

# Quantized mixed tensor space and Schur–Weyl duality I

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## Abstract

This paper studies a  $q$ -deformation,  $\mathfrak{B}_{r,s}^n(q)$ , of the walled Brauer algebra (a certain subalgebra of the Brauer algebra) and shows that the centralizer algebra for the action of the quantum group  $\mathbf{U}_R(\mathfrak{gl}_n)$  on mixed tensor space  $(R^n)^{\otimes r} \otimes (R^n)^{* \otimes s}$  is generated by the action of  $\mathfrak{B}_{r,s}^n(q)$  for any commutative ring  $R$  with one and an invertible element  $q$ .

*Key words:* Schur-Weyl duality, walled Brauer algebra, mixed tensor space  
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## Introduction

Schur-Weyl duality is a special case of a bicentralizer property. For algebras  $A$  and  $B$ , we say that an  $A$ - $B$ -bimodule  $T$  satisfies the *bicentralizer property* if  $\text{End}_A(T) = \rho(B)$  and  $\text{End}_B(T) = \sigma(A)$ , where  $\rho$  and  $\sigma$  are the corresponding representation maps. The classical Schur-Weyl duality due to Schur ([18]) is the bicentralizer property for the representations of the group algebras of the symmetric group  $\mathfrak{S}_m$  and the general linear group  $GL(n)$  on the *ordinary tensor space*  $(\mathbb{C}^n)^{\otimes m}$ . Schur-Weyl duality plays an important role in representation theory, since it brings together the representation theory of the general linear group and the symmetric group. This duality has been generalized to algebras related to these group algebras and deformations of them.

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This paper studies Schur-Weyl duality on the mixed tensor space in a very general setting. Let  $R$  be a commutative ring with 1,  $q$  an invertible element of  $R$ ,  $n$  a positive and  $r, s$  nonnegative integers. The quantum group  $\mathbf{U}$  of the general linear group is a  $q$ -deformation of the hyperalgebra of the general linear group and acts on the *mixed tensor space*  $V^{\otimes r} \otimes V^{*\otimes s}$ , where  $V$  is the natural  $\mathbf{U}$ -module and  $V^*$  its dual. We show that  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  is the image of the representation of a  $q$ -deformation  $\mathfrak{B}_{r,s}^n(q)$  of the walled Brauer algebra. This was previously known only in cases  $R = \mathbb{C}(q)$  when  $q$  is an indeterminate or  $R = \mathbb{C}$ .

The other half of the Schur-Weyl duality for mixed tensor space, namely that  $\text{End}_{\mathfrak{B}_{r,s}^n(q)}(V^{\otimes r} \otimes V^{*\otimes s})$  is equal to the image of the representation of  $\mathbf{U}$ , will be shown by completely different methods in a forthcoming paper by the authors (see [6]).

Schur-Weyl duality for mixed tensor space has been investigated in [1] for  $R = \mathbb{C}$  and  $q = 1$  and in [14, 15] for  $R = \mathbb{C}(q)$ . Our results hold for general  $R$  and invertible parameter  $q$ . It therefore covers the quantum case as well as the classical case for hyperalgebras and hence the case of the general group over an infinite field. If the algebras are semisimple, then it suffices to decompose the tensor space into irreducible modules. We obtain more general results by defining an integral isomorphism between  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  and  $\text{End}_{\mathbf{U}}(V^{\otimes r+s})$ .

To prove our results, it is helpful to find a simple description of endomorphisms of  $\mathbf{U}$ -modules. For  $q = 1$  these objects are the walled Brauer diagrams (see [1]) which generate the walled Brauer algebra. Leduc introduced in [15] a  $q$ -deformation  $\mathcal{A}_{r,s}$  of the walled Brauer algebra by (non diagrammatic) generators and relations and defined an epimorphism from a certain tangle algebra to  $\mathcal{A}_{r,s}$ . We alter this description to get it precise: instead of nonoriented tangles, we take oriented tangles generating a suitable deformation of the walled Brauer algebra in terms of diagrams. These oriented tangles allow us to prove our results by a simple argument, once it is shown that they act on the mixed tensor space.

The paper is organized as follows: in Section 1, the case  $q = 1$  is studied. This special case provides the main ideas for the quantized case, but it is much easier than the quantized case. The algebra  $\mathfrak{B}_{r,s}^n(1)$  is the walled Brauer algebra, a subalgebra of the Brauer algebra which is described in terms of Brauer diagrams. By a simple argument we show that for  $q = 1$  the result for the mixed tensor space  $V^{\otimes r} \otimes V^{*\otimes s}$  is equivalent to the classical result for the ordinary tensor space  $V^{\otimes r+s}$ . Sections 2 and 3 provide

information about the quantum group, the Hecke algebra and Schur–Weyl duality for the ordinary tensor space. Sections 4 and 5 are devoted to find an isomorphism of  $R$ -modules between  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  and  $\text{End}_{\mathbf{U}}(V^{\otimes r+s})$ . Unless  $q = 1$ , this isomorphism is not given by simply commuting components of a tensor product. Each element of the Hecke algebra  $\mathcal{H}_{r+s}$  induces an element of  $\text{End}_{\mathbf{U}}(V^{\otimes r+s})$ , but even for small integers  $r, s, n$  it is nontrivial to compute the images of the generators of the Hecke algebra under the above isomorphism. This problem is solved by the description with oriented tangles which replace the walled Brauer diagrams. In Section 6 these tangles are defined, in Section 7 an action of tangles on the mixed tensor space is defined (coinciding with the action of  $\mathcal{A}_{r,s}$ ), and the relationship between the isomorphism  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \text{End}_{\mathbf{U}}(V^{\otimes r+s})$  and the description by tangles is considered. Section 8 finally proves the main results as an easy consequence of the preceding sections.

## 1. The case $q = 1$

Let  $R$  be a commutative ring with 1,  $n$  a positive integer,  $r, s$  nonnegative integers and  $m = r + s$ . The general linear group  $G = GL_n(R)$  acts naturally on  $V = R^n$ , and thus acts diagonally on  $V^{\otimes m}$ . This action of  $G$  obviously commutes with the action of the symmetric group  $\mathfrak{S}_m$  acting on  $V^{\otimes m}$  by place permutations, so we have a natural algebra map

$$\sigma_m : R\mathfrak{S}_m \rightarrow \text{End}_G(V^{\otimes m})$$

from the group algebra  $R\mathfrak{S}_m$  into the algebra  $\text{End}_G(V^{\otimes m})$  of  $R$ -linear endomorphisms commuting with the  $G$ -action. The famed classical Schur–Weyl duality which is due to Schur states that the representation maps  $\sigma_m$  and  $R\mathfrak{S}_m \rightarrow \text{End}_{R\mathfrak{S}_m}(V^{\otimes m})$  are surjective for  $R = \mathbb{C}$ .

The general linear group  $G$  also acts on the dual space  $V^*$  via  $(gf)(v) = f(g^{-1}v)$  for  $f \in V^*, v \in V, g \in G$  and thus acts on  $V^{\otimes r} \otimes V^{*\otimes s}$ . Analogous to the action of the group algebra  $R\mathfrak{S}_m$  on  $V^{\otimes m}$  we have an algebra  $\mathfrak{B}_{r,s}^n$  acting on  $V^{\otimes r} \otimes V^{*\otimes s}$ , called the walled Brauer algebra, such that this action commutes with the action of  $G$ . Before we introduce this algebra, we will first recall the definition of the Brauer algebra in [3].

A Brauer  $m$ -diagram is an undirected graph consisting of  $m$  vertices in a top row,  $m$  vertices in a bottom row, and  $m$  edges connecting the vertices, such that each vertex is the endpoint of exactly one edge. The *Brauer algebra*

$\mathfrak{B}_m^n$  is the free  $R$ -module over the set of Brauer  $m$ -diagrams with the following multiplication: For two Brauer  $m$ -diagrams  $d_1$  and  $d_2$  we let  $d_1 d_2 = n^k d_3$ , where  $d_3$  and  $k$  are given by the following procedure: place  $d_1$  above  $d_2$  and identify the vertices along the bottom row of  $d_1$  in order with the vertices along the top row of  $d_2$ . Remove the identified vertices and also the closed cycles, if any, and retain the remaining vertices and paths. This results in a new Brauer  $m$ -diagram  $d_3$ . Then let  $k$  be the number of discarded cycles. Here is an example illustrating the multiplication procedure:

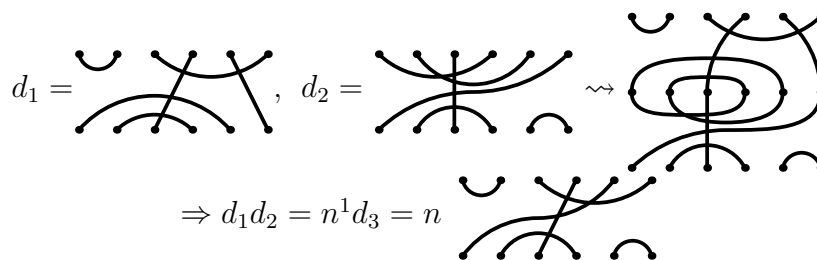


Figure 1: Mutlification in the Brauer algebra

An edge in a Brauer diagram  $d$  is said to be *propagating* if its endpoints lie on opposite rows of  $d$ ; the diagram  $d$  is *totally propagating* if all of its edges are propagating. The subalgebra of  $\mathfrak{B}_m^n$  generated as an  $R$ -module by the totally propagating diagrams is isomorphic with  $R\mathfrak{S}_m$ .

Now we take  $m = r + s$ , and let  $F$  be the set of vertices in the first  $r$  positions in each row of a Brauer  $m$ -diagram. Let  $L$  be the set consisting of the remaining vertices (those occurring in the last  $s$  positions in each row). Consider the subalgebra of  $\mathfrak{B}_m^n$  generated as an  $R$ -module by the diagrams  $d$  such that:

1. the endpoints of propagating edges are both in  $F$  or both in  $L$ ;
2. one endpoint of a non-propagating edge is in  $F$ , the other one in  $L$ .

One easily verifies that this is indeed a subalgebra. It is known as the *walled Brauer algebra*  $\mathfrak{B}_{r,s}^n$  introduced in [1].

Brauer ([3]) defined an action of the Brauer algebra, and thus of its subalgebras, on the tensor space  $V^{\otimes m}$ . To explain this action, we fix the natural basis  $v_1, \dots, v_n$  of  $V = R^n$ . Then  $V^{\otimes m}$  has a basis  $v_{\mathbf{i}} := v_{i_1} \otimes \dots \otimes v_{i_m}$  indexed by *multi-indices*  $\mathbf{i} = (i_1, \dots, i_m)$  with entries  $i_j \in \{1, \dots, n\}$ . To determine the coefficient  $a_{\mathbf{i}, \mathbf{j}}$  of the matrix representing the (right-)action of a Brauer diagram, one writes the entries of  $\mathbf{i}$  and  $\mathbf{j}$  in order along the top and

bottom rows of vertices in the diagram. Then  $a_{i,j}$  is 1 if both ends of each edge carry the same index, and 0 if there is an edge with ends indexed by two different numbers. Note that a totally propagating Brauer diagram permutes the components of the tensor product and thus we regain the action of the symmetric group.

By identifying  $V$  with its linear dual  $V^*$  via  $v_i \mapsto v_i^*$ , the Brauer algebra resp. its subalgebras also acts on  $V_1 \otimes V_2 \otimes \dots \otimes V_m$ , where  $V_i$  is  $V$  or  $V^*$ . In particular, we have actions of  $\mathfrak{S}_{r+s}$  on  $V^{\otimes r+s}$  and  $\mathfrak{B}_{r,s}^n$  on  $V^{\otimes r} \otimes V^{*\otimes s}$ . In [1] it is shown that the representation map

$$\sigma_{r,s} : \mathfrak{B}_{r,s}^n \rightarrow \text{End}_G(V^{\otimes r} \otimes V^{*\otimes s})$$

is surjective for  $R = \mathbb{C}$ . With a simple argument, we will reprove this result and generalize it to arbitrary commutative rings with one, for which  $\sigma_{r+s}$  is surjective. De Concini and Procesi showed in [4] that this surjectivity holds for any commutative ring  $R$  with 1 which satisfies the following condition: whenever  $f$  is a polynomial of degree  $r+s$  and  $f(x) = 0$  for all  $x \in R$ , then  $f = 0$ . In particular,  $\sigma_{r+s}$  is surjective for infinite fields and for  $\mathbb{Z}$ .

**Theorem 1.1.** *For any commutative ring  $R$  with 1 we have*

1.  $\sigma_{r,s}$  is surjective if and only if  $\sigma_{r+s}$  is surjective.
2. The action of  $\mathfrak{B}_{r,s}^n$  is faithful if and only if  $n \geq r+s$ .
3. The annihilator of  $\mathfrak{B}_{r,s}^n$  on  $V^{\otimes r} \otimes V^{*\otimes s}$  is free over  $R$  if and only if the annihilator of  $R\mathfrak{S}_{r+s}$  on  $V^{\otimes r+s}$  is free over  $R$ . If so, they are of the same rank.

*Proof.* For  $G$ -modules  $W_1$  and  $W_2$ , the  $G$ -modules  $W_1 \otimes W_2$  and  $W_2 \otimes W_1$  are isomorphic, the isomorphism is given by interchanging the components of the tensor product. If  $W_1$  and  $W_2$  are free over  $R$  of finite rank, then  $\text{Hom}(W_1, W_2)$  is canonically isomorphic to  $W_1^* \otimes W_2$ . Thus we have a sequence of isomorphisms of  $R$ -modules

$$\begin{aligned} \text{End}_R(V^{\otimes r+s}) &\cong V^{*\otimes r+s} \otimes V^{\otimes r+s} \cong V^{*\otimes s} \otimes V^{*\otimes r} \otimes V^{\otimes r} \otimes V^{\otimes s} \\ &\cong V^{\otimes s} \otimes V^{*\otimes r} \otimes V^{\otimes r} \otimes V^{*\otimes s} \cong \text{End}_R(V^{\otimes r} \otimes V^{*\otimes s}). \end{aligned}$$

Since the  $G$ -linear homomorphisms in  $\text{Hom}(W_1, W_2)$  are mapped exactly to the  $G$ -invariants of  $W_1^* \otimes W_2$ , we obtain an  $R$ -linear isomorphism between  $\text{End}_G(V^{\otimes r+s})$  and  $\text{End}_G(V^{\otimes r} \otimes V^{*\otimes s})$ .

Let  $d$  be a totally propagating Brauer diagram. Thus  $\sigma_{r+s}(d)$  is an element of  $\text{End}_G(V^{\otimes r+s})$ . The above isomorphism permutes components in a tensor product, that is permutes rows and columns in the matrix representing the element of  $\text{End}_G(V^{\otimes r+s})$ . Therefore,  $\sigma_{r+s}(d)$  maps to  $\sigma_{r,s}(d')$ , where  $d'$  is the Brauer diagram obtained by interchanging the vertices to the right of the wall, a top vertex with the corresponding bottom vertex, as depicted in Figure 2. Let  $R\mathfrak{S}_{r+s} \rightarrow \mathfrak{B}_{r,s}^n$  be the  $R$ -linear isomorphism given by  $d \mapsto d'$ .

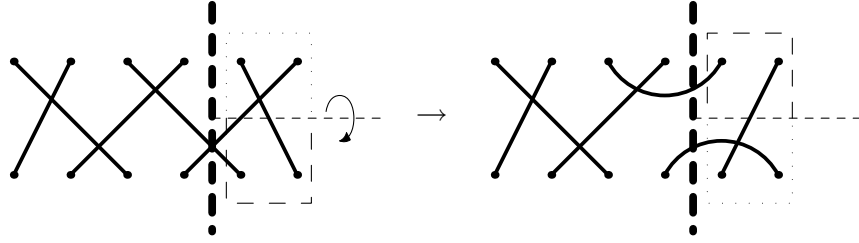


Figure 2: interchanging vertices on one side of the wall

Thus we have the following commuting diagram of  $R$ -linear maps:

$$\begin{array}{ccc}
 R\mathfrak{S}_{r+s} & \xrightarrow{\cong} & \mathfrak{B}_{r,s}^n \\
 \sigma_{r+s} \downarrow & & \downarrow \sigma_{r,s} \\
 \text{End}_G(V^{\otimes r+s}) & \xrightarrow{\cong} & \text{End}_G(V^{\otimes r} \otimes V^{*\otimes s})
 \end{array}$$

The first and the third part of the claim follow easily. For the second part note that the group algebra  $R\mathfrak{S}_{r+s}$  acts faithfully on  $V^{\otimes r+s}$  if and only if  $n \geq r + s$ .  $\square$

The rest of the paper treats a  $q$ -analogue of this theorem. For general (invertible)  $q \in R$  the result is much more difficult to obtain, but in outline the method parallels the proof in the  $q = 1$  case.

## 2. The Hopf algebra $U_R$

Now, let  $R$  be a commutative ring with 1 and  $q$  an invertible element of  $R$ . In this section, we introduce the quantized enveloping algebra of the general linear Lie algebra  $\mathfrak{gl}_n$  over  $R$  with parameter  $q$  and summarize some well known results; see for example [12, 13, 16].

