Finiteness Properties of Chevalley Groups over a Polynomial Ring over a Finite Field

Kai-Uwe Bux Ralf Gramlich Stefan Witzel

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Abstract

It is known from work by H. Abels and P. Abramenko that for a classical $\mathbb{F}_q$-group $\mathcal{G}$ of rank n the arithmetic lattice $\mathcal{G}(\mathbb{F}_q[t])$ of $\mathbb{F}_q[t]$-points is of type F_{n-1} provided that q is large enough. We show that the statement is true without any assumption on q and for any isotropic, absolutely almost simple group $\mathcal{G}$ defined over $\mathbb{F}_q$.

Let k be a global function field and let $\mathcal{G}$ be a connected, noncommutative, absolutely almost simple k-group of positive rank. Let $\mathcal{O}_S$ be the ring of S-integers in k. For each place $p \in S$, there is an associated euclidean building X_p acted upon by $\mathcal{G}(k_p) \supseteq \mathcal{G}(\mathcal{O}_S)$. The dimension of the building X_p is the local rank of $\mathcal{G}$ at the place p. In [BW07, Theorem 1.2], K. Wortman and the first author have shown that $\mathcal{G}(\mathcal{O}_S)$ is not of type F_d, where d is the sum of local ranks of $\mathcal{G}$ at the places in S. This settles the negative part of the following:

Rank Conjecture (see [Behr98] or [BW07]). *The group $\mathcal{G}(\mathcal{O}_S)$ is of type F_{d-1} but not of type F_d.*

Results in favor of the rank conjecture include [Stuh80] in which U. Stuhler shows that it holds for $\mathrm{SL}_2(\mathcal{O}_S)$. This result has been generalized by Wortman and the first author in [BW08] to arbitrary $\mathcal{G}$ of global rank one. Concerning higher ranks, H. Abels [Abel91] and P. Abramenko [Abra87] independently proved the rank conjecture for $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ provided that q is sufficiently large. Abramenko has better bounds, but they still grow exponentially with n. In [Abra96], Abramenko has verified the rank conjecture for $\mathcal{G}(\mathbb{F}_q[t])$ for classical groups $\mathcal{G}$ again under the hypothesis that q is sufficiently large with a bound depending only on the rank of $\mathcal{G}$. We generalize the last result.

Theorem A. *Let $\mathcal{G}$ be an absolutely almost simple $\mathbb{F}_q$-group of rank $n \geq 1$. Then the group $\mathcal{G}(\mathbb{F}_q[t])$ is of type F_{n-1} but not of type F_n.*

Most of our argument will be purely geometric, and we shall deduce Theorem A from:

Theorem B. *Let $X := (X_+, X_-)$ be a thick, locally finite, irreducible euclidean twin building, and let G be a group acting strongly transitively on X such that any pair (c_+, c_-) of chambers has a finite stabilizer. Fix a chamber $d_- \in X_-$. Then the stabilizer $\Gamma := \mathrm{Stab}_G(d_-)$ is of type F_{n-1} but not of type F_n where $n := \dim(X_+)$.*

We shall reduce Theorem A to Theorem B in Section 10. We start in Section 1 with an outline of the geometric argument for Theorem B. Here, we also indicate why we have to confine ourselves to euclidean twin buildings. This is a trade off: Abramenko [Abra96] was able to allow the compact hyperbolic case but had to exclude small values of q. The strategy proposed by H. Behr in [Behr04] to eliminate the restriction on q has not yet been carried out successfully.

Only the positive part of the claims are new. This is clear for Theorem A as the fact that $\mathcal{G}(\mathbb{F}_q[t])$ is not of type F_n follows from [BW07, Theorem 1.2]. For Theorem B, one can use the classification of euclidean buildings [Weis08, Chapter 28] and Margulis' Arithmeticity Theorem [Marg91, Chapter IX] to see that Γ is an S-arithmetic group. Since p-adic euclidean buildings do not arise in twin buildings [MuVM09, Corollary 18], the group Γ is arithmetic over a global function field and [BW07, Theorem 1.2] applies.

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1 A geometric heuristic for Theorem B

The method of [Abra96] is to use a filtration on the positive partner in a twin building induced by numerical codistance to a fixed chamber in the negative building. We shall use a metric version of this idea. Our first task, therefore, will be to define the notion of metric codistance that relates to ordinary metric distance as W-valued codistance relates to W-valued distance.

For the following, we assume that we are given a twin building $X = (X_+, X_-)$ that has a geometric realization. In particular, for any two points $x_\pm$ and $y_\pm$ in $X_\pm$, we have a metric distance between them, denoted by $\mu(x_\pm, y_\pm)$. We say for short, that X is a metric twin building. The metric structure is obvious for euclidean buildings, where apartments are euclidean spaces. The compact hyperbolic case is also straightforward: here apartments have the geometry of hyperbolic space. In general, one could use the Davis realization turning the building, and each apartment, into a CAT(0) space. Note that under any such interpretation, isometries in twin buildings induce isometries in the geometric sense.

Observation 1.1. *Let Σ and Σ' be two twin apartments both containing the chamber c. Then, the retraction [AbBr08, Exercise 5.185]*

$$\rho := \rho_{\Sigma,c} : X \longrightarrow \Sigma$$

restricts to an isometry from Σ' to Σ. Moreover, this isometry fixes the intersection $\Sigma \cap \Sigma'$ pointwise. (Recall [AbBr08, Exercise 5.163] that isometries of twin apartments also preserve codistances and in particular the opposition relation.) □

Lemma 1.2. *Let τ_+ be a cell in X_+, and let τ_- be a cell in X_-. Any two twin apartments Σ and Σ' that both contain τ_+ and τ_- are isometric via an isometry fixing τ_+ and τ_- pointwise.*

Proof. Let $c_\pm$ be a chamber in $\Sigma_\pm$ containing $\tau_\pm$, and let $c'_\pm$ be a chamber in $\Sigma'_\pm$ also containing $\tau_\pm$. Let Σ'' be a twin apartment containing c_+ and c'_-. Then Observation 1.1 implies that Σ'' is isometric to Σ on the one hand and to Σ' on the other via isometries fixing τ_+ and τ_- pointwise. □

Let x_+ be a point in X_+ and x_- be a point in X_-. The <u>metric codistance</u> $\mu_*(x_+, x_-)$ is the metric distance of x_+ to the point $x_-^{\mathrm{op}_\Sigma}$ opposite to x_- in some twin apartment Σ containing x_+ and x_-. Note that the metric codistance does not depend on the choice of Σ, since any other twin apartment Σ' also containing x_+ and x_- is isometric to Σ via an isometry fixing x_+ and x_-. Since isometries respect opposition, the isometry takes $x_-^{\mathrm{op}_\Sigma}$ to $x_-^{\mathrm{op}_\Sigma}$.

Observation 1.3. *Let $\Sigma = (\Sigma_+, \Sigma_-)$ be any twin apartment containing x_+ and x_-. On Σ_+, the metric codistance to x_- agrees with the metric distance to $x_-^{\mathrm{op}_\Sigma}$. In particular, level sets of the metric codistance to x_- inside Σ_+ are round spheres.* □

We define the <u>geodesic ray</u> from x_+ to x_- as:

$$[x_+, x_-) := \{ y_+ \in X_+ \mid \mu_*(x_+, x_-) + \mu(x_+, y_+) = \mu_*(y_+, x_-) \}$$

Rays are meaningful mostly if the metric structure on X is euclidean:

Proposition 1.4. *Assume that $X = (X_+, X_-)$ is euclidean and $\mu_*(x_+, x_-) \neq 0$. Then, the geodesic ray $[x_+, x_-)$ truly is a geodesic ray in the euclidean building X_+ – at least inside the star of the cell carrying x_+.*

Proof. Assume that X is euclidean and that $\mu_*(x_+, x_-) \neq 0$. Let $\Sigma = (\Sigma_+, \Sigma_-)$ be any twin apartment containing x_+ and x_-. Then the intersection $\Sigma_+ \cap [x_+, x_-)$ is the geodesic ray through x_+ in Σ_+ pointing away from $x_-^{\mathrm{op}_\Sigma}$.

Moreover, if Σ' is any other twin apartment containing x_+ and x_-, then any isometry from Σ to Σ' fixing x_+ and x_- takes the ray $\Sigma_+ \cap [x_+, x_-)$ to the ray $\Sigma'_+ \cap [x_+, x_-)$.

Since $[x_+, x_-)$ intersects each twin apartment around x_+ and x_- in a ray, the set is a union of rays issuing from x_+. We have to show that there is no branching at x_+.

We want to argue by contradiction: assuming that branching happens, find a twin apartment containing initial segments of both rays. It remains to prove that such a twin apartment can be chosen to also contain x_-.

Let $\mathcal{R}_+$ be the residue in X_+ around the carrier of x_+, let c_- be a chamber in Σ_- containing x_-, and let c_+ be the projection of c_- into $\mathcal{R}_+$. Note that c_+ lies in Σ_+ and contains the initial segment of $[x_+, x_-) \cap \Sigma_+$. By [AbBr08, Exercise 5.186], for any chamber c'_+ in $\mathcal{R}_+$, there is a twin apartment containing c'_+, c_+, and $c_- \ni x_-$. In particular, this holds for any chamber c'_+ in Σ'_+ containing the initial segment of $[x_+, x_-) \cap \Sigma'_+$. Thus, we found a twin apartment, in which we could observe the branching of $[x_+, x_-)$. □

Observation 1.5. *The proof also shows that the initial segment of $[x_+, x_-)$ is contained in any chamber of the residue $\mathcal{R}_+$ around x_+ that arises as a projection of a chamber containing x_-.*

Remark 1.6. One can also observe that in a euclidean twin building, $[x_+, x_-)$ can nowhere branch and actually is a geodesic ray in the metric sense. Also, this holds true not just in euclidean twin buildings but in any CAT(0) twin building where apartments have the unique extension property for geodesic segments and all proper residues are spherical.

Equipped with these geometric tools, we can make a first (albeit failing) attempt to prove Theorem B. Recall the setup: we fix a thick, locally finite euclidean building $X = (X_+, X_-)$, a group G acting strongly transitively on X such that stabilizers of twin chambers (c_+, c_-) are finite, and a chamber d_- in X_-. We want to determine the finiteness properties of the stabilizer $\Gamma := \mathrm{Stab}_G(d_-)$. We also fix a point $z_- \in d_-$.

We shall study the action of Γ on the euclidean building X_+. Note that the stabilizer in Γ of each chamber c_+ in X_+ is finite. Thus, all cell stabilizers of the Γ-action are finite. As G acts strongly transitively, Γ acts transitively on the set of points $\{x_+ \in X_+ \mid \mu_*(x_+, z_-) = 0\}$. For any positive real number R, let $X_+(R)$ be the maximal subcomplex of X_+ contained in the subset $\{x_+ \in X_+ \mid \mu_*(x_+, z_-) \leq R\}$. It follows from transitivity, that Γ acts cocompactly on $X_+(R)$ since X_+ is locally finite.

