THE LINKAGE PRINCIPLE FOR RESTRICTED CRITICAL LEVEL REPRESENTATIONS OF AFFINE KAC–MOODY ALGEBRAS

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Abstract. We study the restricted category $\mathcal{O}$ for an affine Kac–Moody algebra at the critical level. In particular, we prove the first part of the Feigin-Frenkel conjecture: the linkage principle for restricted Verma modules. Moreover, we prove a version of the BGGH-reciprocity principle and we determine the block decomposition of the restricted category $\mathcal{O}$. For the proofs we need a deformed version of the classical structures, so we mostly work in a relative setting.

1. Introduction

The representation theory of an affine Kac–Moody algebra at the critical level is of central importance in the approach towards the geometric Langlands program that was proposed by Edward Frenkel and Dennis Gaitsgory in [FG06]. While there is already a good knowledge on the connection between critical level representations and the geometry of the associated affine Grassmannian, central problems, as for example the determination of the critical simple highest weight characters, still remain open. In this paper we continue our approach towards a description of the critical level category $\mathcal{O}$ that we started in the paper [AF08].

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody algebra associated to a finite dimensional, simple complex Lie algebra $\mathfrak{g}$ (for the specialists we point out that we add the derivation operator to the centrally extended loop algebra). We study the corresponding highest weight category $\mathcal{O}$.

The Lie algebra $\widehat{\mathfrak{g}}$ has a one dimensional center and we let $K \in \widehat{\mathfrak{g}}$ be one of its generators. The center acts semisimply on each object of $\mathcal{O}$, so $\mathcal{O}$ decomposes according to the eigenvalue of the action of K. We say that an object M of $\mathcal{O}$ has level $k \in \mathbb{C}$ if K acts on M as multiplication with k, and we let $\mathcal{O}_k$ be the full subcategory of $\mathcal{O}$ that consists of all modules of level k. There is one special value, $k = c$, which is called the *critical* level.

For all levels $k \neq c$ the categorical structure of $\mathcal{O}_k$ is well-known and admits a description in terms of the affine Hecke algebra associated to $\widehat{\mathfrak{g}}$, in analogy to the case of the category $\mathcal{O}$ for a finite dimensional simple complex Lie algebra (cf. [Fi06]). However, for $k = c$ the structure changes drastically. In fact, Lusztig anticipated in his ICM address in 1990 that the representation theory at the critical level resembles the representation theory of a small quantum group or a modular Lie algebra (cf. [L91]). In particular, it should not be the affine Hecke algebra that governs the structure of $\mathcal{O}_c$, but its periodic module. The Feigin–Frenkel conjecture on the simple critical characters (cf. [AF08]) points in this direction as well. So one might hope that there is a description of the critical level representation theory that closely resembles the one given for small quantum groups and modular Lie algebras by Andersen, Jantzen and Soergel in [AJS94].

The main result in this paper is another step towards such a description (following the paper [AF08]). We prove the restricted linkage principle, i.e. we show that a simple module occurs in a restricted Verma module only if their highest weights lie in the same orbit under the associated integral Weyl group (cf. Theorem 6.1). Moreover, we study restricted projective objects, prove a BGGH-reciprocity result (cf. Theorem 5.5) and describe the corresponding block decomposition (cf. Theorem 6.2). Our results are in close analogy to the quantum group and the modular case, hence they strongly support the above conjectures. In the remainder of the introduction we explain the statements in more detail.

1.1. **Restricted representations of $\widehat{\mathfrak{g}}$.** The action of the derivation operator of $\widehat{\mathfrak{g}}$ allows us to consider $\mathcal{O}$ as a graded category, i.e. there is a naturally defined shift equivalence $T\colon \mathcal{O} \to \mathcal{O}$. We call an object M of $\mathcal{O}$ *restricted* if for each $n \in \mathbb{Z}$, $n \neq 0$, and each natural transformation $\phi\colon \mathrm{id} \to T^n$, the induced homomorphism $\phi^M\colon M \to T^n M$ is the zero homomorphism.

The shift equivalence T preserves the subcategories $\mathcal{O}_k$ for any k. The critical level, however, is the only level for which T preserves even each indecomposable block. Each object of non-critical level is restricted, as there is no non-trivial natural transformation $\mathrm{id} \to T^n$ for $n \neq 0$. In the critical level, however, our definition singles out a very interesting subcategory $\overline{\mathcal{O}}_c$ of $\mathcal{O}_c$. For many problems, as for example the computation of simple characters, it is sufficient (and very convenient) to work with $\overline{\mathcal{O}}_c$ instead of $\mathcal{O}_c$. It is also $\overline{\mathcal{O}}_c$ that should resemble the representation category of a small quantum group or a modular Lie algebra.

1.2. **Verma modules, simple and projective objects.** We denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$ and by $\widehat{\mathfrak{h}}^\star$ its complex dual vector space. For each $\lambda \in \widehat{\mathfrak{h}}^\star$ we denote by $\Delta(\lambda)$ the Verma module for $\widehat{\mathfrak{g}}$ with highest weight λ, and by $L(\lambda)$ its simple quotient. We call a subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ *open* if it contains with any element also each element that is smaller with respect to the usual partial order "$\leqslant$" on $\widehat{\mathfrak{h}}^\star$. We call $\mathcal{J}$ *(locally) bounded* if for any element of $\mathcal{J}$ there is only a finite number of elements in $\mathcal{J}$ that are larger. For such an open and bounded subset $\mathcal{J}$ we consider the full subcategory $\mathcal{O}^\mathcal{J}$ of $\mathcal{O}$ that contains all objects whose set of weights is contained in $\mathcal{J}$.

The categories $\mathcal{O}^\mathcal{J}$ have the advantage that they contain enough projectives. We have $L(\lambda) \in \mathcal{O}^\mathcal{J}$ if and only if $\lambda \in \mathcal{J}$, and there exists a projective cover $P^\mathcal{J}(\lambda) \to L(\lambda)$ in $\mathcal{O}^\mathcal{J}$. In general, it depends on $\mathcal{J}$. Each $P^\mathcal{J}(\lambda)$ admits a Verma flag, i.e. a finite filtration such that the subquotients are isomorphic to Verma modules. For the corresponding multiplicities holds the BGGH-reciprocity formula

$$(P^\mathcal{J}(\lambda) : \Delta(\mu)) = \begin{cases} [\Delta(\mu) : L(\lambda)], & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{else.} \end{cases}$$

(By $[M : L(\lambda)]$ we denote the Jordan-Hölder multiplicity of $L(\lambda)$ in M, whenever this makes sense, cf. Section 2.10.)

1.3. **The block decomposition and the Kac–Kazhdan theorem.** Let "$\sim$" be the equivalence relation on $\widehat{\mathfrak{h}}^\star$ generated by the relations $\lambda \sim \mu$ if there is some $\mathcal{J}$ such that $P^\mathcal{J}(\lambda)$ contains a subquotient isomorphic to $L(\mu)$. For an equivalence class $\Lambda \in \widehat{\mathfrak{h}}^\star/{\sim}$ let $\mathcal{O}_\Lambda \subset \mathcal{O}$ be the full subcategory generated by the $P^\mathcal{J}(\lambda)$ for arbitrary bounded open subsets $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ and $\lambda \in \Lambda$. From the definitions one can almost immediately deduce the block decomposition: the functor

$$\begin{aligned} \prod_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} \mathcal{O}_\Lambda &\to \mathcal{O}, \\ (M_\Lambda) &\mapsto \bigoplus_\Lambda M_\Lambda \end{aligned}$$

is an equivalence.

If $\lambda \in \mathcal{J}$, then $P^\mathcal{J}(\lambda)$ contains $\Delta(\lambda)$ as a subquotient. Hence if $[\Delta(\lambda) : L(\mu)] \neq 0$, then $\lambda \sim \mu$. As each $P^\mathcal{J}(\lambda)$ admits a Verma flag, we deduce that "$\sim$" is also generated by the relations $\lambda \sim \mu$ for all pairs λ, μ such that $[\Delta(\lambda) : L(\mu)] \neq 0$. The main result in [KK79] now gives us a rather explicit description of such pairs (λ, μ). In order to formulate the Kac–Kazhdan theorem we need the following definition:

let $(\cdot,\cdot)\colon \widehat{\mathfrak{h}}^\star \times \widehat{\mathfrak{h}}^\star \to \mathbb{C}$ be a non-degenerate, invariant bilinear form and denote by "$\preceq$" the partial order on $\widehat{\mathfrak{h}}^\star$ generated by the relations $\mu \preceq \lambda$ if there exists a positive root β of $\widehat{\mathfrak{g}}$ and a number $n \in \mathbb{N}$ such that $2(\lambda+\rho,\beta) = n(\beta,\beta)$ and $\mu = \lambda - n\beta$. (Here $\rho \in \widehat{\mathfrak{h}}^\star$ is an arbitrary Weyl vector, i.e. an element that takes the value 1 on each simple coroot.)

Theorem 1.1. [KK79] *We have* $[\Delta(\lambda) : L(\mu)] \neq 0$ *if and only if* $\mu \preceq \lambda$.

As a corollary we obtain that the equivalence relation "$\sim$" is generated by the partial order "$\preceq$".

1.4. **The linkage principle.** The construction of the partial order "$\preceq$" motivates the following definition: for $\lambda \in \widehat{\mathfrak{h}}^\star$ we denote by $\widehat{\mathcal{W}}(\lambda)$ be the subgroup of the Weyl group $\widehat{\mathcal{W}}$ associated to $\widehat{\mathfrak{g}}$ that is generated by the reflections s_β for all *real* roots β such that $2(\lambda+\rho,\beta) \in \mathbb{Z}(\beta,\beta)$. It is called the *integral Weyl group* associated to λ. If $\lambda \sim \mu$, then we have $\widehat{\mathcal{W}}(\lambda) = \widehat{\mathcal{W}}(\mu)$. The non-restricted linkage principle is the following.

- Suppose that λ is non-critical, i.e. $\lambda(K) \neq c$. Then $[\Delta(\lambda) : L(\mu)] \neq 0$ implies $\mu \in \widehat{\mathcal{W}}(\lambda).\lambda$ and $\mu \leq \lambda$.
- Suppose that λ is critical, i.e. $\lambda(K) = c$. Then $[\Delta(\lambda) : L(\mu)] \neq 0$ implies $\mu \in \widehat{\mathcal{W}}(\lambda).\lambda + \mathbb{Z}\delta$ and $\mu \leq \lambda$.

Here $\delta \in \widehat{\mathfrak{h}}^\star$ denotes the smallest positive imaginary root.

1.5. **The restricted versions.** A Verma module $\Delta(\lambda)$ of critical level is never restricted. It possesses, however, a unique maximal restricted quotient $\overline{\Delta}(\lambda)$, which is called the *restricted Verma module* of highest weight λ. On the other side, a simple module $L(\lambda)$ is always restricted. As before we define for any bounded open subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ the truncated subcategory $\overline{\mathcal{O}}^{\mathcal{J}}$ of $\overline{\mathcal{O}}$.

In this paper we prove that there is a restricted projective cover $\overline{P}^{\mathcal{J}}(\lambda) \to L(\lambda)$ in $\overline{\mathcal{O}}^{\mathcal{J}}$ for each simple module $L(\lambda)$ with $\lambda \in \mathcal{J}$. We prove that each $\overline{P}^{\mathcal{J}}(\lambda)$ admits a restricted Verma flag, i.e. a finite filtration with subquotients isomorphic to restricted Verma modules, and that for the multiplicities holds the following version of the BGGH-reciprocity formula: we have

$$(P^{\mathcal{J}}(\lambda) : \overline{\Delta}(\mu)) = \begin{cases} [\overline{\Delta}(\mu) : L(\lambda)], & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{else.} \end{cases}$$

Then we prove the restricted linkage principle:

- Suppose that λ is critical. Then $[\overline{\Delta}(\lambda) : L(\mu)]$ implies $\mu \in \widehat{\mathcal{W}}(\lambda).\lambda$ and $\mu \leq \lambda$.

For the proof of the linkage principle we need a deformation theory, i.e. we have to replace the field of complex number by a *deformation algebra* A. The main technical point in this paper is to study the deformed restricted category $\overline{\mathcal{O}}_A$, in particular its projective objects, and to prove the BGGH-reciprocity in this relative setting.

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2. Affine Kac–Moody algebras and the deformed category $\mathcal{O}$

In this section we recall the construction of the deformed category $\mathcal{O}$ associated to an affine Kac-Moody algebra. Our main reference is [Fi03].

2.1. **Affine Kac-Moody algebras.** We fix a finite dimensional, complex, simple Lie algebra $\mathfrak{g}$ and denote by $\widehat{\mathfrak{g}}$ the corresponding affine Kac–Moody algebra. As a vector space we have $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}D$ and the Lie bracket is given by

$$\begin{aligned} [K, \widehat{\mathfrak{g}}] &= 0, \\ [D, x \otimes t^n] &= nx \otimes t^n, \\ [x \otimes t^m, y \otimes t^n] &= [x, y] \otimes t^{m+n} + m\delta_{m,-n} k(x, y)K \end{aligned}$$

for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Here $k \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denotes the Killing form for $\mathfrak{g}$.

Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ inside $\mathfrak{b}$. The corresponding Cartan and Borel subalgebras of $\widehat{\mathfrak{g}}$ are

$$\begin{aligned} \widehat{\mathfrak{h}} &:= \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D \\ \widehat{\mathfrak{b}} &:= (\mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[t] + \mathfrak{b} \otimes_{\mathbb{C}} \mathbb{C}[t]) \oplus \mathbb{C}K \oplus \mathbb{C}D. \end{aligned}$$

2.2. **Roots of $\widehat{\mathfrak{g}}$.** The decomposition $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ allows us to embed $\mathfrak{h}^\star$ in $\widehat{\mathfrak{h}}^\star$ using the map that is dual to the projection $\widehat{\mathfrak{h}} \to \mathfrak{h}$. Let $\delta, \kappa \in \widehat{\mathfrak{h}}^\star$ be the elements dual to D and K, resp., with respect to the direct decomposition, so we have $\delta(\mathfrak{h} \oplus \mathbb{C}K) = \kappa(\mathfrak{h} \oplus \mathbb{C}D) = 0$ and $\delta(D) = \kappa(K) = 1$. Then $\widehat{\mathfrak{h}}^\star = \mathfrak{h}^\star \oplus \mathbb{C}\kappa \oplus \mathbb{C}\delta$.

Let $R \subset \mathfrak{h}^\star$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ the root space decomposition. The set of roots of $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$ then is $\widehat{R} = \widehat{R}^{\mathrm{re}} \cup \widehat{R}^{\mathrm{im}}$, where

$$\begin{aligned}\widehat{R}^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in R, n \in \mathbb{Z}\},\\ \widehat{R}^{\mathrm{im}} &= \{n\delta \mid n \in \mathbb{Z}, n \neq 0\}.\end{aligned}$$

The sets $\widehat{R}^{\mathrm{re}}$ and $\widehat{R}^{\mathrm{im}}$ are called the sets of real and of imaginary roots, resp. The corresponding root spaces are

$$\begin{aligned}\widehat{\mathfrak{g}}_{\alpha+n\delta} &= \mathfrak{g}_\alpha \otimes t^n,\\ \widehat{\mathfrak{g}}_{n\delta} &= \mathfrak{h} \otimes t^n.\end{aligned}$$

The positive roots $\widehat{R}^+ \subset \widehat{R}$ are those that correspond to roots of $\widehat{\mathfrak{b}}$. Explicitely, we have

$$\widehat{R}^+ = \{\alpha + n\delta \mid \alpha \in R, n > 0\} \cup \{\alpha \mid \alpha \in R^+\},$$

where $R^+ \subset R$ denotes the roots of $\mathfrak{b} \subset \mathfrak{g}$. We set $\widehat{R}^{+,\mathrm{re}} := \widehat{R}^+ \cap \widehat{R}^{\mathrm{re}}$ and $\widehat{R}^{+,\mathrm{im}} := \widehat{R}^+ \cap \widehat{R}^{\mathrm{im}}$. We denote by $\Pi \subset R$ the set of simple roots corresponding to our choice of $\mathfrak{b}$. The set of simple affine roots is

$$\widehat{\Pi} := \Pi \cup \{-\gamma + \delta\},$$

where $\gamma \in R^+$ is the highest root.

2.3. **The Weyl group and the bilinear form.** To any real root $\alpha \in \widehat{R}^{\mathrm{re}}$ there is an associated coroot $\alpha^\vee \in \widehat{\mathfrak{h}}$ and a linear isomorphism $s_\alpha \colon \widehat{\mathfrak{h}}^\star \to \widehat{\mathfrak{h}}^\star$ given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. The affine Weyl group associated to our data is the subgroup $\widehat{\mathcal{W}}$ of $\mathrm{GL}(\widehat{\mathfrak{h}}^\star)$ generated by the isomorphisms s_α with $\alpha \in \widehat{R}^{\mathrm{re}}$.

