Poincaré series and Coxeter functors for Fuchsian singularities

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ABSTRACT We consider Fuchsian singularities of arbitrary genus and prove, in a conceptual manner, a formula for their Poincaré series. This uses Coxeter elements involving Eichler-Siegel transformations. We give geometrical interpretations for the lattices and isometries involved, lifting them to triangulated categories.

Introduction

A Fuchsian singularity is the affine surface singularity obtained from the cotangent bundle of the upper half plane by taking the quotient by a Fuchsian group of the first kind and collapsing the zero section. In particular, it has a good $\mathbb{C}^*$-action. The surface can be compactified in a natural manner, leading to additional cyclic quotient singularities of type A_μ on the boundary. After resolving the singularities on the bundary, one gets a star-shaped configuration of rational (-2)-curves with a central curve of genus g and self-intersection number $2g - 2$.

In the case $g = 0$, the dual graph of this configuration determines a Coxeter element. It was shown in [EP] that the Poincaré series of the graded coordinate ring of the singularity is the quotient of the characteristic polynomials of two suitable extensions of this Coxeter element.

Here we treat Fuchsian singularities of arbitrary genus g. If $g > 0$, there is no longer a reflection defined by the homology class of the central curve. Therefore, one has to modify the definition of the Coxeter element. We replace the product of two reflections by an Eichler-Siegel transformation. With this change, we prove a result analogous to the one stated above, along the lines of [EP]. We also give a geometrical and categorical interpretation of the Coxeter elements, thereby explaining where the Eichler-Siegel transformation comes from and why the methods applied before have to break down.

1 Eichler-Siegel transformations

We first recall the definition of the Eichler-Siegel transformations. Let $(V, \langle\ ,\ \rangle)$ be an even integral lattice and denote by $\mathrm{O}(V)$ the group of isometries of this lattice. Define a map $\Psi\colon V \otimes V \to \mathrm{End}(V)$, $\sum_i u_i \otimes a_i \mapsto \mathrm{id} - \sum_i \langle \cdot, a_i \rangle u_i$.

Let $a \in V$ be arbitrary and $u \in V$ isotropic and orthogonal to a, i.e. $\langle u, u \rangle = \langle a, u \rangle = 0$. The *Eichler-Siegel transformation* corresponding to u and a is defined as

$$\psi_{u,a} := \Psi((\tfrac{1}{2}\langle a, a \rangle u - a) \otimes u)\ \Psi(u \otimes a).$$

It is easily checked that $\psi_{u,a}$ is an isometry, using the formula

$$\psi_{u,a}(x) = x + \langle x, u \rangle a - \langle x, a \rangle u - \tfrac{1}{2} \langle a, a \rangle \langle x, u \rangle u.$$

Example 1. Let $a \in V$ be a root, i.e. $\langle a, a \rangle = -2$. Then $\Psi(-a \otimes a) = s_a$ is the reflection corresponding to a, given by $s_a(v) = v + \langle v, a \rangle a$ for all $v \in V$. Furthermore, for $u \in V$ with $\langle u, u \rangle = \langle a, u \rangle = 0$, one easily sees $\psi_{u,a} = \Psi((-u-a) \otimes u) \, \Psi(u \otimes a) = s_a s_{a-u}$.

Example 2. Let V_- be any even lattice. Denote by U the unimodular hyperbolic plane with a symplectic basis u, w and the symmetric bilinear form $\langle u, w \rangle = 1$ and $\langle u, u \rangle = \langle w, w \rangle = 0$. Let $V_+ := V_- \oplus U$ be the orthogonal direct sum and define a group homomorphism $m \colon V_- \to \mathrm{O}(V_+)$, $a \mapsto m_a := \psi_{u,a}$.

The isometry m_a is given by $m_a(u) = u$, $m_a(v) = v - \langle v, a \rangle u$ for any $v \in V_-$ and $m_a(w) = w + a - \tfrac{1}{2} \langle a, a \rangle u$. This example appears in [Ei, I, §3]. (M. Eichler notes that these automorphisms first occurred in a paper of C. L. Siegel.)

2 The result: Poincaré series of Fuchsian singularities

Let (X, x) be a normal surface singularity with a good $\mathbb{C}^*$-action. This means that $X = \mathrm{Spec}(A)$ is a normal two-dimensional affine algebraic variety over $\mathbb{C}$ which is smooth outside its *vertex* x. Its coordinate ring A has the structure of a graded $\mathbb{C}$-algebra $A = \bigoplus_{k=0}^{\infty} A_k$, $A_0 = \mathbb{C}$, and x is defined by the maximal ideal $\mathfrak{m} = \bigoplus_{k=1}^{\infty} A_k$.

A natural compactification of X is given by $\overline{X} := \mathrm{Proj}(A[t])$, where t has degree 1 for the grading of $A[t]$ (see [P1]). This is a normal projective surface with $\mathbb{C}^*$-action, and $\overline{X}$ may acquire additional singularities on the boundary $\overline{X}_\infty := \overline{X} \setminus X = \mathrm{Proj}(A)$ which itself is a smooth, projective curve.

A normal surface singularity (X, x) with good $\mathbb{C}^*$-action is called *Fuchsian* if the canonical sheaf of $\overline{X}$ is trivial. In this case, the singularities on the boundary are all of type A_μ. The genus of the Fuchsian singularity is defined as the genus $g = g(\overline{X}_\infty)$ of the boundary.