Should it turn out that $X_+(R)$ is $(n-2)$-connected for some R, [Bro87, Proposition 1.1 and Proposition 3.1] would imply that Γ is of type F_{n-1}. For the topological analysis, we use the Morse function

$$\begin{array}{rccc} h' : & X_+ & \longrightarrow & \mathbb{R}_{\geq 0} \\ & x_+ & \mapsto & \mu_*(x_+, z_-) \end{array}$$

The building X_+ is contractible. By standard arguments from combinatorial Morse theory, connectivity properties of sublevel complexes can be deduced from the same connectivity properties of descending links. It remains to argue that descending links are $(n-2)$-connected.

Here, Observation 1.4 is useful. It says that the geodesic ray $[x_+, z_-)$ determines a direction in $\mathrm{lk}(x_+)$, which we may think of as the gradient $\nabla_{x_+} h'$. Because of Observation 1.3, the descending link at sufficiently high vertices should be the set of all those directions in $\mathrm{lk}(x_+)$ that span an obtuse angle with $\nabla_{x_+}h'$ (gradient criterion): large spheres are almost flat. Such subcomplexes of the spherical building $\mathrm{lk}(x_+)$ are called hemisphere complexes and sufficiently highly connected by results of B. Schulz [Schu05].

This strategy *almost* succeeds. Generically, descending links are hemisphere complexes of the right dimension and connectivity. However, there are certain bad regions in X_+ (inside a single apartment, they look like corridors) where the descending link is not correctly detected by the gradient criterion. Thus, the main technical difficulty will be to perturb the Morse function h' so that the descending links inside bad corridors are improved without destroying connectivity of descending links in other regions.

The remainder of this paper is organized as follows. After some preliminaries on zonotopes in Section 2 and on subcomplexes of spherical buildings in Section 3, we define in Section 4 a primary Morse function h (the height), which is Γ-invariant and has cocompact sublevel complexes. The main results about this improved version of h' are Proposition 4.4, which ensures that gradients for h can be defined, and Proposition 4.6, which says that the descending links with respect to h are never inconsistent with the gradients of h. However, we cannot avoid that there are h-horizontal edges (i.e., edges whose endpoints are of equal height). In order to break ties, we introduce a secondary and even a tertiary Morse function in Section 7 using the depth borrowed from [BW08] and described in Section 6. We analyse the descending links arising from this Morse function in Section 8. The final two sections are devoted to the proofs of Theorem B and Theorem A, respectively.

We note that Abramenko has examples in the compact hyperbolic case showing that one cannot expect the analogously defined stabilizer Γ to be of type F_{n-1} in this case. Thus, it might be useful to conclude this section with an explanation why our strategy breaks down in the compact hyperbolic case.

Roughly, we filter the building X_+ by metric codistance to z_-. Inside apartments, the filtration coincides with the filtration by metric balls centered at some point (opposite to z_-). Huge balls approximate horoballs. It is a feature of euclidean space that horospheres are hyperplanes. So if a huge circle runs through a vertex in a euclidean Coxeter complex, it will cut its star roughly in half. This is reason why we hope that, at least where the Morse function is large, we can expect relative links of our filtration to look like hemisphere complexes in spherical buildings.

This heuristic fails in the hyperbolic case. A horosphere through a vertex of a hyperbolic Coxeter complex need not split its star into two halves of equal size. Figure 1 shows a horoball and the center vertex lies on its boundary circle. Only two vertices in the link are inside the horoball. This explains why relative links in the hyperbolic case are genuinely smaller (at least for filtrations based on the metric approach).

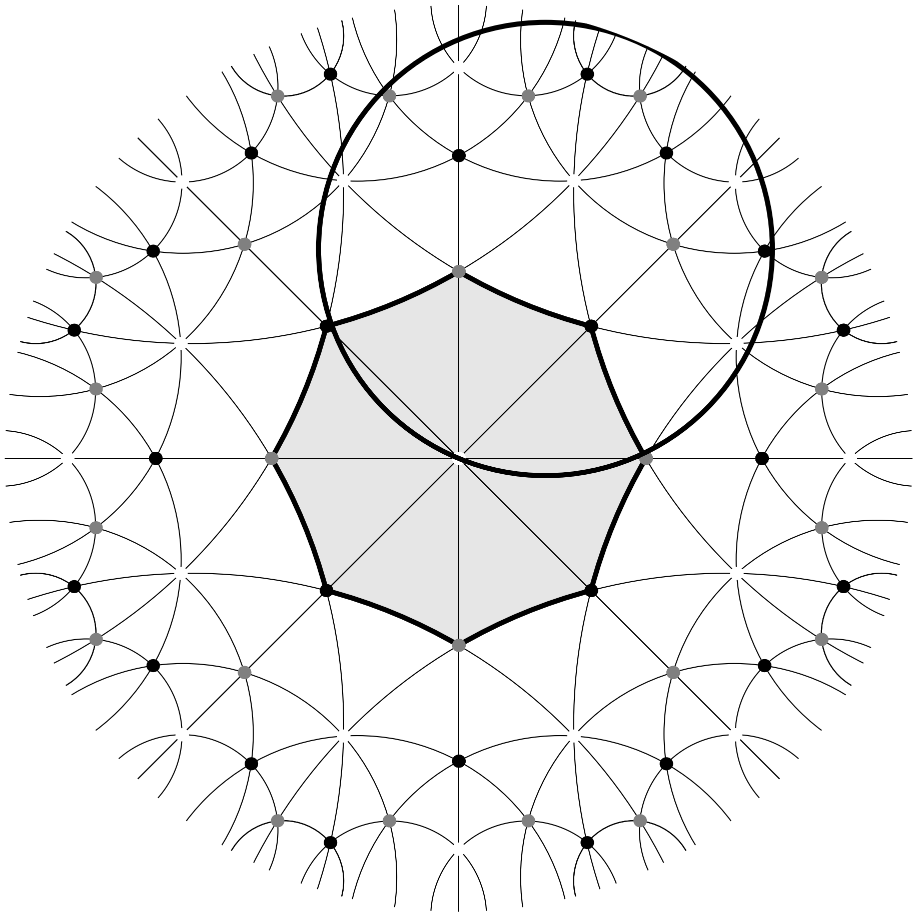


Figure 1: a star in a hyperbolic Coxeter complex

2 Some euclidean geometry

Throughout this section, $\mathbb{E}$ is a fixed euclidean space with inner product $\langle -, - \rangle$ and origin 0. Also, we fix a finite reflection group W with 0 as a global fixed point.

Let F be a face of some (convex and compact) polytope $Z \subset \mathbb{E}$. The normal cone

$$\mathrm{N}(F) := \left\{ \mathbf{n} \in \mathbb{E} \,\middle|\, \langle \mathbf{n}, \mathbf{z} \rangle = \max_{\mathbf{z}' \in Z} \langle \mathbf{n}, \mathbf{z}' \rangle \text{ for all } \mathbf{z} \in F \right\}$$

is the set of all $\mathbf{n} \in \mathbb{E}$ such that the function $\langle \mathbf{n}, - \rangle$ restricted to Z assumes its maximum on the points in F. It is a closed convex cone. We think of the vectors in $\mathrm{N}(F)$ as directions because, for any point $\mathbf{x} \in \mathbb{E}$, the closest point projection onto Z satisfies

$$\mathrm{pr}_Z(\mathbf{x}) \in F \quad \text{if and only if} \quad \mathbf{x} - \mathrm{pr}_Z(\mathbf{x}) \in \mathrm{N}(F) \,.$$

Our reason to consider an inner product space instead of using a vector space and its dual is to ease phrasing of claims such as the following:

Lemma 2.1. *Let Z be W-invariant and let F be a face of Z. For any $\mathbf{f} \in F$ and $\mathbf{n} \in \mathrm{N}(F)$, there is no wall with respect to W that separates two of the three vectors $\mathbf{f}$, $\mathbf{n}$, and $\mathbf{f} + \mathbf{n}$.*

Proof. It suffices to show that no wall H with respect to W separates $\mathbf{f}$ from $\mathbf{n}$. So assume to the contrary that H does separate $\mathbf{f}$ from $\mathbf{n}$. Since Z is W-invariant, the point $\sigma_H(\mathbf{f})$ lies in Z. Note that $\sigma_H(\mathbf{f}) - \mathbf{f}$ is orthogonal to H and lies on the same side as $\mathbf{n}$ (the side opposite to $\mathbf{f}$). It follows that $\langle \mathbf{n}, \sigma_H(\mathbf{f}) - \mathbf{f} \rangle > 0$, whence

$$\langle \mathbf{n}, \sigma_H(\mathbf{f}) \rangle = \langle \mathbf{n}, \mathbf{f} \rangle + \langle \mathbf{n}, \sigma_H(\mathbf{f}) - \mathbf{f} \rangle > \langle \mathbf{n}, \mathbf{f} \rangle.$$

This is a contradiction as $\langle \mathbf{n}, \mathbf{f} \rangle$ is the maximum value of the function $\langle \mathbf{n}, - \rangle$ on Z. □

For a finite set $D \subset \mathbb{E}$, the convex, compact polytope

$$Z(D) := \left\{ \sum_{\mathbf{d} \in D} \alpha_{\mathbf{d}} \mathbf{d} \;\middle|\; 0 \leq \alpha_{\mathbf{d}} \leq 1 \text{ for all } \mathbf{d} \in D \right\}$$

is called the zonotope spanned by D. This construction ensures:

Observation 2.2. *Through every point $\mathbf{z} \in Z(D)$ and every $\mathbf{d} \in D$, there is a line segment parallel to $[0, \mathbf{d}]$ inside $Z(D)$.* □

The faces of a zonotope are translated zonotopes. More precisely:

Lemma 2.3. *For any direction $\mathbf{n} \in \mathbb{E}$, the faces of $Z(D)$ maximal with respect to the property of being orthogonal to $\mathbf{n}$ are translates of the zonotope $Z(D_{\mathbf{n}})$ where $D_{\mathbf{n}} := \{ \mathbf{d} \in D \mid \langle \mathbf{n}, \mathbf{d} \rangle = 0 \}$. More precisely, for any point $\mathbf{x} \in \mathbb{E}$, the maximal face of $Z(D)$ containing $\mathrm{pr}_{Z(D)}(\mathbf{x})$ and orthogonal to $\mathbf{n} := \mathbf{x} - \mathrm{pr}_{Z(D)}(\mathbf{x})$ is given by*

$$F_{\mathbf{x}} := \left(\sum_{\substack{\mathbf{d} \in D \\ \langle \mathbf{n}, \mathbf{d} \rangle > 0}} \mathbf{d} \right) + Z(D_{\mathbf{n}}) \, .$$

Proof. We just observed that $F_{\mathbf{x}}$ consists precisely of those points in $Z(D)$ on which the function $\langle \mathbf{n}, - \rangle$ restricted to $Z(D)$ is maximal. It follows that $F_{\mathbf{x}}$ is a face, that is is maximal among the faces orthogonal to $\mathbf{n}$, and that $\mathbf{x} - \mathrm{pr}_{Z(D)}(\mathbf{x}) = \mathbf{n} \in \mathrm{N}(F_{\mathbf{x}})$. The last statement implies $\mathrm{pr}_{Z(D)}(\mathbf{x}) \in F_{\mathbf{x}}$. □

We close this section with a consideration of the distance to Z, i.e., the function $\mu(Z, -)$. As Z is convex, so is associated distance function. In particular, for any simplex $\sigma \subset \mathbb{E}$ the subset of points farthest away from Z contains a vertex. For points in σ closest to Z, we have:

Proposition 2.4. *Let $\sigma \subset \mathbb{E}$ be a simplex. Suppose that D is sufficiently rich: $\mathbf{v} - \mathbf{v}' \in D$, for any two vertices $\mathbf{v}$ and $\mathbf{v}'$ in σ. Then the subset of points on σ closest to $Z := Z(D)$ contains a vertex.*

Proof. Consider a point $\mathbf{x} \in \sigma$ minimizing the distance to Z. Without loss of generality, we may assume that $\mathbf{x}$ lies in the relative interior of σ. Then, $\mathbf{n} := \mathbf{x} - \mathrm{pr}_Z(\mathbf{x})$ is orthogonal to σ. The point $\mathrm{pr}_Z(\mathbf{x})$ lies in Z and by richness of D, there is a parallel translate of the zonotope $Z(D')$ through $\mathrm{pr}_Z(\mathbf{x})$ that lies entirely in Z. Here $D' \subseteq D$ consists of all vectors representing differences of adjacent vertices in σ. Note that $\mathrm{pr}_Z(\mathbf{x})$ does not have to lie in the center of this translate.

