THE SPHERICITY OF THE PHAN GEOMETRIES OF TYPE B_n AND C_n AND THE PHAN-TYPE THEOREM OF TYPE F_4

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Abstract. We adapt and refine the methods developed in [Abr96] and [DGM] in order to establish the sphericity of the Phan geometries defined in [BGHS07] (type B_n) and [GHS03] (type C_n), and their generalizations. As applications of this sphericity we determine the topological finiteness length of the unitary form of the group $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t,t^{-1}])$ (Theorem 7.1) and give the first published proof of the Phan-type theorem of type F_4 (Theorem 7.11). Apart from that we reproduce the topological finiteness length of the group $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t])$ and the Phan-type theorems of types B_n and C_n. In the theory of arithmetic groups our result on the topological finiteness length of the unitary form of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t,t^{-1}])$ is another example supporting the rank conjecture, see [Beh98, p. 80]. Within the revision of the classification of the finite simple groups this publication of the Phan-type theorem of type F_4 concludes the revision of Phan's theorems [Pha77a], [Pha77b] and their extension to the non-simply laced diagrams; cf. [AB08, Section 14.2] on page 656 and [GLS05] on page 333.

1. Introduction

In this paper we prove the following theorem (for the exact statement see Theorem 4.1).

Main Theorem. *A generalized Phan geometry of type B_n or C_n is $(n-1)$-spherical provided the defining field is sufficiently large. In fact, it is even Cohen-Macaulay.*

As the name suggests, the class of generalized Phan geometries contains the class of Phan geometries. These have been introduced in [BGHS07] (type B_n) and [GHS03] (type C_n) in order to prove Phan-type theorems, that is, analogs of Phan's group-theoretic recognition results in [Pha77a] and [Pha77b]. The Phan-type theorems state that the unitary forms of the groups $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ and $\mathrm{Spin}_{2n+1}(\mathbb{F}_{q^2})$, i.e., their subgroups fixed by the involution that acts as the field involution on $\mathbb{F}_{q^2}$ and takes $\mathbb{F}_q$-rational group elements to their transpose inverse, are the universal enveloping group of the amalgam of their fundamental subgroups of rank one and two. Using Tits' Lemma [Tit86, Corollaire 1] this amalgamation result follows from the simple connectedness of the corresponding Phan geometries, which was established in [BGHS07] and [GHN07] in the case B_n and in [GHS03] and [GHN06] in the case C_n. This article provides alternative proofs of the Phan-type theorems of type B_n and C_n and, based on the local-to-global approach via the filtration described in

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[DM07], gives a much shorter alternative to the unpublished proof of the Phan-type theorem of type F_4 by Gramlich, Hoffman, Mühlherr, and Shpectorov.

Another class of geometries contained in the class of generalized Phan geometries are the geometries opposite one fixed chamber in a spherical building. The sphericity of these geometries (in spherical buildings of classical type) has been established by Abels and Abramenko in [AA93], [Abr96] in order to determine the topological finiteness lengths of certain arithmetic subgroups of affine Kac-Moody groups which are commensurable to the Borel lattice described in [CG99], [Rém99]; cf. [Abe91], [Abr96].

Our strategy is an adaption and generalization of that work. One of our main motivations for this article is to determine the finiteness length of unitary forms of certain affine Kac-Moody groups which are commensurable to the flip lattice described in [Gra]. Concretely, if G is a semisimple algebraic group scheme of spherical type X_n, then $G(\mathbb{F}_{q^2}[t,t^{-1}])$ acts on a twin building (Δ_+,Δ_-) of type $\tilde{X}_n$, see [AB08, Chapter 11]. Let θ be the map that takes g to $(g^\sigma)^{-T}$, where σ acts as the field involution on $\mathbb{F}_{q^2}$ and exchanges t and t^{-1}. We are interested in the group $K(\mathbb{F}_{q^2}[t,t^{-1}])$ of θ-fixed elements of $G(\mathbb{F}_{q^2}[t,t^{-1}])$. The group $K(\mathbb{F}_{q^2}[t,t^{-1}])$ is an arithmetic subgroup of $G(\mathbb{F}_{q^2}(t))$ whose local rank over $\mathbb{F}_{q^2}((t))$ or, equivalently, $\mathbb{F}_{q^2}((t^{-1}))$ equals n. So, according to the rank conjecture, see [Beh98, p. 80], the group $K(\mathbb{F}_{q^2}[t,t^{-1}])$ should be of finiteness type F_{n-1}, but not FP_n, which we confirm in the present article.

The group $K(\mathbb{F}_{q^2}[t,t^{-1}])$ naturally acts on the subcomplex Δ_θ of the building Δ_+ of chambers that are mapped to opposite ones by θ. We use a filtration $(\Delta_i)_i$ of Δ_+ with $\Delta_0=\Delta_\theta$ from [DM07], whose relative links will turn out (Theorem 6.8) to be generalized Phan geometries (cf. [DGM, Fact 5.1]). The types of generalized Phan geometries that occur are the types of spherical residues of the building. So by showing the sphericity of generalized Phan geometries, we show that the filtration preserves a certain degree of connectedness. Brown's criterion [Bro87, Corollary 3.3] then allows us to determine the topological finiteness length of $K(\mathbb{F}_{q^2}[t,t^{-1}])$.

Generalized Phan geometries of type A_n have been investigated by Devillers, Gramlich, and Mühlherr in [DGM]. Since all irreducible spherical residues of a building of type $\tilde{A}_n$ are of type A_m, the procedure described above allowed the authors of loc. cit. to determine the finiteness length of the unitary form of $\mathrm{SL}_{n+1}(\mathbb{F}_{q^2}[t,t^{-1}])$. The Main Result of [DGM] together with our Main Result allows us to determine the finiteness length of the unitary form of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t,t^{-1}])$, see Theorem 7.1. Both results affirm the rank conjecture in their respective cases.

The paper is organized as follows: The definition of generalized Phan geometries of type B_n and C_n is given in Section 2. In Section 3 we recall and collect some topological facts that are used later. Section 4 contains the precise statement of the Main Theorem and a proof of how it can be deduced from some technical lemmas. These technical lemmas are then proved in Section 5. In Section 6 we will show that generalized Phan geometries occur as relative links in the filtration mentioned above. Finally, Section 7 contains two applications, namely the computation of the topological finiteness length of the unitary form of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t,t^{-1}])$ mentioned above, and the local recognition of groups that admit a weak Phan system of type F_4 (Theorem 7.11).