Let (X, x) be a Fuchsian singularity of genus g. According to [Dol] and [Lo, (1.2) Proposition], there exists a finitely generated cocompact Fuchsian group of the first kind $\Gamma \subset PSL(2, \mathbb{R})$ which acts properly discontinuously on the upper half plane $\mathbb{H}$ such that the action of Γ lifts to the cotangent bundle $T_{\mathbb{H}}^{-1}$ and $A_k = H^0(\mathbb{H}, T_{\mathbb{H}}^{-k})^\Gamma$. The quotient $Z = \mathbb{H}/\Gamma$ is a compact Riemann surface. Let D_0 be a canonical divisor of Z. By [P1, Theorem 5.1] there exist points $p_1, \ldots, p_r \in Z$ and integers $\alpha_i > 1$ for $i = 1, \ldots, r$ such that

$$A_k = L(D^{(k)}), \quad D^{(k)} := kD_0 + \sum_{i=1}^{r} \left[k \frac{\alpha_i - 1}{\alpha_i} \right] p_i \quad \text{for } k \geq 0.$$

Here, $[x]$ denotes the largest integer $\leq x$, and $L(D) := H^0(Z, \mathcal{O}_Z(D))$ for a divisor D on Z denotes the linear space of meromorphic functions f on Z such that $(f) \geq -D$. The genus g of Z coincides with the genus of $\overline{X}_\infty$. The degree of D_0 is $2g - 2$.
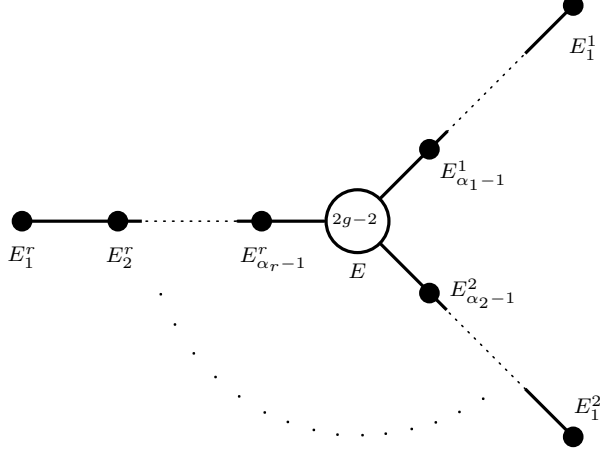
Figure 1: Dual graph of $\mathcal{E} = \widetilde{X}_\infty$

We enumerate the points p_i so that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r$. The variety $\overline{X}$ has cyclic quotient singularities of type $A_{\alpha_1-1}, \ldots, A_{\alpha_r-1}$ along $\overline{X}_\infty := \overline{X} \setminus X$. Let $\pi \colon S \to \overline{X}$ be the minimal normal crossing resolution of all singularities of $\overline{X}$. The preimage $\mathcal{E} := \widetilde{X}_\infty$ of $\overline{X}_\infty$ under π consists of the strict transform E of $\overline{X}_\infty$ and r chains $E_1^i, \ldots, E_{\alpha_i-1}^i$, $i = 1, \ldots, r$, of rational curves of self-intersection number -2 which intersect according to the dual graph shown in Figure 1. The central curve E is a curve of genus g with self-intersection number $2g - 2$.

We consider the *Poincaré series* of the algebra A

$$p_A(t) = \sum_{k=0}^{\infty} \dim(A_k)\ t^k.$$

In order to give a description for p_A, we need some definitions.

Let V_- be the lattice generated by the irreducible components of $\widetilde{X}_\infty = \mathcal{E}$ with bilinear form $\langle -, - \rangle$ given by the intersection numbers. Define two more lattices as orthogonal directs sums by $V_0 := V_- \oplus \mathbb{Z}u$ and $V_+ := V_- \oplus U$ where $\langle u, u \rangle = 0$. For both V_0 and V_+, denote by $\psi_{u,E}$ the Eichler-Siegel transformation corresponding to u and E, as introduced in Example 2. Define isometries $\tau_0 \in O(V_0)$ and $\tau_+ \in O(V_+)$ by

$$\tau_0 := s_{E_1^1} \cdots s_{E_{\alpha_1-1}^1} \cdots s_{E_1^r} \cdots s_{E_{\alpha_r-1}^r} \psi_{u,E},$$

$$\tau_+ := s_{E_1^1} \cdots s_{E_{\alpha_1-1}^1} \cdots s_{E_1^r} \cdots s_{E_{\alpha_r-1}^r} \psi_{u,E} s_{u-w}.$$

Let $\Delta_0(t) = \det(1 - \tau_0^{-1}t)$ and $\Delta_+(t) = \det(1 - \tau_+^{-1}t)$ be the characteristic polynomials of τ_0 and τ_+ respectively, using a suitable normalization.

Theorem 3. *For a Fuchsian singularity we have* $p_A = \dfrac{\Delta_+}{\Delta_0}$.

Remark 4. In [Eb], the following formula is proved: $p_A = \Delta_+/\psi_A$ where $\psi_A(t) = (1-t)^{2-2g-r} \prod_{i=1}^{r}(1 - t^{\alpha_i})$. Note that Eichler-Siegel transformations are used. In view

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of the approach of this work, the denominator ψ_A amounts to a factorisation of Δ_0. In [Eb, Remark 1] it is also observed that $\psi_A = \Delta_0$ if $g = 0$.