Nonetheless, it follows that at least one vertex of σ is within distance at most $|\mathbf{n}|$ from this parallel copy of $Z(D')$. The claim follows by choice of $\mathbf{x}$ as a point on σ of lowest height. □

3 Some subcomplexes of spherical buildings

To deduce finiteness properties, we use the well-established technique of filtering a complex upon which the group acts. The main task, as usual, is to control the homotopy type of relative links that arise in the filtration. In this section, we collect the results concerning connectivity properties of those subcomplexes of spherical buildings that we will encounter.

Let M be euclidean or hyperbolic space or a round sphere. We call an intersection of a non-empty family of closed half-spaces (or hemispheres in the latter case) <u>demi-convex</u>. We call a subset of M <u>fat</u> if it has non-empty interior. Note that a proper open convex subset of M is contained in an open hemisphere.

Observation 3.1. *Let $A \subset M$ be fat and demi-convex and let $B \subset M$ be proper, open, and convex. If A and B intersect, then $A \setminus B$ strongly deformation retracts onto the boundary part $\partial(A) \setminus B$.*

Proof. Note that B intersects the interior of A since every boundary point of the convex set A is an accumulation point of interior points because A is fat. Choose x in the intersection. Note that A is star-like with regard to x, and the geodesic projection away from x restricts to the deformation retraction we need. □

Iterated application of the same projection trick yields:

Proposition 3.2. *Suppose that L is a geometric CW-complex, i.e., its cells carry a spherical, euclidean, or hyperbolic structure in which they are demi-convex (i.e., each cell is an intersection of half-spaces in the model space). Let B be an open subset of L that intersects each cell in a convex set. Then there is a strong deformation retraction*

$$\rho_L : L \setminus B \longrightarrow (L \setminus B)^\circ =: L^B$$

of $L \setminus B$ onto its maximal subcomplex.

Proof. First, we assume that L has finite dimension. Let τ be a maximal cell of L. If $\tau \subseteq B$, the cell τ does not intersect $L \setminus B$ and we do not need to do anything. If τ avoids B, the map ρ must be the identity on τ. Otherwise, let x be a point in the intersection $\tau \cap B$ chosen in the relative interior of τ. Projecting away from x, as in Observation 3.1, deformation retracts $\tau \setminus B$ onto $\partial(\tau) \setminus B$. The maps constructed for two maximal cells agree on their intersection. Hence we can paste all these maps together to get a deformation retraction of $L \setminus B$ onto $L' \setminus B$ where L' is L with the interiors of all maximal cells intersecting B removed.

Now, L' has other maximal cells, which might intersect B. Using the same construction for L', we obtain another deformation retraction $L' \setminus B \to L'' \setminus B$. We keep going, removing more and more cells intersecting B. Since the dimension of L is finite, the process terminates after finitely many steps. The composition of the maps thus obtained is the strong deformation retraction from $L \setminus B$ onto L^B. This proves the claim for finite dimensional L.

Note that the construction is local: what it does on a cell is only determined by the intersection of this cell with the set B. Hence, the deformation retraction is compatible with subcomplexes. More precisely, if K is a subcomplex of L, then the deformation retractions ρ_L and ρ_K from above are constructed such that ρ_K is the restriction of ρ_L to K. It follows that the pair $(L \setminus B, K \setminus B)$ is homotopy equivalent to (L^B, K^B). Applying this observation to pairs of skeleta, it the claim follows by standard arguments in the case that L has infinite dimension. □

Let Δ be a spherical building. We regard Δ as a metric space with the angular metric. So each apartment is a round sphere of radius 1. When Δ is a finite building, the topology induced by the metric agrees with the weak topology it carries as a simplicial complex. For infinite buildings, both topologies differ and we will use the weak topology throughout for the building and all its subcomplexes.

Proposition 3.3. *Let Δ be a spherical building and fix a chamber C in Δ. Let $B \subset \Delta$ be a subset such that, for any apartment Σ containing C the intersection $B \cap \Sigma$ is a proper, open, and convex subset of the sphere Σ. Then the space $Y := \Delta \setminus B$ and its maximal subcomplex Δ^B are both $(\dim(\Delta) - 1)$-connected. The complex Δ^B has dimension $\dim(\Delta)$ and hence is spherical of this dimension.*

Remark 3.4. Using $B = \emptyset$ in Proposition 3.3, we obtain the Solomon-Tits Theorem as a special case. Satz 3.5 of [Schu05], whose proof inspired the argument given below, is the special case where B is open, convex, and of diameter strictly less than π.

Proof of Proposition 3.3. We observe first that Proposition 3.2 implies that the subset Y and its maximal subcomplex Δ^B are homotopy equivalent. Therefore, it suffices to prove that Y is $(\dim(\Delta) - 1)$-connected.

We have to contract spheres of dimensions up to $\dim(\Delta) - 1$. Let $S \subseteq Y$ be such a sphere. Since S is compact in Δ, it is covered by a finite family of apartments and we can apply [v.He03, Lemma 3.5]: there is a finite sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ such that (a) each Σ_i contains C, (b) the sphere S is contained in the union $\bigcup_i \Sigma_i$, and most importantly, (c) for each $i \geq 2$ the intersection $\Sigma_i \cap (\Sigma_1 \cup \cdots \cup \Sigma_{i-1})$ is a union of closed half-apartments, each of which contains C. Put $L_i := \Sigma_1 \cup \cdots \cup \Sigma_i$ and observe that L_i is obtained from L_{i-1} by gluing in the closure $A_i := \overline{\Sigma_i \setminus (\Sigma_1 \cup \cdots \cup \Sigma_{i-1})}$ along the boundary $\partial(A_i)$ of A_i in Σ_i. Note that A_i is fat and demi-convex.

Now, we can build $L_k \setminus B$ inductively. We begin with $L_1 \setminus B$, which is contractible. The space $L_i \setminus B$ is obtained from $L_{i-1} \setminus B$ by gluing in $A_i \setminus B$ along $\partial(A_i) \setminus B$. If A_i and B are disjoint, this is a cellular extension of dimension $\dim(\Delta)$ as A_i is fat. Otherwise, Observation 3.1 implies that $A_i \setminus B$ deformation retracts onto $\partial(A_i) \setminus B$, whence $L_i \setminus B$ and $L_{i-1} \setminus B$ are homotopy equivalent in this case. In the end, the sphere S can be contracted inside $L_k \setminus B$. □

An interesting special case, also already noted in [Schu05], is obtained when B is chosen as the open $\frac{\pi}{2}$-ball around a fixed point $n \in \Delta$, which we think of as the north pole. Then the complex $\Delta^{\geq\frac{\pi}{2}}(n) := \Delta^B$ is a closed hemisphere complex and $\dim(\Delta)$-spherical by Proposition 3.3. The argument fails badly if B is chosen as the *closed ball* of radius $\frac{\pi}{2}$ around n. In fact, the open hemisphere complex $\Delta^{>\frac{\pi}{2}}(n)$ spanned by all vertices avoiding the closed ball B generally is not $\dim(\Delta)$-spherical: the dimension of $\Delta^{>\frac{\pi}{2}}(n)$ might be too small. A main result of Schulz is to show that this is the only obstruction.

Proposition 3.5 (see [Schu05, Page 27]). *The open hemisphere complex $\Delta^{>\frac{\pi}{2}}(n)$ is spherical of dimension $\dim(\Delta_{\mathrm{ver}})$. If Δ is thick, then neither open nor closed hemisphere complexes in Δ are contractible.*

The subcomplex $\Delta_{\mathrm{ver}}(n)$ is defined as follows: The equator $\Delta^{=\frac{\pi}{2}}(n)$ is the subcomplex spanned by those points in Δ of distance $\frac{\pi}{2}$ from n. Recall that Δ decomposes as a join of unique irreducible factors. The horizontal part $\Delta_{\mathrm{hor}}(n)$ is the join of all factors fully contained in the equator. The complex $\Delta_{\mathrm{ver}}(n)$ is the join of the other irreducible factors. In particular,

$$\Delta = \Delta_{\mathrm{hor}}(n) * \Delta_{\mathrm{ver}}(n)\,. \tag{1}$$

4 The primary Morse function

We now begin the proof of Theorem B proper. First, let us fix a euclidean twin building $X = (X_+, X_-)$ and a chamber $d_- \in X_-$. We also fix a point $z_- \in d_-$.

Let $\tilde{\Sigma}$ be the euclidean Coxeter complex upon which the apartments in X_+ are modeled. We denote by $\mathbb{E}$ the underlying euclidean space where the origin 0 shall be a special vertex in $\tilde{\Sigma}$. We let W denote the spherical Weyl group generated by the walls through 0. Finally, for this section, we choose a finite subset $D \subset \mathbb{E}$. For the moment, we just require that it is invariant under the finite group W, but in the course of this work, we shall impose stronger restrictions upon D. The W-invariance is inherited by the zonotope $Z := Z(D)$. In particular, Lemma 2.1 applies.