There is a bilinear form $(\cdot,\cdot) \colon \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \mathbb{C}$ that is, up to multiplication with a non-zero scalar, uniquely defined by being non-degenerate, symmetric and invariant, i.e. it satisfies $([x,y],z) = (x,[y,z])$ for $x,y,z \in \widehat{\mathfrak{g}}$. Its restriction to $\widehat{\mathfrak{h}} \times \widehat{\mathfrak{h}}$ is non-degenerate as well and hence induces a non-degenerate bilinear form on $\widehat{\mathfrak{h}}^\star \times \widehat{\mathfrak{h}}^\star$ that we denote again by $(\cdot,\cdot)$. It is explicitely given by the following formulas:

$$\begin{aligned}(\alpha,\beta) &= k(\alpha,\beta),\\ (\kappa, \mathfrak{h}^\star \oplus \mathbb{C}\kappa) &= 0,\\ (\delta, \mathfrak{h}^\star \oplus \mathbb{C}\delta) &= 0,\\ (\kappa,\delta) &= 1,\end{aligned}$$

for $\alpha, \beta \in \mathfrak{h}^\star$ (here we denote by $k\colon \mathfrak{h}^\star \times \mathfrak{h}^\star \to \mathbb{C}$ the bilinear form induced by the Killing form). Moreover, it is invariant under the action of $\widehat{\mathcal{W}}$, i.e. for $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ and $w \in \widehat{\mathcal{W}}$ we have

$$(\lambda, \mu) = (w(\lambda), w(\mu)).$$

2.4. **The deformed category $\mathcal{O}$.** Let $S := S(\mathfrak{h})$ and $\widehat{S} := S(\widehat{\mathfrak{h}})$ be the symmetric algebras over the complex vector spaces $\mathfrak{h}$ and $\widehat{\mathfrak{h}}$. The projection $\widehat{\mathfrak{h}} \to \mathfrak{h}$ along the decomposition $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ yields an algebra homomorphism $\widehat{S} \to S$. We think from now on of S as an $\widehat{S}$-algebra via this homomorphism.

Let A be a commutative, associative, noetherian, unital S-algebra. In the following we call such an algebra a *deformation algebra*. Using the homomorphism $\widehat{S} \to S$ from above we can consider A as an $\widehat{S}$-algebra as well. We denote by $\tau\colon \widehat{\mathfrak{h}} \to A$ the composition of the canonical map $\widehat{\mathfrak{h}} \to \widehat{S}$ with the structure homomorphism $\widehat{S} \to A$, $f \mapsto f \cdot 1_A$. Note that $\tau(D) = \tau(K) = 0$.

For any complex Lie algebra $\mathfrak{l}$ we denote by $\mathfrak{l}_A := \mathfrak{l} \otimes_{\mathbb{C}} A$ the A-linear Lie algebra obtained from $\mathfrak{l}$ by base change. An $\mathfrak{l}_A$-module is then an A-module endowed with an operation of $\mathfrak{l}$ that is A-linear. We denote by $U(\mathfrak{l}_A)$ the universal enveloping algebra of the A-Lie algebra $\mathfrak{l}_A$.

Definition 2.1. Let M be a $\widehat{\mathfrak{g}}_A$-module.

(1) We say that M is a *weight module* if $M = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^\star} M_\lambda$, where
$$M_\lambda := \left\{ m \in M \mid H.m = (\lambda(H).1_A + \tau(H))m \text{ for all } H \in \widehat{\mathfrak{h}} \right\}.$$
We call M_λ the *weight space* of M corresponding to λ (even though its weight is rather $\lambda + \tau$).

(2) We say that M is *locally $\widehat{\mathfrak{b}}_A$-finite*, if for each $m \in M$ the space $U(\widehat{\mathfrak{b}}_A).m$ is a finitely generated A-module.

We define $\mathcal{O}_A$ as the full subcategory of $\widehat{\mathfrak{g}}_A$-mod that consists of locally $\widehat{\mathfrak{b}}_A$-finite weight modules.

One checks easily that $\mathcal{O}_A$ is an *abelian* subcategory of the category of all $\widehat{\mathfrak{g}}_A$-modules. In the following we write $\mathcal{O}$ for the non-deformed category, i.e. for the category $\mathcal{O}_{\mathbb{C}}$ that is defined by giving $\mathbb{C}$ the structure of a deformation algebra by identifying it with $S/\mathfrak{m}S$, where $\mathfrak{m} \subset S$ is the ideal generated by $\mathfrak{h} \subset S$.

Suppose that $A = k$ is a field. Then we can consider $\widehat{\mathfrak{h}}_k$ and $\widehat{\mathfrak{b}}_k$ as Cartan and Borel subalgebras of $\widehat{\mathfrak{g}}_k$. The $\mathbb{C}$-linear map $\tau\colon \widehat{\mathfrak{h}} \to k$ induces a k-linear map $\widehat{\mathfrak{h}}_k \to k$ that we denote by τ as well and which we consider as an element in the dual space $\widehat{\mathfrak{h}}_k^\star = \mathrm{Hom}_k(\widehat{\mathfrak{h}}_k, k)$. Moreover, each $\lambda \in \widehat{\mathfrak{h}}^\star$ induces a k-linear map $\widehat{\mathfrak{h}}_k \to k$, hence we can consider $\widehat{\mathfrak{h}}^\star$ as a subset of $\widehat{\mathfrak{h}}_k^\star$. Then $\mathcal{O}_k$ is the full subcategory of the usual category $\mathcal{O}$ over $\widehat{\mathfrak{g}}_k$ that consists of modules with the property that all weights lie in the affine space $\tau + \widehat{\mathfrak{h}}^\star \subset \widehat{\mathfrak{h}}_k^\star$.

2.5. **The level.** Suppose that M is a weight module. Since $\tau(K) = 0$, the element K acts on a weight space M_λ by multiplication with the scalar $\lambda(K) \in \mathbb{C}$. For $k \in \mathbb{C}$ we denote by M_k the eigenspace of the action of K on M with eigenvalue k. Since K is central each eigenspace M_k is a submodule of M and we have $M = \bigoplus_{k\in\mathbb{C}} M_k$. In the case $M = M_k$ we call k the *level* of the module M and we let $\mathcal{O}_{A,k} \subset \mathcal{O}_A$ be the full subcategory whose objects are those of level k.

It turns out that there is a distinguished level $c \in \mathbb{C}$ which is critical in the sense that the structure of $\mathcal{O}_{A,c}$ differs drastically from the structure of $\mathcal{O}_{A,k}$ for all $k \neq c$. For the definition of c see Section 2.16.

2.6. **Base change - part 1.** Let A and A' be two deformation algebras and consider A' as an A-algebra via a homomorphism $A \to A'$ of unital algebras.

Lemma 2.2. *The functor $\cdot \otimes_A A'$ induces a functor $\mathcal{O}_A \to \mathcal{O}_{A'}$ and for any $k \in \mathbb{C}$ a functor $\mathcal{O}_{A,k} \to \mathcal{O}_{A',k}$.*

Proof. For any $\lambda \in \widehat{\mathfrak{h}}^\star$ and $M \in \mathcal{O}_A$ we have $(M \otimes_A A')_\lambda = M_\lambda \otimes_A A'$, hence $M \otimes_A A'$ is a weight module. If for $m \in M$ the A-module $U(\widehat{\mathfrak{b}}_A).m$ is generated by $v_1, \ldots, v_n$, then the A'-module $U(\widehat{\mathfrak{b}}_{A'}).(m \otimes 1)$ is generated by $v_1 \otimes 1, \ldots, v_n \otimes 1$. Since A' is assumed to be Noetherian, every A'-submodule of the latter module is finitely generated as well. From this it follows that $M \otimes_A A'$ is locally $\widehat{\mathfrak{b}}_{A'}$-finite. Hence the functor $\cdot \otimes_A A'$ sends an object of $\mathcal{O}_A$ to an object of $\mathcal{O}_{A'}$. It is clear that it preserves the level. $\square$

2.7. **The duality.** For $M \in \mathcal{O}_A$ we define

$$M^\star := \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^\star} \mathrm{Hom}_A(M_\lambda, A).$$

Then $M^\star$ carries an action of $\widehat{\mathfrak{g}}$ that is given by $(X.\phi)(m) = \phi(-\omega(X).m)$ for $X \in \widehat{\mathfrak{g}}$, $\phi \in M^\star$ and $m \in M$. Here $\omega\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is the Chevalley-involution (cf. [K90, Section 1.3]). It has the property that it maps the root space $\widehat{\mathfrak{g}}_\alpha$ corresponding to $\alpha \in \widehat{R}$ to $\widehat{\mathfrak{g}}_{-\alpha}$. Together with the obvious A-module structure, $M^\star$ is an object in $\mathcal{O}_A$, and if M is of level k, then $M^\star$ is also of level k.

2.8. **The deformed Verma modules.** For $\lambda \in \widehat{\mathfrak{h}}^\star$ we denote by A_λ the $\widehat{\mathfrak{b}}_A$-module that is free of rank one as an A-module and on which $\widehat{\mathfrak{b}}$ acts via the character $\lambda + \tau$: this means that $H \in \widehat{\mathfrak{h}}$ acts as multiplication with the scalar $\lambda(H).1_A + \tau(H)$ and each $X \in [\widehat{\mathfrak{b}}, \widehat{\mathfrak{b}}]$ acts by the zero homomorphism. The *deformed Verma-module* with highest weight λ is

$$\Delta_A(\lambda) := U(\widehat{\mathfrak{g}}_A) \otimes_{U(\widehat{\mathfrak{b}}_A)} A_\lambda.$$

The *deformed dual Verma module* associated to λ is

$$\nabla_A(\lambda) := \Delta_A(\lambda)^\star.$$

Both $\Delta_A(\lambda)$ and $\nabla_A(\lambda)$ are locally $\widehat{\mathfrak{b}}_A$-finite weight modules, hence are contained in $\mathcal{O}_A$. If $A \to A'$ is a homomorphism of deformation algebras, then we have isomorphisms

$$\Delta_A(\lambda) \otimes_A A' \cong \Delta_{A'}(\lambda), \quad \nabla_A(\lambda) \otimes_A A' \cong \nabla_{A'}(\lambda).$$

2.9. **Simple objects in $\mathcal{O}_A$.** Now suppose that A is a local deformation algebra with maximal ideal $\mathfrak{m} \subset A$ and residue field $k = A/\mathfrak{m}$. The residue field inherits the structure of an S-algebra and is, as such, a deformation algebra as well. The canonical map $A \to k$ gives us a base change functor $\cdot \otimes_A k \colon \mathcal{O}_A \to \mathcal{O}_k$ by Lemma 2.2.

As we have observed before, the category $\mathcal{O}_k$ is just a direct summand of the usual category $\mathcal{O}$ for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_k$. Its objects are those whose non-zero weight spaces correspond to weights in the affine hyperplane $\tau + \widehat{\mathfrak{h}}^\star \subset \widehat{\mathfrak{h}}_k^\star$. By the classical theory, the simple isomorphism classes in $\mathcal{O}_k$ are parametrized by the set of their highest weights. This set is $\tau + \widehat{\mathfrak{h}}^\star$ and we denote by $L_k(\lambda)$ a representative corresponding to $\tau + \lambda$.

In [Fi03, Proposition 2.1] we showed the following.

Proposition 2.3. *Suppose that A is a local deformation algebra with residue field k. Then the functor $\cdot \otimes_A k$ yields a bijection*

$$\left\{ \begin{matrix} \textit{simple isomorphism} \\ \textit{classes of } \mathcal{O}_A \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{simple isomorphism} \\ \textit{classes of } \mathcal{O}_k \end{matrix} \right\}.$$

We denote by $L_A(\lambda)$ the simple object corresponding to $L_k(\lambda)$ under the above bijection.

2.10. **Jordan–Hölder multiplicities.** Suppose now that $A = k$ is a field. In this case we consider the full subcategory $\mathcal{O}_k^f$ of $\mathcal{O}_k$ that consists of objects M such that each weight space M_λ is finite dimensional as a k-vector space and such that there exist $\mu_1, \dots, \mu_n \in \widehat{\mathfrak{h}}^\star$ with the property that $M_\lambda \neq 0$ implies $\lambda \leq \mu_i$ for some i.

We denote by "$\leqslant$" the usual partial order on $\widehat{\mathfrak{h}}^\star$ defined by $\lambda \leqslant \mu$ if $\mu - \lambda$ is a sum of positive roots of $\widehat{\mathfrak{g}}$. Let $\mathbb{Z}[\widehat{\mathfrak{h}}^\star] = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^\star} \mathbb{Z}e^\lambda$ be the group algebra of the additive group $\widehat{\mathfrak{h}}^\star$ and $\widehat{\mathbb{Z}[\widehat{\mathfrak{h}}^\star]} \subset \prod_{\lambda \in \widehat{\mathfrak{h}}^\star} \mathbb{Z}e^\lambda$ its completion with respect to the partial order: An element in $\widehat{\mathbb{Z}[\widehat{\mathfrak{h}}^\star]}$ is an element $\sum_{\lambda \in \widehat{\mathfrak{h}}^\star} f_\lambda e^\lambda$ such that there exist $\mu_1, \dots, \mu_n \in \widehat{\mathfrak{h}}^\star$ with the property that $f_\lambda \neq 0$ implies $\lambda \leq \mu_i$ for some i.

For each $M \in \mathcal{O}_k^f$ we can then define its character

$$\operatorname{ch} M := \sum_{\lambda \in \widehat{\mathfrak{h}}^\star} \dim_k M_\lambda \cdot e^\lambda \in \widehat{\mathbb{Z}[\widehat{\mathfrak{h}}^\star]}.$$

Now each simple object $L_k(\lambda)$ belongs to $\mathcal{O}_k^f$ and there are well defined number $a_\mu \in \mathbb{N}$ with

$$\operatorname{ch} M = \sum_{\mu \in \widehat{\mathfrak{h}}^\star} a_\mu \operatorname{ch} L_k(\mu).$$

(cf. [DGK82]). Note that the sum on the right hand side is in general an infinite sum. We define the multiplicity of $L_k(\mu)$ in M as

$$[M : L_k(\mu)] := a_\mu.$$

2.11. **Truncation.** Our next aim is to study the projective objects in $\mathcal{O}_A$. Unfortunately not all of the $L_A(\lambda)$ admit a projective cover. In order to overcome this slight technical problem, we introduce certain truncated subcategories of $\mathcal{O}_A$ in which a projective cover exist for each of its simple objects.

Let $\mathcal{J}$ be a subset of $\widehat{\mathfrak{h}}^\star$. We call $\mathcal{J}$ *open* if for all $\lambda \in \mathcal{J}$, $\mu \in \widehat{\mathfrak{h}}^\star$ with $\mu \leq \lambda$ we have $\mu \in \mathcal{J}$. This indeed defines a topology on $\widehat{\mathfrak{h}}^\star$. Note that a subset $\mathcal{I} \subset \widehat{\mathfrak{h}}^\star$ is closed in this topology if $\lambda \in \mathcal{I}$, $\mu \in \widehat{\mathfrak{h}}^\star$ with $\mu \geq \lambda$ implies $\mu \in \mathcal{I}$.

We now construct a functorial filtration on each object of $\mathcal{O}_A$ that is indexed by the set of closed subsets of $\widehat{\mathfrak{h}}^\star$ and, dually, a functorial cofiltration indexed by the set of open subsets of $\widehat{\mathfrak{h}}^\star$.

Definition 2.4. Suppose that $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ is open and let $\mathcal{I} := \widehat{\mathfrak{h}}^\star \setminus \mathcal{J}$ be its closed complement. Let $M \in \mathcal{O}_A$.

(1) We define $M_\mathcal{I} \subset M$ as the $\widehat{\mathfrak{g}}_A$-submodule generated by the weight spaces corresponding to weights in $\mathcal{I}$, i.e.

$$M_\mathcal{I} := U(\widehat{\mathfrak{g}}_A).\bigoplus_{\lambda \in \mathcal{I}} M_\lambda.$$

(2) We define
$$M^{\mathcal{J}} := M/M_{\mathcal{I}}.$$
Let $\mathcal{O}_{A,\mathcal{I}} \subset \mathcal{O}_A$ be the full subcategory of objects M with $M = M_{\mathcal{I}}$ and $\mathcal{O}_A^{\mathcal{J}} \subset \mathcal{O}_A$ the full subcategory of objects M with $M = M^{\mathcal{J}}$.

Note that an object M of $\mathcal{O}_A$ belongs to $\mathcal{O}_{A,\mathcal{I}}$ if and only if it is generated by its weight spaces corresonding to weights in $\mathcal{I}$. Dually, M belongs to $\mathcal{O}_A^{\mathcal{J}}$ if and only if $M_\lambda \neq 0$ implies that $\lambda \in \mathcal{J}$.