3 The proof

The proof of Theorem 3 consists of two steps. First, we consider a general even lattice V_- possessing a basis with at most one non-root. For such lattices, we develop the rational function Δ_+/Δ_0 into a formal power series. This is inspired by Lenzing's approach [Len] but different both in details and in spirit; for the latter, see Remark 13 below. In the second step, we show that this power series coincides with the Poincaré series if we start with the lattice coming from a Fuchsian singularity.

3.1 Hilbert-Poincaré series for even lattices almost generated by roots

Let $(V_-, \langle -, - \rangle)$ be an even lattice with basis $e_1, \ldots, e_{n-1}, e$ where $e_1, \ldots, e_{n-1}$ are roots. Here $n \geq 1$ and $V_- = \mathbb{Z}e$ if $n = 1$. Define $g \in \mathbb{Z}$ by $\langle e, e \rangle = 2g - 2$. Consider the lattices $V_0 := V_- \oplus \mathbb{Z}u$ and $V_+ := V_- \oplus U$ defined as before. Let $\tau_0 = s_{e_1} \cdots s_{e_{n-1}} \psi_{u,e}$ and $\tau_+ = \tau_0 s_{u-w}$. Note that τ_0 can be seen as an isometry of V_+ or as an isometry of V_0. We write $\tau_0|_{V_0}$ if we mean the latter.

Define a Hilbert-Poincaré series corresponding to V_0 and e as follows:

$$P_{(V_0,e)}(t) := \sum_{k=0}^{\infty} \left(1 - g + \sum_{\ell=0}^{k-1} \langle e, \tau_0^\ell(e) \rangle \right) t^k.$$

Proposition 5. *We have* $\det(1 - \tau_+^{-1}t) \det(1 - \tau_0|_{V_0}^{-1}t)^{-1} = P_{(V_0,e)}(t) + g + t.$

Proof. Borrowing an idea of [Len, §18], we use formal power series to invert $1 - \tau_0^{-1}t$. Consider the linear operator on $V_+[[t]] := \mathbb{Z}[[t]] \otimes V_+$ given by

$$h := (1 - \tau_+^{-1}t)(1 - \tau_0^{-1}t)^{-1} = (1 - s_{u-w}\tau_0^{-1}t)\left(\sum_{k \geq 0} \tau_0^{-k}t^k \right)$$

$$= 1 - \sum_{k \geq 1} \langle \cdot, \tau_0^k(u-w) \rangle t^k (u-w),$$

where the last equation follows from an easy computation unravelling s_{u-w}. For all $v \in V_+$, we thus find $h(v) \in v + \mathbb{Z}[[t]](u-w)$, hence $h(u-w) = \det(h)(u-w)$. Furthermore invoking $\det(1 - \tau_0^{-1}t) = (1 - t)\det(1 - \tau_0|_{V_0}^{-1}t)$, we see that

$$\det(1 - \tau_+^{-1}t) \det(1 - \tau_0|_{V_0}^{-1}t)^{-1} = (1 - t)\det(h)$$

$$= (1 - t)\left(1 - \sum_{k \geq 1} \langle u - w, \tau_0^k(u-w) \rangle t^k \right)$$

$$= 1 + t - \sum_{k \geq 1} \langle u - w, (\tau_0^k - \tau_0^{k-1})(u-w) \rangle t^k.$$

Plugging in the definition of $\psi_{u,e}$ now yields

$$\langle u - w, (\tau_0^k - \tau_0^{k-1})(u-w) \rangle = \tfrac{1}{2}\langle e, e \rangle - \langle u - w, \tau_0^k(e) \rangle$$

and $\langle \tau_0^k(e), u - w \rangle = \sum_{\ell=0}^{k-1} \langle e, \tau_0^\ell(e) \rangle$. Putting these pieces together gives the claim. $\square$

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3.2 Poincaré series of Fuchsian singularities

Let now V_0 be the lattice introduced in Section 2, generated by u and the components of $\mathcal{E}$. Theorem 3 follows at once from Propositions 5 and 6.

Proposition 6. *The Poincaré series of a Fuchsian singularity is* $P_{(V_0,E)}(t) + g + t$.

Proof. The element u spans the radical $\mathrm{rad}(V_0)$ of the lattice V_0. Let $\overline{V}_0 = V_0/\mathrm{rad}(V_0)$. For an automorphism σ of V_0, we denote by the same letter the induced automorphism $\sigma\colon \overline{V}_0 \to \overline{V}_0$. By more abuse of notation, we will denote elements of V_0 and their classes in $\overline{V}_0$ by the same letter. In order to compute the series $P_{(V_0,E)}(t)$, it suffices to consider the automorphism τ_0 of $\overline{V}_0$. Now one can easily see that, on the quotient space, $\psi_{u,E} = \mathrm{id}_{\overline{V}_0}$. Therefore we have, again on the quotient $\overline{V}_0$,

$$\tau_0 = \tau_1 \cdots \tau_r \text{ where } \tau_i := s_{E_1^i} \cdots s_{E_{\alpha_i-1}^i} \text{ for } i = 1, \ldots, r.$$