Let  $P^\vee$  be the free  $\mathbb{Z}$ -module with basis  $h_1, \dots, h_n$  and let  $\varepsilon_1, \dots, \varepsilon_n \in P^{\vee*}$  be the corresponding dual basis:  $\varepsilon_i$  is given by  $\varepsilon_i(h_j) := \delta_{i,j}$  for  $j = 1, \dots, n$ , where  $\delta$  is the usual Kronecker symbol. For  $i = 1, \dots, n-1$  let  $\alpha_i \in P^{\vee*}$  be defined by  $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ .

**Definition 2.1.** The quantum general linear algebra  $U_q(\mathfrak{gl}_n)$  is the associative  $\mathbb{Q}(q)$ -algebra with 1 generated by the elements  $e_i, f_i$  ( $i = 1, \dots, n-1$ ) and  $q^h$  ( $h \in P^\vee$ ) with the defining relations

$$\begin{aligned} q^0 &= 1, & q^h q^{h'} &= q^{h+h'} \\ q^h e_i q^{-h} &= q^{\alpha_i(h)} e_i, & q^h f_i q^{-h} &= q^{-\alpha_i(h)} f_i, \\ e_i f_j - f_j e_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, & \text{where } K_i &:= q^{h_i - h_{i+1}}, \\ e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0 & \text{for } |i - j| = 1, \\ f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0 & \text{for } |i - j| = 1, \\ e_i e_j &= e_j e_i, & f_i f_j &= f_j f_i & \text{for } |i - j| > 1. \end{aligned}$$

$U_q(\mathfrak{gl}_n)$  is a Hopf algebra with comultiplication  $\Delta$ , counit  $\varepsilon$  and antipode  $S$  defined by

$$\begin{aligned} \Delta(q^h) &= q^h \otimes q^h, \\ \Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, & \Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i, \\ \varepsilon(q^h) &= 1, & \varepsilon(e_i) &= \varepsilon(f_i) = 0, \\ S(q^h) &= q^{-h}, & S(e_i) &= -e_i K_i, & S(f_i) &= -K_i^{-1} f_i. \end{aligned}$$

Note that  $\Delta$  and  $\varepsilon$  are homomorphisms of algebras and  $S$  is an invertible anti-homomorphism of algebras. Let  $V$  be a free  $\mathbb{Q}(q)$ -module with basis  $\{v_1, \dots, v_n\}$ . We make  $V$  into a  $U_q(\mathfrak{gl}_n)$ -module via

$$\begin{aligned} q^h v_j &= q^{\varepsilon_j(h)} v_j \text{ for } h \in P^\vee, j = 1, \dots, n, \\ e_i v_j &= \begin{cases} v_i & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases} & f_i v_j &= \begin{cases} v_{i+1} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We call  $V$  the *vector representation* of  $U_q(\mathfrak{gl}_n)$ .

Let  $[l]_q$  (in  $\mathbb{Z}[q, q^{-1}]$  resp. in  $R$ ) be defined by  $[l]_q := \sum_{i=0}^{l-1} q^{2i-l+1}$ ,  $[l]_q! := [l]_q [l-1]_q \dots [1]_q$  and let  $e_i^{(l)} := \frac{e_i^l}{[l]_q!}$ ,  $f_i^{(l)} := \frac{f_i^l}{[l]_q!}$ . Let  $\mathbf{U}$  be the  $\mathbb{Z}[q, q^{-1}]$

subalgebra of  $U_q(\mathfrak{gl}_n)$  generated by the  $q^h$  and the divided powers  $e_i^{(l)}$  and  $f_i^{(l)}$  for  $l \geq 0$ .  $\mathbf{U}$  is a Hopf algebra and we have

$$\begin{aligned}\Delta(e_i^{(l)}) &= \sum_{k=0}^l q^{k(l-k)} e_i^{(l-k)} \otimes K_i^{k-l} e_i^{(k)} \\ \Delta(f_i^{(l)}) &= \sum_{k=0}^l q^{-k(l-k)} f_i^{(l-k)} K_i^k \otimes f_i^{(k)} \\ S(e_i^{(l)}) &= (-1)^l q^{l(l-1)} e_i^{(l)} K_i^l \\ S(f_i^{(l)}) &= (-1)^l q^{-l(l-1)} K_i^{-l} f_i^{(l)} \\ \varepsilon(e_i^{(l)}) &= \varepsilon(f_i^{(l)}) = 0.\end{aligned}$$

Furthermore, the  $\mathbb{Z}[q, q^{-1}]$ -lattice  $V_{\mathbb{Z}[q, q^{-1}]}$  in  $V$  generated by the  $v_i$  is invariant under the action of  $\mathbf{U}$ . Now, make the transition from  $\mathbb{Z}[q, q^{-1}]$  to an arbitrary commutative ring  $R$  with 1: Let  $q \in R$  be invertible and consider  $R$  as a  $\mathbb{Z}[q, q^{-1}]$ -module via specializing  $q \in \mathbb{Z}[q, q^{-1}] \mapsto q \in R$ . Let  $\mathbf{U}_R := R \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbf{U}$ . Then  $\mathbf{U}_R$  inherits a Hopf algebra structure from  $\mathbf{U}$  and  $V_R := R \otimes_{\mathbb{Z}[q, q^{-1}]} V_{\mathbb{Z}[q, q^{-1}]}$  is a  $\mathbf{U}_R$ -module.

For nonnegative integers  $k, l$ , let  $\begin{bmatrix} l \\ k \end{bmatrix}_q := \frac{[l]_q!}{[k]_q! ([l-k]_q)!}$ .

**Lemma 2.2.** *For  $l > 0$  we have*

$$\begin{aligned}\sum_{k=0}^{l-1} (-q^{l-1})^{-k} (f_i^{(k)} \otimes 1) \cdot \Delta(f_i^{(l-k)}) &= (K_i^l \otimes 1) \cdot (1 \otimes f_i^{(l)} - S(f_i^{(l)}) \otimes 1), \\ \sum_{k=0}^{l-1} (-q^{l-1})^k (e_i^{(k)} \otimes K_i^k) \cdot \Delta(e_i^{(l-k)}) &= 1 \otimes e_i^{(l)} - (S(e_i^{(l)}) \otimes 1)(K_i^{-l} \otimes K_i^{-l}).\end{aligned}$$



*Proof.* The following formulas can be shown by induction on  $j \geq 0$ :

$$\begin{aligned}
& \sum_{k=0}^j (-q^{l-1})^{-k} (f_i^{(k)} \otimes 1) \cdot \Delta(f_i^{(l-k)}) \\
&= K_i^l \otimes f_i^{(l)} + \sum_{k=0}^{l-j-1} (-1)^j q^{-(j+k)(l-k)} \begin{bmatrix} l-k-1 \\ j \end{bmatrix}_q f_i^{(l-k)} K_i^k \otimes f_i^{(k)} \\
& \sum_{k=0}^j (-q^{l-1})^k (e_i^{(k)} \otimes K_i^{-k}) \cdot \Delta(e_i^{(l-k)}) \\
&= 1 \otimes e_i^{(l)} + \sum_{k=0}^{l-j-1} (-1)^j q^{(j+k)(l-k)} \begin{bmatrix} l-k-1 \\ j \end{bmatrix}_q e_i^{(l-k)} \otimes K_i^{k-l} e_i^{(k)}
\end{aligned}$$

and one gets the lemma by setting  $j = l - 1$ .  $\square$

### 3. The Hecke algebra and Schur–Weyl duality

If no ambiguity arises, we shall henceforth write  $V$  instead of  $V_R$  and  $\mathbf{U}$  instead of  $\mathbf{U}_R$ . Let  $r, s$  be given nonnegative integers. For convenience of notation we will set  $m = r + s$  throughout the paper. Note that if  $W, W_1$  and  $W_2$  are  $\mathbf{U}$ -modules, one can turn  $W^* = \text{Hom}_R(W, R)$  and  $W_1 \otimes W_2$  into  $\mathbf{U}$ -modules setting  $(xg)(w) = g(S(x)w)$  and  $x(w_1 \otimes w_2) = \Delta(x)(w_1 \otimes w_2)$  for  $g \in W^*, x \in \mathbf{U}, w \in W$  and  $w_i \in W_i$ . Our main object of study is the mixed tensor space defined below.

- Definition 3.1.**
1. The  $\mathbf{U}$ -module  $V^{\otimes m}$  is called *tensor space* or *ordinary tensor space*.
  2. The  $\mathbf{U}$ -module  $V^{\otimes r} \otimes V^{*\otimes s}$  is called *mixed tensor space*.

Let  $I(n, m)$  be the set of  $m$ -tuples with entries from  $\{1, \dots, n\}$ . The elements of  $I(n, m)$  are called *multi indices*. We define an element  $v_{\mathbf{i}} := v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_m} \in V^{\otimes m}$  for each multi index  $\mathbf{i} = (i_1, i_2, \dots, i_m) \in I(n, m)$ . Then  $\{v_{\mathbf{i}}, \mathbf{i} \in I(n, m)\}$  is an  $R$ -basis of ordinary tensor space  $V^{\otimes m}$ . Let  $\rho_{ord}$  resp.  $\rho_{mxd}$  be the representations of  $\mathbf{U}$  on the ordinary resp. mixed tensor space.

**Definition 3.2.** Let  $\mathcal{H}_m$  be the associative  $R$ -algebra with identity generated by  $T_1, \dots, T_{m-1}$  subject to the relations

$$\begin{aligned} (T_i + q)(T_i - q^{-1}) &= 0 && \text{for } i = 1, \dots, m-1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } i = 1, \dots, m-2 \\ T_i T_j &= T_j T_i && \text{for } |i - j| \geq 2 \end{aligned}$$

where  $q \in R$  is invertible. The algebra  $\mathcal{H}_m$  is called the *Hecke algebra*.

We remark that the first defining relation  $(T_i + q)(T_i - q^{-1}) = 0$  above is often replaced by a relation of the form  $(\tilde{T}_i + \tilde{q})(\tilde{T}_i - 1) = 0$ . The two approaches are easily seen to be equivalent, via the transformation given by  $\tilde{T}_i = qT_i$ ,  $\tilde{q} = q^2$ .

The symmetric group  $\mathfrak{S}_m$  acts on  $I(n, m)$  by place permutation, i. e.  $\mathbf{i}.s_k = (i_1, \dots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \dots, i_m)$  for a Coxeter generator  $s_k$ . The Hecke algebra  $\mathcal{H}_m$  acts on  $V^{\otimes m}$  by

$$v_{\mathbf{i}} T_k = \begin{cases} q^{-1} v_{\mathbf{i}} & \text{if } i_k = i_{k+1} \\ v_{\mathbf{i}.s_k} & \text{if } i_k < i_{k+1} \\ v_{\mathbf{i}.s_k} + (q^{-1} - q)v_{\mathbf{i}} & \text{if } i_k > i_{k+1} \end{cases}$$

and one easily checks that this action commutes with the action of  $\mathbf{U}$  defined earlier.

Green showed in [10], that  $\rho_{ord}(\mathbf{U})$  is the  $q$ -Schur algebra defined in [5, 8]. Let  $\sigma : \mathcal{H}_m \rightarrow \text{End}_R(V^{\otimes m})$  be the corresponding representation of the Hecke algebra. The next theorem not only shows that  $\text{End}_{\mathbf{U}}(V^{\otimes m})$  is an epimorphic image of the Hecke algebra  $\mathcal{H}_m$  (this was proved in full generality in [9, Theorem 6.2]), but also that some kind of converse is true ([7, Theorem 3.4]). In the literature, this property is called Schur–Weyl duality.

**Theorem 3.3** (Schur–Weyl duality for the ordinary tensor space [9, 7]).

$$\begin{aligned} \text{End}_{\mathbf{U}}(V^{\otimes m}) &= \sigma(\mathcal{H}_m) \\ \text{End}_{\mathcal{H}_m}(V^{\otimes m}) &= \rho_{ord}(\mathbf{U}). \end{aligned}$$

Our goal is to prove similar statements in which the tensor space  $V^{\otimes m}$  is replaced by the mixed tensor space  $V^{\otimes r} \otimes V^{*\otimes s}$  and the Hecke algebra is replaced by the ‘quantized walled Brauer algebra’ given in Definition 6.3 ahead.

#### 4. The finite duals of $\mathbf{U}$ -modules

Let  $W$  be an  $R$ -free  $\mathbf{U}$ -module of finite rank. Recall that the dual space  $W^* = \text{Hom}_R(W, R)$  carries a  $\mathbf{U}$ -module structure via  $(xg)(v) = g(S(x)v)$  for  $x \in \mathbf{U}, g \in W^*, v \in W$ . There is a different  $\mathbf{U}$ -action on the dual space of  $W$  given by  $(xg)(v) = g(S^{-1}(x)v)$ . This  $\mathbf{U}$ -module will be denoted by  $W'$ . In fact,  $W^*$  and  $W'$  are isomorphic as  $\mathbf{U}$ -modules, even in a natural way, but the isomorphism is not the identity on the dual space. Similarly,  $W$  and its double dual  $W^{**}$  are naturally isomorphic as  $\mathbf{U}$ -modules, but the isomorphism is not the usual isomorphism between an  $R$ -module and its double dual.

**Lemma 4.1** ([12], Lemma 3.5.1). *The usual isomorphism between an  $R$ -free module  $W$  of finite rank and its double dual induces  $\mathbf{U}$ -module isomorphisms*

$$W'^* \cong W \cong W^{**}.$$

The tensor product of two  $\mathbf{U}$ -modules is a  $\mathbf{U}$ -module via the comultiplication  $\Delta$ . The map sending  $\sum_{g \in W', v \in W} g \otimes v$  to the  $R$ -endomorphism of  $W$  given by  $(\sum_{g,v} g \otimes v)(w) = \sum_{g,v} g(w) \cdot v$  for  $w \in W$  defines an  $R$ -linear isomorphism of  $W' \otimes W$  onto  $\text{End}_R(W)$ . The set of  $\mathbf{U}$ -invariants of  $W$  is given by

$$W^{\mathbf{U}} := \{w \in W : xw = \varepsilon(x)w \text{ for all } x \in \mathbf{U}\}.$$

**Lemma 4.2.**

$$\text{End}_{\mathbf{U}}(W) \cong (W' \otimes W)^{\mathbf{U}}$$

under the above isomorphisms.

*Proof.* Suppose  $\sum_{g,v} g \otimes v \in \text{End}_{\mathbf{U}}(W) \subset W' \otimes W$ , which means that for all  $x \in \mathbf{U}$  and for all  $w \in W$  we have  $x((\sum_{g,v} g \otimes v)(w)) = (\sum_{g,v} g \otimes v)(xw)$ . The left hand side is

$$x \left( \left( \sum g \otimes v \right) (w) \right) = x \sum g(w)v = \sum g(w)xv = \left( \sum g \otimes xv \right) (w).$$

The right hand side can be written as

$$\left( \sum g \otimes v \right) (xw) = \sum g(xw)v = \sum (S(x)g)(w)v = \left( \sum S(x)g \otimes v \right) (w).$$

This means that  $\sum_{g,v} g \otimes v \in \text{End}_{\mathbf{U}}(W)$  if and only if  $\sum_{g,v} g \otimes xv = \sum_{g,v} S(x)g \otimes v$  for all  $x \in \mathbf{U}$ .

For  $l \geq 0$ , let  $\mathbf{U}_l$  be the algebra generated by the  $q^h$  together with the divided powers  $e_i^{(k)}, f_i^{(k)}$  with  $k \leq l$ . We claim that  $\mathbf{U}_l$ -invariants and  $\mathbf{U}_l$ -endomorphisms coincide. If  $x = q^h$  then

$$\begin{aligned} \sum_{g,v} g \otimes xv &= \sum_{g,v} S(x)g \otimes v \iff \sum_{g,v} g \otimes q^h v = \sum_{g,v} q^{-h} g \otimes v \\ &\iff \sum_{g,v} q^h g \otimes q^h v = \sum_{g,v} g \otimes v \\ &\iff q^h \sum_{g,v} g \otimes v = \varepsilon(q^h) \sum_{g,v} g \otimes v. \end{aligned}$$

This shows the claim for  $l = 0$ . Suppose the claim holds for  $\mathbf{U}_{l-1}$  and we have an element  $\sum_{g,v} g \otimes v$  that already commutes with  $\mathbf{U}_{l-1}$ , i. e.  $\Delta(f_i^{(j)}) \sum_{g,v} g \otimes v = \Delta(e_i^{(j)}) \sum_{g,v} g \otimes v = 0$  for all  $j < l$  and  $\Delta(q^h) \sum_{g,v} g \otimes v = \sum_{g,v} g \otimes v$ . Lemma 2.2 shows that

$$\begin{aligned} \Delta(f_i^{(l)}) \sum_{g,v} g \otimes v = 0 &\iff \sum_{g,v} g \otimes f_i^{(l)} v - S(f_i^{(l)}) g \otimes v = 0 \text{ and} \\ \Delta(e_i^{(l)}) \sum_{g,v} g \otimes v = 0 &\iff \sum_{g,v} g \otimes e_i^{(l)} v - S(e_i^{(l)}) g \otimes v = 0. \end{aligned}$$

Since  $f_i^{(l)}, e_i^{(l)}$  and  $\mathbf{U}_{l-1}$  generate  $\mathbf{U}_l$ , the lemma follows.  $\square$

For  $R$ -modules  $V$  and  $W$  let  $\tau : V \otimes W \rightarrow W \otimes V$  be the  $R$ -homomorphism defined by  $\tau(v \otimes w) = w \otimes v$ .