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also point out that the original unpublished proof of the Phan-type theorem of type F_4 was found by Gramlich, Hoffman, Mühlherr, and Shpectorov during an RiP stay in Oberwolfach during the summer of 2005.

2. Generalized Phan geometries of type B_n and C_n

In this section we give the precise definition of the central objects of study of this paper, the generalized Phan geometries. In order to do so, we need to recall the notion of transversality.

Let V be a vector space over a field $\mathbb{F}$. For $U, W \leq V$, we say that U is *transversal* to W and write $U \pitchfork W$, if $U \cap W = 0$ or $\langle U, W\rangle = V$. Note that $U \pitchfork W$ if and only if $\dim(U \cap W) = \max\{0, \dim U + \dim W - \dim V\}$. For a flag $F = (0 = V_0 \leq \ldots \leq V_k = V)$ and a subspace $U \leq V$ we say that U is *transversal* to F and write $U \pitchfork F$, if $U \pitchfork V_i$ for $0 \leq i \leq k$. This is the case if and only if $\langle U, V_{k_U}\rangle = V$ where $k_U = \min\{i \mid U \cap V_j \neq \{0\}\}$.

Given a flag $F = (0 = V_0 \leq \ldots \leq V_k = V)$ we call a family $(\omega_i)_{1\leq i\leq k}$ of σ-hermitian forms $\omega_i \colon V_i \times V_i \to \mathbb{F}$ *compatible with* F if $\operatorname{Rad} \omega_i = V_{i-1}$.

Definition 2.1. Let F be as above and let $\omega = (\omega_i)_i$ be a family of compatible σ-hermitian forms. For $U \leq V$ we say that U is *transversal* to (F, ω), if U is transversal to F and $U \cap V_{k_U}$ is ω_{k_U}-non-degenerate. In this case we write $U \pitchfork (F, \omega)$.

Definition 2.2. Let V be a vector space of dimension $2n + 1$ equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of Witt index n. Let $F = (0 = V_0 \leq \ldots \leq V_k = V)$ be a flag satisfying $F^\perp = F$. Let ω be a family of σ-hermitian forms compatible with F and assume that there is an ω_k-non-isotropic vector that is $(\cdot, \cdot)$-isotropic. The generalized Phan-geometry of type B_n defined by (F, ω) consists of all subspaces U of V that are totally $(\cdot, \cdot)$-isotropic and transversal to (F, ω).

Definition 2.3. Let V be a vector space of dimension $2n$ equipped with a non-degenerate alternating bilinear form $(\cdot, \cdot)$. Let $F = (0 = V_0 \leq \ldots \leq V_k = V)$ be a flag satisfying $F^\perp = F$. Let ω be a family of σ-hermitian forms compatible with F and assume that there is an ω_k-non-isotropic vector. The generalized Phan-geometry of type C_n defined by (F, ω) consists of all subspaces U of V that are totally $(\cdot, \cdot)$-isotropic and transversal to (F, ω).

Remark 2.4.
1. A non-zero σ-hermitian form over an $\mathbb{F}$-vector space admits a non-isotropic vector unless $\sigma = \mathrm{id}$ and $\mathbb{F}$ has characteristic 2. So the technical condition that there be an ω_k-non-isotropic vector in the above definitions is actually quite weak.
2. A closer look reveals that half of the forms ω_i actually do not play any role because a totally isotropic subspace U that is transversal to F cannot meet any of the V_i with $\dim V_i \leq n$. However, taking this into account would not simplify anything in the present article, but would instead make the definition of a generalized Phan geometry even more cumbersome.

3. Basics

We now recall and collect some topological facts, that we will need later. For an introduction to CW complexes we refer the reader to [Bre93, Section IV.9] and for a discussion of simplicial complexes to [Spa66, Chapter 3].

In particular we use the following notions. If K is a simplicial complex, $|K|$ is its *realization*. If $s \in K$ is a simplex, we then the *star of* s is $\operatorname{st} s = \{t \in K \mid t \cup s \in K\}$. The *link of* s is the subcomplex $\operatorname{lk} s = \{t \in \operatorname{st} s \mid t \cap s = \emptyset\}$. The subcomplex generated by a subset $T \subseteq K$ is the complex $\overline{T} = \{s \in K \mid s \leq t \text{ for some } t \in T\}$. We denote the boundary $\overline{\{s\}} \setminus \{s\}$ of a simplex s by ∂s. If L is another simplicial complex, then $K * L$ denotes the *join* of the two, i.e. the complex with simplices $s \sqcup t, s \in K, t \in L$. This corresponds to the topological join in so far that $|K * L|$ is naturally homeomorphic to $|K| * |L|$.

The *n-sphere* S^n is the space $\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ (with the induced topology) where $\|\cdot\|$ is the standard Euclidean norm. Similarly, the *n-disc* D^n is the space $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. A space X is said to be *n-connected* for $n \geq -1$, if every map $S^k \to X$ extends to a map $D^{k+1} \to X$ for $k \leq n$ (note that S^{-1} is the empty set, so (-1)-connected means non-empty). This is equivalent to saying that X is non-empty and $\pi_k(X) = 1$ for $k \leq n$.

By the Hurewicz Theorem [Bre93, Chapter VII, Theorem 10.7], a space X is n-connected for $n \geq 1$ if and only if it is simply connected and $H_k(X) = 0$ for $k \leq n$. An n-dimensional CW complex is *n-spherical* if it is $(n-1)$-connected. It is *properly n-spherical*, if it is n-spherical and $\pi_n(X) \neq 1$. Again by the Hurewicz Theorem a space is properly n-spherical for $n \geq 2$, if it is n-spherical and $H_n(X) \neq 0$.

A *regular* CW complex is a CW complex in which the characteristic maps of the cells can be chosen to be homeomorphisms. The realization of a simplicial complex is clearly a regular CW complex. We will make repeated use of the following fact.