For τ_i we have $\tau_i(E) = E + \sum_{j=1}^{\alpha_i-1} E_j^i$, $\tau_i(E_j^i) = E_{j+1}^i$ for $j = 1, \ldots, \alpha_i - 2$, $\tau_i(E_{\alpha_i-1}^i) = -\sum_{j=1}^{\alpha_i-1} E_j^i$, and τ_i is the identity on all other basis elements of $\overline{V}_0$. Therefore, if $\alpha_i > 2$ (otherwise $\tau_i^2(E) = E$)

$$\tau_i(E) = E + \sum_{j=1}^{\alpha_i-1} E_j^i, \quad \tau_i^2(E) = E + \sum_{j=2}^{\alpha_i-1} E_j^i, \quad \ldots, \quad \tau_i^{\alpha_i}(E) = E.$$

This implies, for $k > 0$,

$$\Big\langle E, \sum_{\ell=0}^{k-1} \tau_0^\ell E \Big\rangle = \Big\langle E, kE + \sum_{i=1}^{r} \big[k \tfrac{\alpha_i-1}{\alpha_i}\big] E_{\alpha_i-1}^i \Big\rangle = \big\langle E, E^{(k)} \big\rangle = \deg D^{(k)}. \tag{1}$$

Here

$$E^{(k)} := kE + \sum_{1 \le i \le r} \sum_{1 \le j < \alpha_i} \big[\tfrac{kj}{\alpha_i}\big] E_j^i$$

is the total transform of the Weil divisor $k\overline{X}_\infty$ under $\pi : S \to \overline{X}$ [P2, 6.4]. We would like to emphasize that the last equality in (1) is not just numerical but stems from an isomorphism $\mathcal{O}_S(E^{(k)})|_E \cong \mathcal{O}_E(D^{(k)})$, see [P2, §6].

We have $\deg D^{(k)} > 2g - 2$ for any $k > 1$. This is obvious if $g > 1$; in the remaining cases it follows from the relation $\sum_{i=1}^{r} \frac{1}{\alpha_i} < r + 2g - 2$ of [Lo, (1.7)]. As the degree of $D^{(k)}$ is large enough, we get $\dim L(D^{(k)}) = 1 - g + \deg D^{(k)}$ by Riemann-Roch. Finally, the two series coincide for $t = 0$ as well since $\dim L(D^{(1)}) = \dim L(D_0) = g$. $\qquad\square$

4 The interpretation: Lifting Coxeter elements to functors

The lattice V_- is constructed from the geometry, but its extensions V_0 and V_+ are purely algebraic, as is the Eichler-Siegel transformation. Similar to our approach in [EP], we are interested in a geometric interpretation of these invariants. Roughly speaking,

this is achieved by constructing, in a natural manner out of the geometry, triangulated categories whose numerical K-groups yield these lattices. In turn, the Coxeter elements can be lifted to invertible functors. The idea of Coxeter functors is certainly not new, see Remark 13 below. For principal reasons, our method only applies to a certain (large) class of Fuchsian singularities, those which are negatively smoothable.

4.1 Negatively smoothable Fuchsian singularities

Let $\pi\colon S \to \overline{X}$ be the minimal normal crossing resolution of $\overline{X}$. It contains the total transform of the curve at infinity, $\mathcal{E} = \widetilde{X}_\infty := \pi^{-1}(\overline{X}_\infty)$. This is a configuration of curves, all but one of which are smooth, rational (-2)-curves.

We assume that X is *negatively smoothable* (for the definition see [P2]). This implies that there is a deformation of $\overline{X}$ with the following properties: all members of the family are partial resolutions of $\overline{X}$ (namely, the singularities at infinity are resolved); each member contains the curve configuration $\mathcal{E}$; the generic fibre is smooth. By [P2, Proposition 6.13], the generic fibre is a smooth K3 surface.

For example, if (X, x) is an isolated hypersurface or complete intersection singularity, then it is negatively smoothable. On the other hand, since the rank of the Néron-Severi group of a K3 surface is at most 20, a necessary condition for negative smoothability is $\sum_i \alpha_i \leq 19 + r$.

In the sequel, we assume that (X, x) is negatively smoothable. Let Y be a generic fibre, it is a smooth K3 surface containing the configuration $\mathcal{E}$.

4.2 Lattices from the K-groups

Let $\mathrm{Coh}(Y)$ be the abelian category of coherent sheaves on Y and $K(Y)$ its Grothendieck K-group. The Euler pairing on $K(Y)$, defined as $\chi(A, B) = \sum_i(-1)^i \dim \mathrm{Ext}^i_Y(A, B)$ for coherent sheaves A and B on Y, is symmetric by Serre duality and the fact that Y is a K3 surface. Note that $\chi(A) := \chi(\mathcal{O}_Y, A)$ is the Euler characteristic of a sheaf $A \in \mathrm{Coh}(Y)$. We equip the K-group (and all groups derived from it) with the *negative* Euler pairing. Let $N(Y)$ be the numerical K-group which is obtained from $K(Y)$ by dividing out the radical of the Euler form.

Denote by $\mathrm{Coh}_\mathcal{E}(Y)$ the abelian subcategory of $\mathrm{Coh}(Y)$ consisting of sheaves whose support is contained in $\mathcal{E}$ and let $K_\mathcal{E}(Y)$ be its K-group. Let $N_\mathcal{E}(Y)$ be the image of $K_\mathcal{E}(Y)$ under $K(Y) \to N(Y)$.