**Lemma 4.3.** *The following diagrams commute:*

1.

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{S} & \mathbf{U} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathbf{U} \otimes \mathbf{U} & \xrightarrow{\tau} \mathbf{U} \otimes \mathbf{U} \xrightarrow{S \otimes S} & \mathbf{U} \otimes \mathbf{U} \end{array}$$

that is  $\Delta \circ S = (S \otimes S) \circ \tau \circ \Delta$ . Similarly

2.

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{S^{-1}} & \mathbf{U} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathbf{U} \otimes \mathbf{U} & \xrightarrow{\tau} \mathbf{U} \otimes \mathbf{U} \xrightarrow{S^{-1} \otimes S^{-1}} & \mathbf{U} \otimes \mathbf{U} \end{array}$$

that is  $\Delta \circ S^{-1} = (S^{-1} \otimes S^{-1}) \circ \tau \circ \Delta$ .

This is easily checked.

**Lemma 4.4.** *Let  $W_1$  and  $W_2$  be  $\mathbf{U}$ -modules which are  $R$ -free of finite rank. Then there are natural isomorphisms of  $\mathbf{U}$ -modules*

$$(W_1 \otimes W_2)^* \cong W_2^* \otimes W_1^*, \quad (W_1 \otimes W_2)' \cong W_2' \otimes W_1'.$$

The isomorphisms  $\phi^\square : W_2^\square \otimes W_1^\square \rightarrow (W_1 \otimes W_2)^\square$  (with  $\square = * \text{ or } '$ ) are given by

$$\phi^\square(g \otimes h)(v \otimes w) := h(v)g(w)$$

for  $g \in W_2^\square, h \in W_1^\square, v \in W_1, w \in W_2$ .

*Proof.* Let  $x \in \mathbf{U}$ . We write  $\Delta(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}$  in the Sweedler notation. By Lemma 4.3 we have  $\Delta \circ S(x) = (S \otimes S) \circ \tau(\sum_{(x)} x_{(0)} \otimes x_{(1)}) = \sum_{(x)} S(x_{(1)}) \otimes S(x_{(0)})$  and  $\Delta \circ S^{-1}(x) = \sum_{(x)} S^{-1}(x_{(1)}) \otimes S^{-1}(x_{(0)})$ . With this notation we have

$$\begin{aligned} x(\phi(g \otimes h))(v \otimes w) &= \phi(g \otimes h)(S^{\pm 1}(x)(v \otimes w)) \\ &= \phi(g \otimes h)(\Delta(S^{\pm 1}(x))(v \otimes w)) \\ &= \phi(g \otimes h)\left(\sum_{(x)} S^{\pm 1}(x_{(1)})v \otimes S^{\pm 1}(x_{(0)})w\right) \\ &= \sum_{(x)} h(S^{\pm 1}(x_{(1)})v) g(S^{\pm 1}(x_{(0)})w), \end{aligned}$$

where  $\pm$  is  $+$  for  $\square = *$  and  $-$  for  $\square = '$ . On the other hand

$$\begin{aligned} \phi(x(g \otimes h))(v \otimes w) &= \phi(\Delta(x)(g \otimes h))(v \otimes w) \\ &= \phi\left(\sum_{(x)} x_{(0)}g \otimes x_{(1)}h\right)(v \otimes w) \\ &= \sum_{(x)} (x_{(1)}h)(v) (x_{(0)}g)(w) \\ &= \sum_{(x)} h(S^{\pm 1}(x_{(1)})v) g(S^{\pm 1}(x_{(0)})w) \end{aligned}$$

and the claim is proved.  $\square$

## 5. The isomorphisms $V' \otimes V \cong V \otimes V^*$ and $V^* \otimes V \cong V \otimes V^*$

The following lemma exhibits isomorphisms  $V' \otimes V \rightarrow V \otimes V^*$  and  $V^* \otimes V \rightarrow V \otimes V^*$ . As mentioned in the remarks preceding Lemma 4.1 we have an isomorphism  $V^* \cong V'$  as  $\mathbf{U}$ -modules. We will need the following two isomorphisms generalizing the interchange of the components of a tensor product. If we want to emphasize that  $v_i^*$  is considered as an element of  $V'$ , we write  $v'_i$  instead of  $v_i^*$ .

**Lemma 5.1.** *Let  $\psi' : V' \otimes V \rightarrow V \otimes V^*$ , and  $\psi : V^* \otimes V \rightarrow V \otimes V^*$  be the isomorphism of  $R$ -modules defined by*

$$\begin{aligned} \psi'(v'_i \otimes v_k) &= \begin{cases} q^{n+1-2i} v_k \otimes v_i^* & k \neq i \\ q^{n+1-2i} \left( q^{-1} v_i \otimes v_i^* + (q^{-1} - q) \sum_{l=1}^{i-1} v_l \otimes v_l^* \right) & k = i \end{cases} \\ \psi(v_i^* \otimes v_k) &= \begin{cases} v_k \otimes v_i^* & k \neq i \\ q^{-1} v_i \otimes v_i^* + (q^{-1} - q) \sum_{l=1}^{i-1} v_l \otimes v_l^* & k = i \end{cases} \end{aligned}$$

The isomorphisms  $\psi'$  and  $\psi$  are isomorphisms of  $\mathbf{U}$ -modules. Note that they are bijective for any choice of the ring  $R$ , since  $q$  is invertible. If  $q = 1$ , then all the isomorphisms are given by permuting the components of the tensor product.

The lemma can be easily verified. For positive integers  $r$  and  $s$ , consider the following isomorphisms:

$\text{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$  is isomorphic as an  $R$ -module to  $(V^{\otimes r} \otimes V^{*\otimes s})' \otimes (V^{\otimes r} \otimes V^{*\otimes s})$ , such that  $\mathbf{U}$ -endomorphisms correspond to  $\mathbf{U}$ -invariants by Lemma 4.2. Lemma 4.4 together with Lemma 4.1 shows that  $(V^{\otimes r} \otimes V^{*\otimes s})' \otimes (V^{\otimes r} \otimes V^{*\otimes s})$  is isomorphic as a  $\mathbf{U}$ -module to  $V^{\otimes s} \otimes V'^{\otimes r} \otimes V^{\otimes r} \otimes V^{*\otimes s} = V^{\otimes s} \otimes V'^{\otimes r-1} \otimes V' \otimes V \otimes V^{\otimes r-1} \otimes V^{*\otimes s}$ . Applying  $\psi'$  to the middle part, this is isomorphic as a  $\mathbf{U}$ -module to  $V^{\otimes s} \otimes V'^{\otimes r-1} \otimes V \otimes V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}$ . By a similar argument as above, we have  $R$ -isomorphisms

$$V^{\otimes s} \otimes V'^{\otimes r-1} \otimes V \otimes V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s} \cong \text{End}_R(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s})$$

and again  $\mathbf{U}$ -invariants map to  $\mathbf{U}$ -endomorphisms. Successive application of  $\psi$  shows that  $V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}$  and  $V^{\otimes r-1} \otimes V^{*\otimes s+1}$  are isomorphic  $\mathbf{U}$ -modules (and thus isomorphic as  $R$ -modules), so we have

$$\begin{aligned} \text{End}_R(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}) &\cong \text{End}_R(V^{\otimes r-1} \otimes V^{*\otimes s+1}) \\ \text{End}_{\mathbf{U}}(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}) &\cong \text{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1}). \end{aligned}$$

This proves the following proposition:

**Proposition 5.2.** *There is an isomorphism of  $R$ -modules*

$$\text{End}_R(V^{\otimes r} \otimes V^{*\otimes s}) \rightarrow \text{End}_R(V^{\otimes r-1} \otimes V^{*\otimes s+1})$$

*mapping  $\mathbf{U}$ -endomorphisms to  $\mathbf{U}$ -endomorphisms. Repeated application leads to an isomorphism of  $R$ -modules from  $\text{End}_{\mathbf{U}}(V^{\otimes r+s})$  to  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$ .*

## 6. Oriented tangles

It is time to introduce the quantized walled Brauer algebra. This is a generalization of the algebra defined in [14]; we give a graphical description based on tangles.

A *tangle* is a knot diagram in a rectangle (with two opposite edges designated “top” and “bottom”) contained in the plane  $\mathbb{R}^2$ , consisting of  $m$  vertices along the top edge,  $m$  vertices along the bottom edge, and  $m$  strands in  $\mathbb{R}^3$  connecting the vertices, such that each vertex is an endpoint of exactly one strand, along with a finite number of closed cycles. Two tangles are regularly isotopic, if they are related by a sequence of Reidemeister’s moves RMII and RMIII, together with isotopies fixing the vertices:

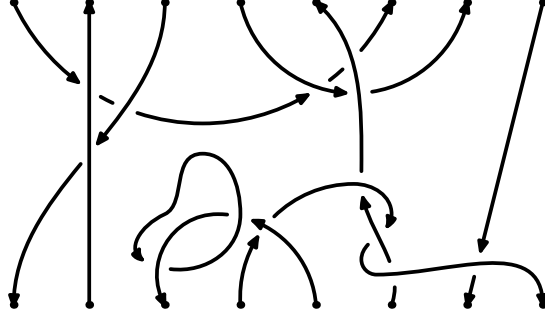
$$\text{RMII: } \begin{array}{c} \text{---} \\ \text{---} \end{array} \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) = \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \left( \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \quad \text{RMIII: } \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

We fix the following notations: the vertices in the top row are denoted by  $t_1, t_2, \dots, t_m$ , the vertices in the bottom row by  $b_1, b_2, \dots, b_m$ . An *oriented tangle* is a tangle such that each of its strands has a direction. Let  $T$  be an oriented tangle. To  $T$  we associate sequences  $I = (I_1, \dots, I_m)$  and  $J = (J_1, \dots, J_m)$  of symbols ‘ $\downarrow$ ’ and ‘ $\uparrow$ ’ by the following rules:

1. If there is a strand starting in  $t_k$  then let  $I_k = \downarrow$ , if there is a strand ending in  $t_k$ , let  $I_k = \uparrow$ .
2. If there is a strand starting in  $b_k$  then let  $J_k = \uparrow$ , if there is a strand ending in  $b_k$ , let  $J_k = \downarrow$ .

Observe that for a given oriented tangle the number of entries equal to  $\downarrow$  in  $I$  necessarily is the same as the number of entries in  $J$  equal to  $\downarrow$ . We henceforth assume that the indices  $I, J$  satisfy this condition and call  $T$  an oriented tangle of *type*  $(I, J)$ . We indicate multiple occurrences of one symbol next to each other by exponents.

**Example 6.1.**



is an oriented tangle of type  $((\downarrow, \uparrow, \downarrow^2, \uparrow^3, \downarrow), (\downarrow, \uparrow, \downarrow, \uparrow^3, \downarrow^2))$ .

If  $v$  is a vertex, let  $s(v)$  be the strand starting or ending in  $v$ . If  $s$  is a strand, let  $b(s)$  be the beginning vertex and  $e(s)$  the ending vertex of  $s$ . Note that strands begin and end at vertices and are not closed cycles.

**Definition 6.2.** Let  $\mathcal{U}_{I,J}$  be the  $R$ -module generated by all oriented tangles of type  $(I, J)$  up to regular isotopy and the following relations which can be applied to a local disk in a tangle

$$\begin{aligned}
 (O1) \quad & \left( \begin{array}{c} \swarrow \quad \searrow \\ \downarrow \quad \uparrow \end{array} - \begin{array}{c} \swarrow \quad \searrow \\ \uparrow \quad \downarrow \end{array} = (q^{-1} - q) \right) \left( \begin{array}{c} \swarrow \\ \downarrow \end{array} \right) \\
 (O2) \quad & \left( \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} = \begin{array}{c} \circlearrowright \\ \downarrow \end{array} = [n]_q \right) \\
 (O3) \quad & \left( \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = \begin{array}{c} \uparrow \\ \circlearrowright \end{array} = q^n \right) \\
 (O4) \quad & \left( \begin{array}{c} \uparrow \\ \circlearrowright \end{array} = \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = q^{-n} \right)
 \end{aligned}$$

Let  $S$  be a tangle of type  $(I, J)$  and  $T$  be a tangle of type  $(J, K)$ . Then all the strands at the bottom vertices of  $S$  have the same direction (up or down) as the corresponding strands at the top row of  $T$ . Thus if one places  $S$  above  $T$  and identifies the bottom row of vertices in  $S$  with the top row in  $T$ , one gets a tangle of type  $(I, K)$ . Denote this concatenated tangle by  $S/T$ . Concatenation induces an  $R$ -bilinear map  $\mathcal{U}_{I,J} \times \mathcal{U}_{J,K} \rightarrow \mathcal{U}_{I,K}$ . In particular,  $\mathcal{U}_{I,I}$  is an associative  $R$ -algebra, the multiplication given by concatenation of tangles.

**Definition 6.3.** Let  $\mathfrak{B}_{r,s}^n(q) := \mathcal{U}_{(\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s)}$ . Then  $\mathfrak{B}_{r,s}^n(q)$  is an  $R$ -algebra which we call the *quantized walled Brauer algebra*.



Let us imagine a plane  $P$  orthogonal to the plane of the enclosing rectangle of the tangle, such that the first  $r$  vertices along the top and bottom edges are on one side of  $P$  and the remaining  $s$  vertices along the top and bottom edges are on the other side of  $P$ . The plane  $P$  is sometimes called the *wall*. If  $s$  is a strand of a tangle belonging to  $\mathcal{U}_{(\downarrow r \uparrow s), (\downarrow r \uparrow s)}$  and if  $b(s)$ ,  $e(s)$  lie on the same edge (top or bottom) of the enclosing rectangle, then necessarily  $s$  crosses the wall  $P$ . Moreover, if  $b(s)$ ,  $e(s)$  lie on the opposite edges of the enclosing rectangle, then the strand  $s$  may be positioned in such a way that it does not intersect the wall  $P$ . This geometric understanding of the tangles in  $\mathfrak{B}_{r,s}^n(q)$  explain the terminology ‘walled Brauer algebra’.

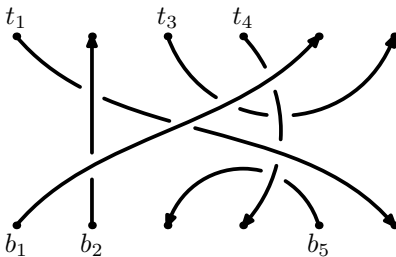
Note that for  $q = 1$  relation (O1) means that we don’t have to distinguish over- and under-crossings, relations (O3) and (O4) are just the Reidemeister Moves I. We will show that  $\mathcal{U}_{I,J}$  has an  $R$ -basis indexed by certain tangles which are the  $q$ -analogues of Brauer diagrams in the classical case  $q = 1$ . If  $q$  is not necessarily 1, then using relation (O1) one can switch crossings in an oriented tangle (modulo a linear combination of oriented tangles with fewer crossings), in particular one can move a strand on top of all other strands, a second strand on top of the remaining strands and so on. The relations (O1), (O3) and (O4) can be used to untangle the strands and closed cycles, and finally relation (O2) can be used to eliminate the unknotted cycles. This serves as motivation for the following definition of descending tangles, since each oriented tangle can be written as a linear combination of descending ones (with respect to a given total ordering on the starting vertices).

**Definition 6.4.** Let  $T$  be an oriented tangle of type  $(I, J)$ . Chose a total ordering  $\preceq$  on the starting vertices of  $T$ . We say that  $T$  is *descending* with respect to  $\preceq$  if the following conditions hold:

1. No strand crosses itself.
2. Two strands cross at most once.
3. There are no closed cycles.
4. If two strands  $s_1$  and  $s_2$  with  $b(s_1) \prec b(s_2)$  cross, then  $s_1$  over-crosses  $s_2$ .

This means that a strand  $s$  lies on top of all strands  $t$  with beginning vertex  $b(t) \succ b(s)$ . Note that each tangle satisfying 1, 3 and 4 is regularly isotopic to a descending tangle.

**Example 6.5.** Let  $T$  be the tangle



The strand  $s(b_1)$  lies on top of all other strands,  $s(b_2)$  lies on top of all strands except for  $s(b_1)$  etc. Thus  $T$  is descending with respect to the total orderings  $b_1 \prec b_2 \prec t_1 \prec t_4 \prec t_3 \prec b_5$  and  $b_1 \prec b_2 \prec t_1 \prec t_4 \prec b_5 \prec t_3$ .

We will show that the descending tangles (up to regular isotopy) with respect to some fixed total ordering on the starting vertices form a basis for  $\mathcal{U}_{I,J}$ .