If $\mathcal{J}' \subset \mathcal{J}$ is another open subset with complement $\mathcal{I}' \supset \mathcal{I}$, then we have a natural inclusion $M_{\mathcal{I}} \subset M_{\mathcal{I}'}$ and a natural quotient $M^{\mathcal{J}} \to M^{\mathcal{J}'}$. For $\lambda \in \widehat{\mathfrak{h}}^\star$, each of the modules $\Delta_A(\lambda)$, $\nabla_A(\lambda)$ and $L_A(\lambda)$ is contained in $\mathcal{O}_A^{\mathcal{J}}$ if and only if $\lambda \in \mathcal{J}$. Note that $M \to M_{\mathcal{I}}$ defines a functor from $\mathcal{O}_A$ to $\mathcal{O}_{A,\mathcal{I}}$ that is right adjoint to the inclusion $\mathcal{O}_{A,\mathcal{I}} \subset \mathcal{O}_A$. Dually, $M \mapsto M^{\mathcal{J}}$ defines a functor from $\mathcal{O}_A$ to $\mathcal{O}_A^{\mathcal{J}}$ that is left adjoint to the inclusion $\mathcal{O}_A^{\mathcal{J}} \subset \mathcal{O}_A$.

Lemma 2.5. *Suppose that $\mathcal{J}$ is an open subset in $\widehat{\mathfrak{h}}^\star$ and that P is a projective object in $\mathcal{O}_A^{\mathcal{J}}$. Then for any open subset $\mathcal{J}' \subset \mathcal{J}$, the object $P^{\mathcal{J}'}$ is projective in $\mathcal{O}_A^{\mathcal{J}'}$.*

Proof. This follows immediately from the fact that the functor $(\cdot)^{\mathcal{J}'} : \mathcal{O}_A^{\mathcal{J}} \to \mathcal{O}_A^{\mathcal{J}'}$ is left adjoint to the (exact) inclusion functor $\mathcal{O}_A^{\mathcal{J}'} \to \mathcal{O}_A^{\mathcal{J}}$. □

Lemma 2.6. *Let $M \in \mathcal{O}_A$. Suppose that $\mathcal{J}' \subset \mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ are open subsets. Then there is a canonical isomorphism $M^{\mathcal{J}'} \xrightarrow{\sim} (M^{\mathcal{J}})^{\mathcal{J}'}$.*

Proof. We denote by a the canonical homomorphism $M \to M^{\mathcal{J}'}$ and by b the composition of the homomorphisms $M \to M^{\mathcal{J}}$ and $M^{\mathcal{J}} \to (M^{\mathcal{J}})^{\mathcal{J}'}$ and prove the claim by showing that the kernels of a and b coincide. First, note that the kernel of a is the submodule of M generated by its weight spaces corresponding to weights in $\widehat{\mathfrak{h}}^\star \setminus \mathcal{J}'$. The kernel of $M \to M^{\mathcal{J}}$ is generated by weights in $\widehat{\mathfrak{h}}^\star \setminus \mathcal{J} \subset \widehat{\mathfrak{h}}^\star \setminus \mathcal{J}'$, and the kernel of $M^{\mathcal{J}} \to (M^{\mathcal{J}})^{\mathcal{J}'}$ is generated by weights in $\widehat{\mathfrak{h}}^\star \setminus \mathcal{J}'$. Hence $\ker a = \ker b$. □

Lemma 2.7. *Let $0 \to M \to N \to O \to 0$ be an exact sequence in $\mathcal{O}_A$. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and let $\mathcal{I}$ be its closed complement. Then the following holds.*

(1) *The sequence $0 \to M_{\mathcal{I}} \to N_{\mathcal{I}} \to O_{\mathcal{I}} \to 0$ is exact at $M_{\mathcal{I}}$ and at $O_{\mathcal{I}}$ (but not necessarily at $N_{\mathcal{I}}$).*
(2) *The sequence $0 \to M^{\mathcal{J}} \to N^{\mathcal{J}} \to O^{\mathcal{J}} \to 0$ is exact at $N^{\mathcal{J}}$ and at $O^{\mathcal{J}}$ (but not necessarily at $M^{\mathcal{J}}$).*
(3) *The first sequence is exact at $N_{\mathcal{I}}$ if and only if the second sequence is exact at $M^{\mathcal{J}}$.*

That the two sequences referred to in the lemma need not be exact is shown by the example $0 \to \Delta(-2) \to \Delta(0) \to L(0) \to 0$ of modules in the usual category $\mathcal{O}$ over the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Proof. From the very definition of the functors it follows that the map $M_{\mathcal{I}} \to N_{\mathcal{I}}$ is injective and that the map $N_{\mathcal{I}} \to O_{\mathcal{I}}$ is surjective, hence (1). Now (2) and (3) follow from some chasing in the diagram

$$\begin{tikzcd} 0 \arrow[r] & M_{\mathcal{I}} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M^{\mathcal{J}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N_{\mathcal{I}} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N^{\mathcal{J}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & O_{\mathcal{I}} \arrow[r] & O \arrow[r] & O^{\mathcal{J}} \arrow[r] & 0 \end{tikzcd}$$

which is commutative with exact rows and a short exact middle column. □

2.12. **Verma flags.**

Definition 2.8. Let M be an object in $\mathcal{O}_A$. We say that M *admits a Verma flag* if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for $i = 1, \ldots, n$, M_i/M_{i-1} is isomorphic to $\Delta_A(\mu_i)$ for some $\mu_i \in \widehat{\mathfrak{h}}^\star$.

Suppose that $M \in \mathcal{O}_A$ admits a Verma flag. For each $\mu \in \widehat{\mathfrak{h}}^\star$, the number of occurences of $\Delta_A(\mu)$ as a subquotient of a Verma flag of M is independent of the chosen filtration. We denote this number by $(M : \Delta_A(\mu))$.

Let $\mu \in \widehat{\mathfrak{h}}^\star$ and $M \in \mathcal{O}_A$. The set $\mathcal{J} = \{\nu \in \widehat{\mathfrak{h}}^\star \mid \nu \leq \mu\}$ is open and we define $M^{\leqslant\mu} := M^{\mathcal{J}}$. We define $M^{<\mu}$ likewise. Then we set

$$M_{[\mu]} := \ker\left(M^{\leqslant\mu} \to M^{<\mu}\right).$$

Note that $M_{[\mu]}$ is generated by its μ-weight space. One can show that M admits a Verma flag if and only of $M_{[\mu]}$ is non-zero for only finitely many μ and for such μ it is isomorphic to a finite direct sum of copies of $\Delta_A(\mu)$.

2.13. **Projective objects in $\mathcal{O}_A$.** As in Proposition 2.3 we assume that A is a local deformation algebra with residue field k. For general λ the simple module $L_A(\lambda)$ admits a projective cover in $\mathcal{O}_A$ only if we restrict the set of allowed weights from above. So let us call a subset

$\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ *bounded* (rather *locally bounded from above*) if for any $\lambda \in \mathcal{J}$ the set $\{\mu \in \mathcal{J} \mid \lambda \leq \mu\}$ is finite.

Theorem 2.9. *Suppose that A is a local deformation algebra with residue field k. Let $\mathcal{J}$ be a bounded open subset of $\widehat{\mathfrak{h}}^\star$.*

(1) *For each $\lambda \in \mathcal{J}$ there exists a projective cover $P_A^{\mathcal{J}}(\lambda)$ of $L_A(\lambda)$ in $\mathcal{O}_A^{\mathcal{J}}$. It admits a Verma flag and we have*

$$(P_A^{\mathcal{J}}(\lambda) : \Delta_A(\mu)) = \begin{cases} [\nabla_k(\mu) : L_k(\lambda)], & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{else.} \end{cases}$$

(2) *If $\mathcal{J}' \subset \mathcal{J}$ is open as well, then*

$$P_A^{\mathcal{J}}(\lambda)^{\mathcal{J}'} \cong P_A^{\mathcal{J}'}(\lambda).$$

(3) *If $A \to A'$ is a homomorphism of local deformation algebras and $P \in \mathcal{O}_A^{\mathcal{J}}$ is projective, then $P \otimes_A A' \in \mathcal{O}_{A'}^{\mathcal{J}}$ is projective.*
(4) *We have $P_A^{\mathcal{J}}(\lambda) \otimes_A k \cong P_k^{\mathcal{J}}(\lambda)$.*
(5) *Suppose that P is a finitely generated projective object in $\mathcal{O}_A^{\mathcal{J}}$ and that $A \to A'$ is a homomorphism of local deformation algebras. For any $M \in \mathcal{O}_A^{\mathcal{J}}$ the natural map*

$$\mathrm{Hom}_{\mathcal{O}_A}(P, M) \otimes_A A' \to \mathrm{Hom}_{\mathcal{O}_{A'}}(P \otimes_A A', M \otimes_A A')$$

is an isomorphism.

Proof. The proofs of the above statements are all contained in [Fi03]. So here we give only a short sketch. First one shows, under the assumption that $\mathcal{J}$ is bounded, that for each $\lambda \in \mathcal{J}$ there is an object $Q_A^{\mathcal{J}}(\lambda)$ in $\mathcal{O}_A^{\mathcal{J}}$ that represents the functor $M \mapsto M_\lambda$. One can find the idea of its construction in [RCW82]. As each object of $\mathcal{O}_A$ is a weight module, this functor is exact, hence $Q_A^{\mathcal{J}}(\lambda)$ is projective. Moreover, it is clear that there is a surjection $Q_A^{\mathcal{J}}(\lambda) \to L_A(\lambda)$, so we can take for $P_A^{\mathcal{J}}(\lambda)$ any indecomposable direct summand of $Q_A^{\mathcal{J}}(\lambda)$ that maps surjectively onto $L_A(\lambda)$.

As each simple object is a quotient of one of the $Q_A^{\mathcal{J}}(\lambda)$, the set of objects $\{Q_A^{\mathcal{J}}(\lambda)\}_{\lambda \in \mathcal{J}}$ generates $\mathcal{O}_A^{\mathcal{J}}$ in the sense that each object is a quotient of a direct sum of various $Q_A^{\mathcal{J}}(\lambda)$'s. Moreover, from the construction it follows quite easily that there is an isomorphism $Q_A^{\mathcal{J}}(\lambda) \otimes_A A' \cong Q_{A'}^{\mathcal{J}}(\lambda)$ for any homomorphism $A \to A'$ of deformation algebras.

Property (5) now holds for the objects $Q_A^{\mathcal{J}}(\lambda)$ as they represent the functor $M \mapsto M_\lambda$. From this one deduces (5) for any finitely generated projective object P, as any projective object is a direct summand of a direct sum of certain $Q_A^{\mathcal{J}}(\lambda)$'s. In an analogous fashion, part (3) is first proven for $Q_A^{\mathcal{J}}(\lambda)$ and follows for arbitrary projectives P in $\mathcal{O}_A^{\mathcal{J}}$.

Part (4) is proven using the idempotent lifting lemma and property (5), in [Fi03, Proposition 2.6]. Then part (1) is shown to follow from part (4) (cf. [Fi03, Theorem 2.7], note that there the multiplicity is stated in terms of a Verma module, not a dual Verma module, but the characters coincide and it is more natural to use the dual Verma module). Clearly, (1) implies (2) as the truncation functor preserves projectivity and is exact on Verma flags. □

2.14. **The block decomposition of $\mathcal{O}_A$.** Let A be a local deformation algebra with residue field k. We let $\sim_A$ be the equivalence relation on $\widehat{\mathfrak{h}}^\star$ that is generated by the relations $\lambda \sim_A \mu$ for all $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ for which there exists an open bounded subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ such that $L_A(\mu)$ is a subquotient of $P_A^{\mathcal{J}}(\lambda)$.

Lemma 2.10. *The equivalence relation $\sim_A$ is also generated by either of the following sets of relations:*

1. $\lambda \sim_A \mu$ *if there exists an open bounded subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ such that* $(P_A^{\mathcal{J}}(\lambda) : \Delta_A(\mu)) \neq 0$.
2. $\lambda \sim_A \mu$ *if* $[\Delta_k(\lambda) : L_k(\mu)] \neq 0$.

For an equivalence class $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A$ we define the full subcategory $\mathcal{O}_{A,\Lambda}$ of $\mathcal{O}_A$ that contains all objects M that have the property that each highest weight of a subquotient lies in Λ. Note that it is the subcategory generated by the objects $P_A^{\mathcal{J}}(\lambda)$ for all $\lambda \in \Lambda$ and all bounded open subsets $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ that contain λ.

Then we have the following result on the decomposition of $\mathcal{O}_A$.

Theorem 2.11 ([Fi03, Proposition 2.8])**.** *The functor*

$$\prod_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A} \mathcal{O}_{A,\Lambda} \to \mathcal{O}_A$$

$$(M_\Lambda) \mapsto \bigoplus_\Lambda M_\Lambda$$

is an equivalence of categories.

2.15. **The Kac–Kazhdan theorem, integral roots and the integral Weyl group.** The Kac-Kazhdan theorem gives a rather explicit description of the set of pairs (λ, μ) such that $[\Delta_k(\lambda) : L_k(\mu)] \neq 0$. By the lemma above, these pairs generate the equivalence relation "$\sim_A$".

Recall the bilinear form $(\cdot,\cdot)\colon \widehat{\mathfrak{h}}^\star \times \widehat{\mathfrak{h}}^\star \to \mathbb{C}$. For any deformation algebra A we set $\widehat{\mathfrak{h}}^\star_A := \widehat{\mathfrak{h}}^\star \otimes_{\mathbb{C}} A = \mathrm{Hom}_{\mathbb{C}}(\widehat{\mathfrak{h}}, A)$ and denote by

$(\cdot,\cdot)_A\colon \widehat{\mathfrak{h}}^\star_A \times \widehat{\mathfrak{h}}^\star_A \to A$ the A-bilinear continuation of $(\cdot,\cdot)$. The structure map $\tau\colon \widehat{\mathfrak{h}} \to A$ can be considered as an element in $\widehat{\mathfrak{h}}^\star_A$. Let $\rho \in \widehat{\mathfrak{h}}^\star$ be an element with $(\rho,\alpha) = 1$ for any simple affine root $\alpha \in \widehat{\Pi}$.

Now we can state the result of Kac and Kazhdan (we slightly reformulate their original theorem in terms of equivalence classes):

Theorem 2.12 ([KK79])**.** *The relation "$\sim_A$" is generated by $\lambda \sim_A \mu$ for all pairs λ,μ such that there exists a root $\alpha \in \widehat{R}$ and $n \in \mathbb{Z}$ with $2(\lambda+\rho,\alpha)_k = n(\alpha,\alpha)_k$ and $\lambda - \mu = n\alpha$.*

For $\lambda \in \widehat{\mathfrak{h}}^\star$ we define the set of *integral roots* (with respect to λ) by

$$\widehat{R}_A(\lambda) := \{\alpha \in \widehat{R} \mid 2(\lambda+\rho,\alpha)_k \in \mathbb{Z}(\alpha,\alpha)_k\}$$

and the corresponding *integral Weyl group* by

$$\widehat{\mathcal{W}}_A(\lambda) := \langle s_\alpha \mid \alpha \in \widehat{R}(\lambda) \cap \widehat{R}^{\mathrm{re}}_A\rangle \subset \widehat{\mathcal{W}}.$$

Let $\Lambda \subset \widehat{\mathfrak{h}}^\star$ be an equivalence class with respect to "$\sim_A$". It follows from the Kac–Kazhdan theorem that we have $\widehat{R}_A(\lambda) = \widehat{R}_A(\mu)$ and $\widehat{\mathcal{W}}_A(\lambda) = \widehat{\mathcal{W}}_A(\mu)$ for all $\lambda,\mu \in \Lambda$. Hence we can denote these two objects by $\widehat{R}_A(\Lambda)$ and $\widehat{\mathcal{W}}_A(\Lambda)$.

2.16. **The critical level.** Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A$ be an equivalence class. For each $\lambda,\mu \in \Lambda$ we then have $\lambda(K) = \mu(K)$, hence there is a certain $k = k(\Lambda) \in \mathbb{C}$ such that each object in $\mathcal{O}_{A,\Lambda}$ is of level k. We call this k also the *level of* Λ.

Lemma 2.13. *Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A$ be an equivalence class. The following are equivalent.*

(1) *We have $\lambda(K) = -\rho(K)$ for some $\lambda \in \Lambda$.*
(2) *We have $\lambda(K) = -\rho(K)$ for all $\lambda \in \Lambda$.*
(3) *We have $\lambda + \delta \in \Lambda$ for all $\lambda \in \Lambda$.*
(4) *We have $n\delta \in \widehat{R}_A(\Lambda)$ for some $n \neq 0$.*
(5) *We have $n\delta \in \widehat{R}_A(\Lambda)$ for all $n \neq 0$.*

The level $c := -\rho(K)$ is called the *critical* level.

3. The graded center

In this section we recall one of the most significant structures that we encounter for the category $\mathcal{O}$ of an affine Kac-Moody algebra at the critical level. Recall that we add the derivation operator D to the central extension of the loop algebra corresponding to $\mathfrak{g}$. This allows us to consider $\mathcal{O}$ (and the deformed versions $\mathcal{O}_A$) as graded categories, i.e. there is a natural shift functor T on $\mathcal{O}$ and the associated *graded* *center* $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \operatorname{End}(\mathrm{id}, T^n)$ of $\mathcal{O}$. We use this graded center to define the *restricted representations* of $\widehat{\mathfrak{g}}$.