Using the notation of Figure 1 and choosing a point $p \in E$, we will consider the following sheaves supported on $\mathcal{E}$

$$F_j^i := \mathcal{O}_{E_j^i}(-1), \qquad F \in \mathrm{Pic}^{g-1}(E) \text{ with } H^0(F) = 0, \qquad \widetilde{F} := F(p).$$

So F is a line bundle of degree $g - 1$ supported on E. The condition $H^0(F) = 0$ implies $H^1(F) = 0$ by Riemann-Roch. Line bundles without global sections make up the complement of the theta-divisor in $\mathrm{Pic}^{g-1}(E)$. In particular, such line bundles are not unique except for $F = \mathcal{O}_E(-1)$ if $g = 0$.

The classes in $N_{\mathcal{E}}(Y)$ of these sheaves form a basis. This is well-known, but see Subsection 4.4 for details. In this setting, we define three sublattices of $N(Y)$. Recall that we have equipped the numerical K-groups with the negative Euler pairing.

$$
\begin{aligned}
V_0' &:= N_{\mathcal{E}}(Y) & &\text{with } u' := [F] - [\widetilde{F}] = -[k(p)], \\
V_+' &:= N_{\mathcal{E}}(Y) \oplus \mathbb{Z}[\mathcal{O}_Y] & &\text{with } w' := [\mathcal{O}_Y] + u' = [\mathcal{I}_p], \\
V_-' &:= U'^{\perp} = N_{\mathcal{E}}(Y) \cap [\mathcal{O}_Y]^{\perp} & &\text{with } U' := \mathbb{Z}u' + \mathbb{Z}w'.
\end{aligned}
$$

Note that the class $-u'$ is represented by the skyscraper sheaf $k(p)$ of p and that w' is represented by the ideal sheaf $\mathcal{I}_p$ of p. In the derived category, the class u' is thus given by the shift $k(p)[1]$. It is worth pointing out that the structure sheaves of different points are in general not identified in $K(Y)$ but that they all represent the same class in $N(Y)$. Similarly, all choices for F lead to the same class in $N(Y)$. The alternative description of V_-' follows from the next lemma which also shows that these lattices are indeed isometric to the ones used before.

Lemma 7. *The map $\eta_-\colon V_- \to V_-'$ defined by $E_j^i \mapsto F_j^i$ and $E \mapsto F$ is an isometry. The extensions $\eta_0\colon V_0 \to V_0'$ and $\eta_+\colon V_+ \to V_+'$ of η_- mapping $u \mapsto u'$ and $w \mapsto w'$ are isometries.*

Proof. First, U' is a unimodular hyperbolic plane: $\operatorname{Ext}^0(k(p), k(p)) = \mathbb{C}$ implies via Serre duality $\operatorname{Ext}^2(k(p), k(p)) = \mathbb{C}$, and $\operatorname{Ext}^1(k(p), k(p)) = \mathbb{C}^2$ as Y is a smooth surface. This shows $\chi(u', u') = 0$. From $w' = [\mathcal{O}_Y] + u'$ we get $-\chi(u', w') = 1$. Finally, $\chi(w', w') = 0$ follows from $\chi(\mathcal{O}_Y) = 2$ for the K3 surface Y.

We now show that η_- is well-defined, i.e. takes values in $U'^{\perp}$. Let C be any irreducible component of $\mathcal{E}$. The short exact sequence $0 \to \mathcal{O}_Y(-C) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$ yields $\chi(\mathcal{O}_C, u') = 0$. Next, $[\mathcal{O}_C(D)] = [\mathcal{O}_C] - \deg(D)u'$ for any $D \in \operatorname{Div}(C)$ and hence $\chi(\mathcal{O}_C(D), u') = 0$. Furthermore, $\chi(\mathcal{O}_C) = 1 - g_C$, so that $\deg(D) = g_C - 1$ implies $\chi(\mathcal{O}_C(D)) = 0$ and so $\chi(\mathcal{O}_C(D), w') = 0$.

The map η_- is a bijection as it is injective and V_-' and V_- are free abelian groups of the same rank. It only remains to show that ψ_- respects the pairings. For any two irreducible curves C and C' on Y, their intersection number can be computed as $C.C' = -\chi(\mathcal{O}_C, \mathcal{O}_{C'})$. This is immediate if C and C' are transversal, for then the only non-vanishing summand in the Euler pairing is $\dim \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_{C'})$, which is the number of intersection points. In case $C' = C$, the short exact sequence from above yields $-\chi(\mathcal{O}_C, \mathcal{O}_C) = \chi(\mathcal{O}_Y(C)|_C) - \chi(\mathcal{O}_C) = \deg(\mathcal{O}_C(C))$, the last term being the self-intersection number by definition. To conclude, just observe that the quantity $\chi(\mathcal{O}_C(D), \mathcal{O}_{C'}(D'))$ is not affected by the choice of $D \in \operatorname{Div}(C)$, $D' \in \operatorname{Div}(C')$, as above. $\square$

Remark 8. As in [EP], we can consider the group $N_{\mathcal{E}}(S)$ where $\pi\colon S \to \overline{X}$ is the minimal normal crossing resolution of the compactification $\overline{X}$. Then the lattice V_0 is also isometric to this group endowed with the negative Euler pairing. In particular, the proof of Proposition 6 was carried out with this realisation of V_0, which does not rely on an assumption of negative smoothability. However, it is not possible to extend the isometry $V_0 \xrightarrow{\sim} N_{\mathcal{E}}(S)$ to an isometry of V_+ with a sublattice of $N(S)$.