Proposition 3.1. *Let X be a regular CW complex that can be written as $X = B \cup \bigcup \mathcal{A}$ where B and $A \in \mathcal{A}$ are subcomplexes. Assume that B is m-connected, that A is m-connected, that $B \cap A$ is $(m-1)$-connected, and that $A \cap A' \leq B$ for $A, A' \in \mathcal{A}, A \neq A'$. Then X is m-connected.*

Proof. By [Bjö95, Lemma 10.2] each of the spaces $B \cup A$, $A \in \mathcal{A}$, is m-connected. So we can apply [Bjö95, Theorem 10.6] to the family $(B \cup A)_{A \in \mathcal{A}}$ in order to conclude that X is m-connected if and only if the nerve is. But the nerve of this family is a simplex, hence contractible. $\square$

Proposition 3.2. *Let X be k-connected and Y be l-connected. Then $X * Y$ is $(k + l + 1)$-connected.*

Proof. It is clear that $X * Y$ is path-connected. We have to see that $X * Y$ is simply connected, if X and Y both are path-connected or X is non-empty and Y is simply-connected. Since $X * Y$ and $S(X \wedge Y)$ are homotopy equivalent (see [Hat01, Chapter 0, exercise 24]) it suffices to see that $X \wedge Y$ is connected in the two cases above. This is not hard. Now the result follows from [Bjö95, (9.12)] taking into account our remarks above. $\square$

We also need to fix some notation concerning buildings and twin buildings. Since we study topological properties of buildings and their sub-geometries, we consider buildings as simplicial complexes as introduced for example [AB08, Chapter 4].

If Δ is a building of type I, then the chamber systems of $\operatorname{st} s$ and $\operatorname{lk} s$ are isomorphic and describe the residue of type $I \setminus \operatorname{typ}(s)$ of any chamber in $\operatorname{st} s$. However $\operatorname{lk} s$ is the more natural model in so far as it is the simplicial complex naturally obtained from the chamber system. On the other hand $\operatorname{st} s$ has the virtue that its chambers actually are (and not only correspond to) chambers of K. This simplifies notation when dealing with projections. We will therefore consider both complexes as residues and make the correspondence implicit.

4. The Main Theorem

In this section we show how the following theorem, which is the exact statement of our Main Theorem, can be deduced from four rather technical lemmas that will be proven in the following section.

Theorem 4.1. *Let $m, n > 0$ be integers, let $\mathbb{F}$ be a field, and let σ be an automorphism of $\mathbb{F}$ of order 1 or 2. If $\mathbb{F}$ is finite, assume that $|\mathbb{F}| \geq 4^{n-1}2m$, if $\sigma = \mathrm{id}$, and that $|\mathbb{F}| \geq 4^{n-1}(q+1)m$, if $\mathbb{F} = \mathbb{F}_{q^2}$ and $\sigma \neq \mathrm{id}$. For any family $(\Gamma_j)_{1 \leq j \leq m}$ of σ-hermitian generalized Phan geometries, all of type B_n or C_n defined over $\mathbb{F}$, their intersection $\Gamma = \bigcap_j \Gamma_j$ is $(n-1)$-spherical.*

Let V be a vector space equipped with a bilinear form $(\cdot, \cdot)$ and let Γ be a generalized Phan geometry of type B_n or C_n given by a flag $F = (0 = V_0 \lneq \ldots \lneq V_k = V)$ and a family $\omega = (\omega_i)_i$ of σ-hermitian forms $\omega_i \colon V_i \times V_i \to \mathbb{F}$.

We use the following strategy due to Abramenko [Abr96, Chapter II] to prove the theorem. Let p be a one-dimensional space that is non-degenerate with respect to ω_k. Let $Z = \{U \in \Gamma \mid \langle p, U \rangle \pitchfork (F, \omega)\}$. Let $Y_0 = \{U \in Z \mid U \cap p^\perp, \langle U \cap p^\perp, p \rangle \in \Gamma\}$. For $1 \leq i \leq n$ let

$$Y_i = Y_{i-1} \cup \{U \in Z \mid \dim(U) = i\}$$

and for $n+1 \leq i \leq 2n$ let

$$Y_i = Y_{i-1} \cup \{U \in \Gamma \mid \dim(U) = 2n+1-i\} \ .$$

We are going to show inductively that each Y_i is $(n-1)$-spherical. For this purpose it suffices (see Proposition 3.1) to show that the residue of a $U \in Y_i \setminus Y_{i-1}$ in Y_i is $(n-2)$-spherical. This in turn will follow if we can prove the following lemmas. Let $U \in \Gamma$ be of dimension k.

Lemma 4.2. *If $U \in Y_i \setminus Y_{i-1}$ for $1 \leq i \leq n$, then $Y_i^{>U} := \{W \in Y_i \mid W > U\} = \{W \in Y_0 \mid W > U\}$ is an intersection of at most four generalized Phan geometries of type B_{n-k} respectively C_{n-k}.*

Lemma 4.3. *If $U \in Y_i \setminus Y_{i-1}$ for $n+1 \leq i \leq 2n$, then $Y_i^{>U} = \{W \in Y_i \mid W > U\} = \{W \in \Gamma \mid W > U\}$ is a generalized Phan geometry of type B_{n-k} respectively C_{n-k}.*

Lemma 4.4. *The set $Y_i^{<U} := \{W \in Y_i \mid W < U\} = \{W \in Z \mid W < U\}$ is an intersection of at most two generalized Phan geometries of type A_{k-1}.*

Lemma 4.5. *Let Γ be the intersection of m generalized Phan geometries, all of type either B_1 or C_1, defined over $\mathbb{F}$ by σ-hermitian forms. If $\mathbb{F}$ is finite, assume that $|\mathbb{F}| \geq 2m$, if $\sigma = \mathrm{id}$, and that $|\mathbb{F}| \geq (q+1)m$, if $\sigma \neq \mathrm{id}$ and $\mathbb{F} = \mathbb{F}_{q^2}$. Then Γ contains at least one point.*

Proof. (Proof of Theorem 4.1) To simplify notation define $C := 2$ in case $\sigma = \mathrm{id}$ and $C := q+1$ in case $\sigma \neq \mathrm{id}$, so that the condition on the field in the statement of the theorem can be written as $|\mathbb{F}| \geq 4^{n-1}Cm$.

We proceed by an induction on n. To be 0-spherical, the space has to be (-1)-connected, i.e. non-empty, so the induction basis is just Lemma 4.5. Assume that the statement is true for all $k < n$ and consider the filtration $(Y_i)_i$ described above.

We now proceed by an induction on i. The geometry Y_0 is contractible because $U \mapsto U \cap p^\perp \mapsto \langle U \cap p^\perp, p \rangle \mapsto p$ is a deformation retraction to one point.