3.1. **Tensor products.** Suppose that M is a $\widehat{\mathfrak{g}}_A$-module and that L is a $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{\mathbb{C}}$-module. Then $M \otimes_{\mathbb{C}} L$ acquires the structure of a $\widehat{\mathfrak{g}}_A$-module such that $\widehat{\mathfrak{g}}$ acts via the usual tensor product action ($X(m \otimes l) = Xm \otimes l + m \otimes Xl$ for $X \in \widehat{\mathfrak{g}}$, $m \in M$, $l \in L$) and A acts only on the first factor.

Lemma 3.1. (1) *If M is locally $\widehat{\mathfrak{b}}_A$-finite and L is locally $\widehat{\mathfrak{b}}$-finite, then $M \otimes_{\mathbb{C}} L$ is locally $\widehat{\mathfrak{b}}_A$-finite.*
(2) *If M and L are weight modules, then $M \otimes_{\mathbb{C}} L$ is a weight module.*
(3) *If $M \in \mathcal{O}_A$ and $L \in \mathcal{O}$, then $M \otimes_{\mathbb{C}} L \in \mathcal{O}_A$.*

Proof. In order to prove (1) it is enough to show that the $\widehat{\mathfrak{b}}_A$-submodule of $M \otimes_{\mathbb{C}} L$ that is generated by an element $m \otimes l$ with $m \in M$, $l \in L$, is finitely generated over A. This follows from the fact that $U(\widehat{\mathfrak{b}}_A).m$ is a finitely generated A-submodule of M and $U(\widehat{\mathfrak{b}}).l$ is a finite dimensional $\mathbb{C}$-subvector space of L.

In the situation of (2) we have $(M \otimes_{\mathbb{C}} L)_\lambda = \bigoplus_{\mu\in\widehat{\mathfrak{h}}^\star} M_\mu \otimes_{\mathbb{C}} L_{\lambda-\mu}$, hence $M \otimes_{\mathbb{C}} L$ is a weight module. Now (3) is implies by (1) and (2). $\square$

3.2. **A shift functor.** Note that $\delta \in \widehat{\mathfrak{h}}^\star$ is the smallest positive imaginary root of $\widehat{\mathfrak{g}}$. The simple module $L(\delta) = L_{\mathbb{C}}(\delta)$ is one-dimensional. The $\widehat{\mathfrak{g}}$-module $L(\delta) \otimes_{\mathbb{C}} L(-\delta) \cong L(0)$ is the trivial module. In particular, the *shift functor*

$$\begin{aligned} T\colon \mathcal{O}_A &\to \mathcal{O}_A \\ M &\mapsto M \otimes_{\mathbb{C}} L(\delta) \end{aligned}$$

is an equivalence with inverse $T^{-1} = \cdot \otimes_{\mathbb{C}} L(-\delta)$.

Let $n \in \mathbb{Z}$ and consider the space $\mathcal{A}_A^n := \operatorname{Hom}(\mathrm{id}, T^n)$ of natural transformations between the identity functor on $\mathcal{O}_A$ and the functor T^n. Recall that an element $\phi \in \operatorname{Hom}(\mathrm{id}, T^n)$ associates a homomorphism $\phi^M\colon M \to T^nM$ to any object $M \in \mathcal{O}_A$ such that for any homomorphism $f\colon M \to N$ in $\mathcal{O}_A$ the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi^M"'] & N \arrow[d, "\phi^N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. Note that $\mathcal{A}_A^n$ carries a natural structure of an A-module: for $a \in A$, $\phi \in \mathcal{A}_A^n$ and $M \in \mathcal{O}_A$ we let $(a.\phi)^M$ be the homomorphism $a.(\phi^M)$.

There is an A-bilinear map

$$\begin{aligned}\mathcal{A}_A^n \times \mathcal{A}_A^m &\to \mathcal{A}_A^{m+n}\\ (\phi,\psi) &\mapsto \big(M \mapsto (T^m\phi^M)\circ\psi^M\big)\end{aligned}$$

which makes $\mathcal{A}_A := \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_A^n$ into a graded A-algebra. It is called the *graded center* of $\mathcal{O}_A$.

Note that $\mathcal{A}_A$ is a commutative algebra. Since $L(\delta)$ has level 0 the shift functor T preserves the subcategories $\mathcal{O}_{A,k}$, i.e. we get induced autoequivalences $T\colon \mathcal{O}_{A,k} \to \mathcal{O}_{A,k}$ for each k. Accordingly, $\mathcal{A}_A$ splits into the direct product of the A-algebras $\mathcal{A}_{A,k}$ with $k \in \mathbb{C}$.

Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A$ be an equivalence class. The corresponding block $\mathcal{O}_{A,\Lambda}$ is preserved by the functor T if and only if for each $\lambda \in \Lambda$ we have $\lambda + \delta \in \Lambda$, hence if and only if Λ is critical (cf. Lemma 2.13). So if $k \neq c$ then $\mathcal{A}_{A,k}^n = 0$ for all $n \neq 0$. In contrast we have $\mathcal{A}_{A,c}^n \neq 0$ for all n. Most of the results in the following make sense in arbitrary level, but contain no information if the level is not critical.

For some deformation algebras A and non-critical k the algebra $\mathcal{A}_{A,k} = \mathcal{A}_{A,k}^0$, i.e. the ordinary center of $\mathcal{O}_{A,k}$, is calculated in [Fi03].

3.3. **Base change - part 2.** Let $\mathcal{P}_A \subset \mathcal{O}_A$ be the full subcategory consisting of objects that are isomorphic to an arbitrary direct sum of various $P_A^{\mathcal{J}}(\lambda)$'s for arbitrary open bounded subsets $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ and $\lambda \in \mathcal{J}$. Then $\mathcal{P}_A$ generates the category $\mathcal{O}_A$, which means that for any $M \in \mathcal{O}_A$ there is an exact sequence $P' \to P \to M \to 0$ with $P, P' \in \mathcal{P}_A$.

Let $n \in \mathbb{Z}$. We have a natural homomorphism

$$\begin{aligned}\mathcal{A}_A^n &\to \prod_{\substack{\mathcal{J}\subset\widehat{\mathfrak{h}}^\star \text{ open}\\ \lambda\in\mathcal{J}}} \operatorname{Hom}(P_A^{\mathcal{J}}(\lambda), T^n P_A^{\mathcal{J}}(\lambda))\\ \phi &\mapsto (\phi^{P_A^{\mathcal{J}}(\lambda)})\end{aligned}$$

of A-modules. Since $\mathcal{P}_A$ generates $\mathcal{O}_A$ we have the following:

Proposition 3.2. *The above homomorphism is injective and identifies $\mathcal{A}_A^n$ with the subspace $\mathcal{E}_A \subset \prod_{\mathcal{J},\lambda\in\mathcal{J}} \operatorname{Hom}(P_A^{\mathcal{J}}(\lambda), T^n P_A^{\mathcal{J}}(\lambda))$ of tuples $(\psi_{\lambda,\mathcal{J}})$ such that for any homomorphism $f\colon P_A^{\mathcal{J}}(\lambda) \to P_A^{\mathcal{J}'}(\lambda')$ in $\mathcal{P}_A$ the following diagram commutes:*

$$\begin{tikzcd} P_A^{\mathcal{J}}(\lambda) \arrow[r, "f"] \arrow[d, "\psi_{\lambda,\mathcal{J}}"'] & P_A^{\mathcal{J}'}(\lambda') \arrow[d, "\psi_{\lambda',\mathcal{J}'}"] \\ T^n P_A^{\mathcal{J}}(\lambda) \arrow[r, "T^n f"] & T^n P_A^{\mathcal{J}'}(\lambda'). \end{tikzcd}$$

We use the above proposition now to construct a base change homomorphism.

Proposition 3.3. *Suppose that $A \to A'$ is a homomorphism of local deformation algebras. Then there is a unique homomorphism $\Theta\colon \mathcal{A}_A \to \mathcal{A}_{A'}$ of graded algebras such that for any $M \in \mathcal{O}_A$ and $\phi \in \mathcal{A}_A$ we have*

$$\Theta(\phi)^{M\otimes_A A'} = \phi^M \otimes \mathrm{id}\colon M \otimes_A A' \to M \otimes_A A'.$$

Proof. By Theorem 2.9 we have for any $P \in \mathcal{P}_A$ that $P \otimes_A A' \in \mathcal{P}_{A'}$ and for any $P, P' \in \mathcal{P}_A$ we have that

$$\mathrm{Hom}(P, P') \otimes_A A' = \mathrm{Hom}(P \otimes_A A', P' \otimes_A A').$$

Now the statement follows from Proposition 3.2. □

3.4. **A duality on the graded center.** Suppose that $M \in \mathcal{O}_A$ is reflexive, i.e. that the natural homomorphism $M \to (M^\star)^\star$ is an isomorphism. Let $n \in \mathbb{Z}$ and $\phi \in \mathcal{A}_A^n$. Let us apply this element to the dual of M: we get a homomorphism $\phi^{M^\star}\colon M^\star \to T^n M^\star$. Taking the dual gives a homomorphism $(\phi^{M^\star})^\star\colon (T^n M^\star)^\star = T^n M \to (M^\star)^\star = M$. After applying the functor T^{-n} we get a homomorphism $M \to T^{-n}M$, which we denote by $(D\phi)^M$.

It is clear that $D\phi$ defines thus a homomorphism $\mathrm{id} \to T^{-n}$ between the functors restricted to the subcategory of reflexive objects. As each weight space of an indecomposable projective object in $\mathcal{O}_A^{\mathcal{J}}$ is free over A of finite rank, each indecomposable projective is reflexive and we deduce from Proposition 3.2 that Dz defines a natural transformation $\mathrm{id} \to T^{-n}$ on the whole of $\mathcal{O}_A$. The duality hence gives us a map

$$D\colon \mathcal{A}_A^n \to \mathcal{A}_A^{-n}.$$

3.5. **The Feigin–Frenkel center.** Let $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \subset \widehat{\mathfrak{g}}$ be the centrally extended loop algebra. For $k \in \mathbb{C}$ we denote by $U(\widetilde{\mathfrak{g}})_k$ the quotient of the universal enveloping algebra of $\widetilde{\mathfrak{g}}$ by the ideal generated by $K - k$. Following [Fr05] we define the completion

$$U'_k := \varprojlim_N U_k(\widetilde{\mathfrak{g}})/U_k(\widetilde{\mathfrak{g}}) \cdot (\mathfrak{g} \otimes t^N \mathbb{C}[t]).$$

Note that the action of $\widetilde{\mathfrak{g}}$ on each object in $\mathcal{O}_{A,k}$ naturally extends to an action of $U'(\widetilde{\mathfrak{g}})_k$, by the local $\widehat{\mathfrak{b}}$-finiteness condition.

If $k = c$ is the critical value, then U'_c acquires a large center $\mathcal{Z}_c$. The grading operator D defines a $\mathbb{Z}$ grading $\mathcal{Z}_c = \bigoplus_{n\in\mathbb{Z}} \mathcal{Z}_c^n$ and we can view each element $z \in \mathcal{Z}_c^n$ as a natural transformation $\mathrm{id}_{\mathcal{O}_{A,k}} \to T^n$. Hence we obtain a graded homomorphism $\mathcal{Z}_c \to \mathcal{A}_{A,c}$. Moreover, this homomorphism is compatible with the base change maps $\mathcal{A}_{A,c} \to \mathcal{A}_{A',c}$ associated to a homomorphism $A \to A'$ of deformation algebras, in the sense that the composition $\mathcal{Z}_c \to \mathcal{A}_{A,c} \to \mathcal{A}_{A',c}$ is the natural map $\mathcal{Z}_c \to \mathcal{A}_{A',c}$ defined above.

4. Restricted representations

Let A be a deformation algebra. For the moment, it need not be local.

Definition 4.1. Let $M \in \mathcal{O}_A$. We say that M is *restricted* if for all $n \neq 0$ and all $\phi \in \mathcal{A}_A^n$ we have $\phi^M(M) = 0$.

We denote by $\overline{\mathcal{O}}_A$ the full subcategory of $\mathcal{O}_A$ that consists of restricted representations. For an open subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ we set $\overline{\mathcal{O}}_A^{\mathcal{J}} = \overline{\mathcal{O}}_A \cap \mathcal{O}_A^{\mathcal{J}}$. If k is not critical, each object $M \in \mathcal{O}_{A,k}$ is restricted as $\mathcal{A}_{A,k}^n = 0$ for $n \neq 0$.

For $M \in \mathcal{O}_A$ and $n \in \mathbb{Z}$ let $\mathcal{A}_A^n M$ be the submodule of M generated by the images of all homomorphisms $\phi^{T^{-n}M} \colon T^{-n}M \to M$. We define

$$\begin{aligned} M_{\mathrm{res}} &:= \{m \in M \mid \phi^M(m) = 0 \text{ for all } \phi \in \mathcal{A}_A^n,\, n \neq 0\}, \\ M^{\mathrm{res}} &:= M/\sum_{n\in\mathbb{Z}, n\neq 0} \mathcal{A}_A^n M. \end{aligned}$$

Both M_{res} and M^{res} are restricted objects in $\mathcal{O}_A$. We get functors $M \mapsto M_{\mathrm{res}}$ and $M \mapsto M^{\mathrm{res}}$ from $\mathcal{O}_A$ to $\overline{\mathcal{O}}_A$ that are right resp. left adjoint to the inclusion functor $\overline{\mathcal{O}}_A \to \mathcal{O}_A$.

4.1. **Restriction, truncation and base change.**

We now collect some first results on the restriction functor.

Lemma 4.2. *Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open. For each $M \in \mathcal{O}_A$ there is a natural isomorphism*

$$(M^{\mathrm{res}})^{\mathcal{J}} \cong (M^{\mathcal{J}})^{\mathrm{res}}.$$

Proof. Let us consider the compositions $a \colon M \xrightarrow{a_1} M^{\mathrm{res}} \xrightarrow{a_2} (M^{\mathrm{res}})^{\mathcal{J}}$ and $b \colon M \xrightarrow{b_1} M^{\mathcal{J}} \xrightarrow{b_2} (M^{\mathcal{J}})^{\mathrm{res}}$ of the canonical quotient maps. We show that the kernels of a and b coincide, which implies the statement of the lemma.

The kernel of a is generated by $a_1^{-1}(M_\mu^{\mathrm{res}})$ with $\mu \notin \mathcal{J}$, hence it is generated by the subspaces $\mathcal{A}_A^n M$ for $n \neq 0$ and the weight spaces M_μ for $\mu \notin \mathcal{J}$. The kernel of b is generated by $b_1^{-1}(\mathcal{A}_A^n M^{\mathcal{J}})$ for $n \neq 0$, hence it is generated by the subspaces $\mathcal{A}_A^n M$ for $n \neq 0$ and the weight spaces M_μ for $\mu \notin \mathcal{J}$ as well. $\square$

Lemma 4.3. *Let $M \in \mathcal{O}_A$ and fix a homomorphism $A \to A'$ of deformation algebras. Then there is a canonical isomorphism*

$$(M \otimes_A A')^{\mathrm{res}} \to (M^{\mathrm{res}} \otimes_A A')^{\mathrm{res}}.$$

Proof. We consider the homomorphisms $a\colon M \otimes_A A' \to (M \otimes_A A')^{\mathrm{res}}$ and $b\colon M \otimes_A A' \to M^{\mathrm{res}} \otimes_A A' \to (M^{\mathrm{res}} \otimes_A A')^{\mathrm{res}}$ and we show that $\ker a = \ker b$. Note that the kernel of a is generated by the subspaces $\mathcal{A}^n_{A'}(M \otimes_A A')$ for $n \neq 0$ and the kernel of b is generated by the spaces $\mathcal{A}^n_{A'}(\mathcal{A}^m_A M \otimes_A A')$ for $m \neq 0$, $n \neq 0$. From Proposition 3.3 we deduce that

$$\sum_{m\neq 0, n\neq 0} \mathcal{A}^n_{A'}(\mathcal{A}^m_A M \otimes_A A') = \sum_{n \neq 0} \mathcal{A}^n_{A'}(M \otimes_A A'),$$

hence $\ker a = \ker b$ and the lemma is proven. □

4.2. **Restricted Verma modules.** For $\lambda \in \widehat{\mathfrak{h}}^\star$ we define the restricted Verma module by

$$\overline{\Delta}_A(\lambda) := \Delta_A(\lambda)^{\mathrm{res}}$$

and the restricted dual Verma module by

$$\overline{\nabla}_A(\lambda) := \nabla_A(\lambda)_{\mathrm{res}}.$$

Lemma 4.4. *Suppose that $X \in \overline{\mathcal{O}}_A$ is a restricted module and $\lambda \in \widehat{\mathfrak{h}}^\star$ is maximal with $X_\lambda \neq 0$. Then each surjective map $X \to \overline{\Delta}_A(\lambda)$ splits.*