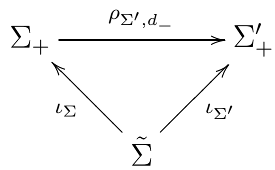
Consider any twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ where Σ_- contains the chamber d_-. Then, there is a unique point $z_-^{\mathrm{op}_\Sigma} \in \Sigma_+$ opposite to z_-. Identifying Σ_+ with $\mathbb{E}$, we define

$$Z_\Sigma := z_-^{\mathrm{op}_\Sigma} + Z$$

Observe that Z_Σ is well-defined since any identification of Σ_+ with $\mathbb{E}$ that respects the structure of the underlying labeled Coxeter complexes gives rise to the same set Z_Σ since Z is W-invariant. The same consideration shows:

Observation 4.1. *Let $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be another twin apartment containing d_-. Then the isometry $\rho = \rho_{\Sigma', d_-} : \Sigma \to \Sigma'$ from Observation 1.1 takes Z_Σ to $Z_{\Sigma'}$.* □

Remark 4.2. One can do even better. The positive partners in twin apartments as above are parameterized by the chambers in X_+ opposite to d_-. It follows that the isometries fixing d_- from Observation 1.1 are canonical and allow one to identify all such apartments with $\tilde{\Sigma}$ in a compatible way, i.e., there is a famillly of isometries $\iota_\Sigma : \tilde{\Sigma} \to \Sigma_+$ such that the diagrams



all commute. Finally, identifying $\tilde{\Sigma}$ and $\mathbb{E}$, the zonotope Z_Σ would be well-defined even for zonotopes $Z \subset \mathbb{E}$ that are not W-invariant.

Given any point $x_+ \in X_+$, we chose a twin apartment $\Sigma = (X_+, X_-)$ containing x_+ and d_-. We define the height of x_+ to be the metric distance

$$h(x_+) := \mu(Z_\Sigma, x_+)$$

from x_+ to the convex, compact polytope Z_Σ. Observation 4.1 implies that $h(x_+)$ is independent of the chosen twin apartment Σ.

Observation 4.3. *Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment containing d_-. The restriction of h to Σ_+ is a convex function as the metric distance to a convex compact polytope. In particular, at least one highest point on any simplex is a vertex.* □

Turning to gradients of h, we can first define the ray $[x_+, \infty)_\Sigma$ relative to the twin apartment Σ as the direction of the geodesic ray in Σ_+ through x_+ away from $\mathrm{pr}_{Z_\Sigma}(x_+)$. Here, we assume that $h(x_+) > 0$.

Proposition 4.4. *Let $x_+ \in \Sigma_+$ be a point with $h(x_+) > 0$. Let $\Sigma = (\Sigma_+, \Sigma_-)$ and $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be two twin apartments both containing x_+ and d_-. Then the two geodesic rays $[x_+, \infty)_\Sigma$ and $[x_+, \infty)_{\Sigma'}$ coincide.*

Proof. Let c_+ be the projection of d_- into the residue around the carrier of x_+. Since Σ contains x_+ and d_-, it also contains c_+. By Observation 1.5, the gradient $\nabla_{x_+} h'$ is carried by c_+.

We identify Σ_+ with $\mathbb{E}$ so that $Z_\Sigma \subset \Sigma_+$ corresponds to $Z \subset \mathbb{E}$. In particular, $\mathrm{op}_\Sigma\, z_-$ corresponds to 0. Let $\mathbf{x} \in \mathbb{E}$ be the vector corresponding to $x_+ \in \Sigma_+$. Then $\mathbf{f} := \mathrm{pr}_Z(\mathbf{x})$ corresponds to $\mathrm{pr}_{Z_\Sigma}(x_+)$. Put $\mathbf{n} := \mathbf{x} - \mathbf{f}$. By Lemma 2.1, any W-chamber containing $\mathbf{x} = \mathbf{n} + \mathbf{f}$ also contains $\mathbf{n}$. Note that the vector $\mathbf{x}$ is parallel to $\nabla_{x_+} h'$. Hence, the chamber c_+ also contains an initial segment of $[x_+, \infty)_\Sigma$, which is parallel to $\mathbf{n}$.

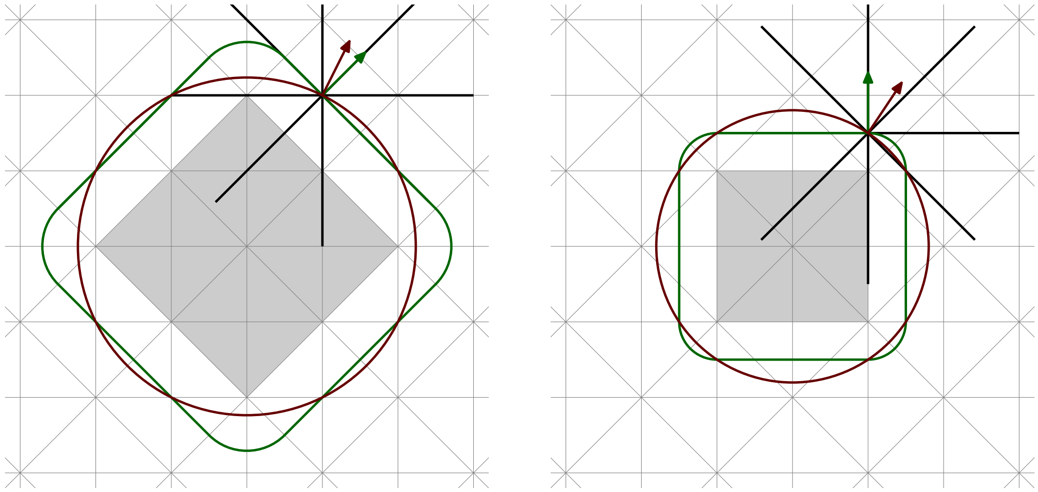


Figure 2: comparing $\nabla_{x_+}h'$ and $\nabla_{x_+}h$

In each figure, the circle is a level set for h', and the perpendicular arrow indicates $\nabla_{x_+}h'$. The other closed curve is a level set for h. The W-chamber for the dihedral group of order 8 based at x_+ and containing $\nabla_{x_+}h'$ also contains $\nabla_{x_+}h$. Consequently, any chamber of the underlying Coxeter complex supporting $\nabla_{x_+}h'$ also carries $\nabla_{x_+}h$. Note that x_+ does not need to be a vertex.

The same argument applies to Σ'. Hence both apartments contain c_+ and at least the initial segments of $[x_+,\infty)_\Sigma$ and $[x_+,\infty)_{\Sigma'}$ agree.

The intersection of the two rays is a closed set (in each of the rays) and for any point y_+ in the intersection, we have:

$$[y_+,\infty)_\Sigma \subseteq [x_+,\infty)_\Sigma \qquad \text{and} \qquad [y_+,\infty)_{\Sigma'} \subseteq [x_+,\infty)_{\Sigma'}$$

Therefore, the argument from above also shows that the rays $[x_+,\infty)_\Sigma$ and $[x_+,\infty)_{\Sigma'}$ share a little open segment around y_+. Thus, the intersection of the two rays is open in each of the rays. Hence the intersection is connected and the claim follows. □

Proposition 4.4 implies that we can define the flow line $[x_+,\infty)$ through x_+ as the geodesic ray $[x_+,\infty)_\Sigma$ in any twin apartment containing x_+. We define the gradient $\nabla_{x_+}h$ as the direction of $[x_+,\infty)$ at x_+.

A good deal of our analysis regards the interplay of the simplicial structure on X_+ and the height h. We start with the following:

Observation 4.5. *Let τ_+ be a simplex in X_+ and let x_+ be a point in τ_+. If $\nabla_{x_+}h$ is orthogonal to τ_+, then x_+ is a point where h assumes its minimum value on τ_+.*

Proof. We choose a twin apartment $\Sigma = (\Sigma_+,\Sigma_-)$ containing τ_+ and d_-. In Σ_+, h is given as the function $\mu(Z_\Sigma,-)$. The claim now follows since Z_Σ is convex and τ_+ spans an affine subspace. □

Recall that $\mathbb{E}$ is identified with the model apartment $\tilde{\Sigma}$. We say that a finite subset $D \subset \mathbb{E}$ is almost rich if it is W-invariant and contains all differences of adjacent

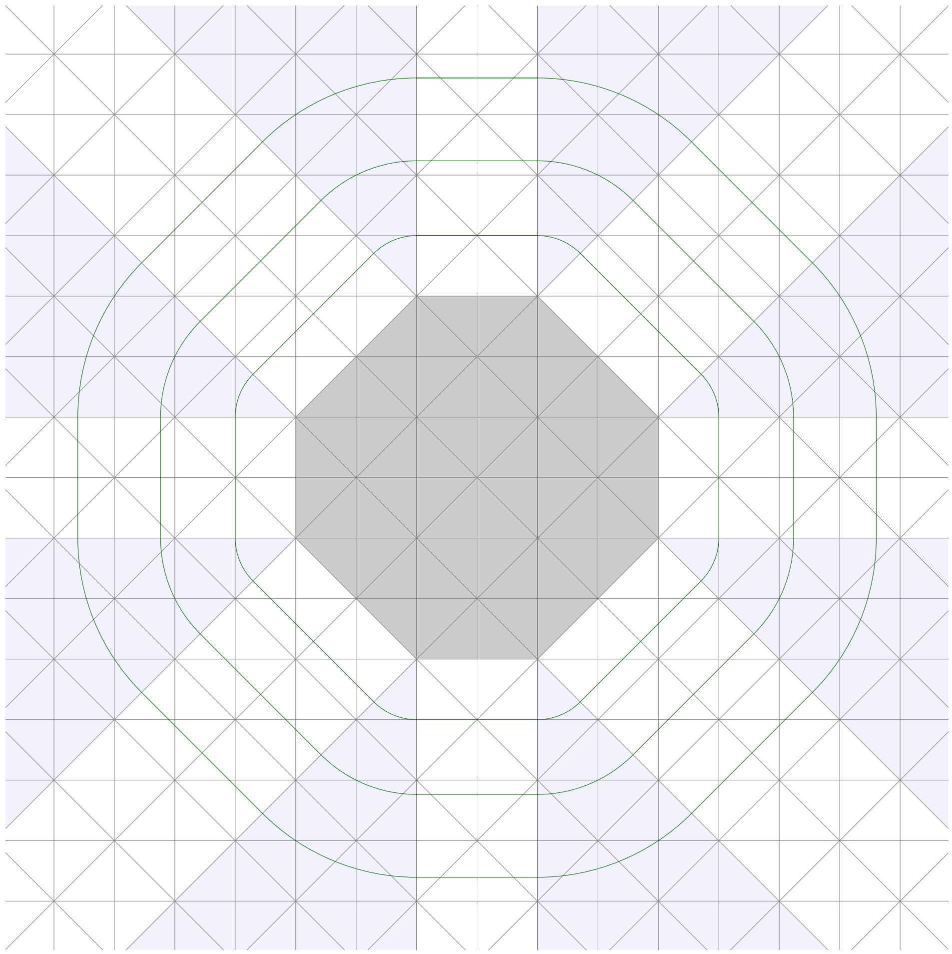


Figure 3: an almost rich zonotope for $\tilde{\mathsf{B}}_2$

This figure illustrated some h-level. The lightly shaded areas and the white corridors are the normal cones for the zonotope.

vertices in $\tilde{\Sigma}$. If D is almost rich, h does not suffer from the obvious deficiencies of h'.