A *Brauer diagram* is a graph consisting of  $2m$  vertices arranged in two rows and  $m$  edges, such that each vertex belong to exactly one edge. If  $T$  is a tangle, let  $c(T)$  be the Brauer diagram obtained by connecting the vertices which are connected by a strand in the tangle  $T$ , so closed circles are ignored. We call  $c(T)$  the *connector* of  $T$ . We say, that a Brauer diagram is of type  $(I, J)$ , if it is the connector of a tangle of type  $(I, J)$ . Note that for a given Brauer diagram  $c$ , there are always different indices, such that  $c$  is the connector of tangles of these types. Indeed, all other types are obtained by changing the orientation on strands in one tangle with connector  $c$  in all possible ways.

Furthermore, there are precisely  $m!$  Brauer diagrams of type  $(I, J)$ : For a given type  $(I, J)$  there are  $m$  starting vertices and  $m$  end vertices. In a Brauer diagram of type  $(I, J)$  the edges always connect a starting vertex with an end vertex. Thus the Brauer diagrams of type  $(I, J)$  are in one-to-one correspondence with bijections from the set of starting vertices to the set of end vertices, and there are  $m!$  such bijections.

Fix some total ordering on the set of starting vertices. For each Brauer diagram  $c$  of type  $(I, J)$  choose a tangle  $T_c$  of type  $(I, J)$  with connector  $c$ , which is descending with respect to the fixed total ordering. Consider the strand in  $T_c$  starting at the first starting vertex. The Brauer diagram  $c$  determines the end vertex of this strand. Furthermore, this strand does not cross itself, thus if  $T'_c$  is another descending tangle with connector  $c$ ,

one can use the Reidemeister Moves II and III to move the top strand of  $T'_c$  to the position of the top strand of  $T_c$ . A similar argument works for the other strands, thus  $T_c$  and  $T'_c$  are regularly isotopic and we have a one-to-one correspondence between the set of Brauer diagrams of type  $(I, J)$  and the set of isotopy classes of descending tangles of type  $(I, J)$  with respect to a given total ordering of the starting vertices.

**Lemma 6.6.** ([19]) *Fix a total ordering on the set of starting vertices and let  $\mathcal{B} = \{T_c\}$  where  $c$  runs through the set of Brauer diagrams of type  $(I, J)$ .*

1. *Let  $\Lambda = \mathbb{Z}[a, \lambda^{\pm 1}, a^{-1}(\lambda^{-1} - \lambda)]$  and let  $\mathcal{U}'_{I,J}$  be the  $\Lambda$ -module generated by the tangles of type  $(I, J)$  with relations*

$$\begin{aligned}
 (O1') & \quad \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = a \quad \left( \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \\
 (O2') & \quad \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = a^{-1}(\lambda^{-1} - \lambda) = \delta \\
 (O3') & \quad \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \lambda \begin{array}{c} \uparrow \\ \uparrow \end{array} \\
 (O4') & \quad \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \lambda^{-1} \begin{array}{c} \uparrow \\ \uparrow \end{array}
 \end{aligned}$$

*Then  $\mathcal{U}'_{I,J}$  is  $\Lambda$ -free with basis  $\mathcal{B}$ .*

2.  *$\mathcal{U}_{I,J}$  is  $R$ -free with basis  $\mathcal{B}$ . In particular,  $\mathcal{U}_{I,J}$  has  $R$ -rank  $m!$ .*

*Proof.* In his diploma thesis [19], F. Weimer showed the special case  $I = J = (\downarrow^r, \uparrow^s)$  of the first assertion. Actually, the restriction to  $(\downarrow^r, \uparrow^s)$  is not essential, if  $I$  and  $J$  have  $r$  entries equal to  $\downarrow$  and  $s$  entries equal to  $\uparrow$ , then  $\mathcal{U}_{I,J}$  and  $\mathcal{U}_{(\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s)}$  are isomorphic as  $R$ -modules: let  $S$  and  $T$  be descending tangles (with respect to some total ordering) without horizontal strands of type  $((\downarrow^r, \uparrow^s), I)$  and  $(J, (\downarrow^r, \uparrow^s))$  respectively. Let  $S'$  and  $T'$  be the tangles obtained by taking the mirror images of  $S$  and  $T$  and reversing the orientation of all strands. Then  $S/S'$ ,  $S'/S$ ,  $T/T'$  and  $T'/T$  are oriented tangles with non-crossing vertical strands, and  $S/_-/T : \mathcal{U}_{I,J} \rightarrow \mathcal{U}_{(\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s)}$  and  $S'/_-/T' : \mathcal{U}_{(\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s)} \rightarrow \mathcal{U}_{I,J}$  are homomorphisms of  $R$ -modules which are inverse to each other.

The proof follows exactly the proof of Morton and Wassermann ([17]) for the Birman–Wenzl–Murakami algebra. Instead of the knot invariant Morton and Wassermann used, this proof needs a knot invariant of oriented knots, whose existence is assured in [11].

The second assertion follows directly by change of base rings.  $\square$

**Corollary 6.7.** *For each Brauer diagram  $c$  of type  $(I, J)$  choose an oriented tangle  $S_c$  of type  $(I, J)$  with connector  $c$  without self-crossings and closed cycles such that two strands cross at most once. Then the  $S_c$  form a basis of  $\mathcal{U}'_{I,J}$  resp. of  $\mathcal{U}_{I,J}$ .*

*Proof.* If  $T$  is a tangle such that two strands cross at most once, then each tangle which is regular isotopic to  $T$  has at least as many crossings as  $T$ , since crossings can only be eliminated by the Reidemeister Moves II and III if two strands cross more than once. Choose a total ordering of the starting vertices and let the elements of the corresponding basis be denoted by  $T_c$ . Using relation (O1)' resp. (O1),  $S_c$  can be rewritten as  $T_c$  plus a linear combination of tangles with strictly fewer crossings, which in turn can be written as a linear combination of descending tangles  $T_c$  with fewer crossings. Thus the base change matrix is an upper diagonal matrix (if the basis elements are ordered compatible with the number of crossings), the entries on the diagonal are 1, thus this matrix is invertible, and  $\{S_c\}$  is a basis.  $\square$

Note that if  $I$  and  $J$  have the same numbers of up-arrows and down-arrows, then  $\mathcal{U}_{I,I}$  and  $\mathcal{U}_{J,J}$  are isomorphic as  $R$ -algebras. The isomorphism is given by conjugation with a descending tangle without horizontal strands of type  $(I, J)$ .

**Corollary 6.8.**  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  and  $\mathcal{H}_m$  are isomorphic as  $R$ -algebras.

*Proof.* The generators  $\downarrow \cdots \downarrow$ ,  $\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$ ,  $\downarrow \cdots \downarrow$  of  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  satisfy the relations of the  $T_i \in \mathcal{H}_m$ . Thus  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  is an epimorphic image of  $\mathcal{H}_m$ . For  $w \in \mathfrak{S}_m$  let  $w = s_{i_1} s_{i_2} \dots s_{i_l}$  be a reduced expression, and let  $T_w = T_{i_1} T_{i_2} \dots T_{i_l}$ . It is known that  $T_w$  does not depend on the reduced expression and  $\{T_w \mid w \in \mathfrak{S}_m\}$  is a basis of  $\mathcal{H}_m$ .

The starting vertices in the top row can be ordered from left to right. Then a descending tangle with respect to this ordering corresponds to one of the  $T_w$ , which shows that  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  and  $\mathcal{H}_m$  are isomorphic.  $\square$

The quantized walled Brauer algebra  $\mathfrak{B}_{r,s}^n(q)$  can be given by generators and relations as well.

**Definition 6.9** ([15]). Let  $\mathcal{A}_{r,s}$  be the  $\Lambda$ -algebra generated by the elements

$g_1, \dots, g_{r-1}, g_1^*, \dots, g_{s-1}^*$  and  $D$  subject to the following relations

$$\begin{array}{ll}
\text{(i)} & g_i g_j = g_j g_i, |i - j| \geq 2 \\
\text{(ii)} & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \\
\text{(iii)} & g_i^2 + a g_i - 1 = 0 \\
\text{(iv)} & g_i g_j^* = g_j^* g_i \\
\text{(v)} & D g_i = g_i D, 2 \leq i \leq r - 1 \\
\text{(vi)} & D g_1 D = \lambda^{-1} D \\
\text{(vii)} & D^2 = \delta D \\
\text{(viii)} & D g_1^{-1} g_1^* D g_1 = D g_1^{-1} g_1^* D g_1^* \\
\text{(i}^*) & g_i^* g_j^* = g_j^* g_i^*, |i - j| \geq 2 \\
\text{(ii}^*) & g_j^* g_{j+1}^* g_j^* = g_{j+1}^* g_j^* g_{j+1}^* \\
\text{(iii}^*) & g_j^{*2} + a g_j^* - 1 = 0 \\
\text{(v}^*) & D g_j^* = g_j^* D, 2 \leq j \leq s - 1 \\
\text{(vi}^*) & D g_1^* D = \lambda^{-1} D \\
\text{(viii}^*) & g_1 D g_1^{-1} g_1^* D = g_1^* D g_1^{-1} g_1^* D
\end{array}$$

Note that (iii) implies that  $g_1$  is invertible with  $g_1^{-1} = g_1 + a$ .

If  $f : \Lambda \rightarrow R$  is a specialization homomorphism, let  $\mathcal{A}_{r,s}(R)$  be the algebra given by the same generators and relations with specialized coefficients.

The definition of Leduc in [15] used a different set of parameters. One may recover Leduc's algebra as the specialization  $\Lambda \rightarrow \mathbb{C}(z, q) : a \mapsto q - q^{-1}, \lambda \mapsto z^{-1}, \delta \mapsto (z - z^{-1})/(q - q^{-1})$ .

**Corollary 6.10.**  $\mathcal{U}'_{(\downarrow r \uparrow s), (\downarrow r \uparrow s)}$  and  $\mathcal{A}_{r,s}$  are isomorphic as  $\Lambda$ -algebras. The same is true for any specialization, in particular for the specialization  $f : \Lambda \rightarrow R : a \mapsto q^{-1} - q, \lambda^{\pm 1} \mapsto q^{\pm n}, \delta \mapsto [n]_q$  we have  $\mathcal{A}_{r,s} \cong \mathfrak{B}_{r,s}^n(q)$  as  $R$ -algebras.

*Proof.* It is easy to verify that

$$\mathcal{A}_{r,s} \rightarrow \mathcal{U}'_{(\downarrow r \uparrow s), (\downarrow r \uparrow s)} : \begin{array}{l}
1 \mapsto \downarrow \cdots \downarrow \uparrow \cdots \uparrow \\
g_i \mapsto \downarrow \cdots \downarrow \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \downarrow \cdots \downarrow \uparrow \cdots \uparrow \\
g_j^* \mapsto \downarrow \cdots \downarrow \uparrow \cdots \uparrow \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \uparrow \cdots \uparrow \\
D \mapsto \downarrow \cdots \downarrow \begin{array}{c} \frown \\ \smile \end{array} \uparrow \cdots \uparrow
\end{array}$$

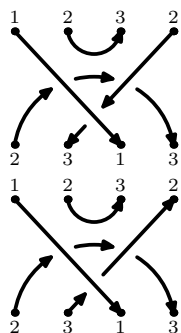
where the crossings affect the  $r - i$ -th and  $r - i + 1$ -st node and the  $j$ -th and  $j + 1$ -st node respectively, induces a well defined homomorphism of algebras. The same arguments as in [15] show that  $\mathcal{A}_{r,s}$  is generated as an  $\Lambda$ -module by so called standard monomials. Indeed, our parameters in the relations differ from those in [15] but the transformations can be done in the same way. These standard monomials map to oriented tangles without closed components and self-crossings with different connectors, such that two strands cross at most once. By Corollary 6.7 the images of the standard monomials are a basis of  $\mathcal{U}'_{(\downarrow r \uparrow s), (\downarrow r \uparrow s)}$ , thus a generating system maps to a basis. The claim follows.  $\square$

Note that in the classical case, the walled Brauer algebra is a subalgebra of the Brauer algebra. There is also a  $q$ -deformation of the Brauer algebra, the BMW-algebra (Birman-Murakami-Wenzl algebra, see [2]), which has also a tangle description. Unlike the quantized walled Brauer algebra, the tangles used for the BMW-algebra are not oriented and relation  $(O1)$  involves an additional term. Thus, the quantized walled Brauer algebra is not a subalgebra of the BMW-algebra, at least not in a canonical way.

## 7. The action of oriented tangles

In this section we will give an action of  $\mathfrak{B}_{r,s}^n(q)$  on the mixed tensor space coinciding with the action of  $\mathcal{A}_{r,s}$  given in [15]. However, we will give a procedure to calculate the entries of the matrix of an arbitrary tangle, and not only of a generator. Furthermore, this procedure assigns to each oriented tangle of type  $(I, J)$  an  $R$ -linear map.

Let  $\mathbf{i}, \mathbf{j} \in I(n, m)$ . Recall that  $I(n, m)$  is the set of  $m$ -tuples with entries from 1 to  $n$ . Write the entries of  $\mathbf{i}$  resp. of  $\mathbf{j}$  in order from left to right along the vertices of the top resp. bottom row of the tangle. If  $\beta$  is a vertex, let  $\beta(\mathbf{i}, \mathbf{j})$  be the entry written at the vertex  $\beta$ . We say that  $T$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$  if  $T$  is descending with respect to some total ordering  $\prec$  on the starting vertices, such that  $\beta \prec \gamma$  implies  $\beta(\mathbf{i}, \mathbf{j}) \leq \gamma(\mathbf{i}, \mathbf{j})$ , that is  $\prec$  is a refining of the ordering on starting vertices by their attached indices. So this means, that the strands starting from vertices  $\beta$  with  $\beta(\mathbf{i}, \mathbf{j}) = 1$  are on top of the strands with starting index 2, etc. For example for  $n = 3, m = 4$ ,



is descending with respect to  $((1, 2, 3, 2), (2, 3, 1, 3))$ .

The total ordering on the starting vertices is  $t_1 \prec t_4 \prec b_1 \prec t_2, t_1 \prec t_4 \prec t_2 \prec b_1$  or  $t_1 \prec t_2 \prec t_4 \prec b_1$ .

is not descending with respect to  $((1, 2, 3, 2), (2, 3, 1, 3))$ .

To each  $m$ -tuple  $I$  with entries in  $\{\uparrow, \downarrow\}$  we associate an  $m$ -fold tensor product  $V_I$ , such that the  $k$ -th component of  $V_I$  is  $V$  if  $I_k = \downarrow$  and  $V^*$  if  $I_k = \uparrow$ . Recall that  $\{v_1, \dots, v_n\}$  is a basis of  $V$ . Let  $\{v_1^*, \dots, v_n^*\}$  be the dual basis. For  $\mathbf{i} = (i_1, \dots, i_m) \in I(n, m)$  let  $v_{\mathbf{i}}^I = x_{i_1} \otimes \dots \otimes x_{i_m} \in V_I$  where  $x_{i_\rho}$  is either  $v_{i_\rho} \in V$  or  $v_{i_\rho}^* \in V^*$ . Then  $\{v_{\mathbf{i}}^I \mid \mathbf{i} \in I(n, m)\}$  is a basis of  $V_I$ . We will now

assign to each oriented tangle  $T$  of type  $(I, J)$  a linear map  $\psi_T$  from  $V_I$  to  $V_J$  acting on  $V_I$  from the right by giving a procedure to calculate the entries of the corresponding matrix with rows and columns indexed by  $I(n, m)$ . We let these matrices act from the right as well.

**Procedure 7.1.** *Let  $(I, J)$  be a type and  $\mathbf{i}, \mathbf{j} \in I(n, m)$  be fixed. Choose a total ordering on the set of starting vertices compatible with the partial order on the starting vertices by the double index  $(\mathbf{i}, \mathbf{j})$ .*

*Let  $T$  be an oriented tangle  $T$  of type  $(I, J)$  which is descending with respect to this total ordering. To  $T$  we define a value  $\psi_{\mathbf{i}, \mathbf{j}}(T) \in R$  as follows. If there is a strand in  $T$  whose starting and end vertex are labeled by different indices from the double index  $(\mathbf{i}, \mathbf{j})$ , let  $\psi_{\mathbf{i}, \mathbf{j}}(T) = 0$ . If for all strands in  $T$  the starting and end vertex carry the same label, we may assign this label to the strand as well. Then let  $\psi_{\mathbf{i}, \mathbf{j}}(T)$  be the product of the following factors (the empty product being 1 as usual):*

- *we have a factor  $q^{-1}$  for each crossing  $\begin{array}{c} \swarrow \\ \searrow \end{array}$  of strands labeled by the same number and with  $q$  for each such crossing  $\begin{array}{c} \searrow \\ \swarrow \end{array}$ ,*
- *in addition, for each horizontal strand in the upper row from left to right labeled by  $i$  we have a factor  $q^{2i-n-1}$ , and a factor  $q^{-2i+n+1}$  for each horizontal strand in the bottom row from left to right labeled by  $i$ .*

*Note that if  $T$  and  $T'$  are regularly isotopic, then  $\psi_{\mathbf{i}, \mathbf{j}}(T) = \psi_{\mathbf{i}, \mathbf{j}}(T')$ , thus  $\psi_{\mathbf{i}, \mathbf{j}}$  is a well defined map from a basis of  $\mathcal{U}_{I, J}$  to  $R$  and extends uniquely to an  $R$ -linear map  $\mathcal{U}_{I, J} \rightarrow R$ .*

Note that this procedure involves a choice, namely the total ordering on the starting vertices compatible with the partial ordering given by  $(\mathbf{i}, \mathbf{j})$ . Before we show that the resulting maps  $\psi_{\mathbf{i}, \mathbf{j}}$  do not depend on this choice we illustrate the procedure by an example.