4.3 Lifting Coxeter elements to functors

Denote by $\mathcal{D}^b(Y)$ the bounded derived category of coherent sheaves on Y. Clearly, a triangle autoequivalence $\varphi \colon \mathcal{D}^b(Y) \xrightarrow{\sim} \mathcal{D}^b(Y)$ descends to isometries $\varphi^K \in \mathrm{O}(K(Y))$ and $\varphi^N \in \mathrm{O}(N(Y))$ between (numerical) K-groups. We employ two types of geometrically defined functors in order to lift $\tau_0 \in \mathrm{O}(V_0)$ and $\tau_+ \in \mathrm{O}(V_+)$ to autoequivalences of $\mathcal{D}^b(Y)$.

Spherical twists. A coherent sheaf G on a K3 surface is *spherical* if $\mathrm{Hom}(G,G) = \mathbb{C}$ and $\mathrm{Ext}^1(G,G) = 0$. For such a sheaf, the functor T_G defined by distinguished triangles $\mathrm{Hom}^\bullet(G,A) \otimes G \to A \to \mathsf{T}_G(A)$ for any $A \in \mathcal{D}^b(Y)$, is an autoequivalence of $\mathcal{D}^b(Y)$. (For a proof and the correct definition of spherical in the general context, see [Huy, §8.1].) It is easy to see that $\mathsf{T}_G|_{G^\perp} = \mathrm{id}$ where $G^\perp = \{A \in \mathcal{D}^b(Y) \mid \mathrm{Hom}^\bullet(G,A) = 0\}$ and that $\mathsf{T}_G(G) \cong G[-1]$. Hence, the spherical twist induces the reflection $\mathsf{T}_G^K = s_{[G]}$ where $[G] \in K(Y)$ is by sphericality a root (for the negative Euler pairing); analogously $\mathsf{T}_G^N = s_{[G]}$.

Note that $\mathcal{O}_Y$ is a spherical sheaf. If $i \colon C \hookrightarrow Y$ is the embedding of a smooth, rational (-2)-curve, then $i_* \mathcal{O}_C(n)$, abusively denoted by $\mathcal{O}_C(n)$, is spherical for any $n \in \mathbb{Z}$, since C is rigid.

Line bundle twists. A line bundle $L \in \mathrm{Pic}(Y)$ gives rise to the autoequivalence $\mathsf{M}_L \colon \mathcal{D}^b(Y) \to \mathcal{D}^b(Y)$, $A \mapsto L \otimes A$. Decompose $[L] = w + \ell + du$ in $N(Y)$, where $d \in \mathbb{Z}$ and $\ell \in N(Y)$ with $\chi(w,\ell) = \chi(u,\ell) = 0$. We claim that $\mathsf{M}_L^N = \psi_{u,\ell} = m_\ell$ is an Eichler-Siegel transformation.

First, as Y is a K3 surface, we have $\chi(L,L) = \chi(\mathcal{O}_Y) = 2$. Together with $\chi(L,L) = \chi(w + \ell + du, w + \ell + du) = \chi(\ell,\ell) - 2d$, this shows $d = \frac{1}{2}\chi(\ell,\ell) - 1$. The claim follows from $\mathsf{M}_L^N(u) = u$, $\mathsf{M}_L^N(w) = w + \ell + \frac{1}{2}\chi(\ell,\ell)u$ and $\mathsf{M}_L^N(v) = v + \chi(\ell,v)u$ for $v \in U^\perp$. The first two equations are obvious. For the third, without loss of generality assume $v = [D]$ with $D \in \mathrm{Pic}(C)$, $\deg(D) = g_C - 1$ and write $L = \mathcal{O}_Y(A - A')$ with ample, effective divisors A and A', both meeting C transversally. The sequences $0 \to L|_C \to \mathcal{O}_Y(A)_C \to \mathcal{O}_{A' \cap C} \to 0$ and $0 \to \mathcal{O}_C \to \mathcal{O}_Y(A)|_C \to \mathcal{O}_{A \cap C} \to 0$ are exact by the transversality assumptions, leading to $[L \otimes \mathcal{O}_C] = [\mathcal{O}_C] + ku$ and $[L^\vee] = w - \ell + d'u$ for some $k, d' \in \mathbb{Z}$. Hence, $\mathsf{M}_L^N(v) = v + k'u$ and the coefficient k' is readily computed as $-k' = \chi(w, v + k'u) = \chi(\mathcal{O}_Y, L \otimes D) = \chi(L^\vee, D) = \chi(w - \ell + d'u, v) = -\chi(\ell,v)$.

As an example, $[\mathcal{O}_Y(E)] = w + [F] + du$ for the smooth curve E of genus g in Y. Hence $\mathsf{M}_{\mathcal{O}_Y(E)}^N = m_E$, identifying $\eta_-(E) = [F]$.