We want to apply Proposition 3.1 to the setup $X = |Y_i|$, $B = |Y_{i-1}|$, and $\mathcal{A} = \{|\mathrm{st}_{Y_i}(U)| \mid U \in Y_i \setminus Y_{i-1}\}$. The space $|Y_i|$ is $(n-1)$-spherical by induction. To see that $|\mathrm{st}_{Y_i}(U)| \cap |\mathrm{st}_{Y_i}(U')| \subseteq |Y_{i-1}|$, let $W \in \mathrm{st}_{Y_i}(U) \cap \mathrm{st}_{Y_i}(U')$, i.e., W is incident with U and U'. Then its dimension is not $\dim(U) = \dim(U')$, whence $W \in Y_{i-1}$. The space $|\mathrm{st}_{Y_i}(U)|$ is clearly contractible, as it is a cone over $|\mathrm{lk}_{Y_i}(U)|$. So it remains to see that, for $U \in Y_i \setminus Y_{i-1}$ of dimension k, the space $|\mathrm{st}(U) \cap Y_{i-1}|$ is $(n-2)$-spherical.

In order to do so, we remark that $\mathrm{st}(U) \cap Y_{i-1} = Y_i^{<U} * Y_i^{>U}$. Now by Lemma 4.4, the complex $Y_i^{<U}$ is the intersection of at most $2m$ generalized Phan geometries of type A_{k-1}. Since $2^{k-1}C2m \leq 4^{n-1}Cm \leq q$, the Main Theorem from [DGM] implies that $Y_i^{<U}$ is $(k-1)$-spherical. Similarly, Lemmas 4.2 or 4.3 imply that $Y_i^{>U}$ is an intersection of at most $4m$ generalized Phan geometries of type B_{n-k} or C_{n-k}. Now $4^{k-2}C4m \leq 4^{n-1}Cm \leq q$, so by induction $Y_i^{>U}$ is $(n-k-1)$-spherical. Hence $Y_i^{<U} * Y_i^{>U}$ is $(n-1)$-spherical by Proposition 3.2. $\square$

5. Proof of the lemmas

Before proceeding to the proof of the lemmas let us first recall some notions from multilinear algebra. For an $\mathbb{F}$-vector space V, an automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$ of order 1 or 2, and $\varepsilon \in \{-1, 1\}$, a form $(\cdot,\cdot)\colon V \times V \to \mathbb{F}$ is called *(σ,ε)-hermitian*, if it is biadditive and $(\lambda x, \mu y) = \lambda\mu^\sigma(x,y)$ and $(x,y) = \varepsilon(y,x)^\sigma$ for all $x, y \in V$, $\lambda, \mu \in \mathbb{F}$, $x, y \in V$. It is *σ-hermitian*, if it is $(\sigma, 1)$-hermitian. Clearly symmetric bilinear, σ-hermitian and alternating forms are (σ,ε)-hermitian for some σ, ε. The form is *non-degenerate* if $V^\perp = 0$.

Lemma 5.1. *Let V be a vector space and let A, B and C be subspaces.*

(1) *Then* $\dim(\langle A, B\rangle) = \dim(A) + \dim(B) - \dim(A \cap B)$.
(2) *Let $(\cdot,\cdot)$ be a (σ,ε)-hermitian form. Then $A^\perp \cap B^\perp = \langle A, B\rangle^\perp$. If in addition $(\cdot,\cdot)$ is non-degenerate, then also $A^{\perp\perp} = A$, and $\langle A^\perp, B^\perp \rangle = (A \cap B)^\perp$.*

We start with the proof of the induction basis, Lemma 4.5. Here is a restatement (in fact a slightly stronger version) that is obtained by unfolding the definitions.

Lemma 5.2 (restatement of Lemma 4.5)**.** *Let $\mathbb{F}$ be a field and*

(1) *V be an $\mathbb{F}$-vector space of dimension 2 and $(\cdot,\cdot)$ be a non-degenerate alternating bilinear form (case C_1), or*
(2) *V be an $\mathbb{F}$-vector space of dimension 3 and $(\cdot,\cdot)$ be a non-degenerate symmetric bilinear form of Witt index 1 (case B_1).*

Let ω_i, $1 \leq i \leq m$, be σ-hermitian forms. If $\mathbb{F}$ is finite, assume that $|\mathbb{F}| > 2m$, if $\sigma = \mathrm{id}$, and $q^2 > (q+1)m$, if $\mathbb{F} = \mathbb{F}_{q^2}$ and $\sigma \neq \mathrm{id}$. Assume that for every i there is a vector that is $(\cdot,\cdot)$-isotropic and ω_i-non-isotropic. Then there is a vector that is $(\cdot,\cdot)$-isotropic and ω_i-non-isotropic for $1 \leq i \leq m$.

Proof. We proceed by induction on the number of maps. The induction basis holds by hypothesis. Assume the statement to be true up to $m-1$.

Case C_1: First we consider the alternating case. The condition that a vector be $(\cdot,\cdot)$-isotropic is empty, so we are left with the condition on the forms ω_i. Assume that $\sigma = \mathrm{id}$ and let y, z form a basis for V. If y is non-isotropic with respect to all of the ω_i, then there is nothing to prove. Otherwise, consider the vectors $u_\alpha = \alpha y + z$ with $\alpha \in \mathbb{F}$. Then

$$\omega_i(u_\alpha, u_\alpha) = \alpha^2 \omega_i(y,y) + \omega_i(z,z) + 2\alpha\omega_i(y,z)\ , \quad 1 \le i \le m, \tag{5.1}$$

are non-zero polynomials in α that are at most quadratic, and at least one of the polynomials (the one with $\omega_i(y,y) = 0$) is at most linear. Each polynomial has at most two zeroes (and the one with $\omega_i(y,y) = 0$ has at most one zero), so if $|\mathbb{F}| \ge 2m$, there exists an α such that $\omega_i(u_\alpha, u_\alpha) \ne 0$ for all i.

Now assume that $\sigma \ne \mathrm{id}$. If $\mathbb{F}$ is infinite, let $\mathbb{F}^\sigma$ denote the field of its σ-fixed elements. Let V' be the $\mathbb{F}^\sigma$-span of an $\mathbb{F}$-basis of V. Then V' is a 2-dimensional $\mathbb{F}^\sigma$-vector space on which the ω_i are symmetric bilinear. By the argument above there exists a vector $u \in V'$ that is ω_i-non-isotropic.

If $\mathbb{F} = \mathbb{F}_{q^2}$ and $\sigma \ne \mathrm{id}$, then $\alpha^\sigma = \alpha^q$ for $\alpha \in \mathbb{F}$. The argument now works as above with (5.1) replaced by

$$\omega_i(u_\alpha, u_\alpha) = \alpha^{q+1}\omega_i(y,y) + \omega_i(z,z) + (\alpha + \alpha^q)\omega_i(y,z)\ , \quad 1 \le i \le m\ . \tag{5.2}$$

The condition becomes that $|\mathbb{F}| \ge (q+1)m$.