Proposition 4.6. *Assume that D is almost rich. Let ϵ be an edge in X_+ connecting the vertices ν and ν'. Then the following hold:*

1. *The function h is monotonic on ϵ.*
2. *The angle $\angle_\nu(\epsilon, \nabla_\nu h) > \frac{\pi}{2}$ if and only if $h(\nu) > h(\nu')$.*

Proof. The function h is convex and attains its minimum at a boundary point by Proposition 2.4. This proves the first claim. If $\angle_\nu(\epsilon, \nabla_\nu h) \neq \frac{\pi}{2}$ the height h changes when one moves from the vertex ν infinitesimally into the edge ϵ. If the angle is obtuse, the height decreases; if the angle is acute, the height increases. Accordingly, ν must be the highest or lowest point, respectively, as h is monotonic on ϵ. Observation 4.5 covers the remaining case that $\nabla_\nu h$ is orthogonal to ϵ. □

5 Simplices of constant height

A simplex τ_+ in X_+ is <u>h-horizontal</u> if h restricts to a constant function on τ_+.

Observation 5.1. *Let τ_+ be an h-horizontal simplex. Then all flow lines issuing in τ_+ are pairwise parallel and orthogonal to τ_+.*

Proof. The claim is clear for points in the relative interior of τ_+. It follows for points on the boundary by continuity. □

It follows that we can talk about the <u>gradient</u> $\nabla_{\tau_+} h$ for an h-horizontal simplices τ_+, which we identify with the gradient at the barycenter of τ_+.

Let x_+ be a point carried by the simplex τ_+ in X_+. We think of the <u>link</u> $\mathrm{lk}(x_+)$ as the space of directions issuing from x_+. The link of $\mathrm{lk}(\tau_+)$ is the space of directions at x_+ orthogonal to τ_+. It does not depend on the particular point x_+ carried by τ_+. Both links are spherical buildings and can be regarded as metric spaces via the angular metric. The point link splits as a spherical join

$$\mathrm{lk}(x_+) = \partial(\tau_+) * \mathrm{lk}(\tau_+) \tag{2}$$

where $\partial(\tau_+)$ is the round sphere of directions at x_+ that do not leave τ_+. Note that $\partial(\tau_+)$ has an obvious simplicial structure being the boundary of a simplex. The spherical building $\mathrm{lk}(\tau_+)$ also has a simplicial structure, whose face poset corresponds to the poset of cofaces of τ_+.

Now we specialize to the case where σ_+ is a horizontal simplex in X_+. For concreteness, we think of $\mathrm{lk}(\sigma_+)$ as centered at the barycenter σ_+° of σ_+. Note that $\nabla_{\sigma_+^\circ} h \in \mathrm{lk}(\sigma_+)$ as a consequence of Observation 5.1. We regard the distinguished point $\nabla_{\sigma_+^\circ} h$ as the north pole in the spherical building $\mathrm{lk}(\sigma_+)$. Thus for any horizontal simplex, the link decomposes as in (1):

$$\mathrm{lk}(\sigma_+) = \mathrm{lk}_{\mathrm{hor}}(\sigma_+) * \mathrm{lk}_{\mathrm{ver}}(\sigma_+) \tag{3}$$

of the link into the horizontal and vertical part of $\mathrm{lk}(\sigma_+)$ with respect to the north pole $\nabla_{\sigma_+^\circ} h$. We call the horizontal part $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$ the <u>horizontal link</u>, and we call the vertical part $\mathrm{lk}_{\mathrm{ver}}(\sigma_+)$ the <u>vertical link</u> of σ_+. Beware that the vertical link can contain equatorial simplices; and consequently, not every h-horizontal coface of σ_+ defines a simplex in $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$.

6 The depth of horizontal simplices

Horizontal simplices are the main obstacle for the analysis of the cocompact filtration of X_+ by height. We will use the method of [BW08] to cope with this difficulty. Here, we mostly follow [BW08, Section 5].

Let σ_+ be an h-horizontal simplex in X_+. By Observation 5.1, the flow lines starting in σ_+ are pairwise parallel geodesic rays in X_+ and therefore, they define a point $\mathrm{e}(\sigma_+)$ in the spherical building at infinity. Let β be a Busemann function centered at that point. Since the flow lines are orthogonal to σ_+, the function β is constant on σ_+, i.e., the simplex σ_+ is β-horizontal. The notion of the horizontal and vertical link of σ_+ defined above agree with the notions in [BW08, Section 5], whence we can use some results therein directly.

Lemma 6.1. *For any h-horizontal simplex σ_+, there is a unique face $\sigma_+^{\min}$ such that for any proper face $\sigma'_+ < \sigma_+$, the following equivalence holds*

$$\sigma_+ \setminus \sigma'_+ \in \mathrm{lk}_{\mathrm{hor}}(\sigma'_+) \quad \text{if and only if} \quad \sigma_+^{\min} \leq \sigma'_+.$$

Proof. The statement is [BW08, Lemma 5.2] for β-horizontal simplices. Since we can find a Busemann function β, for which σ_+ and all its faces are β-horizontal, the claim follows. □

In the same way, the following lemma is an immediate consequence of [BW08, Observation 5.3].

Lemma 6.2. *Suppose $\sigma_+^{\min} \leq \sigma'_+ \leq \sigma_+$. Then $\sigma_+^{\min} = {\sigma'_+}^{\min}$.* □

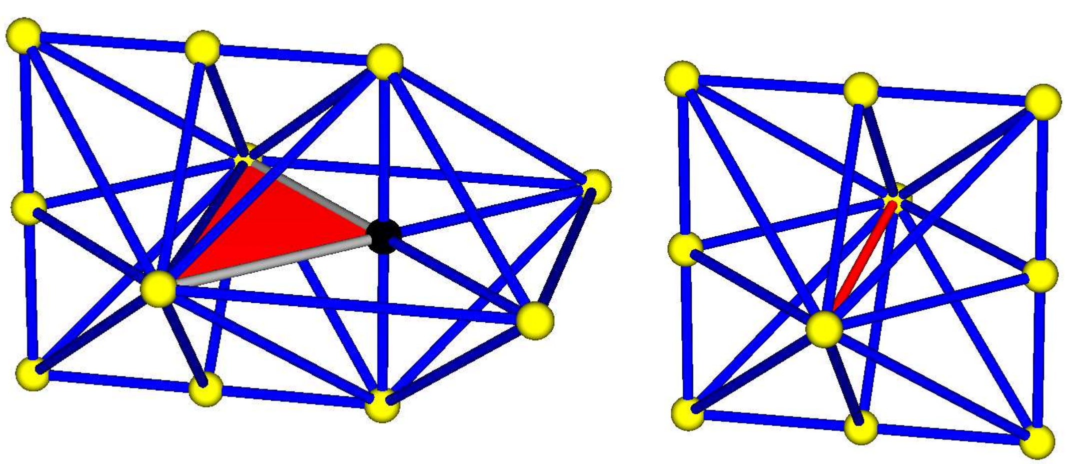


Figure 4: the face $\sigma_+^{\min}$

Both figures take place inside the Coxeter complex $\tilde{\mathsf{B}}_3$. In the picture on the left hand side, the black vertex is the face $\sigma_+^{\min}$ of the horizontal solidly colored 2-simplex σ_+. The two edges of σ_+ containing $\sigma_+^{\min}$ illustrate Lemma 6.2. In the picture on the right, the horizontal simplex σ_+ is the center edge. Here, we have $\sigma_+ = \sigma_+^{\min}$.

For any two h-horizontal simplices σ_+ and σ'_+, we define going up as

$$\sigma'_+ \nearrow \sigma_+ \quad :\Longleftrightarrow \quad \sigma'_+ = \sigma_+^{\min} \neq \sigma_+$$

and going down as

$$\sigma_+ \searrow \sigma'_+ \quad :\Longleftrightarrow \quad \sigma_+^{\min} \not\leq \sigma'_+ < \sigma_+.$$

We define a move as either going up or going down.

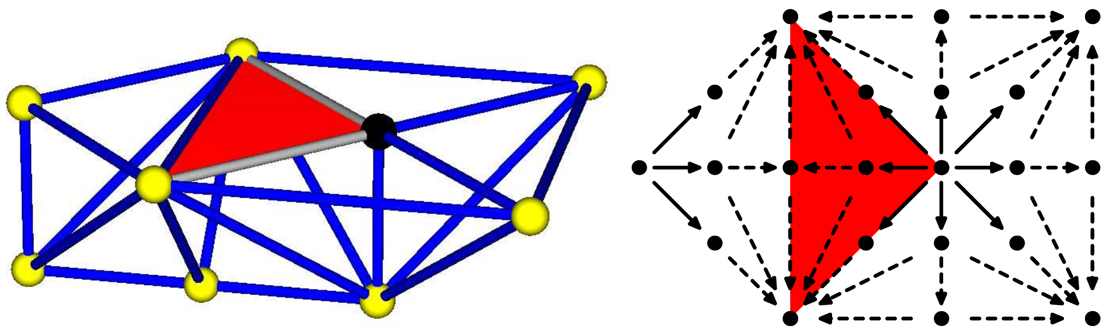


Figure 5: sequences of moves

This figure continues Figure 4; it also takes place inside a Coxeter complex of type $\tilde{\mathsf{B}}_3$. It shows a possible patch of horizontal 2-simplices. Each dot in the picture on the right represents a simplex; for orientation, one horizontal 2-simplex has been filled in. Arrows indicate moves: solid arrows represent going up, whereas dashed arrows represent going down. Note that there are no moves between triangles and short edges.

Observation 6.3. *If there is a move from σ_+ to σ'_+, then either σ_+ is a face of σ'_+ or vice versa σ'_+ is a face of σ_+. In either case, we have $\mathrm{e}(\sigma_+) = \mathrm{e}\big(\sigma'_+\big)$.* □

Proposition 6.4. *There is a uniform upper bound, depending only on the building X_+, on the length of any sequence of moves.*

Proof. Consider a sequence of moves starting at the horizontal simplex σ_+. By Observation 6.3, for any simplex σ'_+ encountered along that sequence, we have $\mathrm{e}(\sigma_+) = \mathrm{e}\big(\sigma'_+\big)$. Let β be a Busemann function centered at $\mathrm{e}(\sigma_+)$. It follows that the given sequence of moves is a "β-sequence" consisting of "β-moves" as considered in [BW08, Proposition 5.4], where the existence of a uniform bound (depending only on the dimension of X_+) on the length of any such sequence is proved. □

We define the <u>depth</u> $\mathrm{dp}(\sigma_+)$ of an h-horizontal simplex σ_+ as the maximum length of a sequence of moves starting at σ_+.

Remark 6.5. Since not every β-move is a legal h-move, the depth as defined here will generally be lower than the depth used in [BW08].

7 Subdividing along horizontal simplices

In this section, we assume that the zonotope Z is defined via an almost rich set D. Then, being connected by an h-horizontal edge is an equivalence relation on the vertices of a given simplex τ_+ and the equivalence classes correspond to the maximal horizontal faces of τ_+. This fact allows us to mimic the subdivision rule used in [BW08, Section 6].