Proof. Let $x \in X_\lambda$ be a preimage of a generator of $\overline{\Delta}_A(\lambda)$. By maximality of λ there is a homomorphism $\Delta_A(\lambda) \to X$ that sends a generator of $\Delta_A(\lambda)$ to x. As X is restricted and since the functor $(\cdot)^{\mathrm{res}}$ is left adjoint to the inclusion functor $\overline{\mathcal{O}}_A \to \mathcal{O}_A$, this homomorphism induces a homomorphism $\overline{\Delta}_A(\lambda) \to X$ which is left invers to our original map up to multiplication with a non-zero scalar. □

Let us denote by $\mathcal{Z}_c^+ := \bigoplus_{n>0} \mathcal{Z}_c^n$ the strictly positive part of the Feigin–Frenkel center. Recall that each $z \in \mathcal{Z}_c^n$ gives us an element in $\mathcal{A}^n_{A,c}$, so for each $M \in \mathcal{O}_{A,c}$ a homomorphism $M \to T^n M$. Let us denote by $\mathcal{Z}_c^+ M \subset M$ the submodule generated by the images of the homomorphisms $z\colon T^{-n}M \to M$ for all $n > 0$. Then we have the following result:

Lemma 4.5. *Let $A = k$ be a field. For each critical $\lambda \in \widehat{\mathfrak{h}}^\star$, the restricted Verma module $\overline{\Delta}_k(\lambda)$ equals the quotient $\Delta_k(\lambda)/\mathcal{Z}_c^+\Delta_k(\lambda)$.*

Proof. As the action of $\mathcal{Z}_c$ on $\Delta_k(\lambda)$ factors over the action of $\mathcal{A}_k$ via the canonical map $\mathcal{Z}_c \to \mathcal{A}_k$, the restricted Verma module $\overline{\Delta}_k(\lambda)$ is a quotient of $\Delta_k(\lambda)/\mathcal{Z}_c^+\Delta_k(\lambda)$. As there is no non-zero homomorphism $\Delta_k(\lambda - n\delta) \to \Delta_k(\lambda)/\mathcal{Z}_c^+\Delta_k(\lambda)$ (by (??)), we can deduce $\overline{\Delta}_k(\lambda) = \Delta_k(\lambda)/\mathcal{Z}_c^+\Delta_k(\lambda)$. □

4.3. **The character of a restricted Verma module.** Let us define the numbers $p(n) \in \mathbb{N}$ for $n \geq 0$ by the following equation (in $\mathbb{Z}[\widehat{\widehat{\mathfrak{h}}^\star}]$)

$$\prod_{l \geq 0} (1 + e^{-l\delta} + e^{-2l\delta} + \dots)^{\mathrm{rk}\mathfrak{g}} = \sum_{n \geq 0} p(n) e^{-n\delta},$$

and the numbers $q(n) \in \mathbb{Z}$, $n \geq 0$ by the corresponding equation for the inverse of the left hand side:

$$\left(\prod_{l \geq 0} (1 + e^{-l\delta} + e^{-2l\delta} + \dots)^{\mathrm{rk}\mathfrak{g}} \right)^{-1} = \prod_{l \geq 0} (1 - e^{-l\delta})^{\mathrm{rk}\mathfrak{g}} = \sum_{n \geq 0} q(n) e^{-n\delta}.$$

Lemma 4.6. *Suppose that $A = k$ is a field. Let $\lambda \in \widehat{\mathfrak{h}}^\star$ be critical.*

(1) *We have*

$$\mathrm{ch}\, \overline{\Delta}_k(\lambda) = e^\lambda \prod_{\alpha \in \widehat{R}^{+,\mathrm{re}}} (1 + e^{-\alpha} + e^{-2\alpha} + \dots).$$

(2) *For all $\mu \in \widehat{\mathfrak{h}}^\star$ we have*

$$[\overline{\Delta}_k(\lambda) : L_k(\mu)] = \sum_{n \geq 0} q(n) [\Delta_k(\lambda - n\delta) : L_k(\mu)].$$

Proof. The first statement is due to Feigin–Frenkel and Frenkel (cf. Theorem 9.5.3 (??) in [Fr07]). Using the well-known character formula for the usual Verma modules we get

$$\begin{aligned} \mathrm{ch}\, \Delta_k(\lambda) &= e^\lambda \prod_{\alpha \in \widehat{R}^+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots)^{\dim \widehat{\mathfrak{g}}_\alpha} \\ &= \prod_{l > 0} (1 + e^{-l\delta} + e^{-2l\delta} + \dots)^{\mathrm{rk}\mathfrak{g}}\, \mathrm{ch}\, \overline{\Delta}_k(\lambda). \end{aligned}$$

(Note that $\dim \widehat{\mathfrak{g}}_\alpha = 1$ for real roots α, and $\dim \widehat{\mathfrak{g}}_{l\delta} = \mathrm{rk}\mathfrak{g}$ for all $l \neq 0$. Dividing this equation by $\prod_{l>0}(1 + e^{-l\delta} + e^{-2l\delta} + \dots)^{\mathrm{rk}\mathfrak{g}}$ yields

$$\begin{aligned} \mathrm{ch}\, \overline{\Delta}_k(\lambda) &= \left(\prod_{l > 0} (1 + e^{-l\delta} + e^{-2l\delta} + \dots)^{\mathrm{rk}\mathfrak{g}} \right)^{-1} \mathrm{ch}\, \Delta_k(\lambda) \\ &= \sum_{n \geq 0} q(n) e^{-n\delta}\, \mathrm{ch}\, \Delta_k(\lambda) \\ &= \sum_{n \geq 0} q(n)\, \mathrm{ch}\, \Delta_k(\lambda - n\delta), \end{aligned}$$

hence (2). $\square$

4.4. **Some results on the structure of restricted Verma modules.** Before having a closer look at the restricted Verma modules we prove the following commutative algebra statement:

Lemma 4.7. *Let A be a local domain with residue field k and quotient field Q. Let M be a finitely generated A-module and suppose that*

$$\dim_k M \otimes_A k = \dim_Q M \otimes_A Q.$$

Then M is a free A-module of rank $\mathrm{rk}_A M = \dim_k M \otimes_A k = \dim_Q M \otimes_A Q$.

Proof. Let $n = \dim_k M \otimes_A k$ and let $v_1, \ldots, v_n \in M$ be preimages of a basis $\overline{v}_1, \ldots, \overline{v}_n$ of $M \otimes_A k$. By Nakayama's lemma, $v_1, \ldots, v_n$ generates M, so we have a surjective map $A^{\oplus n} \to M$. It induces a surjective map $Q^{\oplus n} \to M \otimes_A Q$, which, by our assumption, is an isomorphism. We deduce that $A^{\oplus n} \to M$ is also injective, hence an isomorphism. □

We will apply the above result to our deformation theory at several places. So let A be a local deformation domain with residue field k and quotient field Q.

Lemma 4.8. *Suppose $\lambda \in \widehat{\mathfrak{h}}^\star$ is critical. Then the following holds:*

1. *The restricted Verma module $\overline{\Delta}_A(\lambda)$ coincides with the quotient of $\Delta_A(\lambda)$ by the submodule $\mathcal{Z}_c^+ \Delta_A(\lambda)$.*
2. *For any $\mu \in \widehat{\mathfrak{h}}^\star$ the weight space $\overline{\Delta}_A(\lambda)_\mu$ is a free A-module of rank*

$$\mathrm{rk}_A \overline{\Delta}_A(\lambda)_\mu = \dim_k \overline{\Delta}_k(\lambda)_\mu.$$

Proof. Let us denote by $\overline{\Delta}_A(\lambda)'$ the quotient of $\Delta_A(\lambda)$ by $\mathcal{Z}_c^+ \Delta_A(\lambda)$. Lemma 4.5 together with the base change remark in Section 3.5 shows that we have isomorphisms

$$\overline{\Delta}_A(\lambda)' \otimes_A Q \cong \overline{\Delta}_Q(\lambda), \quad \overline{\Delta}_A(\lambda)' \otimes_A k \cong \overline{\Delta}_k(\lambda).$$

As these isomorphisms induce isomorphisms on any weight space and since the weight space dimensions coincide by Lemma 4.6, we get the statement (2) of the Lemma for the module $\overline{\Delta}_A(\lambda)'$ instead of $\overline{\Delta}_A(\lambda)$.

As $\mathcal{Z}_c^+$ acts on $\Delta_A(\lambda)$ via a homomorphism $\mathcal{Z}_c^+ \to \mathcal{A}_A$, we have a canonical surjective map

$$\overline{\Delta}_A(\lambda)' \to \overline{\Delta}_A(\lambda).$$

After applying the tensor functor $\cdot \otimes_A Q$ we get canonical maps

$$\overline{\Delta}_A(\lambda)' \otimes_A Q \to \overline{\Delta}_A(\lambda) \otimes_A Q \to \overline{\Delta}_Q(\lambda).$$

As the composition is an isomorphism, the kernel of $\overline{\Delta}_A(\lambda)' \to \overline{\Delta}_A(\lambda)$ is a torsion module. As the first module is free, this homomorphism is hence an isomorphism, which proves statement (1) and at the same time completes statement (2). $\square$

Lemma 4.9. *Let $\lambda \in \widehat{\mathfrak{h}}^\star$ be critical. Then we have $\overline{\Delta}_A(\lambda)^\star \cong \overline{\nabla}_A(\lambda)$, $\overline{\nabla}_A(\lambda)^\star \cong \overline{\Delta}_A(\lambda)$.*

Proof. Note that by Lemma 4.8, each weight space of $\overline{\Delta}_A(\lambda)$ is a free A-module of finite rank, so it is reflexive, i.e. $(\overline{\Delta}_A(\lambda)^\star)^\star = \overline{\Delta}_A(\lambda)$. Hence it is enough to prove that $\overline{\Delta}_A(\lambda)^\star \cong \overline{\nabla}_A(\lambda)$.

We consider now the short exact sequence

$$0 \to \bigoplus_{n \neq 0} \mathcal{A}_A^n \Delta_A(\lambda) \to \Delta_A(\lambda) \to \overline{\Delta}_A(\lambda) \to 0.$$

As each weight space of $\Delta_A(\lambda)$ and of $\overline{\Delta}_A(\lambda)$ is a free A-module of finite rank, the sequence above splits as a sequence of A-modules. Hence each weight space of $\bigoplus_{n \neq 0} \mathcal{A}_A^n \Delta_A(\lambda)$ is free and the dual sequence

$$0 \to \overline{\Delta}_A(\lambda)^\star \to \nabla_A(\lambda) \to \left(\bigoplus_{n \neq 0} \mathcal{A}_A^n \Delta_A(\lambda) \right)^\star \to 0$$

is exact as well.

The injective map on the left factors over the inclusion $\overline{\nabla}_A(\lambda) \to \nabla_A(\lambda)$, as $\overline{\Delta}_A(\lambda)^\star$ is restricted. By definition, the composition of $\overline{\nabla}_A(\lambda) \to \nabla_A(\lambda)$ with the surjection $\nabla_A(\lambda) \to (\bigoplus_{n \neq 0} \mathcal{A}_A^n \Delta_A(\lambda))^\star$ is zero. Hence $\overline{\Delta}_A(\lambda)^\star \cong \overline{\nabla}_A(\lambda)$. $\square$

4.5. **Restricted Verma flags.** Now we state the definition of a restricted Verma flag in analogy to Definition 2.8.

Definition 4.10. We say that a module $M \in \mathcal{O}_A$ *admits a restricted Verma flag* if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for each $i = 1, \dots, n$, M_i/M_{i-1} is isomorphic to $\overline{\Delta}_A(\mu_i)$ for some $\mu_i \in \widehat{\mathfrak{h}}^\star$.

Note that we are careful here. We do not assume that a module admitting a restricted Verma flag is restricted itself. The reason for this is the following: if $A \to A'$ is a homomorphism of deformation algebras and if $M \in \mathcal{O}_A$ admits a restricted Verma flag, then $M \otimes_A A'$ admits a restricted Verma flag as well. It is, however, not clear whether $M \otimes_A A'$ is restricted if M is.

Again, if $M \in \mathcal{O}_A$ admits a restricted Verma flag, then for each $\mu \in \widehat{\mathfrak{h}}^\star$ the number of occurences of $\overline{\Delta}_A(\mu)$ is independent of the chosen filtration. We denote this number by $(M : \overline{\Delta}_A(\mu))$.

One can show that $M \in \mathcal{O}_A$ admits a restricted Verma flag if and only if $M_{[\mu]}$ is non-zero only for finitely many μ and for those it is isomorphic to a finite direct sum of copies of $\overline{\Delta}_A(\mu)$.

Lemma 4.11. *Let $M \in \mathcal{O}_A$.*

(1) *Suppose that M admits a restricted Verma flag and let $\{\mu_1, \dots, \mu_l\}$ be an enumeration of the multiset that contains each $\mu \in \widehat{\mathfrak{h}}^\star$ with multiplicity $(M : \overline{\Delta}_A(\mu))$. Suppose furthermore that this enumeration has the property that $\mu_i > \mu_j$ implies $i < j$. Then there is a filtration $M_0 \subset M_1 \subset \cdots \subset M_l = M$ with $M_i/M_{i-1} \cong \overline{\Delta}_A(\mu_i)$ for each $i = 1, \dots, l$.*

Let $\mathcal{J}$ be an open subset of $\widehat{\mathfrak{h}}^\star$ and let $\mathcal{I} := \widehat{\mathfrak{h}}^\star \setminus \mathcal{J}$ be its complement.

(2) *M admits a restricted Verma flag if and only if both $M_{\mathcal{I}}$ and $M^{\mathcal{J}}$ admit restricted Verma flags.*

(3) *If either of the two equivalent properties in part (2) holds, then we have for all $\mu \in \widehat{\mathfrak{h}}^\star$*

$$(M_{\mathcal{I}} : \overline{\Delta}_A(\mu)) = \begin{cases} (M : \overline{\Delta}_A(\mu)), & \text{if } \mu \in \mathcal{I}, \\ 0, & \text{else}, \end{cases}$$

$$(M^{\mathcal{J}} : \overline{\Delta}_A(\mu)) = \begin{cases} (M : \overline{\Delta}_A(\mu)), & \text{if } \mu \in \mathcal{J}, \\ 0, & \text{else}. \end{cases}$$

Proof. Part (1) follows directly from Lemma 4.4. So let us prove (2). Consider the short exact sequence $0 \to M_{\mathcal{I}} \to M \to M^{\mathcal{J}} \to 0$. From the definition it immediately follows that if $M_{\mathcal{I}}$ and $M^{\mathcal{J}}$ admit restricted Verma flags, then so does M. So suppose that M admits restricted Verma flag. By (1) we can find a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ such that $M_i/M_{i-1} \cong \overline{\Delta}_A(\mu_i)$ and such that $\{\mu_1, \dots, \mu_n\} \subset \mathcal{I}$ and $\{\mu_{n+1}, \dots, \mu_l\} \subset \mathcal{J}$ for some $n \geq 0$. We then have $M_{\mathcal{I}} = M_n$, as M_n is generated by its vectors of weights $\mu_1, \dots, \mu_n$ and the weights of M/M_n belong to $\mathcal{J}$. Hence $M^{\mathcal{J}} = M/M_n$ and we deduce that both $M_{\mathcal{I}}$ and $M^{\mathcal{J}}$ admit a restricted Verma flag and that the multiplicity statements in (3) hold as well. □

Lemma 4.12. *Let $0 \to M \to N \to O \to 0$ be a short exact sequence of finitely generated objects in $\mathcal{O}_A$. Suppose that N and O admit restricted Verma flags. Then the following holds:*

(1) *For any bounded open subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ with complement $\mathcal{I} := \widehat{\mathfrak{h}}^\star \setminus \mathcal{J}$ the sequences*

$$0 \to M_{\mathcal{I}} \to N_{\mathcal{I}} \to O_{\mathcal{I}} \to 0$$

and

$$0 \to M^{\mathcal{J}} \to N^{\mathcal{J}} \to O^{\mathcal{J}} \to 0$$

are exact.

(2) *M admits a restricted Verma flag.*

Proof. Note that by Lemma 2.7 the exactness of one sequence in part (1) implies the exactness of the other. Now if $\mathcal{I}$ is such that it contains no weights of N, then $M_{\mathcal{I}} = N_{\mathcal{I}} = O_{\mathcal{I}} = 0$ and there is nothing to show. Hence we can proceed inductively by assuming that our claim is proven for a closed subset $\mathcal{I}'$ and considering the case $\mathcal{I} = \mathcal{I}' \cup \{\lambda\}$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{\mathcal{I}'} \arrow[r] \arrow[d] & N_{\mathcal{I}'} \arrow[r] \arrow[d] & O_{\mathcal{I}'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{\mathcal{I}} \arrow[r] \arrow[d] & N_{\mathcal{I}} \arrow[r] \arrow[d] & O_{\mathcal{I}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{\mathcal{I}}/M_{\mathcal{I}'} \arrow[r] & N_{\mathcal{I}}/N_{\mathcal{I}'} \arrow[r] & O_{\mathcal{I}}/O_{\mathcal{I}'} \arrow[r] & 0
\end{tikzcd}$$

The first row is exact by assumption. In order to prove that the second row is exact as well, it suffices to prove that the third row is exact.