Case B_1: Now assume that $(\cdot,\cdot)$ is a non-degenerate symmetric bilinear form of Witt index 1 on a vector space of dimension 3. By [Cam91, Theorem 6.3.1] the space $(V, (\cdot,\cdot))$ is the direct sum of a 2-dimensional hyperbolic space and a 1-dimensional non-degenerate space. The same is true of $(V, \xi(\cdot,\cdot))$ where ξ is a non-square. Since we are only interested in whether or not vectors are $(\cdot,\cdot)$-isotropic, it does not matter which of the two forms we consider. One of them – we assume without loss of generality, that it be $(\cdot,\cdot)$ – has Gram matrix

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

with respect to a basis e, f, x. The vectors f and $e - \beta^2/2f + \beta x$ with $\beta \in \mathbb{F}$ are up to scalar multiples all isotropic vectors of $(\cdot,\cdot)$.

Again we consider first the case where $\sigma = \mathrm{id}$. If e is non-isotropic with respect to all ω_i, then there is nothing to prove. Otherwise, we consider $u = e - \beta^2/2f + \beta x$ for $\beta \in \mathbb{F}$. As before, $\omega_i(u,u)$ is a non-zero polynomial of degree at most 2 in β, so it has at most 2 zeroes. And if $\omega_i(e,e) = 0$ then it has at most one zero. As before we find that if $|\mathbb{F}| \ge 2m$ we find a vector that is $(\cdot,\cdot)$-isotropic but ω_i-non-isotropic for $1 \le i \le m$.

The case $\sigma \ne \mathrm{id}$ can be derived from the $\sigma = \mathrm{id}$-case just as in the C_1-case. □

Next we prove Lemmas 4.2, 4.3, and 4.4. Throughout this section, we assume that $(V, (\cdot,\cdot))$ is a vector space of dimension $2n$ respectively $2n+1$ equipped with a non-degenerate alternating, respectively symmetric bilinear form $(\cdot,\cdot)$ of Witt index n. Furthermore, we assume that the flag $F = (V_0 \le \ldots \le V_k)$ and the compatible family $\omega = (\omega_i)_{1 \le i \le k}$ of forms satisfy the Definitions 2.2, resp. 2.3 of a generalized Phan geometry of type B_n or C_n.

We start with the proof of Lemma 4.4 as this is essentially contained in [DGM].

Lemma 5.3. *Let $F = (0 = V_0 \lneq \ldots \lneq V_k = V)$ be a flag and let $\omega = (\omega_i)_i$ be σ-hermitian forms compatible with F. Let $U \pitchfork F$ and let p be a one-dimensional*

ω_k-non-degenerate space such that $p \not\leq U$. Then there is a flag F' of U and a family of forms $\omega' = (\omega'_i)_i$ compatible with F' such that

$$\langle p, W\rangle \pitchfork (F,\omega) \quad \textit{if and only if} \quad W \pitchfork (F',\omega')$$

for $W < U$.

Proof. This can be read off the proof of [DGM, Lemma 4.11]. □

Proof. (Proof of Lemma 4.4) Since U is totally isotropic with respect to $(\cdot,\cdot)$, any $W < U$ is also totally isotropic with respect to $(\cdot,\cdot)$. Hence $\{W \in Z \mid W < U\}$ consists of all $W < U$ such that $W \pitchfork (F,\omega)$ and $\langle p, W\rangle \pitchfork (F,\omega)$. Let $F_1 = (V_i \cap U)_{m\leq i\leq k}$, $\omega_1 = (\omega_i|_U)_{m\leq i\leq k}$ where $m = \max\{i \mid V_i \cap U = 0\}$. Moreover, let (F_2,ω_2) be the restriction to U of the flag whose existence is guaranteed by Lemma 5.3. Then $\{W \in Z \mid W < U\}$ is the intersection of the two generalized Phan geometries defined on U by (F_1,ω_1) and (F_2,ω_2). □

Now we proceed to the proof of Lemma 4.3.

Lemma 5.4. *Let F be a flag such that $F = F^\perp$ and let U be transversal to F. Then $U^\perp$ is also transversal to F.*

Proof. By Lemma 5.1 (2) the equality $\langle U, V_i\rangle = V$ is equivalent to $U^\perp \cap V_i^\perp = 0$ and the equality $U \cap V_i = 0$ is equivalent to $\left\langle U^\perp, V_i^\perp\right\rangle = V$. □

Lemma 5.5. *Let V be a vector space, $F = (0 = V_0 \leq \ldots \leq V_k = V)$ be a flag and $\omega = (\omega_i)_i$ be a family of σ-hermitian forms compatible with F. Let $U \lneq U' \leq V$ and assume that $U \pitchfork (F,\omega)$, $U' \pitchfork F$. Let $m_U^{U'} = \max\{i \mid \langle V_i, U\rangle \cap U' = U\}$ and $M_U^{U'} = \min\{i \mid \langle V_i, U\rangle \cap U' = U'\}$. Let $A = (U \cap V_{M_U^{U'}})^{\perp_{\omega_{M_U^{U'}}}} \cap U'$.*

The flag $F' = (V'_{m_U^{U'}} \leq \ldots \leq V'_{M_U^{U'}})$ with $V'_i = (\langle V_i, U\rangle \cap U')/U$ and forms $\omega'_i(x+U, y+U) = \omega_i(x,y)$ for $x, y \in A$ satisfies

$$W \pitchfork (F,\omega) \quad \textit{if and only if} \quad W/U \pitchfork (F',\omega')$$

for $U < W \leq U'$ where $\omega' = (\omega_i)_i$. Moreover ω' is compatible with F'.

Proof. Note that $M_U^{U'} = \min\{i \mid \langle V_i, U\rangle = V\}$: If $\langle V_i, U\rangle = V$, then trivially $\langle V_i, U\rangle \cap U' = U'$. Conversely, if $\langle V_i, U\rangle \cap U' = U'$, then $\langle V_i, U\rangle \geq U', V_i$. But $U' \pitchfork F$, so $V_i \cap U' \neq 0$ (which holds because $U' \gneq U$) implies $\langle V_i, U\rangle \geq \langle U', V_i\rangle = V$. As a consequence $k_U = M_U^{U'}$ or $k_U = M_U^{U'} + 1$. Similarly, using $U \pitchfork F$ one sees that $m_U^{U'} = \max\{i \mid V_i \cap U' = 0\}$, hence $m_U^{U'} = k_{U'} - 1$.

