which is invertible) of $((\mathfrak{r},\mathfrak{s})|(\mathfrak{r}',\mathfrak{s}'))$. $\square$

The following is an immediate consequence of the preceding theorem and Lemma 6.3.

Corollary 6.10. *There exists an R-linear map $\phi : A_q(n, r+(n-1)s) \to A_q(n;r,s)$ given on a basis by $\phi(\mathfrak{t}|\mathfrak{t}') := (-q)^{-c(\mathfrak{t},\mathfrak{t}')}((\mathfrak{r},\mathfrak{s})|(\mathfrak{r}',\mathfrak{s}'))$ if the shape λ of $\mathfrak{t}$ satisfies $\sum_{i=1}^{s} \lambda_i \geq (n-1)s$ and $(\mathfrak{r},\mathfrak{s})$, where $(\mathfrak{r}',\mathfrak{s}')$ are the rational tableaux respectively corresponding to $\mathfrak{t}$ and $\mathfrak{t}'$ under the correspondence of Lemma 6.3, and $\phi(\mathfrak{t}|\mathfrak{t}') := 0$ otherwise. We have*

$$\phi \circ \iota = \mathrm{id}_{A_q(n;r,s)}$$

and thus $\pi = \iota^*$ is surjective.

As noted in Section 2 we now have the main result.

Theorem 6.11 (Schur-Weyl duality for the mixed tensor space, II)**.**

$$S_q(n;r,s) \quad = \quad \mathrm{End}_{\mathfrak{B}_{r,s}(q)}(V^{\otimes r} \otimes V^{*\otimes s}) = \rho_{\mathrm{mxd}}(\mathbf{U}) = \rho_{\mathrm{mxd}}(\mathbf{U}')$$

Furthermore, $S_q(n;r,s)$ is R-free with a basis indexed by standard rational bitableau.

Proof. The first assertion follows from the surjectivity of π, the second assertion is obtained by dualizing the basis of $A_q(n;r,s)$. □

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