Now set $\mathcal{J}' = \widehat{\mathfrak{h}}^\star \setminus \mathcal{I}' = \mathcal{J} \cup \{\lambda\}$. For any $X \in \mathcal{O}_A$ the cokernel of the inclusion $X_{\mathcal{I}'} \hookrightarrow X_{\mathcal{I}}$ identifies with the kernel of the quotient $X^{\mathcal{J}'} \to X^{\mathcal{J}}$. This kernel is the submodule generated by the weight space $(X^{\mathcal{J}'})_\lambda$. Hence we have to show that the sequence

$$0 \to U(\widehat{\mathfrak{g}}_A).(M^{\mathcal{J}'})_\lambda \to U(\widehat{\mathfrak{g}}_A).(N^{\mathcal{J}'})_\lambda \to U(\widehat{\mathfrak{g}}_A).(O^{\mathcal{J}'})_\lambda \to 0$$

is exact.

Now $0 \to M^{\mathcal{J}'} \to N^{\mathcal{J}'} \to O^{\mathcal{J}'} \to 0$ is exact by our inductive assumption, hence $U(\widehat{\mathfrak{g}}_A).(M^{\mathcal{J}'})_\lambda \to U(\widehat{\mathfrak{g}}_A).(N^{\mathcal{J}'})_\lambda$ is injective and $U(\widehat{\mathfrak{g}}_A).(N^{\mathcal{J}'})_\lambda \to U(\widehat{\mathfrak{g}}_A).(O^{\mathcal{J}'})_\lambda$ is surjective. As both $U(\widehat{\mathfrak{g}}_A).(N^{\mathcal{J}'})_\lambda$ and $U(\widehat{\mathfrak{g}}_A).(O^{\mathcal{J}'})_\lambda$ are isomorphic to finite direct sums of copies of $\overline{\Delta}_A(\lambda)$ (since $N^{\mathcal{J}'}$ and $O^{\mathcal{J}'}$ admit restricted Verma flags by Lemma 4.11), the kernel of $U(\widehat{\mathfrak{g}}_A).(N^{\mathcal{J}'})_\lambda \to U(\widehat{\mathfrak{g}}_A).(O^{\mathcal{J}'})_\lambda$ is generated by its λ-weight space, from which we deduce the exactness statement in the middle. So we proved part (1).

Now part (1) implies that for each $\mu \in \widehat{\mathfrak{h}}^\star$ the sequence

$$0 \to M_{[\mu]} \to N_{[\mu]} \to O_{[\mu]} \to 0$$

of subquotients is exact. As $N_{[\mu]}$ and $O_{[\mu]}$ are isomorphic to finite direct sums of copies of $\overline{\Delta}_A(\mu)$ this sequence splits, so $M_{[\mu]}$ is isomorphic to finite direct sums of copies of $\overline{\Delta}_A(\mu)$, hence M admits a Verma flag. □

4.6. **Base change - part 3.**

Lemma 4.13. *Let A be a local deformation domain with residue field k and quotient field Q. Suppose that $M \in \mathcal{O}_A$ has the property that both $M \otimes_A k \in \mathcal{O}_k$ and $M \otimes_A Q \in \mathcal{O}_Q$ admit restricted Verma flags and that the multiplicities coincide, i.e. that for all $\mu \in \widehat{\mathfrak{h}}^\star$ we have*

$$(M \otimes_A k : \overline{\Delta}_k(\mu)) = (M \otimes_A Q : \overline{\Delta}_Q(\mu)).$$

Then M admits a restricted Verma flag with $(M : \overline{\Delta}_A(\mu)) = (M \otimes_A k : \overline{\Delta}_k(\mu))$ for all $\mu \in \widehat{\mathfrak{h}}^\star$.

Proof. Let $\mu \in \widehat{\mathfrak{h}}^\star$. From the above equality of multiplicities we deduce that

$$\dim_k M_\mu \otimes_A k = \dim_Q M_\mu \otimes_A Q.$$

By Lemma 4.7, M_μ is a free A-module. In particular, the natural homomorphism $M \to M \otimes_A Q$ is injective.

Now let $\mu \in \widehat{\mathfrak{h}}^\star$ be a maximal weight of M, let $v \in M_\mu$ be a preimage of a non-zero element $\overline{v} \in (M \otimes_A k)_\mu$. Let $M_1 \subset M$ be the $\widehat{\mathfrak{g}}_A$-submodule generated by v. We have a surjective homomorphism $\overline{\Delta}_A(\mu) \to M_1$ that sends a generator of $\overline{\Delta}_A(\mu)$ to v. Now $M_1 \otimes_A Q$ is generated by the non-zero vector $v \otimes 1$ and since $M \otimes_A Q$ admits a Verma flag we have $M_1 \otimes_A Q \cong \overline{\Delta}_Q(\lambda)$. We deduce that the homomorphism $\overline{\Delta}_A(\mu) \to M_1$ is also injective, hence an isomorphism.

As $M_1 \otimes_A k$ is generated by $\overline{v}$ and $M_1 \otimes_A Q$ is generated by $v \otimes 1$, our assumptions imply that

$$M_1 \otimes_A k \cong \overline{\Delta}_k(\mu) \text{ and } M_1 \otimes_A Q \cong \overline{\Delta}_Q(\mu).$$

Hence we can assume, by induction on the length of the Verma flags of $M \otimes_A k$ and $M \otimes_A Q$, that M/M_1 admits a Verma flag. Hence so does M. □

5. Restricted projective objects

In this section we study the projective objects in the restricted and truncated categories $\overline{\mathcal{O}}_A^{\mathcal{J}}$.

Theorem 5.1. *Suppose that A is a local deformation algebra. For each open bounded subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ and each $\lambda \in \mathcal{J}$ there exists a projective cover $\overline{P}_A^{\mathcal{J}}(\lambda)$ of $L_A(\lambda)$ in $\overline{\mathcal{O}}_A^{\mathcal{J}}$.*

Proof. Let $P \to L_A(\lambda)$ be a projective cover of $L_A(\lambda)$ in $\mathcal{O}_A^{\mathcal{J}}$. We get a projective object P^{res}, as $(\cdot)^{\mathrm{res}}$ is left adjoint to the exact embedding $\overline{\mathcal{O}}_A^{\mathcal{J}} \subset \mathcal{O}_A^{\mathcal{J}}$, and a surjective map $P^{\mathrm{res}} \to L_A(\lambda)$. Now we take for $\overline{P}_A^{\mathcal{J}}(\lambda)$

an indecomposable direct summand of P^{res} that maps surjectively onto $L_A(\lambda)$. $\square$

We will show later that, in the situation of the proof of the above theorem, P^{res} is in fact indecomposable, i.e. we will show that $\overline{P}_A^{\mathcal{J}}(\lambda) \cong P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}}$.

Our next goal is to show that each projective object in $\widehat{\mathcal{O}}_A^{\mathcal{J}}$ admits a restricted Verma flag and that the multiplicities are given, for the indecomposables, by a BGGH-reciprocity formula, at least if A is a local domain. We prove these facts first in the case of a deformation field and then deduce the corresponding statements for local rings.

5.1. **Some homological algebra.** For each bounded open subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ the category $\overline{\mathcal{O}}_A^{\mathcal{J}}$ is abelian and contains enough projectives. So for $M, N \in \overline{\mathcal{O}}_A^{\mathcal{J}}$ we can calculate the group $\mathrm{Ext}^i_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M,N)$ by first choosing a projective resolution

$$\dots \to P_2 \to P_1 \to P_0 \to M \to 0$$

in $\overline{\mathcal{O}}_A^{\mathcal{J}}$ and then calculating the homology of the complex

$$0 \to \mathrm{Hom}_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(P_0,N) \to \mathrm{Hom}_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(P_1,N) \to \mathrm{Hom}_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(P_2,N) \to \dots.$$

If now $\mathcal{J}' \subset \mathcal{J}$ is also open and M admits a restricted Verma flag, then the complex

$$\dots \to P_2^{\mathcal{J}'} \to P_1^{\mathcal{J}'} \to P_0^{\mathcal{J}'} \to M^{\mathcal{J}'} \to 0$$

is a projective resolution of $M^{\mathcal{J}'}$ in $\overline{\mathcal{O}}_A^{\mathcal{J}'}$ (it is exact by Lemma 4.12, as M admits a restricted Verma flag). If, moreover, N is an object in $\overline{\mathcal{O}}_A^{\mathcal{J}'}$, then $\mathrm{Hom}_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(P_i,N) = \mathrm{Hom}_{\overline{\mathcal{O}}_A^{\mathcal{J}'}}(P_i^{\mathcal{J}'},N)$. Hence we get the following:

Lemma 5.2. *Suppose that $\mathcal{J}' \subset \mathcal{J}$ are bounded open subsets of $\widehat{\mathfrak{h}}^\star$ and suppose that $M \in \overline{\mathcal{O}}_A^{\mathcal{J}}$ admits a restriced Verma flag. For all $N \in \overline{\mathcal{O}}_A^{\mathcal{J}'}$ we then have*

$$\mathrm{Ext}^i_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M,N) = \mathrm{Ext}^i_{\overline{\mathcal{O}}_A^{\mathcal{J}'}}(M^{\mathcal{J}'},N)$$

for all $i \geq 0$.

5.2. **A formula for the multiplicities.** We need part (2) of the next proposition in order to prove the BGGH-reciprocity formula for the restricted projective objects.

Proposition 5.3. *Suppose that A is a local deformation algebra and that $M \in \overline{\mathcal{O}}_A^{\mathcal{J}}$ admits a restricted Verma flag. For $\nu \in \mathcal{J}$ the following holds:*

(1) *We have* $\operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M, \overline{\nabla}_A(\nu)) = 0$.
(2) $\operatorname{Hom}(M, \overline{\nabla}_A(\nu))$ *is a free A-module of rank* $(M : \overline{\Delta}_A(\nu))$.

Proof. We first prove part (1) by induction on the length l of the restricted Verma flag of M. If $l = 1$, then $M \cong \overline{\Delta}_A(\lambda)$ for some $\lambda \in \mathcal{J}$. Consider a short exact sequence

$$0 \to \overline{\nabla}_A(\nu) \to X \to \overline{\Delta}_A(\lambda) \to 0.$$

If $\nu \not> \lambda$, then this sequence splits by Lemma 4.4. Each weight space in the above sequence is a free A-module of finite rank, so the duality is involutive and exact on the above sequence. If $\nu > \lambda$, then the dual sequence

$$0 \to \overline{\nabla}_A(\lambda) \to X^\star \to \overline{\Delta}_A(\nu) \to 0$$

splits. Hence $\operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(\overline{\Delta}_A(\lambda), \overline{\nabla}_A(\nu)) = 0$.

Now suppose $l > 1$. Then choose a submodule M_1 of M such that M_1 and M/M_1 are non-zero and admit restricted Verma flags. The short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ induces an exact sequence

$$\operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M/M_1, \overline{\nabla}_A(\nu)) \to \operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M, \overline{\nabla}_A(\nu)) \to \operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M_1, \overline{\nabla}_A(\nu)).$$

By our induction hypothesis, the spaces on the left and on the right vanish, hence so does $\operatorname{Ext}^1_{\overline{\mathcal{O}}_A^{\mathcal{J}}}(M, \overline{\nabla}_A(\nu))$. So part (1) is proven.

Now let us prove part (2). Again we use induction on the length of a restricted Verma flag of M. Suppose that $M \cong \overline{\Delta}_A(\lambda)$. We have $\operatorname{Hom}(\overline{\Delta}_A(\lambda), \overline{\nabla}_A(\nu)) = \operatorname{Hom}(\Delta_A(\lambda), \overline{\nabla}_A(\nu))$. The latter space vanishes if $\lambda \neq \nu$ and it is free of rank 1 if $\lambda = \nu$ (by the statement that is dual to statement (2) in Lemma 4.8). So suppose that $l > 1$ and choose $M_1 \subset M$ as before. By (1) we have an exact sequence

$$0 \to \operatorname{Hom}(M/M_1, \overline{\nabla}_A(\nu)) \to \operatorname{Hom}(M, \overline{\nabla}_A(\nu)) \to \operatorname{Hom}(M_1, \overline{\nabla}_A(\nu)) \to 0$$

and part (2) follows from the induction hypothesis and the additivity of the multiplicities with respect to short exact sequences. $\square$

Our next goal is to prove that each projective in $\overline{\mathcal{O}}_A^{\mathcal{J}}$ admits a restricted Verma flag. First we consider the case that the deformation algebra is a field.

5.3. **The case of a field.**

Theorem 5.4. *Suppose that $A = k$ is a field. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and bounded. Then each projective object P in $\overline{\mathcal{O}}_k^{\mathcal{J}}$ admits a restricted Verma flag.*

Proof. We can assume that P is indecomposable, i.e. that $P = \overline{P}_k^{\mathcal{J}}(\lambda)$ for some $\lambda \in \mathcal{J}$. We prove the statement of the theorem by induction on the number l of elements in the set $\{\mu \in \mathcal{J} \mid \mu \geq \lambda\}$. Suppose that $l = 1$. Then λ is maximal in $\mathcal{J}$, so $\overline{P} \cong \overline{\Delta}_k(\lambda)$ by Lemma 4.4.

So suppose that $l > 1$ and that the claim is proven for all pairs ($\mathcal{J}'$, λ') such that the number of $\mu \in \mathcal{J}'$ with $\mu > \lambda'$ is smaller than l. Let $\mu \in \mathcal{J}$ be maximal with $\mu > \lambda$ and set $\mathcal{J}' = \mathcal{J} \setminus \{\mu\}$. Then $\mathcal{J}'$ is open and bounded as well. Let M be the kernel of the homomorphism $P \to P^{\mathcal{J}'}$ and consider the short exact sequence

$$0 \to M \to P \to P^{\mathcal{J}'} \to 0.$$

Now $P^{\mathcal{J}'}$ is projective in $\overline{\mathcal{O}}_k^{\mathcal{J}'}$ and admits a restricted Verma flag by our induction hypothesis. Hence we have to show that M admits a restricted Verma flag.

We first prove that $\mathrm{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(M, \overline{\nabla}_k(\nu)) = 0$ for all $\nu \in \mathcal{J}$. For $\nu = \mu$ the statement that is dual to Lemma 4.4 (recall that our deformation algebra is a field) gives $\mathrm{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(M, \overline{\nabla}_k(\mu)) = 0$, as all weights of M are smaller or equal to μ. So suppose that $\nu \neq \mu$, i.e. $\nu \in \mathcal{J}'$. The following is a part of the long exact sequence of Ext-groups associated to the short exact sequence above:

$$\begin{aligned} \mathrm{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(P, \overline{\nabla}_k(\nu)) \to{} & \mathrm{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(M, \overline{\nabla}_k(\nu)) \\ & \to \mathrm{Ext}^2_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(P^{\mathcal{J}'}, \overline{\nabla}_k(\nu)). \end{aligned}$$

The first term vanishes as P is projective in $\overline{\mathcal{O}}_k^{\mathcal{J}}$. Since $\overline{\nabla}_k(\nu) \in \overline{\mathcal{O}}_k^{\mathcal{J}'}$ we have, by Lemma 5.2,

$$\mathrm{Ext}^2_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(P^{\mathcal{J}'}, \overline{\nabla}_k(\nu)) = \mathrm{Ext}^2_{\overline{\mathcal{O}}_k^{\mathcal{J}'}}(P^{\mathcal{J}'}, \overline{\nabla}_k(\nu))$$

and the latter Ext-group vanishes by the projectivity of $P^{\mathcal{J}'}$ in $\overline{\mathcal{O}}_k^{\mathcal{J}'}$. Hence $\mathrm{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(M, \overline{\nabla}_k(\nu)) = 0$ for all $\nu \in \mathcal{J}$.

Now M is generated by its μ-weight space. Hence there is a surjection $\overline{\Delta}_k(\mu)^{\bigoplus n} \twoheadrightarrow M$ for some $n > 0$. Let X be its kernel. We assume that n is minimal, i.e. that the weights of X are strictly smaller than μ (again we use the assumption that the deformation algebra is a field). We now prove that $\mathrm{Hom}(X, \overline{\nabla}_k(\nu)) = 0$ for all $\nu \in \mathcal{J}$, which implies that $X = 0$ and hence $M \cong \overline{\Delta}_k(\mu)^{\oplus n}$. We have $\mathrm{Hom}(X, \overline{\nabla}_k(\mu)) = 0$ as all weights of X are smaller than μ. For $\nu \in \mathcal{J}$ with $\nu \neq \mu$ consider the following part of a long exact sequence associated to $0 \to X \to$

$\overline{\Delta}_k(\mu)^{\oplus n} \to M \to 0$:

$$\begin{aligned}\operatorname{Hom}_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(\overline{\Delta}_k(\mu)^{\oplus n}, \overline{\nabla}_k(\nu)) \to \operatorname{Hom}_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(X, \overline{\nabla}_k(\nu)) \\ \to \operatorname{Ext}^1_{\overline{\mathcal{O}}_k^{\mathcal{J}}}(M, \overline{\nabla}_k(\nu)).\end{aligned}$$

The first term vanishes as $\nu \neq \mu$. We have proven above that the third term vanishes also, hence the middle term vanishes, which is what we wanted to verify. □

Now we give an analogue of the BGGH-formula for the multiplicities of restricted Verma modules in a restricted Verma flag of $\overline{P}_k^{\mathcal{J}}(\lambda)$.