Note that A is a complement for $U \cap V_{M_U^{U'}}$ in U' because $U \cap V_{M_U^{U'}}$ is $\omega_{M_U^{U'}}$-non-degenerate and $\langle U, V_{M_U^{U'}}\rangle \cap U' = U'$. Note also that $U' \cap V_i = (A \cap V_i) \oplus_{\omega_i} (U \cap V_i)$ for every $m_U^{U'} \leq i \leq M_U^{U'}$: indeed, either $U \cap V_i = 0$ and $V_i \cap U' \leq \operatorname{Rad}\omega_{k_U} \cap U' \leq A$ or $i = k_U$ and $U' \cap V_{k_U} = (U \cap V_{k_U}) \oplus_{\omega_{k_U}} A$.

We show first that W is transversal to F if and only if W/U is transversal to F'. If $i \geq M_U^{U'}$, i.e., $\langle U, V_i\rangle = V$, then $\langle W, V_i\rangle = V$ and $\langle W, (V_i + U) \cap U'\rangle = U'$. Similarly, if $i \leq m_U^{U'}$, i.e. $U' \cap V_i = 0$, then $W \cap V_i = 0$ and $W \cap \langle V_i, U\rangle = U$. So it suffices, indeed, to restrict attention to the cases $m_U^{U'} \leq i < M_U^{U'}$.

If $W \cap V_i = 0$, then $W/U \cap \langle V_i, U\rangle/U = U/U$. If $\langle W, V_i\rangle = V$, then also $\langle W/U, (\langle V_i, U\rangle \cap U')/U\rangle = U'/U$. Conversely, if $W/U \cap \langle V_i, U\rangle/U = U/U$, then $W \cap V_i \leq U \cap V_i = 0$ (because $W \gneq U$). And if $W/U \cap (\langle V_i, U\rangle \cap U')/U \neq U/U$ and $\langle W/U, (\langle V_i, U\rangle \cap U')/U\rangle = U'/U$, then $\langle V_i, U'\rangle \geq V_i, U'$. As U' is transversal to V_i and $U' \cap V_i \neq 0$, we get that $\langle U', V_i\rangle = V$. The preceding discussion shows also that $k_W = k'_{W/U}$.

Now we show that $W \cap V_{k_W}$ is ω_{k_W}-non-degenerate if and only if $W/U \cap V'_{k_W}$ is ω'_{k_W}-non-degenerate. Let $x \in (W \cap V_{k_W}) \cap (W \cap V_{k_W})^{\perp_{\omega_{k_W}}}$. We write $x = u + a$ with $u \in U \cap V_{k_W}$ and $a \in A \cap V_{k_W}$. For $y \in W \cap V_{k_W} \cap A$ we then have

$$\omega'_{k_W}(x+U, y+U) = \omega_{k_W}(a, y) = \omega_{k_W}(x, y) = 0$$

where the second equality holds because $u \in U \cap V_{k_W}$ and $y \in A \cap V_{k_W}$. Hence $x + U \in \operatorname{Rad}\omega'_{k_W}$.

Conversely, assume that $x \in W \cap V_{k_W} \cap A$ is such that $\omega'_{k_W}(x+U, y+U) = 0$ for all $y \in W \cap V_{k_W} \cap A$. Then, for $z \in W \cap V_{k_W}$ and writing $z = u + w$ with $u \in U \cap V_{k_W}$ and $y \in V_{k_W} \cap A$, we get

$$\omega_{k_W}(x, z) = \omega_{k_W}(x, u) + \omega_{k_W}(x, y) = \omega'_{k_W}(x+U, y+U) = 0,$$

since $x \in V_{k_W} \cap A$ and $u \in V_{k_W} \cap U$. Hence $x \in \operatorname{Rad}\omega_{k_W}$.

To show that $\operatorname{Rad}\omega'_{k_W} \leq V'_{k_W-1}$ let $a \in A \cap V_{k_W}$ such that $a + U \in \operatorname{Rad}\omega'_{k_W}$ and let $y \in V_{k_W}$ be arbitrary. We can write $y = b + u$ with $b \in A \cap V_{k_W}$, $u \in U \cap V_{k_W}$ and have

$$0 = \omega'_{k_W}(a+U, b+U) = \omega_{k_W}(a, b) = \omega_{k_W}(a, b) + \omega_{k_W}(a, u) = \omega_{k_W}(a, y)\ ,$$

so $a \in \operatorname{Rad}\omega_{k_W} = V_{k_W-1}$ and, thus, $a + U \in V'_{k_W-1}$.

Conversely, assume that $x + U \in V'_{k_W-1}$ and write $x = a + u$ for $a \in A \cap V_{k_W}$, $u \in U \cap V_{k_W}$. Then, for $b \in A \cap V_{k_W}$, we have

$$0 = \omega_{k_W}(x, b) = \omega_{k_W}(a, b) + \omega_{k_W}(u, b) = \omega_{k_W}(a, b) = \omega'_{k_W}(a+U, b+U)\ ,$$

whence $x + U = a + U \in \operatorname{Rad}\omega'_{k_W}$. □

Lemma 5.6. *Let V be a vector space equipped with a non-degenerate alternating or σ-hermitian form $(\cdot,\cdot)$. Let $U \leq U' \leq V$ and assume that there is an $H \leq V$ such that $\dim(U'\cap H) - \dim(U\cap H) = \dim(U') - \dim(U)$ and such that $U'\cap H \leq (U\cap H)^\perp$. Let A be a complement for U in U' and define $(\cdot,\cdot)'$ on U'/U by*

$$(a+U, b+U)' = (a, b)\ .$$

Then $(\langle W^\perp, U\rangle \cap U')/U = ((\langle W, U\rangle \cap U')/U)^{\perp'}$.

Proof. We have to show that $(W\cap A)^\perp \cap A = W^\perp \cap A$. The inclusion $\geq$ is trivial, so it suffices to show the converse. By the dimension condition, A is also a complement for $U \cap H$ in $U' \cap H$ so we may just as well assume that $U' \leq U^\perp$. So assume that $a \in A$ is such that $(a, b) = 0$ for $b \in W \cap A$. Let $x \in W$ be arbitrary and write $x = b + u$ with $b \in W \cap A$ and $u \in U$. Then

$$(a, x) = (a, b) + (a, u) = (a, b)$$

because $a \in U' \leq U^\perp$ and $u \in U$. □

Proof. (Proof of Lemma 4.3) The data of Lemma 5.5 by Lemma 5.6 defines a generalized Phan geometry of type B_{n-k} or C_{n-k}. □

For the proof of Lemma 4.2 we will need a somewhat refined version of Lemma 5.5. Namely, we have to weaken the assumptions that U and U' be transversal to F and instead allow deviations by one dimension.