Let X_+° be the simplicial subdivision of X_+ whose vertices are precisely the barycenters σ_+° of h-horizontal simplices σ_+. More precisely, we subdivide each horizontal simplex barycentrically; any simplex is the simplicial join of its maximal horizontal faces and carries the induced subdivision. Note that this rule of subdividing is compatible with inclusion of faces.

Observation 7.1. *The building X_+ is a flag complex, and so is the subdivision X_+°.* □

As in [BW08, Observation 6.1], the link $\mathrm{lk}\big(\sigma_+^\circ\big)$ of a vertex $\sigma_+^\circ \in X_+^\circ$ corresponding to the horizontal simplex σ_+ decomposes as a join

$$\mathrm{lk}\big(\sigma_+^\circ\big) = \mathrm{lk}_\partial\big(\sigma_+^\circ\big) * \mathrm{lk}_\delta\big(\sigma_+^\circ\big) \tag{4}$$

where $\mathrm{lk}_\partial\big(\sigma_+^\circ\big)$ is the barycentric subdivision of $\partial(\sigma_+)$ and $\mathrm{lk}_\delta\big(\sigma_+^\circ\big)$ is the induced subdivision of $\mathrm{lk}(\sigma_+)$. The latter again decomposes as the join of its horizontal and vertical parts (see 3); however, this decomposition is not (immediately) compatible with the simplicial structure on $\mathrm{lk}_\delta\big(\sigma_+^\circ\big)$.

The building X_+ carries the geometric structure of a euclidean simplicial complex. In particular, barycenters have a geometric meaning and simplices in X_+° could be regarded as honest subsets of simplices in X_+. Regarded this way, h already *is* a function on X_+°. However, we only use h to define values on the vertices in X_+°, which is not a problem at all since vertices in X_+° correspond to simplices in X_+ on which h is already constant.

We use the following Morse function on the vertices of X_+°:

$$\begin{aligned} h^\circ : {X_+^\circ}^{(0)} &\longrightarrow \mathbb{R}\times\mathbb{R}\times\mathbb{R} \\ \sigma_+^\circ &\mapsto (h(\sigma_+)\,, \mathrm{dp}(\sigma_+)\,, \dim(\sigma_+)) \end{aligned}$$

We use lexicographic comparison to order $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

Observation 7.2. *There are no h°-horizontal edges, i.e.: if σ_+° and τ_+° are adjacent in X_+°, then $h^\circ\big(\sigma_+^\circ\big) \neq h^\circ\big(\tau_+^\circ\big)$.*

Proof. If there is an edge between σ_+° and τ_+° and $h(\sigma_+) = h(\tau_+)$, then σ_+ is a face of τ_+ or vice versa. In either case, the dimensions differ. □

8 Descending links

Let σ_+° be a vertex in X_+°. The descending link $\mathrm{lk}^\downarrow\big(\sigma_+^\circ\big)$ is the subcomplex of $\mathrm{lk}\big(\sigma_+^\circ\big)$ spanned by all neighbors of σ_+° of strictly lower h°-height. In this section, we shall determine the connectivity of descending links. The argument follows closely the blueprint in [BW08, Section 6].

Observation 8.1. *Links in flag complexes are flag complexes. Hence the decomposition (4) induces a decomposition*

$$\mathrm{lk}^\downarrow\big(\sigma_+^\circ\big) = \mathrm{lk}_\partial^\downarrow\big(\sigma_+^\circ\big) * \mathrm{lk}_\delta^\downarrow\big(\sigma_+^\circ\big) \tag{5}$$

of the descending link. Here $\mathrm{lk}_\partial^\downarrow\big(\sigma_+^\circ\big) := \mathrm{lk}_\partial\big(\sigma_+^\circ\big) \cap \mathrm{lk}^\downarrow\big(\sigma_+^\circ\big)$ *and* $\mathrm{lk}_\delta^\downarrow\big(\sigma_+^\circ\big) := \mathrm{lk}_\delta\big(\sigma_+^\circ\big) \cap \mathrm{lk}^\downarrow\big(\sigma_+^\circ\big)$. □

Lemma 8.2. *For any horizontal simplex σ_+ with $\sigma_+ \neq \sigma_+^{\min}$, the descending link $\mathrm{lk}^{\downarrow}\big(\sigma_+^{\circ}\big)$ is contractible.*

Proof. The argument is the same as in [BW08, Lemma 6.5]. We reproduce the main steps for the convenience of the reader.

Because of the decomposition (5), it suffices to show that $\mathrm{lk}_{\partial}^{\downarrow}\big(\sigma_+^{\circ}\big)$ is contractible. Recall that $\mathrm{lk}_{\partial}\big(\sigma_+^{\circ}\big)$ is the barycentric subdivision of $\partial(\sigma_+)$. As σ_+ is horizontal, h will not decide among its faces whether they define descending directions.

As $\sigma_+^{\min} \neq \sigma_+$, one can go up $\sigma_+^{\min} \nearrow \sigma_+$, whence $\mathrm{dp}\big(\sigma_+^{\min}\big) > \mathrm{dp}(\sigma_+)$. It follows that $\sigma_+^{\min}$ does not define a vertex in $\mathrm{lk}_{\partial}^{\downarrow}\big(\sigma_+^{\circ}\big)$.

For any face $\sigma'_+ < \sigma_+$ with $\sigma_+^{\min} \not\leq \sigma'_+$, one can go down $\sigma_+ \searrow \sigma'_+$. Hence $\mathrm{dp}(\sigma_+) > \mathrm{dp}\big(\sigma'_+\big)$ and the barycenter of σ'_+ belongs to $\mathrm{lk}^{\downarrow}\big(\sigma_+^{\circ}\big)$.

We do not know what happens to barycenters of faces σ'_+ with $\sigma_+^{\min} < \sigma'_+$. Nonetheless, the face part $\mathrm{lk}_{\partial}\big(\sigma_+^{\circ}\big)$ is a sphere. Its descending part is obtained by puncturing the sphere at the barycenter of $\sigma_+^{\min}$. The cofaces of $\sigma_+^{\min}$ may or may not be non-descending, but that only determines the size of the puncture: $\sigma_+^{\min}$ will provide a cone point for the hole; and its complement, the descending face part $\mathrm{lk}_{\partial}^{\downarrow}\big(\sigma_+^{\circ}\big)$, is contractible. □

Now we turn to the descending links of vertices σ_+° with $\sigma_+^{\min} = \sigma_+$. We begin with the face part.

Lemma 8.3. *For any horizontal simplex σ_+ with $\sigma_+ = \sigma_+^{\min}$, the face part $\mathrm{lk}_{\partial}\big(\sigma_+^{\circ}\big)$ is completely descending.*

Proof. For any proper face $\sigma'_+ < \sigma_+$, one goes down $\sigma_+ \searrow \sigma'_+$ since $\sigma_+^{\min} = \sigma_+ \not\leq \sigma'_+ < \sigma_+$. Thus, $\mathrm{dp}(\sigma_+) > \mathrm{dp}\big(\sigma'_+\big)$. □

The coface part is more difficult. Ignoring some subdivision issues for the moment, it decomposes as the join of its vertical and horizontal part. It turns out that the depth of simplices behaves oppositely in both regions.

Lemma 8.4. *Let σ_+ be an h-horizontal simplex and let τ_+ be an h-horizontal coface of σ_+, i.e., assume $\sigma_+ < \tau_+$. If $\sigma_+^{\min} = \sigma_+$ and $\tau_+ \setminus \sigma_+$ is a simplex not completely contained in the horizontal part $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$, then $\mathrm{dp}(\tau_+) > \mathrm{dp}(\sigma_+)$. In particular, the conclusion holds if $\tau_+ \setminus \sigma_+$ lies in the vertical link $\mathrm{lk}_{\mathrm{ver}}(\sigma_+)$.*

Proof. By Lemma 6.1, we have $\tau_+^{\min} \not\leq \sigma_+ < \tau_+$. Hence we go down $\tau_+ \searrow \sigma_+$, whence $\mathrm{dp}(\tau_+) > \mathrm{dp}(\sigma_+)$. □

Lemma 8.5. *Let σ_+ be an h-horizontal simplex and let τ_+ be an h-horizontal coface of σ_+, i.e., assume $\sigma_+ < \tau_+$. If $\sigma_+^{\min} = \sigma_+$ and $\tau_+ \setminus \sigma_+$ is a simplex completely contained in the horizontal part $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$, then $\mathrm{dp}(\sigma_+) > \mathrm{dp}(\tau_+)$.*

Proof. By Lemma 6.1, we have $\tau_+^{\min} \leq \sigma_+ < \tau_+$. By Lemma 6.2, we conclude $\tau_+^{\min} = \sigma_+^{\min}$. Hence we go up $\sigma_+ \nearrow \tau_+$, whence $\mathrm{dp}(\sigma_+) > \mathrm{dp}(\tau_+)$. □

To summarize: in the vertical link, the depth is always biased toward being ascending; in the horizontal link, the depth is always biased in favor of descent.

This also helps with the subdivision issues alluded to above. The link $\mathrm{lk}_\delta(\sigma_+^\circ)$ is a subdivision (inherited from X_+°) of $\mathrm{lk}(\sigma_+)$, and the later decomposes as a join of its horizontal and vertical part. This decomposition is *not* compatible with the subdivision. The problem is that there can be an h-horizontal coface τ_+ of σ_+ that has vertices in the horizontal and vertical part. The barycentric subdivision of τ_+ does not respect the join decomposition. However, Lemma 8.4 implies that for such τ_+, the barycenter τ_+° cannot belong to $\mathrm{lk}_\delta^\downarrow(\sigma_+^\circ)$. Thus, we see the following:

Lemma 8.6. *The coface part $\mathrm{lk}_\delta^\downarrow(\sigma_+^\circ)$ of the descending link decomposes as a join:*

$$\mathrm{lk}_\delta^\downarrow(\sigma_+^\circ) = \mathrm{lk}_{\mathrm{hor}}^{\circ\downarrow}(\sigma_+) * \mathrm{lk}_{\mathrm{ver}}^{\circ\downarrow}(\sigma_+)$$

Here $\mathrm{lk}_{\mathrm{hor}}^{\circ\downarrow}(\sigma_+)$ is the subdivision of $\mathrm{lk}_{\mathrm{hor}}^{\downarrow}(\sigma_+)$ obtained from barycentrically subdividing h-horizontal simplices; and $\mathrm{lk}_{\mathrm{ver}}^{\circ\downarrow}(\sigma_+)$ is defined mutatis mutandis. □

Knowing the depth component of h°, we turn to the height component. It behaves nicely on the vertical component.

Lemma 8.7. *Let σ_+ be an h-horizontal simplex. The* <u>*down set*</u>

$$L^\downarrow := \{\nu \in \mathrm{lk}(\sigma_+) \mid h(\nu) < h(\sigma_+)\}$$

of strictly lower vertices in its link spans the open hemisphere complex in $\mathrm{lk}_{\mathrm{ver}}(\sigma_+)$ with respect to the north pole $\nabla_{\sigma_+^\circ} h$.