Theorem 5.5. *Suppose that $A = k$ is a field. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and bounded and $\lambda \in \mathcal{J}$. Then we have*

$$(\overline{P}_k^{\mathcal{J}}(\lambda) : \overline{\Delta}_k(\mu)) = \begin{cases} [\overline{\nabla}_k(\mu) : L_k(\lambda)], & \text{if } \mu \in \mathcal{J} \\ 0, & \text{else.} \end{cases}$$

Proof. Clearly, $(\overline{P}_k^{\mathcal{J}}(\lambda) : \overline{\Delta}_k(\mu)) = 0$ if $\mu \notin \mathcal{J}$. So suppose that $\mu \in \mathcal{J}$. Using Proposition 5.3 we have

$$\begin{aligned}(\overline{P}_k^{\mathcal{J}}(\lambda) : \overline{\Delta}_k(\mu)) &= \dim_k \operatorname{Hom}(\overline{P}_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\mu)) \\ &= [\overline{\nabla}_k(\mu) : L_k(\lambda)].\end{aligned}$$

The second identity is a consequence of the fact that $\overline{P}_k^{\mathcal{J}}(\lambda)$ is a projective cover of $L_k(\lambda)$ in $\mathcal{O}_k^{\mathcal{J}}$. □

5.4. **Properties of restricted projective objects (in the field case).** From Theorem 5.5 we can now deduce some properties of the restricted projective objects.

Proposition 5.6. *Suppose that the deformation algebra $A = k$ is a field and let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be a bounded open subset.*

(1) *The restriction of an indecomposable projective object in $\mathcal{O}_k^{\mathcal{J}}$ is indecomposable, i.e. for all $\lambda \in \mathcal{J}$ we have*
$$P_k^{\mathcal{J}}(\lambda)^{\mathrm{res}} \cong \overline{P}_k^{\mathcal{J}}(\lambda).$$

(2) *For an open subset $\mathcal{J}'$ of $\mathcal{J}$ we have*
$$\overline{P}_k^{\mathcal{J}'}(\lambda) \cong \overline{P}_k^{\mathcal{J}}(\lambda)^{\mathcal{J}'}.$$

(3) *For any projective object $P \in \mathcal{O}_k^{\mathcal{J}}$ the restriction admits a restricted Verma flag and for the multiplicities holds the following formula:*
$$(P^{\mathrm{res}} : \overline{\Delta}_k(\mu)) = \sum_{n \geq 0} q(n)(P : \Delta_k(\mu - n\delta))$$

for all $\mu \in \mathcal{J}$.

Proof. We start with (1). Recall that $P_k^{\mathcal{J}}(\lambda)^{\text{res}}$ is a projective object in $\overline{\mathcal{O}}_k^{\mathcal{J}}$ and contains $\overline{P}_k^{\mathcal{J}}(\lambda)$ as a direct summand. Both modules admit restricted Verma flags by Theorem 5.4, so it is enough to prove that the corresponding multiplicities coincide. Let $\nu \in \mathcal{J}$. By Proposition 5.3 we have

$$(P_k^{\mathcal{J}}(\lambda)^{\text{res}} : \overline{\Delta}_k(\nu)) = \dim_k \operatorname{Hom}(P_k^{\mathcal{J}}(\lambda)^{\text{res}}, \overline{\nabla}_k(\nu)).$$

Since $\overline{\nabla}_k(\nu)$ is restricted we have

$$\operatorname{Hom}(P_k^{\mathcal{J}}(\lambda)^{\text{res}}, \overline{\nabla}_k(\nu)) = \operatorname{Hom}(P_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)),$$

hence

$$\begin{aligned}(P_k^{\mathcal{J}}(\lambda)^{\text{res}} : \overline{\Delta}_k(\nu)) &= \dim_k \operatorname{Hom}(P_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)) \\ &= [\overline{\nabla}_k(\nu) : L_k(\lambda)] \\ &= (\overline{P}_k^{\mathcal{J}}(\lambda) : \overline{\Delta}_k(\nu)).\end{aligned}$$

The last identity is the BGGH-reciprocity result. Hence the Verma multiplicities of $P_k^{\mathcal{J}}(\lambda)^{\text{res}}$ and of $\overline{P}_k^{\mathcal{J}}(\lambda)$ coincide, hence these modules are isomorphic.

As $P_k^{\mathcal{J}}(\lambda)^{\mathcal{J}'} \cong P_k^{\mathcal{J}'}(\lambda)$ and since truncation commutes with restriction by Lemma 4.2, (1) implies (2).

Now let us prove (3). We have already shown that P^{res} is a projective object in $\overline{\mathcal{O}}_k^{\mathcal{J}}$ (cf. the proof of Theorem 5.1) and that it admits a Verma flag (cf. Theorem 5.4). In order to prove the multiplicity statement, we can assume that $P = P_k^{\mathcal{J}}(\lambda)$ for some λ. Then $P^{\text{res}} \cong \overline{P}_k^{\mathcal{J}}(\lambda)$ by part (1) and the claim is, by the reciprocity results in Theorem 2.9 and Theorem 5.5, equivalent to

$$[\overline{\nabla}_k(\mu) : L_k(\lambda)] = \sum_{n \geq 0} q(n)[\nabla_k(\mu - n\delta) : L_k(\lambda)]$$

which is statement (2) of Lemma 4.6 in terms of the dual Verma modules. □

Next we want to prove the analogous statements in the case of a local deformation algebra. But first we need yet another base change result.

5.5. **Base change - part 4.** Suppose that $A \to A'$ is a homomorphism of local deformation algebras. We have a restricted projective object $\overline{P}_A^{\mathcal{J}}(\lambda)$ and base change gives us an object $\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A'$. Note that we do not know whether this object is restricted or not, i.e. whether it is trivially acted upon by $\mathcal{A}^n_{A'}$ for $n \neq 0$. The following proposition deals with question in the cases of the canonical map $A \to k$, where k is the residue field, and $A \to Q$, where A is a domain and Q its quotient field.

In the proof of the proposition we need an auxiliary category. Recall that we have defined the strictly positive part $\mathcal{Z}^+_c$ of the Feigin–Frenkel center in Section 4.2.

Definition 5.7. We let $\mathcal{O}^+_{A,c}$ be the full subcategory of $\mathcal{O}_{A,c}$ that consists of objects M such that $\mathcal{Z}^+_c M = 0$.

Since $\operatorname{Hom}(T^{-n}\nabla_A(\lambda), \nabla_A(\lambda)) = 0$ for all $\lambda \in \widehat{\mathfrak{h}}^\star$ and $n > 0$, each dual Verma module belongs to $\mathcal{O}^+_{A,c}$ (but $\Delta_A(\lambda)$ does not). Since for all n there is a canonical map $\mathcal{Z}^n_c \to \mathcal{A}^n_{A,c}$ which is compatible with the actions on any object $M \in \mathcal{O}_{A,c}$, we can deduce that $\overline{\mathcal{O}}_{A,c} \subset \mathcal{O}^+_{A,c}$. Moreover, if $A \to A'$ is a homomorphism of deformation algebras and $M \in \overline{\mathcal{O}}_{A,c}$, then $M \otimes_A A' \in \mathcal{O}^+_{A',c}$, since the homomorphism $\mathcal{Z}^n_c \to \mathcal{A}^n_{A,c}$ is compatible with the base change homomorphism $\mathcal{A}^n_{A,c} \to \mathcal{A}^n_{A',c}$, i.e. we get $\mathcal{Z}^n_c \to \mathcal{A}^n_{A',c}$ by composition.

Lemma 5.8. *Suppose that A is a local deformation algebra with residue field k. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and bounded and $\lambda \in \mathcal{J}$.*

(1) *We have $\overline{P}^{\mathcal{J}}_A(\lambda) \otimes_A k \cong \overline{P}^{\mathcal{J}}_k(\lambda)$.*
(2) *The object $P^{\mathcal{J}}_A(\lambda)^{\mathrm{res}}$ is indecomposable, i.e. we have $P^{\mathcal{J}}_A(\lambda)^{\mathrm{res}} \cong \overline{P}^{\mathcal{J}}_A(\lambda)$.*

Proof. If λ is not critical, then there is nothing to prove. So let us suppose that λ is critical. We start with the proof of (1). Note that any module in $\mathcal{O}_k$ can actually be considered as a module in $\mathcal{O}_A$ via the homomorphism $\widehat{\mathfrak{g}}_A \to \widehat{\mathfrak{g}}_k$ of Lie algebras. If $M \in \mathcal{O}_k$ is restricted, then it is also restricted as an object in $\mathcal{O}_A$, since $\mathcal{A}^n_A$ acts on M via a homomorphism $\mathcal{A}^n_A \to \mathcal{A}^n_k$ by Proposition 3.3.

Let $\overline{P}^{\mathcal{J}}_k(\lambda) \to L_k(\lambda)$ be a surjective map. We consider this now as a morphism in $\overline{\mathcal{O}}_A$. By projectivity of $\overline{P}^{\mathcal{J}}_A(\lambda)$ there is a homomorphism $f \colon \overline{P}^{\mathcal{J}}_A(\lambda) \to \overline{P}^{\mathcal{J}}_k(\lambda)$ such that the composition $\overline{P}^{\mathcal{J}}_A(\lambda) \to \overline{P}^{\mathcal{J}}_k(\lambda) \to L_k(\lambda)$ is surjective. Then f is surjective and induces a surjective map $\overline{P}^{\mathcal{J}}_A(\lambda) \otimes_A k \to \overline{P}^{\mathcal{J}}_k(\lambda)$. We want to show that this is an isomorphism. So let X be its kernel.

Now the short exact sequence

$$0 \to X \to \overline{P}^{\mathcal{J}}_A(\lambda) \otimes_A k \to \overline{P}^{\mathcal{J}}_k(\lambda) \to 0$$

is a sequence in $\mathcal{O}_{k,c}^{+}$ (by the above arguments) and we get for all $\nu \in \widehat{\mathfrak{h}}^{\star}$ an exact sequence

$$\begin{aligned}\operatorname{Hom}(\overline{P}_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)) &\to \operatorname{Hom}(\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A k, \overline{\nabla}_k(\nu)) \\ &\to \operatorname{Hom}(X, \overline{\nabla}_k(\nu)) \to \operatorname{Ext}^1_{\mathcal{O}_{k,c}^{+}}(\overline{P}_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu))\end{aligned}$$

The first homomorphism is an isomorphism since it is injective and the dimensions coincide, as we show now: For $\nu \notin \mathcal{J}$, both spaces are the zero spaces. For $\nu \in \mathcal{J}$ we have

$$\begin{aligned}\dim \operatorname{Hom}(\overline{P}_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)) &= [\overline{\nabla}_k(\nu) : L_k(\lambda)] \\ &= \dim \operatorname{Hom}(\overline{P}_A^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)) \\ &= \dim \operatorname{Hom}(\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A k, \overline{\nabla}_k(\nu)).\end{aligned}$$

Now $\overline{P}_k^{\mathcal{J}}(\lambda)$ admits a restricted Verma flag. As in the proof of Proposition 5.3 we can show that this implies that there are no extensions of $\overline{\nabla}_k(\nu)$ with $\overline{P}_k^{\mathcal{J}}(\lambda)$ even in $\mathcal{O}_{k,c}^{+}$, i.e. $\operatorname{Ext}^1_{\mathcal{O}_{k,c}^{+}}(\overline{P}_k^{\mathcal{J}}(\lambda), \overline{\nabla}_k(\nu)) = 0$. Hence $\operatorname{Hom}(X, \overline{\nabla}_k(\nu)) = 0$ for all ν, so $X = 0$ and $\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A k \cong \overline{P}_k^{\mathcal{J}}(\lambda)$.

Now we prove (2). Using Lemma 4.3, Theorem 2.9, (4), and Proposition 5.6, (1), we have

$$\begin{aligned}(P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} \otimes_A k)^{\mathrm{res}} &= (P_A^{\mathcal{J}}(\lambda) \otimes_A k)^{\mathrm{res}} \\ &= P_k^{\mathcal{J}}(\lambda)^{\mathrm{res}} \\ &= \overline{P}_k^{\mathcal{J}}(\lambda).\end{aligned}$$

As the latter module is indecomposable, the first module is indecomposable, hence so is $P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}}$. □

5.6. **The case of a local deformation domain.** We suppose that A is a local deformation algebra which, moreover, is a domain. We want to prove that each projective in $\overline{\mathcal{O}}_A^{\mathcal{J}}$ admits a Verma flag and that the multiplicities are given by a BGGH-type reciprocity formula. We denote by k the residue field of A and by Q its quotient field.

Theorem 5.9. *Suppose that A is a local deformation domain with residue field k. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^{\star}$ be open and bounded and $\lambda \in \mathcal{J}$. Then $\overline{P}_A^{\mathcal{J}}(\lambda)$ admits a restricted Verma flag and we have for all $\mu \in \widehat{\mathfrak{h}}^{\star}$*

$$(\overline{P}_A^{\mathcal{J}}(\lambda) : \overline{\Delta}_A(\mu)) = \begin{cases} [\overline{\nabla}_k(\mu) : L_k(\lambda)], & \text{if } \mu \in \mathcal{J} \\ 0, & \text{else.} \end{cases}$$

Proof. Let us look at the following base change triangle:

$$\begin{tikzcd}[column sep=small] & P_A^{\mathcal{J}}(\lambda)^{\rm res} \arrow[dl] \arrow[dr] & \\ P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A k & & P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q. \end{tikzcd}$$

First we consider the module on the right hand side. Note that we do not know yet if it is restricted, but in any case we have a surjective homomorphism

$$P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q \to (P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q)^{\rm res},$$

where the restriction functor on the right module is the one on $\mathcal{O}_Q$. By Lemma 4.3 we have

$$(P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q)^{\rm res} = (P_A^{\mathcal{J}}(\lambda) \otimes_A Q)^{\rm res}.$$

Now $P_A^{\mathcal{J}}(\lambda) \otimes_A Q$ is a projective object in $\mathcal{O}_Q^{\mathcal{J}}$, hence we can apply part (3) of Proposition 5.6 and deduce that $(P_A^{\mathcal{J}}(\lambda) \otimes_A Q)^{\rm res}$ admits a restricted Verma flag with multiplicities

$$\begin{aligned} ((P_A^{\mathcal{J}}(\lambda) \otimes_A Q)^{\rm res} : \overline{\Delta}_Q(\mu)) &= \sum_{n \geq 0} q(n)(P_A^{\mathcal{J}}(\lambda) \otimes_A Q : \Delta_Q(\mu - n\delta)) \\ &= \sum_{n \geq 0} q(n)(P_A^{\mathcal{J}}(\lambda) : \Delta_A(\mu - n\delta)). \end{aligned}$$

Now let us consider the module $P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A k$ on the left of the above base change triangle. By Lemma 5.8 we have

$$\begin{aligned} P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A k &= \overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A k \\ &= \overline{P}_k^{\mathcal{J}}(\lambda). \end{aligned}$$

Hence this module is already restricted. Proposition 5.6 tells us that $\overline{P}_k^{\mathcal{J}}(\lambda) = P_k^{\mathcal{J}}(\lambda)^{\rm res}$ and that the multiplicities of the latter module are

$$\begin{aligned} (P_k^{\mathcal{J}}(\lambda)^{\rm res} : \overline{\Delta}_k(\mu)) &= \sum_{n \geq 0} q(n)(P_k^{\mathcal{J}}(\lambda) : \Delta_Q(\mu - n\delta)) \\ &= \sum_{n \geq 0} q(n)(P_A^{\mathcal{J}}(\lambda) : \Delta_A(\mu - n\delta)). \end{aligned}$$

We deduce that the Q-dimension of each weight space of $P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q$ is equal or larger than the k-dimension of the resp. weight space of $P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A k$. By Nakayama's Lemma it cannot be larger, so these dimensions coincide, hence we have

$$P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q = (P_A^{\mathcal{J}}(\lambda)^{\rm res} \otimes_A Q)^{\rm res},$$

i.e. that the left hand side is already restricted.