Consider the full triangulated subcategory $\mathcal{D}_{\mathcal{E}}^b(Y)$ consisting of complexes whose support is contained in $\mathcal{E}$ (in other words, which are exact off $\mathcal{E}$). The following full triangulated subcategories of $\mathcal{D}^b(Y)$ will be used:

$$
\begin{aligned}
\mathcal{D}_- &:= \mathcal{D}_{\mathcal{E}}^b(Y) \cap \mathcal{O}_Y^\perp, \\
\mathcal{D}_0 &:= \mathcal{D}_{\mathcal{E}}^b(Y), \\
\mathcal{D}_+ &:= \langle \mathcal{D}_{\mathcal{E}}^b, \mathcal{O}_Y \rangle,
\end{aligned}
$$

i.e. $\mathcal{D}_+$ is the smallest full triangulated subcategory of $\mathcal{D}^b(Y)$ containing $\mathcal{D}_0$ and $\mathcal{O}_Y$ (this decomposition is not semiorthogonal). Note that $E_j^i \in \mathcal{D}_-$ and $F \in \mathcal{D}_-$ by construction.

Lemma 9. *We have:*
 (i) $N(\mathcal{D}_-) = V'_-$ *and* $N(\mathcal{D}_+) = V'_+$.
 (ii) V'_0 *is the image of* $K(\mathcal{D}_0) \hookrightarrow K(\mathcal{D}_+) \to N(\mathcal{D}_+)$.

Proof. (i) is obvious from Lemma 7 and the definitions of V'_-, V'_+ and $\mathcal{D}_-$, $\mathcal{D}_+$, respectively. For (ii), just note that F and $\widetilde{F}$ yield the class of a point in the numerical K-group. Also note that the lattices $N(\mathcal{D}_0)$ and V'_0 are not isomorphic since $[k(p)] \in \mathrm{rad}(K(\mathcal{D}_0))$ by Lemma 7 for the class of (the skyscraper sheaf of) a point. $\qquad\square$

We proceed to define the autoequivalences of $\mathcal{D}^b(Y)$ which lift the Coxeter elements:

$$\varphi_0 := \mathsf{T}_{F_1^1} \cdots \mathsf{T}_{F_{\alpha_1-1}^1} \cdots \mathsf{T}_{F_1^r} \cdots \mathsf{T}_{F_{\alpha_r-1}^r} \mathsf{M}_{\mathcal{O}_Y(E)},$$

$$\varphi_+ := \mathsf{T}_{F_1^1} \cdots \mathsf{T}_{F_{\alpha_1-1}^1} \cdots \mathsf{T}_{F_1^r} \cdots \mathsf{T}_{F_{\alpha_r-1}^r} \mathsf{M}_{\mathcal{O}_Y(E)} \mathsf{T}_{\mathcal{O}_Y} = \varphi_0 \mathsf{T}_{\mathcal{O}_Y}.$$

Theorem 10. *The autoequivalences φ_0 and φ_+ restrict to autoequivalences of $\mathcal{D}_0$ and $\mathcal{D}_+$, respectively, and $\varphi_0^N = \tau_0$ and $\varphi_+^N = \tau_+$.*

Proof. Most of the assertions in the theorem were proven in the preceding discussion. Note that $w - u = [\mathcal{O}_Y]$, so $\mathsf{T}_{\mathcal{O}_Y} = s_{w-u} = s_{u-w}$, as desired. What remains to be shown is $\varphi_0(\mathcal{D}_0) = \mathcal{D}_0$ and $\varphi_+(\mathcal{D}_+) = \mathcal{D}_+$.

For an arbitrary line bundle $L \in \mathrm{Pic}(Y)$, the autoequivalence M_L of $\mathcal{D}^b(Y)$ respects supports. Hence, M_L maps $\mathcal{D}_0$ into $\mathcal{D}_0$. As $\mathcal{O}_Y(E) \in \mathcal{D}_+$, the functor $\mathsf{M}_{\mathcal{O}_Y(E)}$ maps $\mathcal{D}_+$ into $\mathcal{D}_+$.

Turning to the spherical twist functors, the following fact proves the claim: For a full triangulated subcategory $\mathcal{T} \subset \mathcal{D}^b(Y)$ and a spherical object $G \in \mathcal{T}$, the twist T_G restricts to an autoequivalence of $\mathcal{T}$, as follows at once from the triangles defining T_G. $\qquad\square$

Remark 11. There are many other categories that can be used here. For example, instead of $\mathcal{D}_+$ one could as well take the triangulated category generated by the structure sheaves of the surface, the irreducible components of $\mathcal{E}$ and of a point on E. The categories we employ are natural — they do not depend on additional choices. However, note that the Coxeter functors depend on the order of the spherical twists.

Remark 12. The case $g = 0$ has already been treated in [EP]. While making use of the same triangulated categories $\mathcal{D}_0$ and $\mathcal{D}_+$, different functors were presented as lifts of τ_0 and τ_+: since $g = 0$ means that the central curve E is rational, the additional spherical objects $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ can be used to take $\mathsf{T}_{\mathcal{O}_E(-1)} \mathsf{T}_{\mathcal{O}_E}$ in place of $\mathsf{M}_{\mathcal{O}_Y(E)}$.