Proof. First note that a vertex $\nu \in \mathrm{lk}(\sigma_+)$ below σ_+ lies in the vertical link $\mathrm{lk}(\sigma_+)$: if it was horizontal, Observation 4.5 would rule out $h(\nu) < h(\sigma_+)$.

Now let ν be any vertex of $\mathrm{lk}_{\mathrm{ver}}(\sigma_+)$. Fix a vertex ν' in σ_+ and let ϵ be the edge from ν' to ν. Using Proposition 4.6 and the standing assumption that D is almost rich, we have

$$\angle_{\nu'}(\epsilon, \nabla_{\nu'} h) > \frac{\pi}{2} \quad \text{if and only if} \quad \nu \in L^\downarrow.$$

Now note that $\nabla_{\nu'} h$ and $\nabla_{\sigma_+^\circ} h$ are parallel and perpendicular to the line connecting ν' to the barycenter σ_+°. Thus, if ϵ' is the straight line segment connecting σ_+° to ν, we have

$$\angle_{\sigma_+^\circ}(\epsilon', \nabla_{\sigma_+^\circ} h) > \frac{\pi}{2} \quad \text{if and only if} \quad \nu \in L^\downarrow.$$

Hence $L^\downarrow$ is the vertex set of the open hemisphere complex with respect to the north pole $\nabla_{\sigma_+^\circ} h$ in $\mathrm{lk}(\sigma_+)$. □

It remains to study the height h on horizontal links $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$. Recall that the W-invariant subset $D \subset \mathbb{E}$ is almost rich, i.e., it contains all vectors connecting adjacent vertices in $\tilde{\Sigma} \cong \mathbb{E}$. We call D <u>rich</u> if it contains each vector connecting two vertices whose closed stars intersect.

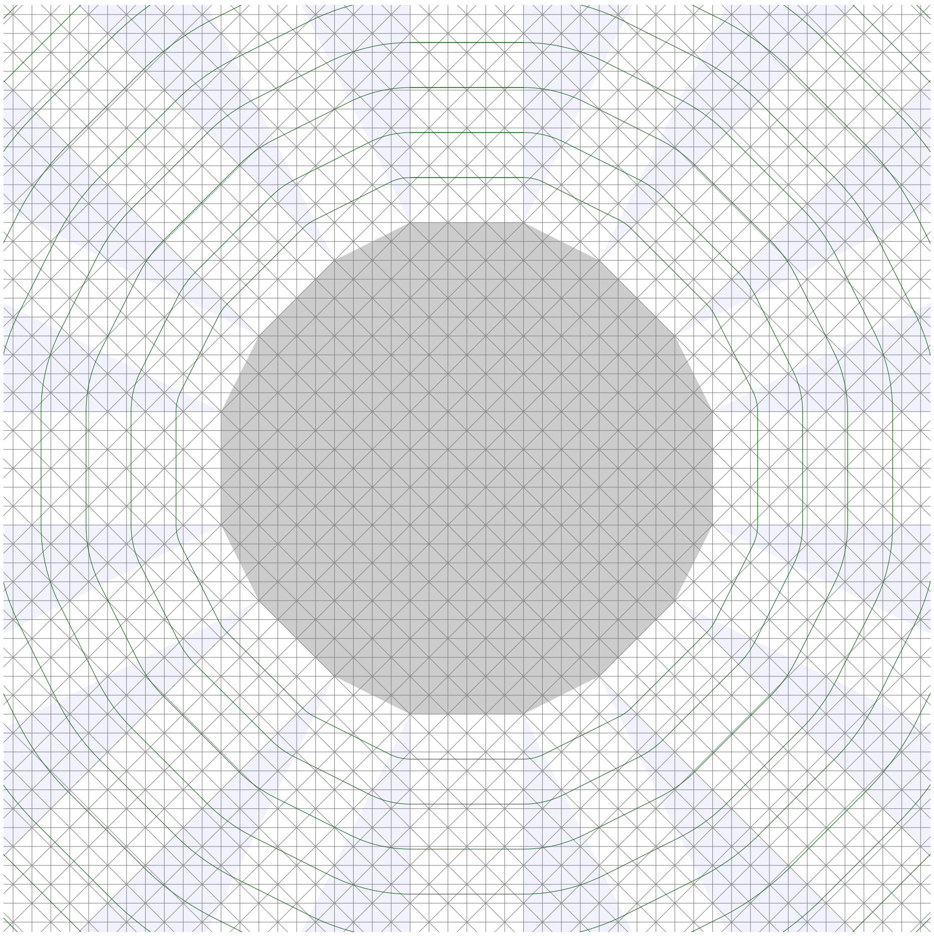


Figure 6: a rich zonotope for $\tilde{\mathsf{B}}_2$

This figure illustrates some h-level sets. The lightly shaded areas and the white corridors are the normal cones for the zonotope.

Lemma 8.8. *Assume that D is rich. Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment containing d_-, let τ_+ be a simplex in Σ_+, and let R be the maximum value of h on τ_+. Then, there is a vertex ν in the set $L^{\uparrow} := \{\nu' \in \mathrm{lk}(\tau_+) \mid h(\nu') > R\}$ of all vertices in the link $\mathrm{lk}(\tau_+)$ strictly higher than τ_+ such that $h(\nu)$ is the minimum value of h on the convex hull of $L^{\uparrow}$. In particular, the convex hull $\overline{L^{\uparrow}}$ is disjoint from τ_+ and therefore $\overline{L^{\uparrow}}$ and τ_+ are separated by a hyperplane in Σ_+.*

Proof. By Proposition 2.4, any simplex spanned by vertices in $L^{\uparrow}$ has a vertex where h assumes its minimum value on that simplex. Since these simplices cover the convex hull of $L^{\uparrow}$ (Carathéordory's Theorem), the claim follows. □

Remark 8.9. One could also include in D difference vectors arising from barycenters of simplices with intersecting stars. This would slightly simplify the proof of Lemma 8.7. (We suggest the name filthy rich for such D.)

Finally, we can paste the pieces together.

Lemma 8.10. *Assume that D is rich. Let σ_+ an h-horizontal simplex with $\sigma_+^{\min} = \sigma_+$. Then the descending link $\mathrm{lk}^{\downarrow}(\sigma_+^{\circ})$ of the barycenter σ_+° is spherical of dimension* $\dim(X_+) - 1$.

Proof. By Observation 8.1 and Lemma 8.6:

$$\mathrm{lk}^{\downarrow}(\sigma_+^{\circ}) = \mathrm{lk}_{\partial}^{\downarrow}(\sigma_+^{\circ}) * \mathrm{lk}_{\mathrm{hor}}^{\circ\downarrow}(\sigma_+) * \mathrm{lk}_{\mathrm{ver}}^{\circ\downarrow}(\sigma_+)$$

The face part $\mathrm{lk}_{\partial}^{\downarrow}(\sigma_+^{\circ})$ is a sphere by Lemma 8.3. The vertical coface part $\mathrm{lk}_{\mathrm{ver}}^{\circ\downarrow}(\sigma_+)$ is a subdivided open hemisphere complex in $\mathrm{lk}(\sigma_+)$ by Lemma 8.7 and Lemma 8.4. By Proposition 3.5, this part is spherical of dimension $\dim(\mathrm{lk}_{\mathrm{ver}}(\sigma_+))$.

It remains to consider the horizontal part $\mathrm{lk}_{\mathrm{hor}}^{\circ\downarrow}(\sigma_+)$. Ultimately, we want to apply Proposition 3.3. So first, we regard $\mathrm{lk}(\sigma_+)$ as a residue in X_+. Let c_+ be the projection of d_- in the residue $\mathrm{lk}(\sigma_+)$. Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment containing c_+ and d_-. Note that c_+ contains $\nabla_{\sigma_+} h'$ and $\nabla_{\sigma_+} h$.

Let $L^{\uparrow}$ be the set of all vertices in $\mathrm{lk}(\sigma_+) \cap \Sigma_+$ strictly higher than σ_+. Lemma 8.8 implies that the subcomplex of the spherical apartment $\mathrm{lk}(\sigma_+) \cap \Sigma_+$ spanned by $L^{\uparrow}$ is convex and has diameter strictly less than π. It follows that there is an open, proper, convex subset M of the round sphere $\mathrm{lk}(\sigma_+) \cap \Sigma_+$ that contains all vertices in $L^{\uparrow}$ and no other vertices of the apartment. E.g., one could take M to be the convex hull of sufficiently small balls around the points in $L^{\uparrow}$.

Any other twin apartment Σ' that contains c_+ and d_- is isometric to Σ via an isometry leaving c_+ fixed. Define $M_{\Sigma'}$ to be the isometric image of M in $\mathrm{lk}(\sigma_+) \cap \Sigma'_+$ As h is compatible with the isometry, $M_{\Sigma'}$ is an open, proper, convex subset containing precisely those vertices of $\mathrm{lk}(\sigma_+) \cap \Sigma'_+$ that are strictly higher than σ_+.

Now put

$$B := \mathrm{lk}_{\mathrm{hor}}(\sigma_+) \cap \bigcup_{\Sigma} M_{\Sigma}$$

where Σ ranges over all twin apartments containing c_+ and d_-. Since c_+ is the projection of d_- into the residue $\mathrm{lk}(\sigma_+)$, any apartment of $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$ that contains the chamber $c_+ \cap \mathrm{lk}_{\mathrm{hor}}(\sigma_+)$ comes from such a twin apartment. Thus, B is an open subset of $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$ that satisfies the hypotheses of Proposition 3.3. It follows that the subcomplex of $\mathrm{lk}_{\mathrm{hor}}(\sigma_+)$ spanned by all those vertices below or at the hight of σ_+ is spherical. By Lemma 8.5, the descending horizontal link $\mathrm{lk}_{\mathrm{hor}}^{\circ\downarrow}(\sigma_+)$ is a subdivision of precisely this spherical complex. □

Combining Lemma 8.2 and Lemma 8.10, we see:

Proposition 8.11. *The descending link $\mathrm{lk}^{\downarrow}(\sigma_+^{\circ})$ of any vertex $\sigma_+^{\circ} \in X_+^{\circ}$ is spherical of dimension* $\dim(X_+) - 1$. □

Of course, the generic vertex will not have neighbors of equal height. It is only along some regions that we encounter strange links. In the generic case, the descending link is an hemisphere complex (in the generic case, open and closed makes no difference). Using Proposition 3.5 for thick buildings, we conclude:

Observation 8.12. *There are arbitrarily high vertices with non-contractible descending links.* □

9 Finiteness properties: proof of Theorem B

Finally, we assume that the twin building X is locally finite. We also assume that the set D, defining the zonotope, is W-invariant and rich. E.g., one could chose D to consist precisely of the difference vectors of any pair of vertices in $\tilde{\Sigma} \cong \mathbb{E}$ whose closed stars intersect.

Observation 9.1. *The group Γ acts simplicially on X_+°. Simplex stabilizers of the action are finite.* □

Observation 9.2. *The function h° is Γ-invariant, and its sublevel complexes are Γ-cocompact.* □

Proof of Theorem B. Given the topological properties of descending links, the deduction of finiteness properties is routine.