**Example 7.2.** Let  $\mathbf{i} = (i_1, i_2, i_3), \mathbf{j} = (j_1, j_2, j_3)$  be multi indices with  $i_1 =$

$2, j_1 = 1, j_3 = 1$  and  $T = \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ \nearrow \quad \nwarrow \\ 1 \end{array}$ . Then we have

$$T = \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ \nearrow \quad \nwarrow \\ 1 \end{array} = \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ \nearrow \quad \nwarrow \\ 1 \end{array} + (q - q^{-1}) \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ \nearrow \quad \nwarrow \\ 1 \end{array},$$

written as a linear combination of descending tangles. We see that  $a_{\mathbf{i}, \mathbf{j}} \neq 0$  only if  $(\mathbf{i}, \mathbf{j}) = ((2, 1, 1), (1, 2, 1))$  or  $(\mathbf{i}, \mathbf{j}) = ((2, 1, 2), (1, 1, 1))$ . If  $(\mathbf{i}, \mathbf{j}) =$

$((2, 1, 1), (1, 2, 1))$ , then  $\psi_{\mathbf{i}, \mathbf{j}}(T) = q^{-1}$  (for the crossing of the strands indexed by 1). If  $(\mathbf{i}, \mathbf{j}) = ((2, 1, 2), (1, 1, 1))$ , then  $\psi_{\mathbf{i}, \mathbf{j}}(T) = (q - q^{-1})q^{n-1} \cdot q^{3-n} = q^3 - q$  (one horizontal strand from left to right on the bottom indexed by 1, one such strand on the top, indexed by 2).

**Lemma 7.3.** *For each  $\mathbf{i}, \mathbf{j} \in I(n, m)$ , the map  $\psi_{\mathbf{i}, \mathbf{j}} : \mathcal{U}_{I, J} \rightarrow R$  defined in Procedure 7.1 is independent of the choice of the total ordering on the starting vertices compatible with the partial ordering given by  $(\mathbf{i}, \mathbf{j})$ .*

*Proof.* Suppose that we have two different total orderings on the starting vertices. We may assume that they only differ in the total ordering of two starting vertices  $\beta_1$  and  $\beta_2$  with the same index lying in two adjacent layers, that is no strand over-crosses the one strand and under-crosses the other strand. Let  $T$  be a descending tangle with respect to the first total ordering. If  $s(\beta_1)$  and  $s(\beta_2)$  do not cross, then  $T$  is descending with respect to the second total ordering. If they do cross, and the crossing is of the kind  $\begin{array}{c} \searrow \\ \nearrow \end{array}$ , we get a factor  $q^{-1}$  in the first total ordering. To write  $T$  as a linear combination of descending tangles with respect to the second total ordering, apply relation (O1). Note that the number of horizontal strands on the top from left to right starting at a vertex with index  $i$  minus the number of such strands on the bottom is the same in the linear combination and in  $T$ . Thus we get a factor  $q$  and a factor  $q^{-1} - q$ , summing up to  $q^{-1}$ . A similar argument works for a crossing  $\begin{array}{c} \nearrow \\ \searrow \end{array}$ .  $\square$

**Definition 7.4.** To each element  $T \in \mathcal{U}_{I, J}$  we associate an element  $\psi_T \in \text{Hom}_R(V_I, V_J)$  acting from the right on  $V_I$  by

$$(v_{\mathbf{i}}^I)\psi_T = \sum_{\mathbf{j} \in I(n, m)} \psi_{\mathbf{i}, \mathbf{j}}(T)v_{\mathbf{j}}^J$$

Clearly,  $\psi : T \mapsto \psi_T$  is an  $R$ -linear map  $\mathcal{U}_{I, J} \rightarrow \text{Hom}_R(V_I, V_J)$ . The  $\psi_{\mathbf{i}, \mathbf{j}}(T)$  are the matrix entries of the matrix corresponding to  $\psi_T$  acting from the right. The next theorem shows that the homomorphism  $\psi$  is compatible with concatenation of tangles.

**Theorem 7.5.** *Let  $T$  be a tangle of type  $(I, J)$  and  $S$  be a tangle of type  $(J, K)$ . Then*

$$\psi_{T/S} = \psi_S \circ \psi_T.$$

*Extending linearly, this formula also holds for  $T \in \mathcal{U}_{I, K}$  and  $S \in \mathcal{U}_{K, J}$ .*



The proof of this theorem is done in several steps. Note that the theorem is equivalent to the formula  $\psi_{\mathbf{i},\mathbf{j}}(T/S) = \sum_{\mathbf{k} \in I(n,m)} \psi_{\mathbf{i},\mathbf{k}}(T)\psi_{\mathbf{k},\mathbf{j}}(S)$  for all  $\mathbf{i}, \mathbf{j} \in I(n, m)$ .

**Definition 7.6.** Let the following tangles be given where the orientation of the vertical strands is omitted and has to be chosen suitably. The  $\rho$  indicates that the horizontal edges and crossings affect the  $\rho$ -th and  $\rho + 1$ -st node.

$$\begin{array}{lll} E_{\rho}^{\rightarrow} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & E_{\rho}^{\leftarrow} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & E_{\rho}^{\rightleftharpoons} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, \\ E_{\rho}^{\leftarrow} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & S_{\rho}^{\times} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \times \\ \times \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & S_{\rho}^{\times} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \times \\ \times \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, \\ S_{\rho}^{\times} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \times \\ \times \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & S_{\rho}^{\times} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{c} \times \\ \times \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array}, & \mathbb{1} = \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \begin{array}{|c|} \hline \cdots \\ \hline \end{array} \end{array}$$

We call these tangles *basic tangles* and call the tangles  $E_{\rho}^{\rightarrow}, E_{\rho}^{\leftarrow}, E_{\rho}^{\rightleftharpoons}, E_{\rho}^{\rightleftharpoons}$  of type e, and the tangles  $S_{\rho}^{\times}, S_{\rho}^{\times}, S_{\rho}^{\times}, S_{\rho}^{\times}$  of type s.

**Lemma 7.7.**  $\mathcal{U}_{I,J}$  is generated by the basic tangles, in the sense that each element of  $\mathcal{U}_{I,J}$  is a linear combination of concatenations of basic tangles.

*Proof.* Note that the basic tangles are in general not elements of  $\mathcal{U}_{I,J}$ . If one changes the crossing in a basic tangle from over-crossing to under-crossing or vice versa, then using relation (O1) the resulting tangle is a linear combination of basic tangles. In Theorem 2.11 of [17] it is shown that a tangle algebra is generated by basic tangles. Although the tangles in [17] are not oriented, the proof can be adopted: If  $T$  is a descending oriented tangle, we can forget the orientation for the moment, write the tangle as a concatenation of basic tangles (possibly with crossings changed from under-crossing to over-crossing or vice versa) as in [17], and then attach suitable orientations to the strands of the basic tangles.  $\square$

Lemma 7.7 shows that it suffices to show Theorem 7.5 for  $S$  a basic tangle.

**Lemma 7.8.** If  $m = 2$  and  $T$  and  $S$  are basic tangles such that  $T/S$  is defined, then we have

$$\psi_{T/S} = \psi_S \circ \psi_T.$$

*Proof.* If  $T/S$  is defined, one can use the relations to write it as a linear combination of basic tangles. We list all possible cases and omit the index  $\rho$  which is always 1.

- $S = \mathbb{1}$  or  $T = \mathbb{1}$ :

It is obvious that  $\psi_{\mathbb{1}}$  is the identity, and concatenation with  $\mathbb{1}$  does not change a tangle.

- $T$  and  $S$  are basic tangles of type e:

The following tangles are defined:

$$E^{\Rightarrow}/E^{\Leftarrow}, E^{\Rightarrow}/E^{\Leftarrow}, E^{\Leftarrow}/E^{\Leftarrow}, E^{\Leftarrow}/E^{\Leftarrow}, \\ E^{\Leftarrow}/E^{\Leftarrow}, E^{\Leftarrow}/E^{\Leftarrow}, E^{\Leftarrow}/E^{\Leftarrow} \text{ and } E^{\Leftarrow}/E^{\Leftarrow}.$$

For example, we show the lemma for  $E^{\Rightarrow}/E^{\Leftarrow} = [n]_q E^{\Rightarrow}$ . We begin with calculating the coefficients of the matrices. We have  $\psi_{\mathbf{i}, \mathbf{k}}(E^{\Rightarrow}) = \delta_{i_1, i_2} \delta_{k_1, k_2} q^{2(i_1 - k_1)}$  and  $\psi_{\mathbf{k}, \mathbf{j}}(E^{\Leftarrow}) = \delta_{k_1, k_2} \delta_{j_1, j_2} q^{-2j_1 + n + 1}$ . Thus

$$\begin{aligned} \sum_{\mathbf{k} \in I(n, 2)} \psi_{\mathbf{i}, \mathbf{k}}(E^{\Rightarrow}) \psi_{\mathbf{k}, \mathbf{j}}(E^{\Leftarrow}) &= \sum_{k_1} \delta_{i_1, i_2} q^{2(i_1 - k_1)} \delta_{j_1, j_2} q^{-2j_1 + n + 1} \\ &= \left( \sum_{k_1} q^{-2k_1 + n + 1} \right) \delta_{i_1, i_2} \delta_{j_1, j_2} q^{2(i_1 - j_1)} \\ &= [n]_q \psi_{\mathbf{i}, \mathbf{j}}(E^{\Rightarrow}) = \psi_{\mathbf{i}, \mathbf{j}}([n]_q E^{\Rightarrow}) \end{aligned}$$

- One of the basic tangles  $T$  and  $S$  is of type e, the other one of type s:

In this case, the following concatenations of tangles are defined:

$$S^{\times}/E^{\Leftarrow}, S^{\times}/E^{\Leftarrow}, S^{\times}/E^{\Rightarrow}, S^{\times}/E^{\Leftarrow}, \\ E^{\Leftarrow}/S^{\times}, E^{\Leftarrow}/S^{\times}, E^{\Rightarrow}/S^{\times} \text{ and } E^{\Leftarrow}/S^{\times}.$$

For example, consider the case  $S^{\times}/E^{\Leftarrow} = q^n E^{\Leftarrow}$ . We have

$$S^{\times} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + (q - q^{-1}) \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$$

and thus

$$\psi_{\mathbf{i}, \mathbf{k}}(S^{\times}) = \begin{cases} q & \text{if } i_1 = i_2 = k_1 = k_2 \\ 1 & \text{if } i_1 = k_2 \neq i_2 = k_1 \\ (q - q^{-1}) q^{2i_1 - 2k_1} & \text{if } i_1 = i_2 > k_1 = k_2 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,  $\psi_{\mathbf{k},\mathbf{j}}(E^{\pm}) = \delta_{k_1,k_2}\delta_{j_1,j_2}$ . Thus

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{\mathbf{i},\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},\mathbf{j}}(E^{\pm}) &= \sum_{k_1} \psi_{\mathbf{i},(k_1,k_1)}(S^{\otimes}) \delta_{j_1,j_2} \\ &= \left( q + (q - q^{-1}) \sum_{k_1 < i_1} q^{2(i_1 - k_1)} \right) \delta_{i_1,i_2} \delta_{j_1,j_2} \\ &= q^{2i_1 - 1} \delta_{i_1,i_2} \delta_{j_1,j_2} = q^n \cdot \psi_{\mathbf{i},\mathbf{j}}(E^{\pm}). \end{aligned}$$

- $T$  and  $S$  are basic tangles of type s:

The following cases have to be considered:  $S^{\otimes}/S^{\otimes}$ ,  $S^{\otimes}/S^{\otimes}$ ,  $S^{\otimes}/S^{\otimes}$  and  $S^{\otimes}/S^{\otimes}$ . For example,  $S^{\otimes}/S^{\otimes} = \mathbf{1} + q^n(q - q^{-1})E^{\pm}$ . We have

$$S^{\otimes} = \begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \searrow \end{array} + (q - q^{-1}) \begin{array}{c} \nearrow \\ \searrow \end{array}$$

and

$$\psi_{\mathbf{k},\mathbf{j}}(S^{\otimes}) = \begin{cases} q & \text{if } k_1 = k_2 = j_1 = j_2 \\ 1 & \text{if } k_1 = j_2 \neq k_2 = j_1 \\ (q - q^{-1}) & \text{if } k_1 = k_2 < j_1 = j_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \psi_{\mathbf{i},\mathbf{j}}(\mathbf{1} + q^n(q - q^{-1})E^{\pm}) &= \begin{cases} 1 + (q - q^{-1})q^{2i_1 - 1} & \text{if } i_1 = i_2 = j_1 = j_2 \\ (q - q^{-1})q^{2i_1 - 1} & \text{if } i_1 = i_2 \neq j_1 = j_2 \\ 1 & \text{if } i_1 = j_1 \neq i_2 = j_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

- If  $i_1 = i_2 = j_1 = j_2 = i$ , then

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{(i,i),\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},(i,i)}(S^{\otimes}) &= \sum_{k \leq i} \psi_{(i,i),(k,k)}(S^{\otimes}) \psi_{(k,k),(i,i)}(S^{\otimes}) \\ &= \left( \sum_{k=1}^{i-1} (q - q^{-1})^2 q^{2i-2k} \right) + q^2 = (q - q^{-1})(q^{2i-1} - q) + q^2 \\ &= q^{2i-1}(q - q^{-1}) + 1 \end{aligned}$$

– If  $i = i_1 = i_2 < j_1 = j_2 = j$ , we have

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{(i,i),\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},(j,j)}(S^{\otimes}) &= \sum_{k \leq i} \psi_{(i,i),(k,k)}(S^{\otimes}) \psi_{(k,k),(j,j)}(S^{\otimes}) \\ &= \left( \sum_{k=1}^{i-1} (q - q^{-1})^2 q^{2i-2k} \right) + q(q - q^{-1}) = q^{2i-1}(q - q^{-1}) \end{aligned}$$

– If  $i = i_1 = i_2 > j_1 = j_2 = j$ , then

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{(i,i),\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},(j,j)}(S^{\otimes}) &= \sum_{k \leq j} \psi_{(i,i),(k,k)}(S^{\otimes}) \psi_{(k,k),(j,j)}(S^{\otimes}) \\ &= \left( \sum_{k=1}^{j-1} (q - q^{-1})^2 q^{2i-2k} \right) + q^{1+2i-2j}(q - q^{-1}) = q^{2i-1}(q - q^{-1}) \end{aligned}$$

– If  $i_1 = j_1 \neq i_2 = j_2$ , then  $\psi_{\mathbf{i},\mathbf{k}}(S^{\otimes})$  and  $\psi_{\mathbf{k},\mathbf{i}}(S^{\otimes})$  do not vanish both only if  $\mathbf{k} = (i_2, i_1)$ . Thus we get

$$\sum_{\mathbf{k}} \psi_{\mathbf{i},\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},\mathbf{i}}(S^{\otimes}) = \psi_{\mathbf{i},(i_2,i_1)}(S^{\otimes}) \psi_{(i_2,i_1),\mathbf{i}}(S^{\otimes}) = 1$$

– In all other cases we have  $\psi_{\mathbf{i},\mathbf{k}}(S^{\otimes}) = 0$  or  $\psi_{\mathbf{k},\mathbf{i}}(S^{\otimes}) = 0$ , and thus

$$\sum_{\mathbf{k}} \psi_{\mathbf{i},\mathbf{k}}(S^{\otimes}) \psi_{\mathbf{k},\mathbf{i}}(S^{\otimes}) = 0.$$

□

**Lemma 7.9.** *If  $T$  and  $S$  are basic tangles for the same  $\rho$ , then  $\psi_{T/S} = \psi_S \circ \psi_T$ .*

*Proof.* Let  $T'$  and  $S'$  be the basic tangles for  $m = 2$  such that  $T$  and  $S$  are obtained from  $T'$  and  $S'$  by adding the same vertical strands to the left and right of the tangles. If  $\mathbf{i}, \mathbf{j} \in I(n, m)$ , let  $\epsilon_{\mathbf{i}, \mathbf{j}} = \delta_{i_1, j_1} \cdots \delta_{i_{\rho-1}, j_{\rho-1}} \delta_{i_{\rho+2}, j_{\rho+2}} \cdots \delta_{i_m, j_m}$  be the product of certain Kronecker deltas. Then

$$\psi_{\mathbf{i}, \mathbf{k}}(T) = \epsilon_{\mathbf{i}, \mathbf{k}} \psi_{(i_{\rho}, i_{\rho+1}), (k_{\rho}, k_{\rho+1})}(T') \text{ and } \psi_{\mathbf{k}, \mathbf{j}}(S) = \epsilon_{\mathbf{k}, \mathbf{j}} \psi_{(k_{\rho}, k_{\rho+1}), (j_{\rho}, j_{\rho+1})}(S')$$

and Lemma 7.8 shows that

$$\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \psi_{\mathbf{k}, \mathbf{j}}(S) = \epsilon_{\mathbf{i}, \mathbf{j}} \psi_{(i_{\rho}, i_{\rho+1}), (j_{\rho}, j_{\rho+1})}(T'/S') = \psi_{\mathbf{i}, \mathbf{j}}(T/S).$$

□

Note that for  $S = \mathbf{1}$ , Theorem 7.5 is obviously true. Before we will prove the rest of the theorem, we will show several lemmas dealing with subcases of the theorem. The setting will always be the following:

- $S$  is a basic tangle of type  $e$  or  $s$ , such that the  $\rho$ -th and the  $\rho + 1$ -st vertices are affected by the horizontal strands or the crossing.  $T$  is a tangle such that  $T/S$  is defined.
- $S'$  is the tangle for  $m = 2$  defined as in the proof of Lemma 7.9.
- $\epsilon_{\mathbf{i}, \mathbf{j}} = \delta_{i_1, j_1} \cdots \delta_{i_{\rho-1}, j_{\rho-1}} \delta_{i_{\rho+2}, j_{\rho+2}} \cdots \delta_{i_m, j_m}$  is defined as in Lemma 7.9.
- If  $\mathbf{i}$  is a multi index and  $k, l \in \{1, \dots, n\}$ , we write  $\mathbf{i}_{k,l}$  for the multi index obtained from  $\mathbf{i}$  by replacing the  $\rho$ -th entry by  $k$  and the  $\rho + 1$ -st entry by  $l$ .