So we have shown that $P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} \otimes_A k$ and $P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} \otimes_A Q$ admit restricted Verma flags and that the multiplicities coincide. From Lemma 4.13 we deduce that $P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}}$ admits a restricted Verma flag with

$$(P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} : \overline{\Delta}_A(\mu)) = (P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} \otimes_A k : \overline{\Delta}_k(\mu)).$$

By Lemma 5.8 we have $\overline{P}_A^{\mathcal{J}}(\lambda) = P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}}$ and $P_A^{\mathcal{J}}(\lambda)^{\mathrm{res}} \otimes_A k \cong \overline{P}_k^{\mathcal{J}}(\lambda)$. So the BGGH-reciprocity statement is a consequence of Theorem 5.5. □

One proves the following corollary with the same arguments as the ones used for the proof of part (3) of Proposition 5.6.

Corollary 5.10. *Let $P \in \mathcal{O}_A^{\mathcal{J}}$ be projective. Then we have for all $\mu \in \mathcal{J}$*

$$(P^{\mathrm{res}} : \overline{\Delta}_A(\mu)) = \sum_{n \geq 0} q(n)(P : \Delta_A(\mu - n\delta)).$$

5.7. **Base change - part 5.** Suppose that $A \to A'$ is a homomorphism of local deformation domains and let $M \in \mathcal{O}_A$. By Proposition 3.3 the natural homomorphism $M \to M \otimes_A A'$ induces a homomorphism $(\mathcal{A}_A^n M) \otimes_A A' \to \mathcal{A}_{A'}^n (M \otimes_A A')$ and hence a natural, surjective map

$$M^{\mathrm{res}} \otimes_A A' \to (M \otimes_A A')^{\mathrm{res}}.$$

Proposition 5.11. *Suppose that $P \in \mathcal{O}_A^{\mathcal{J}}$ is projective. Then the above homomorphism is an isomorphism*

$$P^{\mathrm{res}} \otimes_A A' \cong (P \otimes_A A')^{\mathrm{res}}.$$

In particular, if $P \in \mathcal{O}_A^{\mathcal{J}}$ is restricted and projective, then $P \otimes_A A' \in \mathcal{O}_{A'}^{\mathcal{J}}$ is restricted and projective.

Proof. Note that $P \otimes_A A'$ is projective in $\mathcal{O}_{A'}^{\mathcal{J}}$ and we have

$$(P : \Delta_A(\nu)) = (P \otimes_A A' : \Delta_{A'}(\nu))$$

for all $\nu \in \mathcal{J}$.

Since P^{res} admits a restricted Verma flag, so does $P^{\mathrm{res}} \otimes_A A'$ and the multiplicities coincide. Using Corollary 5.10 (twice) we have for all

$\nu \in \mathcal{J}$

$$
\begin{aligned}
(P^{\mathrm{res}} \otimes_A A' : \overline{\Delta}_{A'}(\nu)) &= (P^{\mathrm{res}} : \overline{\Delta}_A(\nu)) \\
&= \sum_{n \geq 0} q(n)(P : \Delta_A(\mu - n\delta)) \\
&= \sum_{n \geq 0} q(n)(P \otimes_A A' : \Delta_{A'}(\mu - n\delta)) \\
&= ((P \otimes_A A')^{\mathrm{res}} : \overline{\Delta}_{A'}(\nu)).
\end{aligned}
$$

As the multiplicities coincide, the canonical surjective map $P^{\mathrm{res}} \otimes_A A' \to (P \otimes_A A')^{\mathrm{res}}$ has to be an isomorphism. $\square$

6. The restricted linkage principle and the restricted block decomposition

In this section we use the above BGGH-reciprocity to prove our main theorem, the restricted linkage principle:

Theorem 6.1. *For $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ we have $[\overline{\Delta}(\lambda) : L(\mu)] = 0$ if $\lambda \notin \widehat{\mathcal{W}}(\mu).\mu$.*

Note that the above statement refers to the non-deformed objects (i.e. we have $A = \mathbb{C}$ here). However, for its proof we need the deformation theory developed in the main body of this paper. So let A be an arbitrary local deformation domain with residue field k. As a first step we study the restricted block decomposition.

6.1. **The restricted block decomposition.** Let $\sim_A^{\mathrm{res}}$ be the relation on the set $\widehat{\mathfrak{h}}^\star$ that is generated by setting $\lambda \sim_A^{\mathrm{res}} \mu$ if there is some open subset $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ such that $L_A(\mu)$ is isomorphic to a subquotient of $\overline{P}_A^{\mathcal{J}}(\lambda)$. For an equivalence class $\Lambda \in \widehat{\mathfrak{h}}^\star / \sim_A^{\mathrm{res}}$ let $\overline{\mathcal{O}}_{A,\Lambda} \subset \overline{\mathcal{O}}_A$ be the full subcategory that contains all objects M that have the property that if $L_A(\lambda)$ occurs as a subquotient of M, then $\lambda \in \Lambda$. Then the well-known classical arguments yield the following.

Theorem 6.2. *The functor*

$$
\begin{aligned}
\prod_{\Lambda \in \widehat{\mathfrak{h}}^\star / \sim_A^{\mathrm{res}}} \overline{\mathcal{O}}_{A,\Lambda} &\to \overline{\mathcal{O}}_A \\
(M_\Lambda) &\mapsto \bigoplus M_\Lambda,
\end{aligned}
$$

is an equivalence of categories.

6.2. **Critical restricted equivalence classes.** Let us denote by $\overline{\cdot}\colon \widehat{\mathfrak{h}}^\star \to \mathfrak{h}^\star$, $\lambda \mapsto \overline{\lambda}$, the map that is dual to the inclusion $\mathfrak{h} \to \widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}D \oplus \mathbb{C}K$. Note that $\overline{\delta} = \overline{\kappa} = 0$.

Suppose that $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A^{\mathrm{res}}$ is a critical equivalence class. We define the corresponding set of integral finite roots and the finite integral Weyl group by

$$
\begin{aligned}
R_A(\Lambda) &:= \{\alpha \in R \mid 2(\overline{\lambda} + \overline{\rho}, \alpha)_k \in \mathbb{Z}(\alpha, \alpha)_k\},\\
\mathcal{W}_A(\Lambda) &:= \langle s_\alpha \mid \alpha \in R_A(\Lambda)\rangle \subset \mathcal{W}.
\end{aligned}
$$

Lemma 6.3. *Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A^{\mathrm{res}}$ be a critical equivalence class. Then we have*

$$\overline{\Lambda} = \mathcal{W}_A(\Lambda).\overline{\lambda}$$

for all $\lambda \in \Lambda$.

Proof. The inclusion $\overline{\Lambda} \subset \mathcal{W}_A(\Lambda).\overline{\Lambda}$ is clear, for the other direction use finite Verma modules. □

6.3. **Generic and subgeneric equivalence classes.**

Definition 6.4. Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A^{\mathrm{res}}$ be a critical equivalence class. We call Λ

(1) *generic*, if $\overline{\Lambda} \subset \mathfrak{h}^\star$ contains exactly one element,
(2) *subgeneric*, if $\overline{\Lambda} \subset \mathfrak{h}^\star$ contains exactly two elements.

We call a critical element $\lambda \in \widehat{\mathfrak{h}}^\star$ *generic* (*subgeneric*, resp.) if it is contained in a generic (subgeneric, resp.) equivalence class.

Let $\lambda \in \widehat{\mathfrak{h}}^\star$ be a critical element and suppose that $s_\alpha.\lambda \neq \lambda$. Then we have $s_\alpha.\lambda > \lambda$ if and only if $s_{-\alpha+\delta}.\lambda < \lambda$. We define $\alpha \uparrow \lambda$ to be the element in the set $\{s_\alpha.\lambda, s_{-\alpha+\delta}.\lambda\}$ that is bigger than λ. Note that this defines a bijection $\alpha \uparrow \cdot\colon \widehat{\mathfrak{h}}^\star \to \widehat{\mathfrak{h}}^\star$.

Part (1) of the following Theorem is a direct consequence of Theorem 5.9 and Theorem 4.8 in [Fr05] (which states that a generic restricted Verma module is simple) and part (2) is a direct consequence of Theorem 5.9 and Theorem 5.9 in [AF08].

Theorem 6.5. *Let A be a local deformation algebra. Let $\Lambda \in \widehat{\mathfrak{h}}^\star/\sim_A^{\mathrm{res}}$ be a critical equivalence class and fix $\lambda \in \Lambda$. Let $\mathcal{J} \subset \widehat{\mathfrak{h}}^\star$ be open and bounded.*

(1) *Suppose that λ is generic. Then*

$$\overline{P}_A^{\mathcal{J}}(\lambda) \cong \overline{\Delta}_A(\lambda)$$

if $\mathcal{J}$ contains λ.

(2) *Suppose that* $\lambda \in \mathcal{J}$ *is subgeneric and suppose that* $\overline{\Lambda} = \{\overline{\lambda}, s_\alpha.\overline{\lambda}\}$ *for some* $\alpha \in R$. *Then there is a non-split short exact sequence*

$$0 \to \overline{\Delta}_A(\alpha \uparrow \lambda) \to \overline{P}_A^{\mathcal{J}}(\lambda) \to \overline{\Delta}_A(\lambda) \to 0$$

if $\mathcal{J}$ *contains* λ *and* $\alpha \uparrow \lambda$.

Corollary 6.6. *Let* $\Lambda \in \widehat{\mathfrak{h}}^\star/_{\sim_A^{\mathrm{res}}}$ *be an equivalence class.*

(1) *If* Λ *is generic, then* Λ *contains only one element.*
(2) *If* Λ *is subgeneric, then there is some* $\alpha \in R(\Lambda)$ *such that* $\Lambda \subset \widehat{\mathfrak{h}}^\star$ *is an orbit under the action of the subgroup* $\widehat{\mathcal{W}}_\alpha \subset \widehat{\mathcal{W}}$ *that is generated by the reflections* $s_{\alpha+n\delta}$ *for* $n \in \mathbb{Z}$.

For any prime ideal $\mathfrak{p}$ in a commutative ring A we denote by $A_\mathfrak{p}$ the corresponding localization. If A is a domain, then we have canonical inclusions $A \subset A_\mathfrak{p} \subset A_{(0)} = Q$.

Proposition 6.7. *Let* A *be a local deformation domain and let* $\Lambda \subset \widehat{\mathfrak{h}}^\star$ *be a critical equivalence class with respect to* $\sim_A^{\mathrm{res}}$. *If* $A = \bigcap_\mathfrak{p} A_\mathfrak{p}$, *then* $\sim_A^{\mathrm{res}}$ *is the common refinement of all the relations* $\sim_{A_\mathfrak{p}}^{\mathrm{res}}$ *for prime ideals* $\mathfrak{p}$ *of height one, i.e.* $\sim_A^{\mathrm{res}}$ *is generated by the relations* $\lambda \sim_A^{\mathrm{res}} \mu$ *if there is a prime ideal* $\mathfrak{p} \subset A$ *of height one such that* $\lambda \sim_{A_\mathfrak{p}}^{\mathrm{res}} \mu$.

Proof. Let us denote by $\sim'$ the common refinement of the relations $\sim_{A_\mathfrak{p}}^{\mathrm{res}}$ for prime ideals of height one. It suffices to show that if $\lambda, \mu \in \widehat{\mathfrak{h}}^\star$ are critical such that there is an open bounded subset $\mathcal{J}$ of $\widehat{\mathfrak{h}}^\star$ and $(\overline{P}_A^{\mathcal{J}}(\lambda) : \overline{\Delta}_A(\mu)) \neq 0$, then $\lambda \sim' \mu$.

Let us consider the object $\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A Q$. It is an object in $\overline{\mathcal{O}}_Q^{\mathcal{J}}$ and admits a restricted Verma flag. We are going to apply the decomposition result in Theorem 6.2 for the categories $\overline{\mathcal{O}}_Q$ and $\overline{\mathcal{O}}_{A_\mathfrak{p}}$.

Let $\Lambda' \subset \widehat{\mathfrak{h}}^\star$ be the equivalence class under $\sim'$ that contains λ. As Λ' is a union of equivalence classes for $\sim_Q^{\mathrm{res}}$ we can find a unique decomposition

$$\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A Q = X \oplus Y,$$

where X and Y are objects in $\overline{\mathcal{O}}_Q$ admitting a restricted Verma flag such that for all $\nu \in \widehat{\mathfrak{h}}^\star$ we have

$$(X : \overline{\Delta}_Q(\nu)) \neq 0 : \nu \in \Lambda',$$
$$(Y : \overline{\Delta}_Q(\nu)) \neq 0 : \nu \notin \Lambda'.$$

Let $\mathfrak{p} \subset A$ be a prime ideal of height one. As $\sim'$ is coarser than $\sim_{A_\mathfrak{p}}^{\mathrm{res}}$, we deduce that the inclusion $\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A_\mathfrak{p} \to \overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A Q = X \oplus Y$

induces a direct sum decomposition

$$\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A_{\mathfrak{p}} = \left(\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A_{\mathfrak{p}} \cap X\right) \oplus \left(\overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A_{\mathfrak{p}} \cap Y\right).$$

Now each weight space of $\overline{P}_A^{\mathcal{J}}(\lambda)$ is a free A-module of finite rank and we deduce that

$$\overline{P}_A^{\mathcal{J}}(\lambda) = \bigcap_{\mathfrak{p}} \overline{P}_A^{\mathcal{J}}(\lambda) \otimes_A A_{\mathfrak{p}},$$

where the intersection is taken over all prime ideals of height one. Hence we get an induced decomposition

$$\overline{P}_A^{\mathcal{J}}(\lambda) = \left(\overline{P}_A^{\mathcal{J}}(\lambda) \cap X\right) \oplus \left(\overline{P}_A^{\mathcal{J}}(\lambda) \cap Y\right).$$

As $\overline{P}_A^{\mathcal{J}}(\lambda)$ is indecomposable, and since $X \neq 0$ (since the restricted Verma module $\overline{\Delta}_Q(\lambda)$ certainly occurs in X), we get $Y = 0$, i.e. all restricted Verma subquotients of $\overline{P}_A^{\mathcal{J}}(\lambda)$ have highest weights in Λ'. Hence $\sim_A^{\text{res}} = \sim'$. □

6.4. **A special deformation.** Let $\widetilde{S}$ be the localization of S at the maximal ideal $S \cdot \mathfrak{h}$. This is a local deformation domain with the obvious S-algebra structure. Its quotient field is $\mathbb{C} = \widetilde{S}/\widetilde{S} \cdot \mathfrak{h}$ and the category $\mathcal{O}_{\mathbb{C}}$ is identified with the usual category $\mathcal{O}$. For each prime ideal $\mathfrak{p} \subset \widetilde{S}$ we denote by $\widetilde{S}_{\mathfrak{p}}$ the localization of $\widetilde{S}$ at $\mathfrak{p}$. We let $\widetilde{Q} = \widetilde{S}_{(0)}$ be the quotient field of $\widetilde{S}$.

Proposition 6.8. *Let $\mathfrak{p} \subset \widetilde{S}$ be a prime ideal of height one and let $\Lambda \subset \widehat{\mathfrak{h}}^\star$ be an equivalence class for $\sim_{\mathfrak{p}}^{\text{res}}$.*

(1) *If $\alpha^\vee \notin \mathfrak{p}$ for all $\alpha \in R$, then Λ is generic.*
(2) *If $\alpha^\vee \in \mathfrak{p}$ for some $\alpha \in R$, then Λ is either generic or subgeneric. In both cases we have $R(\Lambda) \subset \{\alpha, -\alpha\}$.*

Proof. Let k be the residue field of $\widetilde{S}_{\mathfrak{p}}$. For any $\beta \in R$ we have $(\lambda + \tau, \beta)_k = (\lambda, \beta)_k + (\tau, \beta)_k \in \mathbb{C} \oplus \mathfrak{h}$. Hence we have $2(\lambda + \tau, \beta) \in \mathbb{Z}(\beta, \beta)$ if and only if $2(\lambda, \beta) \in \mathbb{Z}(\beta, \beta)$ and $(\tau, \beta)_k = 0$. The latter equality implies $\beta = \pm\alpha$. From this we deduce both of the above statements. □

Now we can prove our main result, Theorem 6.1.

Proof. We show that $\lambda \sim_{\mathbb{C}}^{\text{res}} \mu$ implies $\mu \in \widehat{\mathcal{W}}(\lambda).\lambda$. Note that by definition we have $\sim_{\mathbb{C}}^{\text{res}} = \sim_{\widetilde{S}}^{\text{res}}$. As we have $\widetilde{S} = \bigcap_{\mathfrak{p}} \widetilde{S}_{\mathfrak{p}}$, where the intersection is taken over all prime ideals of height one, we can apply Proposition

6.7 and deduce that $\sim^{\rm res}_{\widetilde{S}}$ is the common refinement of all $\sim^{\rm res}_{\widetilde{S}_{\mathfrak{p}}}$. Proposition 6.8 shows that the equivalence classes of $\sim^{\rm res}_{\widetilde{S}_{\mathfrak{p}}}$ are either generic or subgeneric. But those we determined in Corollary 6.6: They are orbits under a certain subgroup $\widehat{\mathcal{W}}_\alpha$ of $\widehat{\mathcal{W}}(\lambda)$. Hence λ and μ must be contained in a common $\widehat{\mathcal{W}}(\lambda)$-orbit. □

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