Since Γ acts cocompactly, there are only finitely many Γ-orbits of vertices in X_+° below any given h°-bound in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. In particular, only finitely many elements in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ arise as values of h° below any given bound. Define $X_+^\circ(i)$ to be the subcomplex of X_+° spanned by all vertices σ_+° such that there are at most i elements in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as values of h° and that are strictly below $h^\circ(\sigma_+^\circ)$.

By Observation 7.2, there are no h°-horizontal edges in X_+°. Thus, $X_+^\circ(i+1) \setminus X_+^\circ(i)$ does not contain adjacent vertices.

Recall that n denotes the dimension $\dim(X_+)$. For any vertex $\sigma_+^\circ \in X_+^\circ(i+1) \setminus X_+^\circ(i)$ the relative link $\mathrm{lk}(\sigma_+^\circ) \cap X_+^\circ(i)$ is precisely the descending link $\mathrm{lk}^\downarrow(\sigma_+^\circ)$. By Proposition 8.11, descending links are spherical of dimension $\dim(X_+) - 1$. Thus, the complex $X_+^\circ(i+1)$ is obtained up to homotopy equivalence from $X_+^\circ(i)$ by attaching n-cells. Observation 8.12 ensures that at infinitely many stages the extension is non-trivial.

The group Γ acts on X_+° with finite stabilizers by hypothesis. Thus, all hypotheses of the criterion [Bro87, Corollary 3.3] are satisfied and Γ is of type F_{n-1} but not of type F_n. □

10 Deducing Theorem A from Theorem B

The gap between Theorem A and Theorem B is bridged by the construction of a twin building for the group $\mathcal{G}(\mathbb{F}_q[t, t^{-1}])$. Although certainly known to the experts, we were not able to find a clean reference. For this reason, we outline the construction for connected, simply connected groups (this will be sufficient for the application to finiteness properties). Rémy is going to provide a citable reference in the near future.

First, we deal with split groups.

Proposition 10.1. *Let K be a field of arbitrary characteristic and let $\mathcal{G}$ be an isotropic, connected, simply connected, almost simple, split K-group. Then the functor $\mathcal{G}(-[t, t^{-1}])$ is a Kac-Moody functor.*

We should explain our notation: to any field K', the functor above assigns the group of $K'[t, t^{-1}]$-points of $\mathcal{G}$.

Proof. By [Spri98, Theorem 16.3.2] and [Chev55, §II], an isotropic, connected, simply connected, almost simple K-group that splits over K is a Chevalley group. It follows that the group scheme $\mathcal{G}(-)$ is defined over $\mathbb{Z}$. Hence the functor $\mathcal{G}(-[t, t^{-1}])$ can be defined for all fields.

A Kac-Moody functor is associated to a root datum $\mathcal{D}$, the main part of which is a generalized Cartan matrix A. Classically, this kind of datum classifies reductive groups over the complex numbers. There, the generalized Cartan matrix is not really generalized and defines a finite Coxeter group. Kac-Moody functors were defined by Tits [Tits87] in the case where the generalized Cartan matrix defines an arbitrary Coxeter group.

In order to recognize $\mathcal{G}(-[t, t^{-1}])$ as a Kac-Moody functor, we have to correctly identify its defining datum $\mathcal{D}$. Since the group $\mathcal{G}$ is simply connected, we only have to choose the generalized Cartan matrix A. Here, we use the unique generalized Cartan matrix given by a euclidean Coxeter diagram extending the spherical diagram as defined by $\mathcal{G}$.

To show that $\mathcal{G}(-[t, t^{-1}])$ is the Kac-Moody functor associated to $\mathcal{D}$, one needs to verify the axioms (KMG 1) through (KMG 9) in [Tits87]. All axioms are straight forward to check; however (KMG 5) and (KMG 6) involve the complex Kac-Moody algebra $L(A)$ associated to the given Cartan matrix. To verify these, one needs to know that $L(A)$ is the universal central extension of the Lie algebra $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$ where $\mathfrak{g}$ is the Lie algebra associated to $\mathcal{G}$. See e.g., [Kac90, Theorem 9.11] or [PrSe86, Section 5.2]. □

In [Rémy02], Rémy has extended the construction to non-split groups using the method of Galois descent.

Proposition 10.2. *Let $\mathcal{G}$ be an isotropic, conected, simply connected, almost simple group defined over the finite field $\mathbb{F}_q$. Then the functor $\mathcal{G}(-[t, t^{-1}])$ is an almost split $\mathbb{F}_q$-form of a Kac-Moody group defined over the algebraic closure $\overline{\mathbb{F}_q}$.*

Proof. First, $\mathcal{G}$ splits over $\overline{\mathbb{F}_q}$. Hence, $\mathcal{G}(-[t, t^{-1}])$ is a Kac-Moody functor over $\overline{\mathbb{F}_q}$ by the preceding proposition. Let $\mathcal{D}$ be the associated root datum.

Note that the conditions (KMG 6) through (KMG 9) ensure that the "abstract" and "constructive" Kac-Moody functors associated to $\mathcal{D}$ coincide [Tits87, Theorem 1'], which holds in particular for $\mathcal{G}(-[t, t^{-1}])$. This is relevant as Rémy discusses Galois descent for constructive Kac-Moody functors.

The claim follows from [Rémy02, Section 11] once a list of conditions scattered throughout that section have been verified. Checking individual axioms is straightforward, the hard part (left to the reader) is making sure that no condition is left out. Here is the list:

(PREALG 1) [p. 257] One needs to know that $U_{\mathcal{D}}$ is the $\mathbb{Z}$-form of the universal enveloping algebra of $L(A)$. Its $\mathbb{F}_q$-form is obtained by the Galois action.

(PREALG 2) [p. 257] Straightforward.

(SGR) [p. 266] Straightforward.

(ALG 1) [p. 267] Use Definition 11.2.1 on page 261.

(ALG 2) [p. 267] Straightforward.

(PRD) [p. 273] Observe that the Galois group acts trivially on t and t^{-1}. $\square$

We are finally closing in on twin buildings.

Proposition 10.3. *Let $\mathcal{G}$ be as in Proposition 10.2. The group $\mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ has an RGD system.*

Proof. This follows from [Rémy02, Theorem 12.4.3]; but once again, we need to verify hypotheses. This time, we have to deal with only two:

(DCS_1) [p. 284] This holds as $\mathcal{G}$ splits already over a finite field extension of $\mathbb{F}_q$.

(DCS_2) [p. 284] This follows from $\mathbb{F}_q$ being a finite, and hence perfect field. $\square$

Proposition 10.4. *Let $\mathcal{G}$ be an isotropic, connected, simply connected, almost simple group defined over the finite field $\mathbb{F}_q$ (i.e., $\mathcal{G}$ is as in Proposition 10.2). Then there is a thick, locally finite, irreducible euclidean twin building $X = (X_+, X_-)$ on which $\mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ acts strongly transitively. Moreover, X_+ and X_- are $\mathcal{G}(\mathbb{F}_q[t, t^{-1}])$-equivariantly isomorphic to the euclidean building associated to $\mathcal{G}(\mathbb{F}_q((t)))$ and $\mathcal{G}(\mathbb{F}_q((t^{-1})))$, respectively.*

Proof. By the preceding proposition, the group $\mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ has an RGD system. By [AbBr08, Theorem 8.80 and Theorem 8.81], we find an associated twin building upon which the group acts strongly transitively. Theorem 8.81 also tells us that the root groups act simply transitively, which implies that the twin building is thick and locally finite. That it is irreducible and euclidean is clear as we chose the generalized Cartan matrix A back in in the proof Proposition 10.1 to match the spherical type of $\mathcal{G}$, which is almost simple.

The identification of X_+ and X_- with the euclidean building associated to $\mathcal{G}(\mathbb{F}_q((t)))$, respectively $\mathcal{G}(\mathbb{F}_q((t^{-1})))$, follows from the functoriality of the construction [Rous77, 5.1.2]. $\square$

Remark 10.5. It also follows from [AbBr08, Theorem 8.81] that the building thus constructed is Moufang.

Remark 10.6. For split groups, Abramenko gives the RGD system explicitly in [Abra96, Example 3, page 18]. He also derives RGD systems for groups of the types ${}^2\widetilde{\mathsf{A}}_n$ and ${}^2\widetilde{\mathsf{D}}_n$ in [Abra96, Chapter III.1]. Hence, the only types not covered by his explicit computations are ${}^3\widetilde{\mathsf{D}}_4$ and ${}^2\widetilde{\mathsf{E}}_6$. The marginal gain also explains why we merely sketched the general argument.

Proof of Theorem A using Theorem B. Let $\mathcal{G}$ be as in Theorem A, i.e., $\mathcal{G}$ is an isotropic, almost simple group defined over the finite field $\mathbb{F}_q$. We may assume that $\mathcal{G}$ is connected since the connected component of the identity element has finite index.

Let $\tilde{\mathcal{G}}$ be its "universal cover", i.e., a simply connected, isotropic, almost simple $\mathbb{F}_q$-group which allows for a central isogeny onto $\mathcal{G}$. By [Behr68, Satz 2], the image of $\tilde{\mathcal{G}}(\mathbb{F}_q[t])$ in $\mathcal{G}(\mathbb{F}_q[t])$ under the isogeny has finite index. As the isogeny has finite kernel, the finiteness properties of $\mathcal{G}(\mathbb{F}_q[t])$ and $\tilde{\mathcal{G}}(\mathbb{F}_q[t])$ coincide. Hence we may assume without loss of generality that $\mathcal{G}$ is simply connected.

Now, we can apply Proposition 10.4. Hence there is a thick, locally finite, irreducible euclidean twin building $X = (X_+, X_-)$ on which $G := \mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ acts strongly transitively. The G-equivariant isomorphisms of X_+ and X_- to the euclidean building associated to $\mathcal{G}(\mathbb{F}_q((t)))$ implies first that the $\mathbb{F}_q$-rank n of $\mathcal{G}$ is the dimension of the building. It also implies that stabilizers of pairs (c_+, c_-) of chambers are finite: they are compact and discrete. Finally, the group $\mathcal{G}(\mathbb{F}_q[t])$ is commensurable to the stabilizer $\Gamma := \mathrm{Stab}_G(d_-)$ for some chamber $d_- \in X_-$. Since finiteness properties are invariant under commensurability, Theorem B applies. □

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Kai-Uwe Bux
Fakulät für Mathematik
Universität Bielefeld
Universitätsstraße 25
33501 Bielefeld
Germany
website: `www.kubux.net`

Ralf Gramlich
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
Germany
website: `www.mathematik.tu-darmstadt.de/∼gramlich`
School of Mathematics
University of Birmingham
Edgbaston
Birmingham B15 2TT
United Kingdom
website: `www.mat.bham.ac.uk/staff/gramlichr.shtml`

Stefan Witzel
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstraße 7
64289 Darmstadt
Germany
website: `www.mathematik.tu-darmstadt.de/∼switzel`