**Lemma 7.10.** *Suppose there is a strand in  $T$  connecting the vertices  $b_\rho$  and  $b_{\rho+1}$ . Then there is a tangle  $\tilde{T}$  such that  $T - \tilde{T}$  is a linear combination of tangles with less crossings than  $T$  and  $\psi_{\tilde{T}/S} = \psi_S \circ \psi_{\tilde{T}}$ .*

*Proof.* Fix some total ordering on the set of starting vertices. Then  $T$  is the sum of a tangle which is descending with respect to this ordering and has a strand connecting  $b_\rho$  and  $b_{\rho+1}$  and a linear combination of tangles with less crossings than  $T$ . Thus we may choose a total ordering on the set of starting vertices, assume that  $T$  is descending with respect to this ordering and has a strand connecting  $b_\rho$  and  $b_{\rho+1}$ , and show that  $\psi_{T/S} = \psi_S \circ \psi_T$ .

Fix  $\mathbf{i}, \mathbf{j} \in I(n, m)$ . Let the total ordering be compatible with the double index  $(\mathbf{i}, \mathbf{j})$ , that is let  $T$  be a tangle which is descending with respect to  $(\mathbf{i}, \mathbf{j})$ . Then the strand in  $T$  connecting  $b_\rho$  and  $b_{\rho+1}$  does not cross any other strand, otherwise these strands would cross twice. Thus  $T$  can be divided into two areas such that each strand lies entirely inside one area, namely the horizontal strand connecting  $b_\rho$  and  $b_{\rho+1}$  and the rest of  $T$ . Likewise,  $T/S$  can be divided into two areas. Figure 3 shows  $T$  with the strand connecting  $b_\rho$  and  $b_{\rho+1}$  (the orientation should be chosen appropriately) and  $T/S$ , the bold line dividing the tangles into two areas. The circle inside  $S$  can be interpreted as a crossing or horizontal edges whether  $S$  is of type  $s$  or  $e$ . Note that  $T$  is descending with respect to any multi index of the form  $(\mathbf{i}, \mathbf{j}_{k,l})$ .

To write  $T/S$  as a linear combination of descending tangles with respect to  $(\mathbf{i}, \mathbf{j})$ , it suffices to do transformations in the bottom area. The transformations carried out are quite the same as the transformations used to write

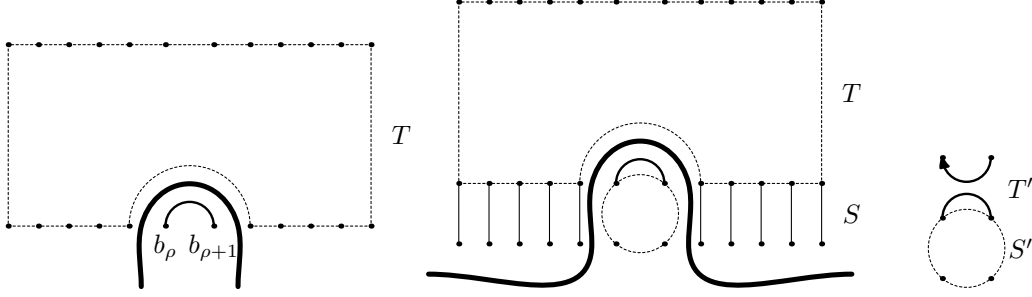


Figure 3:  $T$  has a strand connecting  $b_\rho$  and  $b_{\rho+1}$

$T'/S'$  (see again Figure 3) as a linear combination of descending tangles with respect to  $((11), (j_\rho, j_{\rho+1}))$ , where we set  $T' = E^{\leftarrow}$  or  $E^{\rightarrow}$  depending on whether the strand connecting  $b_\rho$  and  $b_{\rho+1}$  goes from right to left or vice versa. Note that the upper strand in  $T'$  only serves to complete the bottom strand of  $T'$  to a tangle. The direction of this strand has been chosen from right to left since then there is no contribution of this strand to the calculation of  $\psi_{(1,1),(k,l)}(T')$  or  $\psi_{(1,1),(k,l)}(T'/S')$  via Procedure 7.1.

$\psi_{\mathbf{i},\mathbf{k}}(T)$  can be written as a product of two factors, one factor for each area. The factor corresponding to the horizontal strand connecting  $b_\rho$  and  $b_{\rho+1}$  depends only on  $k_\rho$  and  $k_{\rho+1}$ . In fact it coincides with  $\psi_{(1,1),(k_\rho,k_{\rho+1})}(T')$ . Let the other factor be denoted by  $\tau_{\mathbf{i},\mathbf{k}}(T)$ . It is obtained by calculating  $\psi_{\mathbf{i},\mathbf{k}}(T)$ , but omitting the factor coming from the horizontal strand connecting  $b_\rho$  and  $b_{\rho+1}$ .  $\tau_{\mathbf{i},\mathbf{k}}(T)$  does not depend on  $k_\rho$  and  $k_{\rho+1}$  and we have  $\psi_{\mathbf{i},\mathbf{k}}(T) = \tau_{\mathbf{i},\mathbf{k}}(T)\psi_{(1,1),(k_\rho,k_{\rho+1})}(T')$ . The considerations above show that

$$\psi_{\mathbf{i},\mathbf{j}}(T/S) = \tau_{\mathbf{i},\mathbf{j}}(T)\psi_{(1,1),(j_\rho,j_{\rho+1})}(T'/S').$$

By Lemma 7.8 we have

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{\mathbf{i},\mathbf{k}}(T)\psi_{\mathbf{k},\mathbf{j}}(S) &= \sum_{\mathbf{k}} \tau_{\mathbf{i},\mathbf{k}}(T)\psi_{(1,1),(k_\rho,k_{\rho+1})}(T')\epsilon_{\mathbf{k},\mathbf{j}}\psi_{(k_\rho,k_{\rho+1}),(j_\rho,j_{\rho+1})}(S') \\ &= \tau_{\mathbf{i},\mathbf{j}}(T)\psi_{(1,1),(j_\rho,j_{\rho+1})}(T'/S') = \psi_{\mathbf{i},\mathbf{j}}(T/S). \end{aligned}$$

□

**Lemma 7.11.** *Suppose  $T$  is an oriented tangle such that  $b_\rho$  and  $b_{\rho+1}$  are not connected by a strand and the strands ending or starting in  $b_\rho$  and  $b_{\rho+1}$  do not cross. Then there is a tangle  $\tilde{T}$  such that  $T - \tilde{T}$  is a linear combination of tangles with less crossings than  $T$  and  $\psi_{\tilde{T}/S} = \psi_S \circ \psi_{\tilde{T}}$ .*

*Proof.* Similarly as in the proof of Lemma 7.10, one may change the orientation of crossings in  $T$  and then show that  $\psi_{T/S} = \psi_S \circ \psi_T$ . Fix two multi indices  $\mathbf{i}, \mathbf{j} \in I(n, m)$ .

Assume first, that  $S$  is of type  $e$ . By changing the orientation of crossings, we may assume the following: If  $s_1$  is the strand in  $T$  ending in  $b_\rho$  or  $b_{\rho+1}$ ,  $s_2$  is the strand in  $T$  starting at the other vertex and  $k'$  is the label of the starting vertex of  $s_1$  when the vertices of  $T$  are labeled by  $(\mathbf{i}, \mathbf{j})$ , then  $T$  is descending with respect to some total ordering compatible with  $(\mathbf{i}, \mathbf{j}_{k', k'})$  such that the starting vertices of  $s_1$  and  $s_2$  are adjacent in this ordering.

If  $j_\rho \neq j_{\rho+1}$  then both  $\psi_{\mathbf{k}, \mathbf{j}}(S)$  and  $\psi_{\mathbf{i}, \mathbf{j}}(T/S)$  vanish since the horizontal strands connecting  $b_\rho$  and  $b_{\rho+1}$  in  $S$  and in  $T/S$  stay fixed when the tangles are written as a linear combination of descending tangles. So we may assume that  $j_\rho = j_{\rho+1} = j$ . If  $k_\rho \neq k_{\rho+1}$  then  $\psi_{\mathbf{k}, \mathbf{j}}(S) = 0$ . Suppose  $k \neq k'$ . To write  $T$  as a linear combination of descending tangles with respect to  $(\mathbf{i}, \mathbf{j}_{k, k})$ , one can apply the relation (O1) to strands which are all on top of  $s_1$  if  $k < k'$  or below  $s_1$  if  $k > k'$ . Therefore, all tangles appearing in this linear combination have the strand  $s_1$  whose vertices are indexed by  $k$  and  $k'$ , thus  $\psi_{\mathbf{i}, \mathbf{j}_{k, k}}(T) = 0$  and we have

$$\begin{aligned} \sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \psi_{\mathbf{k}, \mathbf{j}}(S) &= \sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \epsilon_{\mathbf{k}, \mathbf{j}} \psi_{(k_\rho, k_{\rho+1}), (j, j)}(S') \\ &= \sum_k \psi_{\mathbf{i}, \mathbf{j}_{k, k}}(T) \psi_{(k, k), (j, j)}(S') = \psi_{\mathbf{i}, \mathbf{j}_{k', k'}}(T) \psi_{(k', k'), (j, j)}(S') \end{aligned}$$

Note that the choice of total ordering on the starting vertices assures that  $T/S$  is regularly isotopic to an oriented tangle which is descending with respect to  $(\mathbf{i}, \mathbf{j})$ . One can compute the value  $\psi_{\mathbf{i}, \mathbf{j}}(T/S)$  directly from  $T/S$ , since the number of positive crossings minus the number of negative crossings is fixed under the Reidemeister Moves. Thus the computation of  $\psi_{\mathbf{i}, \mathbf{j}_{k', k'}}(T)$  differs from that of  $\psi_{\mathbf{i}, \mathbf{j}}(T/S)$  only in the values for the strands  $s_1$  and  $s_2$  in  $T$  and the merged strand and the bottom horizontal strand of the  $S$ -part in  $T/S$ . Clearly,  $\psi_{\mathbf{i}, \mathbf{j}_{k', k'}}(T) \psi_{(k', k'), (j, j)}(S') \neq 0$  if and only if  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) \neq 0$ . The strands in  $T/S$  and in  $S'$  connecting the vertices  $b_\rho$  and  $b_{\rho+1}$  yield the same value. Suppose now that the upper horizontal strand in  $S$  goes from left to right and look at the strands  $s_1$  and  $s_2$  in  $T$  and the merged strand in  $T/S$ , see Figure 4. In either case merging the strands  $s_1$  and  $s_2$  eliminates one horizontal strand from left to right on the bottom row or produces an additional horizontal strand from left to right on the upper row. Thus we

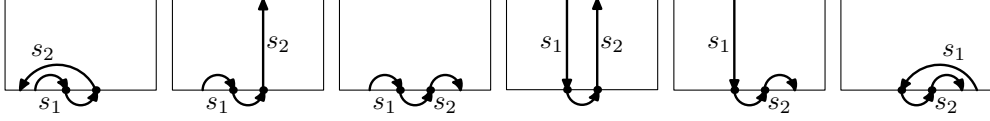


Figure 4: merged strands in  $T/S$

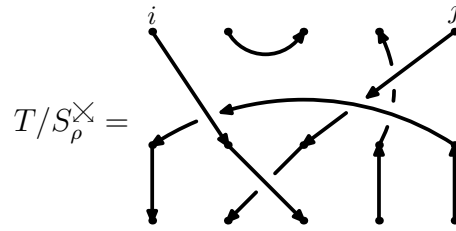
have

$$\psi_{\mathbf{i}, \mathbf{j}_{k', k'}}(T) \psi_{(k', k'), (j, j)}(S') = \psi_{\mathbf{i}, \mathbf{j}}(T/S),$$

which shows the claim. If the upper horizontal strand in  $S$  goes from right to left, then the number of vertical strands from left to right is fixed when merging the strands, and the lemma is also true.

Now, suppose that  $S$  is a basic tangle of type  $s$ . Let  $\mathbf{j}'$  be the multi index obtained from  $\mathbf{j}$  by interchanging the  $\rho$ -th and the  $\rho + 1$ -st entries. We may assume that  $T$  is descending with respect to  $(\mathbf{i}, \mathbf{j}')$ . Let  $\hat{S}$  be the tangle obtained from  $S$  by changing the orientation of the crossing.

If  $S = S_\rho^\times$ , let the vertices of  $T$  be labeled by  $(\mathbf{i}, \mathbf{j})$ . Let  $i$  be the label of the starting vertex of the strand of  $T$  ending in  $b_\rho$ , and  $j$  be the label of the strand ending in  $b_{\rho+1}$ . If  $i < j$  then  $T/S$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$  and we have  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = \psi_{\mathbf{i}, \mathbf{j}'}(T)$ .



Furthermore,  $\psi_{\mathbf{k}, \mathbf{j}}(S) \neq 0$  implies  $\mathbf{k} = \mathbf{j}'$  or  $\mathbf{k} = \mathbf{j}$ .  $\psi_{\mathbf{j}, \mathbf{j}}(S) \neq 0$  implies  $j_\rho > j_{\rho+1}$ , thus we can not have  $i = j_\rho$  and  $j = j_{\rho+1}$ , so  $\psi_{\mathbf{i}, \mathbf{j}}(T) \psi_{\mathbf{i}, \mathbf{j}}(S) = 0$ . We have

$$\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \psi_{\mathbf{k}, \mathbf{j}}(S) = \psi_{\mathbf{i}, \mathbf{j}'}(T) \psi_{\mathbf{j}', \mathbf{j}}(S) = \psi_{\mathbf{i}, \mathbf{j}'}(T) = \psi_{\mathbf{i}, \mathbf{j}}(T/S).$$

If  $i > j$ , then  $T/S = T/\hat{S} + (q^{-1} - q)T$  is a linear combination of descending tangles with respect to  $(\mathbf{i}, \mathbf{j})$  and it suffices to show  $\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \psi_{\mathbf{k}, \mathbf{j}}(\hat{S}) = \psi_{\mathbf{i}, \mathbf{j}}(T/\hat{S})$ . The proof can be copied from that for  $i < j$  by reflecting all tangles at a vertical line.

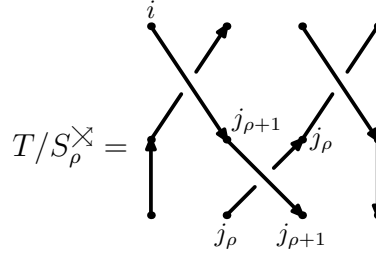


If  $i = j$  then  $T/S$  or  $T/\hat{S}$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$ , in both cases we have  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = q^{-1}\psi_{\mathbf{i}, \mathbf{j}}(T)$ . Furthermore,  $\psi_{\mathbf{i}, \mathbf{k}}(T) \neq 0$  only if  $k_\rho = k_{\rho+1}$  and  $\psi_{\mathbf{k}, \mathbf{j}}(S) \neq 0$  only if  $\mathbf{k} = \mathbf{j}$  or  $\mathbf{k} = \mathbf{j}'$ . if  $j_\rho \neq j_{\rho+1}$  we have  $\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(S) = 0 = \psi_{\mathbf{i}, \mathbf{j}}(T/S)$ , if  $j_\rho = j_{\rho+1}$ , then  $\mathbf{j} = \mathbf{j}'$  and we have

$$\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(S) = \psi_{\mathbf{i}, \mathbf{j}}(T)\psi_{\mathbf{j}, \mathbf{j}}(S) = q^{-1}\psi_{\mathbf{i}, \mathbf{j}}(T) = \psi_{\mathbf{i}, \mathbf{j}}(T/S).$$

If  $S = S_\rho^\times$ , a similar argument holds. Let  $S = S_\rho^\times$ . We have  $S = \hat{S} + (q - q^{-1})E_\rho^\pm$ . We have already seen, that the claim holds for  $E_\rho^\pm$ . Therefore, for the fixed multi indices  $\mathbf{i}$  and  $\mathbf{j}$ , it is enough to show the equation  $\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(S) = \psi_{\mathbf{i}, \mathbf{j}}(T/S)$  or the equation  $\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(\hat{S}) = \psi_{\mathbf{i}, \mathbf{j}}(T/\hat{S})$ .