We remark that $\mathcal{D}_0$ and $\mathcal{D}_+$ are generated by spherical objects if and only if $g = 0$, and then the autoequivalences $\mathsf{T}_{\mathcal{O}_E(-1)} \mathsf{T}_{\mathcal{O}_E}$ and $\mathsf{M}_{\mathcal{O}_Y(E)}$ are genuinely different. For example, $\mathsf{M}_{\mathcal{O}_Y(E)}(\mathcal{O}_E)$ is a sheaf but $\mathsf{T}_{\mathcal{O}_E(-1)} \mathsf{T}_{\mathcal{O}_E}(\mathcal{O}_E)$ has nonzero cohomology in two degrees.

Remark 13. There should be a curve picture of the situation, analogous to the one developed in [KST] for certain hypersurface singularities. More precisely, there should be a lift of the Coxeter element to the graded triangulated category of singularities, $\mathcal{D}_{\mathrm{sg}}^{\mathrm{gr}}(R)$. In the case when $g = 0$ and (X, x) is a hypersurface singularity, it is proved in [KST] that $\mathcal{D}_{\mathrm{sg}}^{\mathrm{gr}}(R)$ is generated by a collection of exceptional objects, turning it into the derived category of a quiver with relations. Nevertheless, if $g > 0$, one cannot expect to have a full exceptional collection but only a differential graded algebra as model. The approach to generating series pursued in [Len] is modelled on the curve case, using non-symmetric forms and roots of length 1.

In [BGP], the Coxeter element (of a root lattice) is lifted to an endofunctor of the category of representations of the (oriented) quiver. This functor is not invertible, which seems to be related to the fact that the category used is abelian and not triangulated.

4.4 Cohomology instead of K-group

We close by pointing out that the lattices V_0 and V_+ can also be obtained from the numerical Chow group or from cohomology. In fact, these two invariants seem to be used more often than the numerical K-group, so we briefly explain the differences. Hitherto, we have opted to work with the (numerical) K-groups exclusively because these are truly intrinsic invariants of the triangulated categories.

The Chern character defines a map ch: $K(Y) \to CH^*(Y) \otimes \mathbb{Q}$ and, by the Riemann-Roch theorem, an isomorphism $K(Y) \otimes \mathbb{Q} \xrightarrow{\sim} CH^*(Y) \otimes \mathbb{Q}$, see [Ful, Corollary 18.3.2]. As Y is a surface with even intersection pairing, the Chern map is already defined without denominators. Next, there is the cycle map $CH^*(Y) \to H^*(Y)$ to singular cohomology with integral coefficients: its image is the algebraic part of cohomology.

As Y is a smooth, projective surface, $CH^1(Y)_{\mathrm{num}}$ is isomorphic to the Néron-Severi group of Y and $CH^2(Y)_{\mathrm{num}}$ is free of rank one, spanned by the class of a point. We find that the Chern map induces an isomorphism $N(Y) \xrightarrow{\sim} CH^*(Y)_{\mathrm{num}}$ which, however, is not an isometry. The cycle map does respect the pairings and yields an isometry $CH^*(Y)_{\mathrm{num}} \xrightarrow{\sim} H^*(Y)_{\mathrm{alg}}$.

Matters can be improved by taking the Mukai vector $v(\cdot) := \mathrm{ch}(\cdot)\sqrt{\mathrm{td}_Y}$ instead of the Chern character (where td_Y is the Todd class of the surface), and by modifying the pairings on Chow ring and cohomology: invert the sign of the unimodular hyperbolic plane spanned by fundamental class and point. Denoting this new pairing by $\langle -, - \rangle$, the Grothendieck-Riemann-Roch theorem gives $\chi(A, B) = -\langle v(A), v(B) \rangle$ for all coherent sheaves on A and B. (As $\mathrm{td}_Y = [Y] - 2u$, where $[Y]$ is the class of the surface and $-u$ is the class of a point, we have $v(A) = \mathrm{ch}(A) - \mathrm{rk}(A)u$ for any $A \in \mathrm{Coh}(Y)$.) See [Huy, §10] for details.

Consequently, we arrive at a chain of lattice isomorphisms

$$N(Y) \xrightarrow{v} CH^*(Y)_{\mathrm{num}} \to H^*(Y)_{\mathrm{alg}}.$$

Note that $v(F_i^j) = \mathrm{ch}(F_i^j) = [E_i^j]$ and $v(F) = \mathrm{ch}(F) = [E]$ as cycles in the numerical Chow group or cohomology. The class of a point is given by $pt = v(\widetilde{F}) - v(F) = -u$.

An autoequivalence $\varphi \in \mathrm{Aut}(\mathcal{D}^b(Y))$ induces isomorphisms φ^{CH} and φ^H of the Chow ring and cohomology, respectively. In contrast to φ^K and φ^N, this is not tautological but relies on Orlov's existence theorem for Fourier-Mukai kernels on smooth, projective varieties; see [Huy, §5] for details. The maps φ^{CH} and φ^H are isometries for the Mukai pairings indicated above. One can easily check that for a spherical sheaf G on Y, $\mathsf{T}_G^H = s_{v(G)}$ is the reflection along its Mukai vector. Given a line bundle L, $\mathsf{M}_L^H = m_{c_1(L)}$ is the Eichler-Siegel transformation for the first Chern class of L; this is also multiplication (using the cup product) with the Chern character of L.

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