Definition 5.7. A space U is said to be *almost transversal* to F, if $U \cap V_i \neq 0$ implies that $\langle U, V_i \rangle$ has codimension at most 1. And U is said to be *almost transversal* to (F, ω), if it is almost transversal and $U \cap V_{k_U}$ is ω_{k_U}-non-degenerate. A space U is said to be *nearly transversal* to F, if $\langle U, V_i \rangle \neq V$ implies that $U \cap V_i$ has dimension at most 1.

Lemma 5.8. *Let V be a vector space, $F = (0 = V_0 \leq \ldots \leq V_k = V)$ be a flag and $\omega = (\omega_i)_i$ be a family of σ-hermitian forms compatible with F. Let $U \lneq U' \leq V$ and assume that U is almost transversal to (F, ω) and U' is nearly transversal to F. Let $m_U^{U'} = \max\{i \mid \langle V_i, U \rangle \cap U' = U\}$ and $M_U^{U'} = \min\{i \mid \langle V_i, U \rangle \cap U' = U'\}$. There is an A such that $U' \cap V_i = A \cap V_i \oplus_{\omega_i} U \cap V_i$. The flag $F' = (V'_{m_U^{U'}} \leq \ldots \leq V'_{M_U^{U'}})$ with $V'_i = (\langle V_i, U \rangle \cap U')/U$ and forms $\omega'_i(x + U, y + U) = \omega_i(x, y)$ for $x, y \in A$ satisfies*

$$W \pitchfork (F, \omega) \quad \text{if and only if} \quad W/U \pitchfork (F', \omega')$$

for $U < W < U'$ where $\omega' = (\omega_i)_i$. Moreover ω' is compatible with F'.

Proof. Note that, if $\dim U' - \dim U \leq 1$, the statement is trivial because there is no W with $U < W < U'$. So we may and do assume for the rest of the proof that $\dim U' - \dim U \geq 2$.

The points at which we used that $U \pitchfork F$ in the preceding proof are the following:

(1) the construction of A,
(2) $m_U^{U'} = \max\{i \mid V_i \cap U' = 0\}$,
(3) if $W/U \cap (\langle V_i, U \rangle \cap U')/U = U/U$, then W transversal F.

The following construction of A works without any assumptions on the transversality of U to F. Let A_1 be an ω_1-orthogonal complement for $U \cap V_1$ in $U' \cap V_1$. Note that $A_1 \leq \operatorname{Rad} \omega_2$, so A_1 is ω_2-orthogonal to $U \cap V_2$ and can be extended to an ω_2-orthogonal complement of $U \cap V_2$ in $U' \cap V_2$. Iterating this process we get finally that $A = A_{M_U^{U'}}$ is as desired.

To see that (2) is still true under the new hypotheses, assume that $\langle V_i, U \rangle \cap U' = U$. Then $V_i \cap U' \leq U$. If $V_i \cap U \neq 0$, then $\langle V_i, U \rangle$ would have codimension at most 1, hence $\langle V_i, U \rangle \cap U'$ would have codimension at most 1 in U'. But U has codimension 2 in U', hence $\langle V_i, U \rangle \cap U' \neq U$ contradicting the assumption. So we see that in fact $V_i \cap U = 0$.

For (3) we only have to show that, if $W \cap \langle V_i, U \rangle = U$, then W is transversal to F. As before we have that $W \cap V_i = U \cap V_i$. If $W \cap V_i = 0$, then there is nothing to show, so we may assume $U \cap V_i \neq 0$ (i.e. $i = k_U$). Then, since U is almost transversal to F, we know that $\langle U, V_i \rangle$ has codimension at most 1. So $W \gneq U$ and $W \cap \langle V_i, U \rangle = U$ imply $\langle V_i, W \rangle = V$.

Note, however, that in this new situation k_W and $k'_{W/U}$ may be distinct. Namely, it may happen that $k'_{W/U} = M_U^{U'}$, but $k_W = k_U$ (and that $M_U^{U'} \neq k_U$). But we have also seen that, if this is the case, then $\langle U, V_{k_U} \rangle$ is a hyperplane, $k_W = k_U$ and $W \cap V_{k_U} = U \cap V_{k_U}$. By assumption $U \cap V_{k_U}$ is ω_{k_U}-non-degenerate, so W is transversal to (F, ω). And W/U, being a complement for the codimension-1-radical of ω'_{k_U}, is ω'_{k_U}-non-degenerate, whence transversal to (F', ω').

The points at which we used that $U' \pitchfork F$ in the preceding proof are the following:

(1) $M_U^{U'} = \min\{i \mid \langle V_i, U \rangle = V\}$,
(2) if $W/U \cap (\langle V_i, U \rangle /U) \neq U/U$ and $\langle W/U, \langle V_i, U \rangle /U \rangle = U'/U$, then $\langle W, V_i \rangle = V$.

To see that (1) is still true under the new hypotheses, assume that $U' = \langle V_i, U\rangle \cap U' = \langle U, V_i \cap U'\rangle$. Since $\dim U' - \dim U > 1$, we see that $V_i \cap U'$ must have dimension at least two. So $\langle V_i, U'\rangle = V$, because U' is nearly transversal to F.

For (2) assume that $W/U \cap \langle V_i, U\rangle /U \neq U$ and $\langle W/U, \langle V_i, U\rangle /U\rangle = U'/U$. Then $\langle W, V_i\rangle \geq V_i, U'$. So, if $\langle W, V_i\rangle \neq V$, then $\dim(U' \cap V_i) = 1$. But this means that $W \cap V_i = U' \cap V_i$. Together with $\langle W, V_i\rangle$ this implies $W = U'$, a contradiction. Hence $\langle W, V_i\rangle = V$. □

Lemma 5.9. *Let V be a finite-dimensional vector space, and let U, U' be subspaces such that $U \lneq U'$. Let H be a hyperplane of V such that $U \not\leq H$. There is an isomorphism of vector spaces $\phi\colon U'/U \to (U' \cap H)/(U \cap H)$ such that*

$$\phi(W/U) = (W \cap H)/(U \cap H)$$

for $U \leq W \leq U'$. Similarly, let p be a one-dimensional subspace of V such that $p \not\leq U'$.