Recall that we assumed that  $T$  is descending with respect to  $(\mathbf{i}, \mathbf{j}')$ . Again let  $i$  be the label of the starting vertex of the strand of  $T$  ending in  $b_\rho$  where  $T$  is labeled by  $(\mathbf{i}, \mathbf{j})$ . If  $i = j_\rho$  we may assume furthermore that the starting vertex of the strand ending in  $b_\rho$  directly precedes  $b_{\rho+1}$  in the corresponding total ordering.



Note that  $\psi_{\mathbf{i}, \mathbf{j}_{l,l}}(T) = 0$  whenever  $l \neq i$ . The transformations used to write  $T$  as a linear combination of descending tangles with respect to  $(\mathbf{i}, \mathbf{j}_{l,l})$  only affect strands on top of the strand ending in  $b_\rho$  if  $l < i$  or strands below this strand if  $l > i$ . Thus all tangles appearing in this linear combination have the strand whose vertices are labeled by  $i$  and  $l$ , thus  $\psi_{\mathbf{i}, \mathbf{j}_{l,l}}(T) = 0$ . Now,  $\psi_{\mathbf{k}, \mathbf{j}}(S) \neq 0$  implies  $\mathbf{k} = \mathbf{j}'$  if  $j_\rho \neq j_{\rho+1}$  and  $\mathbf{k} = \mathbf{j}_{l,l}$  for some  $l$  if  $j_\rho = j_{\rho+1}$ . In either case we have

$$\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(S) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(S)$$

and similarly

$$\sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(\hat{S}) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(\hat{S}).$$

Suppose  $i < j_\rho$ , then  $T/S$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$  and we have  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = \psi_{\mathbf{i}, \mathbf{j}'}(T) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(S)$  if  $j_\rho \neq j_{\rho+1}$  and  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = q\psi_{\mathbf{i}, \mathbf{j}'}(T) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(S)$  if  $j_\rho = j_{\rho+1}$ .

If  $i = j_\rho$  then  $T/S$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$  by the additional assumption on the chosen total ordering. If  $j_\rho \neq j_{\rho+1}$ , then  $i \neq j_{\rho+1}$  and  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = 0 = \psi_{\mathbf{i}, \mathbf{j}'}(T)$ . If  $j_\rho = j_{\rho+1}$  then  $\psi_{\mathbf{i}, \mathbf{j}}(T/S) = q\psi_{\mathbf{i}, \mathbf{j}'}(T) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(S)$ .

If  $i > j_\rho$  then  $T/\hat{S}$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$  and similarly as for  $i < j_\rho$  we have  $\psi_{\mathbf{i}, \mathbf{j}}(T/\hat{S}) = \psi_{\mathbf{i}, \mathbf{j}'}(T)\psi_{\mathbf{j}', \mathbf{j}}(\hat{S})$ . Thus

$$\begin{aligned} \psi_{\mathbf{i}, \mathbf{j}}(T/S) &= \psi_{\mathbf{i}, \mathbf{j}}(T/\hat{S}) + (q - q^{-1})\psi_{\mathbf{i}, \mathbf{j}}(T) \\ &= \sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T) \left( \psi_{\mathbf{k}, \mathbf{j}}(\hat{S}) + (q - q^{-1})\psi_{\mathbf{k}, \mathbf{j}}(\mathbb{1}) \right) = \sum_{\mathbf{k}} \psi_{\mathbf{i}, \mathbf{k}}(T)\psi_{\mathbf{k}, \mathbf{j}}(S). \end{aligned}$$

Finally,  $S = S_\rho^\times$  can be deduced from  $S = S_\rho^\times$  by reflecting all tangles at a vertical line and interchanging  $q$  and  $q^{-1}$ .  $\square$

We are now able to prove Theorem 7.5:

*Proof of Theorem 7.5.* We will proceed by induction on the number of crossings in  $T$ . Suppose there are no crossings in  $T$  then  $b_\rho$  and  $b_{\rho+1}$  are either connected by a strand, or the two strands starting or ending in  $b_\rho$  and  $b_{\rho+1}$  do not cross. Thus we can apply Lemma 7.10 or Lemma 7.11. Note that the linear combination of tangles with less crossings vanishes, since there are no tangles with less than no crossing.

Suppose now, that  $T$  contains at least one crossing. Clearly, we can assume that  $T$  is descending with respect to some ordering, thus two strands cross at most once. If the strands starting or ending in  $b_\rho$  and  $b_{\rho+1}$  coincide or do not cross, we can apply Lemma 7.10 or Lemma 7.11 to write  $T = \tilde{T} + R$  where  $\tilde{T}$  satisfies the theorem and  $R$  is a linear combination of tangles with less crossings than  $T$ , and thus satisfies the theorem as well by the induction hypothesis.

So let  $T$  be a tangle, such that these two strands cross. Since two strands never cross more than once, the Reidemeister Moves III and isotopies fixing the vertices can be used to move the crossing to the bottom, that is  $T$  can be written as a concatenation  $T = \hat{T}/\hat{S}$  where  $\hat{S}$  is a basic tangle with a crossing (or a crossing with changed orientation) affecting the  $\rho$ -th and  $\rho+1$ -st vertices and  $\hat{T}$  is a tangle with strictly less crossings than  $T$ . Thus we may apply the

induction hypothesis to  $\hat{T}$ . Note that  $\hat{S}/S$  is a linear combination of basic tangles. By Lemma 7.9 we get

$$\psi_{T/S} = \psi_{\hat{T}/\hat{S}/S} = \psi_{\hat{S}/S} \circ \psi_{\hat{T}} = \psi_S \circ \psi_{\hat{S}} \circ \psi_{\hat{T}} = \psi_S \circ \psi_{\hat{T}/\hat{S}} = \psi_S \circ \psi_T.$$

□

$\mathcal{U}_{I,I}$  is an  $R$ -algebra with concatenation of tangles as multiplication. The lemma shows that  $V_I$  is a  $\mathcal{U}_{I,I}$ -right module.

- Lemma 7.12.**
1. The action of  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  coincides with the action  $\sigma$  of the Hecke algebra  $\mathcal{H}_m$  (see Definition 3.2 and the following).
  2. The action of  $\mathcal{U}_{(\downarrow^r \uparrow^s),(\downarrow^r \uparrow^s)}$  coincides with the action of  $\mathcal{A}_{r,s}(q^n, q)$  in [15] ( $q$  replaced by  $q^{-1}$ ).
  3. Recall the isomorphism  $\psi$  of Lemma 5.1. Let  $\hat{S} = \downarrow \cdots \downarrow \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \downarrow \cdots \downarrow = S_k^{\times} + (q^{-1} - q)E_k^{\leftarrow}$ , then

$$\psi_{\hat{S}} = \text{id}^{\otimes k-1} \otimes \psi \otimes \text{id}^{\otimes m-k-1}$$

*Proof.* 1. The Hecke algebra  $\mathcal{H}_m$  is generated by  $T_k$ ,  $k = 1, \dots, m-1$ , while  $\mathcal{U}_{(\downarrow^m),(\downarrow^m)}$  is generated by

$$S_k^{\times} = \downarrow \cdots \downarrow \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \downarrow \cdots \downarrow = \downarrow \cdots \downarrow \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} \downarrow \cdots \downarrow + (q^{-1} - q) \downarrow \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \downarrow.$$

Thus it suffices to show that the action of  $T_k$  coincides with the action of  $S = S_k^{\times}$ . We have

$$\psi_{i,j}(s) = \begin{cases} q^{-1} & i_k = i_{k+1} = j_k = j_{k+1}, i_l = j_l \text{ for } l \neq k, k+1 \\ 1 & i_k = j_{k+1} \neq i_{k+1} = j_k, i_l = j_l \text{ for } l \neq k, k+1 \\ q^{-1} - q & i_k = j_k > i_{k+1} = j_{k+1}, i_l = j_l \text{ for } l \neq k, k+1 \\ 0 & \text{otherwise.} \end{cases}$$

and this is exactly the action of  $T_k$ .

2.  $\mathcal{A}_{r,s}(q^n, q)$  is generated by  $g_i$ ,  $i = 1, \dots, r-1$ ,  $g_i^*$ ,  $i = 1, \dots, s-1$  and  $D$  which correspond to the tangles  $S_k^{\times}$  for  $k = r-1, \dots, 1$ ,  $S_k^{\times}$  for  $k = r+1, \dots, r+s-1$  and  $E_r^{\overleftarrow{}}$ . One can easily verify that corresponding generators act the same way.
3. Again, the coefficients  $\psi_{i,j}(\hat{S})$  are exactly the coefficients of the matrix corresponding to  $\psi$ .

□

Lemma 7.12 3 shows that composition with  $\text{id}^{\otimes k-1} \otimes \psi \otimes \text{id}^{\otimes m-k-1}$  corresponds to concatenation with the oriented tangle  $S_k^{\times} + (q^{-1} - q)E_k^{\leftarrow}$ . The next step will be to find a tangle analogue for the isomorphism  $\psi'$ .

Suppose that  $T$  is a tangle of type  $(I, J)$  such that  $I_1 = J_1 = \downarrow$ . As shown in Figure 5, the tangle  $T$  represented by the rectangle can be embedded into a piece of knot diagram (the vertical strands having suitable orientation). The resulting tangle is a tangle of type  $((\uparrow, I_2, \dots, I_m), (\uparrow, J_2, \dots, J_m))$  and will be denoted by  $\curvearrowright T$ .

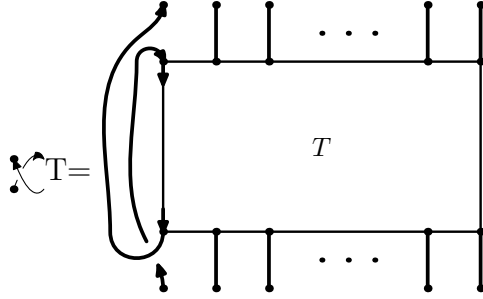


Figure 5:  $T$  embedded into a piece of knot diagram

**Lemma 7.13.** *Let  $T$  be a tangle of type  $(I, J)$  with  $I_1 = J_1 = \downarrow$ . If we identify  $\psi_T$  with an element of an  $2m$ -fold tensor space via the map  $\text{Hom}_R(V_I, V_J) \cong V_I' \otimes V_J$ , we get with  $S = \curvearrowright T$ :*

$$\text{id}^{\otimes m-1} \otimes \psi' \otimes \text{id}^{\otimes m-1}(\psi_T) = \psi_S.$$

*Proof.* For  $\mathbf{i} \in I(n, m)$ , let  $\bar{\mathbf{i}} = (i_2, \dots, i_m) \in I(n, m-1)$ ,  $\bar{\bar{\mathbf{i}}} = (i_3, \dots, i_m) \in I(n, m-2)$  and let  $\hat{\mathbf{i}} = (i_m, \dots, i_2)$  be the multi index  $\bar{\mathbf{i}}$  in reversed order. Let  $\bar{I} = (I_2, \dots, I_m)$ ,  $\hat{I}$ , etc. be defined similarly. Then  $\psi_T$  corresponds to  $\sum_{\mathbf{i}, \mathbf{j}} \psi_{\mathbf{i}, \mathbf{j}}(T) v_{\hat{\mathbf{i}}}^{\hat{I}} \otimes v_{i_1}^{\downarrow} \otimes v_{j_1} \otimes v_{\bar{\mathbf{j}}}^{\bar{J}}$  under the isomorphism  $\text{Hom}_R(V_I, V_J) \cong V_I' \otimes V_J$ . Under  $\text{id}^{\otimes m-1} \otimes \psi' \otimes \text{id}^{\otimes m-1}$ , this  $2m$ -fold tensor maps to

$$\begin{aligned} & \sum_{\mathbf{i}, \mathbf{j}, i_1 \neq j_1} \psi_{\mathbf{i}, \mathbf{j}}(T) q^{n+1-2i_1} v_{\hat{\mathbf{i}}}^{\hat{I}} \otimes v_{j_1} \otimes v_{i_1}^* \otimes v_{\bar{\mathbf{j}}}^{\bar{J}} \\ & + \sum_{\mathbf{i}, \mathbf{j}, i_1 = j_1} \psi_{\mathbf{i}, \mathbf{j}}(T) q^{n+1-2i_1} v_{\hat{\mathbf{i}}}^{\hat{I}} \otimes \left( q^{-1} v_{i_1} \otimes v_{i_1}^* + (q^{-1} - q) \sum_{l=1}^{i_1-1} v_l \otimes v_l^* \right) \otimes v_{\bar{\mathbf{j}}}^{\bar{J}}. \end{aligned}$$

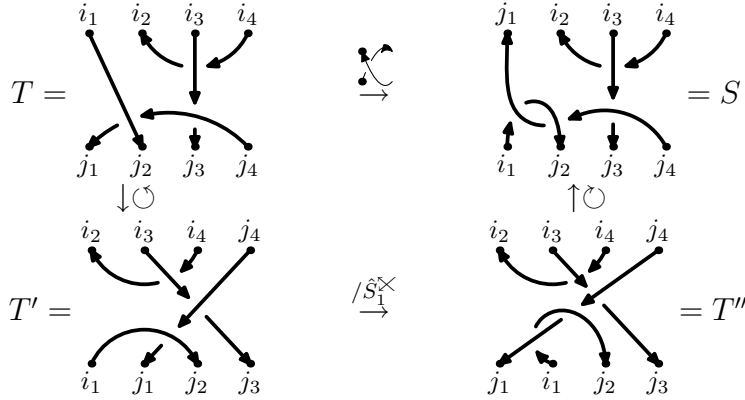
Comparing coefficients, we see that this corresponds to  $\psi_S$  if and only if

$$\psi_{\mathbf{i},\mathbf{j}}(S) = \begin{cases} q^{n+1-2j_1}\psi_{(j_1,\bar{\mathbf{i}}),(i_1,\bar{\mathbf{j}})}(T) & \text{if } i_1 \neq j_1 \\ q^{n-2i_1}\psi_{\mathbf{i},\mathbf{j}}(T) & \\ +(q^{-1}-q)\sum_{k=i_1+1}^n q^{n+1-2k}\psi_{(k,\bar{\mathbf{i}}),(k,\bar{\mathbf{j}})}(T) & \text{if } i_1 = j_1. \end{cases} \quad (*)$$

This is an easy computation for  $m = 1$ , since there is only one descending tangle. So assume that  $m \geq 2$ . We will show Equation (\*) using Theorem 7.5. Let

$$\begin{aligned} (I^\circ, J^\circ) &= ((I_2, \dots, I_m, \bar{J}_m), (\bar{I}_1, J_1, \dots, J_{m-1})), \\ (I^\circ, J^\circ) &= ((\bar{J}_1, I_1, \dots, I_{m-1}), (J_2, \dots, J_m, \bar{I}_m)) \end{aligned}$$

where  $\bar{I}_k = \downarrow$  if and only if  $I_k = \uparrow$ . Let  $(\mathbf{i}^\circ, \mathbf{j}^\circ)$  and  $(\mathbf{i}^\circ, \mathbf{j}^\circ)$  be defined similarly (without the  $\bar{\phantom{x}}$ ). We can define an isomorphism of  $R$ -modules  $\circlearrowleft: \mathcal{U}_{I,J} \rightarrow \mathcal{U}_{I^\circ, J^\circ}$  by rotating the vertices anticlockwise, similarly  $\circlearrowright: \mathcal{U}_{I,J} \rightarrow \mathcal{U}_{I^\circ, J^\circ}$  rotates the vertices clockwise. Let  $\hat{S}_1^{\times} = \begin{array}{c} \nearrow \\ \searrow \end{array} \cdots \begin{array}{c} \searrow \\ \nearrow \end{array} + (q^{-1} - q) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} = S_1^{\times} + (q^{-1} - q)E_1^{\leftarrow}$ . Let  $T' = \circlearrowleft(T)$  (see Figure 6), and  $T'' = T'/\hat{S}_1^{\times}$ , then we have  $\circlearrowright(T'') = \begin{array}{c} \nearrow \\ \searrow \end{array} T =: S$  as illustrated by the following example:



The coefficients  $\psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T')$  are closely related to the coefficients  $\psi_{\mathbf{i}, \mathbf{j}}(T)$ :  $T'$  is descending with respect to  $(\mathbf{i}^\circ, \mathbf{j}^\circ)$  if and only if  $T$  is descending with respect to  $(\mathbf{i}, \mathbf{j})$ . The crossings in both tangles yield the same factors, so we only have to care about the horizontal strands from left to right. The only strands which may change from horizontal to vertical or the other way around, are the strands  $s(t_1)$  and  $s(b_m)$  in  $T$  which turn to  $s(b_1)$  and  $s(t_m)$

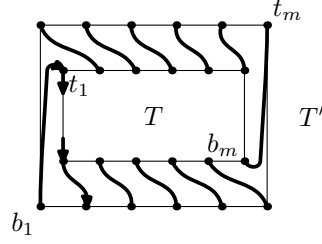


Figure 6: a rotated tangle

in  $T'$ . Suppose first that  $J_m = \uparrow$ . Then  $b_m$  in  $T$  and  $t_m$  in  $T'$  are starting vertices, thus  $s(b_m)$  in  $T$  and  $s(t_m)$  in  $T'$  are never horizontal strands from left to right. If  $s(t_1)$  is a horizontal strand in  $T$ , and thus goes from left to right, then  $s(b_1)$  is vertical in  $T'$ , if  $s(t_1)$  is vertical, then  $s(b_1)$  is a horizontal strand from left to right on the bottom row, thus we have

$$\psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T') = q^{-2i_1+n+1} \psi_{\mathbf{i}, \mathbf{j}}(T) \text{ if } J_m = \uparrow .$$

If  $J_m = \downarrow$  and  $t_1$  and  $b_m$  are connected by a strand in  $T$ , then each horizontal strand remains horizontal and each vertical strand remains vertical. If  $t_1$  and  $b_m$  are not connected by a strand, and  $s(t_1)$  in  $T$  is vertical, then  $s(b_1)$  in  $T'$  is horizontal, if  $s(t_1)$  in  $T$  is horizontal, then  $s(b_1)$  in  $T'$  is vertical. A similar observation can be done for  $s(b_m)$  in  $T$  and  $s(t_m)$  in  $T'$ . In any case we have

$$\psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T') = q^{2(j_m-i_1)} \psi_{\mathbf{i}, \mathbf{j}}(T) \text{ if } J_m = \downarrow .$$

Similarly, one can show that

$$\begin{aligned} \psi_{\mathbf{i}, \mathbf{j}}(S) &= \psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T'') \text{ if } J_m = \uparrow \\ \psi_{\mathbf{i}, \mathbf{j}}(S) &= q^{n+1-2j_m} \psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T'') \text{ if } J_m = \downarrow . \end{aligned}$$

Suppose first, that  $i_1 = j_1$ . Then  $\psi_{\mathbf{k}, \mathbf{j}^\circ}(\hat{S}_1^{\times}) = q^{-1}$  if  $\mathbf{k} = \mathbf{j}^\circ$ ,  $\psi_{\mathbf{k}, \mathbf{j}^\circ}(\hat{S}_1^{\times}) = q^{-1} - q$  if  $\mathbf{k} = (k, k, \mathbf{j}^{\bar{\circ}})$  and  $\psi_{\mathbf{k}, \mathbf{j}^\circ}(\hat{S}_1^{\times}) = 0$  otherwise. If  $J_m = \uparrow$ , then we have

$$\begin{aligned} \psi_{\mathbf{i}, \mathbf{j}}(S) &= \psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T'') = \sum_{\mathbf{k}} \psi_{\mathbf{i}^\circ, \mathbf{k}}(T') \psi_{\mathbf{k}, \mathbf{j}^\circ}(\hat{S}_1^{\times}) \\ &= q^{-1} \psi_{\mathbf{i}^\circ, \mathbf{j}^\circ}(T') + (q^{-1} - q) \sum_{k > i_1} \psi_{\mathbf{i}^\circ, (k, k, \mathbf{j}^{\bar{\circ}})}(T') \\ &= q^{n-2i_1} \psi_{\mathbf{i}, \mathbf{j}}(T) + (q^{-1} - q) \sum_{k > i_1} q^{-2k+n+1} \psi_{(k, \bar{\mathbf{i}}), (k, \bar{\mathbf{j}})}(T). \end{aligned}$$

If  $J_m = \downarrow$  then these equations hold with an additional factor  $q^{-2j_m+n+1}$  in the terms in the middle. Suppose now, that  $i_1 \neq j_1$  and  $J_m = \uparrow$ . Then we have

$$\begin{aligned}\psi_{\mathbf{i},\mathbf{j}}(S) &= \psi_{\mathbf{i} \circ \mathbf{j} \circ}(T'') = \sum_{\mathbf{k}} \psi_{\mathbf{i} \circ, \mathbf{k}}(T') \psi_{\mathbf{k}, \mathbf{j} \circ}(\hat{S}_1^{\times}) \\ &= \psi_{\mathbf{i} \circ, (j_1, i_1, \mathbf{j} \circ)}(T') = q^{-2j_1+n+1} \psi_{(j_1, \bar{\mathbf{i}}), (i_1, \bar{\mathbf{j}})}(T)\end{aligned}$$

Finally, if  $J_m = \downarrow$ , again the same is true with an additional factor  $q^{-2j_m+n+1}$ .  $\square$

## 8. Main results

**Theorem 8.1** (Schur–Weyl duality for the mixed tensor space, I). *Let  $\sigma_{r,s} : \mathfrak{B}_{r,s}^n(q) \rightarrow \text{End}_R(V^{\otimes r} \otimes V^{*\otimes s})$  be the representation of the quantized walled Brauer algebra, then*

$$\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) = \sigma_{r,s}(\mathfrak{B}_{r,s}^n(q)).$$

*Proof.* We fix a nonnegative integer  $m$  and show the result for  $r$  and  $s$  with  $r + s = m$  by induction on  $s$ . For  $s = 0$  the claim follows from Lemma 7.121 and Theorem 3.3. Assume that the theorem holds for  $s$ . Note that  $s < m$ , and thus  $r \geq 1$ , since  $s = m, r = 0$  is already the end of this finite induction. Recall the following sequence of  $R$ -linear isomorphisms (see Proposition 5.2):

$$\begin{aligned}\text{End}_R(V^{\otimes r} \otimes V^{*\otimes s}) &\stackrel{\text{Lemma 4.2}}{\cong} (V^{\otimes r} \otimes V^{*\otimes s})' \otimes (V^{\otimes r} \otimes V^{*\otimes s}) \\ &\stackrel{\text{Lemma 4.4}}{\cong} V^{\otimes s} \otimes V'^{\otimes r} \otimes V^{\otimes r} \otimes V^{*\otimes s} \\ &\stackrel{\text{Lemma 5.1}}{\cong} V^{\otimes s} \otimes V'^{\otimes r-1} \otimes V \otimes V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s} \\ &\stackrel{\text{Lemma 4.2, 4.4}}{\cong} \text{End}_R(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}) \\ &\stackrel{\text{Lemma 5.1}}{\cong} \text{End}_R(V^{\otimes r-1} \otimes V^{*\otimes s+1})\end{aligned}$$

Endomorphisms of  $\mathbf{U}$ -modules correspond to  $\mathbf{U}$ -invariants, thus an element of  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  is mapped to an element of  $\text{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1})$ . Suppose that we are given a tangle  $T$  of type  $((\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s))$  such that  $\psi_T$  is an element of  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  by the induction hypothesis,

and we get all elements of  $\text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  as a linear combination of such  $\psi_T$ 's.

We can split the isomorphism into two isomorphisms

$$\begin{aligned} \text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) &\rightarrow \text{End}_{\mathbf{U}}(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}) \text{ and} \\ \text{End}_{\mathbf{U}}(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s}) &\rightarrow \text{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1}). \end{aligned}$$

Lemma 7.13 shows that  $\psi_T$  is mapped to  $\psi_{\mathcal{X}T} \in \text{End}_R(V^* \otimes V^{\otimes r-1} \otimes V^{*\otimes s})$  under the first isomorphism.

Let  $\psi_k = \text{id}^{\otimes k-1} \otimes \psi \otimes \text{id}^{\otimes m-k-1}$  and  $\hat{S}_k^{\times} = S_k^{\times} + (q^{-1} - q)E_k^{\leftarrow}$ . The second isomorphism maps  $\psi_{\mathcal{X}T}$  to  $\psi_{r-1} \circ \dots \circ \psi_2 \circ \psi_1 \circ \psi_{\mathcal{X}T} \circ \psi_1^{-1} \circ \psi_2^{-1} \circ \dots \circ \psi_{r-1}^{-1}$ . Lemma 7.123 shows that  $\psi_k = \psi_{\hat{S}_k^{\times}}$ . Obviously,  $\hat{S}_k^{\times}/S_k^{\times} = \mathbb{1}$  and  $S_k^{\times}/\hat{S}_k^{\times} = \mathbb{1}$  (the orientation of the non-crossing strands have to be chosen in an appropriate way), thus we have  $\psi_k^{-1} = \psi_{S_k^{\times}}$ .

Putting this all together we get by Theorem 7.5  $\psi_T \in \text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \mapsto \psi_S \in \text{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1})$  with

$$S = T_{r-1}^{-1} / \dots / T_2^{-1} / T_1^{-1} / \mathcal{X}T / T_1 / T_2 \dots / T_{r-1}.$$

Thus  $S$  is obtained from  $T$  by embedding  $T$  into a piece of tangle digram and concatenation of the resulting tangle with other tangles. Instead,  $S$  can be obtained by embedding  $T$  into a piece of knot diagram already containing these tangles, which is done in Figure 7. It also shows, that  $T$  can be obtained from  $S$  in a similar way.  $\square$

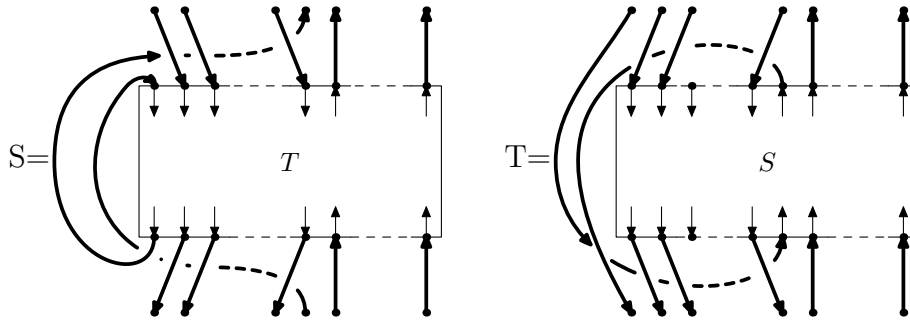


Figure 7: the isomorphism applied to tangles

Now one can easily show



**Corollary 8.2.** 1. Let  $I$  and  $J$  be  $m$ -tuples with entries in  $\{\downarrow, \uparrow\}$ , such that the numbers of entries equal to  $\uparrow$  coincide for  $I$  and  $J$ . Then

$$\mathrm{Hom}_{\mathbf{U}}(V_I, V_J) = \sigma(\mathcal{U}_{I,J})$$

with  $\sigma : \mathcal{U}_{I,J} \rightarrow \mathrm{Hom}_R(V_I, V_J) : T \mapsto \psi_T$ .

2.

$$\mathrm{ann}_{\mathcal{H}_{r+s}}(V^{\otimes r+s}) \cong \mathrm{ann}_{\mathfrak{B}_{r,s}^n(q)}(V^{\otimes r} \otimes V^{*\otimes s})$$

as  $R$ -modules, in particular the action of  $\mathfrak{B}_{r,s}^n(q)$  is faithful if and only if  $\dim(V) \geq r + s$ .

*Proof.* 1.  $V_I$  and  $V_J$  are isomorphic to  $V^{\otimes r} \otimes V^{*\otimes s}$ , the isomorphisms are products of isomorphism  $\psi_k$  and  $\psi_k^{-1}$  as in the previous proof. Multiplying the corresponding tangles from the left resp. right maps  $\mathfrak{B}_{r,s}^n(q)$  to  $\mathcal{U}_{I,J}$ .

2. The proof of Theorem 8.1 shows that there is an  $R$ -linear isomorphism from  $\mathcal{H}_{r+s} \rightarrow \mathfrak{B}_{r,s}^n(q)$  such that  $T \in \mathcal{H}_{r+s} \mapsto S \in \mathfrak{B}_{r,s}^n(q)$  implies  $\psi_T \mapsto \psi_S$  under the isomorphism  $\mathrm{End}_{\mathbf{U}}(V^{\otimes r+s}) \rightarrow \mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$ . Thus if  $T \in \mathcal{H}_{r+s} \mapsto S \in \mathfrak{B}_{r,s}^n(q)$ , then  $\psi_T = 0$  if and only if  $\psi_S = 0$ . The second assertion follows from the well known fact that the action of  $\mathcal{H}_{r+s}$  is faithful if and only if  $\dim(V) \geq r + s$ .

□

**Remark 8.3.** 1. The isomorphism  $\mathrm{End}_R(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathrm{End}_R(V^{\otimes r-1} \otimes V^{*\otimes s+1})$  can be modified such that the tangle analogue of the isomorphism  $\mathrm{End}_{\mathbf{U}}(V^{\otimes r+s}) \cong \mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$  is quite easy to describe. At first, the orientation of the crossings can be changed by modifying the isomorphisms  $\psi$  and  $\psi'$ . Second, concatenation with 'invertible' tangles ( $T$  is invertible if there exists a tangle  $S$  such that  $T/S = \mathbb{1}$  and  $S/T = \mathbb{1}$ ) of suitable type induces an isomorphism  $\mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$ . Composition with the isomorphism  $\mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathrm{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1})$  is again such an isomorphism. Thus there is an isomorphism  $\mathrm{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s}) \cong \mathrm{End}_{\mathbf{U}}(V^{\otimes r-1} \otimes V^{*\otimes s+1})$  mapping  $\psi_T \mapsto \psi_S$  where  $S$  is obtained from  $T$  as in Figure 8. The last part of the strand of  $S$  ending in  $t_{r+1}$  crosses over all other strands, thus one may use the Reidemeister Moves II and III to move it over the other strands of the tangle. The same works for the first part of the strand starting in  $b_{r+1}$  which crosses under all other strands.

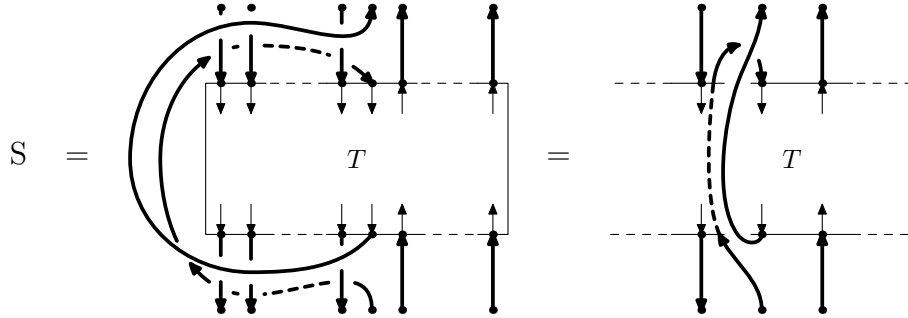


Figure 8: Moving a strand over the tangle

Thus we can get  $S$  from  $T$  as shown in Figure 8. Then the isomorphism  $\text{End}_{\mathbf{U}}(V^{\otimes r+s}) \cong \text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$ , which is a composition of isomorphisms as above, maps  $\psi_T$  to  $\psi_S$  with  $S$  given as in Figure 9. In

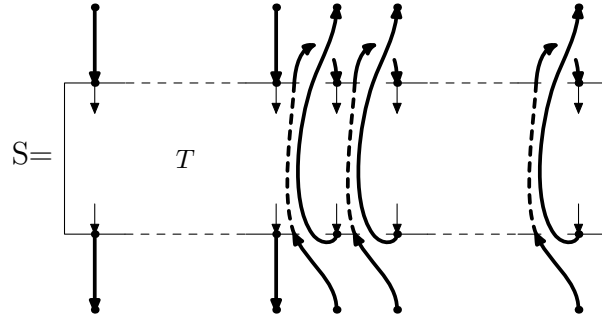


Figure 9:  $\text{End}_{\mathbf{U}}(V^{\otimes r+s}) \cong \text{End}_{\mathbf{U}}(V^{\otimes r} \otimes V^{*\otimes s})$

the classical case  $q = 1$ , over- and under-crossings need not be distinguished, the oriented tangles of type  $((\downarrow^r \uparrow^s), (\downarrow^r \uparrow^s))$  can be replaced by walled Brauer diagrams, that is Brauer diagrams with  $r + s$  vertices in each row and a virtual vertical wall between the first  $r$  vertices and the last  $s$  vertices, such that each horizontal edge does not cross the wall and each vertical edge crosses the wall. Then the isomorphism above turns a diagram of the symmetric group having only vertical edges to a walled Brauer diagram by rotating the right side of the Brauer diagram by  $180^\circ$ , as noted in the introduction and illustrated in Figure 2.

2. If the annihilator of the Hecke algebra  $\mathcal{H}_{r+s}$  on  $V^{\otimes r+s}$  is known, one can easily calculate the annihilator of  $\mathcal{B}_{r,s}^n(q)$  on  $V^{\otimes r} \otimes V^{*\otimes s}$  by writing the elements of the Hecke algebra in terms of tangles and then following the isomorphisms given above.

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