There is an isomorphism of vector spaces $\phi\colon U'/U \to \langle U', p\rangle / \langle U, p\rangle$ such that

$$\phi(W/U) = \langle W, p\rangle / \langle U, p\rangle$$

for $U \leq W \leq U'$.

Proof. Let s be a complement for $H \cap U$ in U and let b_1 be a vector that spans s. Let $b_1, \ldots, b_k$ be a basis for U and let $b_1, \ldots, b_l$ be a basis for U'. Then $b_{k+1} + U, \ldots, b_l + U$ is a basis for U'/U and $b_{k+1} + (U \cap H), \ldots, b_l + (U \cap H)$ is a basis for $(U' \cap H)/(U \cap H)$. The map ϕ that takes $b_i + U$ to $b_i + (U \cap H)$ for $k < i \leq l$ works. □

Proof. (Proof of Lemma 4.2) Note that $U \leq p^\perp$ and $U \in Z$ imply $U \in Y_0$. Hence $U \not\leq p^\perp$. For the same reason, every W in

$$\begin{aligned}\Gamma^{Y_0}_{>U} &= \{W \in Y_0 \mid W > U\} \\ &= \{U < W < U^\perp \mid W \in \Gamma, W \cap p^\perp \in \Gamma, \left\langle W \cap p^\perp, p\right\rangle \in \Gamma, \langle p, W\rangle \pitchfork (F, \omega)\}\end{aligned}$$

satisfies $W \not\leq p^\perp$. Note that $W \leq U^\perp$ implies that $W \cap p^\perp$ is singular if and only if $W = \left\langle W \cap p^\perp, U\right\rangle$ is singular. Furthermore, note that $W \cap p^\perp$ is singular if and only if $\left\langle W \cap p^\perp, p\right\rangle$ is singular. Hence $\Gamma^{Y_0}_{>U}$ consists of the spaces W with $U < W < U^\perp$ that are totally isotropic such that

(1) $W \pitchfork (F, \omega)$,
(2) $\langle W, p\rangle \pitchfork (F, \omega)$,
(3) $W \cap p^\perp \pitchfork (F, \omega)$,
(4) $\left\langle W \cap p^\perp, p\right\rangle \pitchfork (F, \omega)$.

Let A be any complement for U in $U^\perp$ and define $(\cdot, \cdot)'$ on $U^\perp/U$ by

$$(a + U, b + U)' = (a, b)$$

for $a, b \in A$. We want to show that, if W satisfies $U < W < U^\perp$, then W is totally isotropic with respect to $(\cdot, \cdot)$ if and only if W/U is totally isotropic with respect to $(\cdot, \cdot)'$ (which is the case precisely if $W \cap A$ is totally isotropic with respect to $(\cdot, \cdot)$). It is clear that, if W is totally isotropic, then $W \cap A$ is also totally isotropic. Conversely, assume that $W \cap A$ is totally isotropic. Let $x \in W$ be arbitrary and write $x = a + u$ with $a \in A$ and $u \in U$. Then, for $y \in W$ written as $y = b + v$, $b \in A$, $v \in U$, we have

$$(x, y) = (a, b) + (a, v) + (u, b) + (u, v) = (a + U, b + U) = 0$$

where the terms with u or v vanish, because $u, v \in U$ and $W \leq U^\perp$. This shows, in particular, that—although the form $(\cdot,\cdot)'$ depends on the choice of the complement A—whether or not a space $W \geq U$ is totally isotropic with respect to $(\cdot,\cdot)'$ does *not* depend on A.

Let us collect some properties of the spaces mentioned above. Clearly, U and $\langle U, p\rangle$ are transversal to (F,ω), because $U \in Z$. So by Lemma 5.4 the spaces $U^\perp$ and $U^\perp \cap p^\perp$ are transversal to F. As a consequence we find that $\left\langle U^\perp, p\right\rangle$ is nearly transversal to F, and that $U \cap p^\perp$ is almost transversal to (F,ω). Similarly, we see that $\left\langle U^\perp \cap p^\perp, p\right\rangle$ is nearly transversal to F, and that $\left\langle U \cap p^\perp, p\right\rangle = \langle U, p\rangle \cap p^\perp$ is almost transversal to (F,ω).

So, by applying Lemma 5.8 to the pairs $(U_i, U_i')_{1\leq i\leq 4} = ((U, U^\perp), (\langle U, p\rangle, \left\langle U^\perp, p\right\rangle),$ $(U \cap p^\perp, U^\perp \cap p^\perp), (\left\langle U \cap p^\perp, p\right\rangle, \left\langle U^\perp \cap p^\perp, p\right\rangle))$, we obtain flags (F_i', ω_i') on U_i'/U_i such that $W \pitchfork (F,\omega)$ if and only if $W \pitchfork (F_i', \omega_i')$ for $U_i < W < U_i'$.

Using the isomorphisms from Lemma 5.9 we obtain flags (F_i, ω_i) on $U^\perp/U$ such that

(1) $W/U \pitchfork (F_1, \omega_1)$ if and only if $W \pitchfork (F,\omega)$,
(2) $W/U \pitchfork (F_2, \omega_2)$ if and only if $\langle W, p\rangle \pitchfork (F_2', \omega_2')$ if and only if $\langle W, p\rangle \pitchfork (F,\omega)$,
(3) $W/U \pitchfork (F_3, \omega_3)$ if and only if $W \cap U^\perp \pitchfork (F_3', \omega_3')$ if and only if $W \cap U^\perp \pitchfork (F,\omega)$, and
(4) $W/U \pitchfork (F_4, \omega_4)$ if and only if $\left\langle W \cap U^\perp, p\right\rangle \pitchfork (F_4', \omega_4')$ if and only if $\left\langle W \cap U^\perp, p\right\rangle \cap U^\perp \pitchfork (F,\omega)$.

It remains to see that the flags indeed define generalized Phan geometries of type B_n, respectively C_n. More precisely, we have to see that $F_i = F_i^\perp$ for $1 \leq i \leq 4$. This follows from Lemma 5.6 using an arbitrary complement of p as H. □

6. Flip-flop systems and generalized Phan geometries

The aim of this section is to show that generalized Phan geometries arise quite naturally from flip-flop systems. Although it is possible to develop this theory over arbitrary fields, in view of our applications we will concentrate on flip-flop systems over finite fields of square order. This restriction allows us to take some shortcuts that would otherwise not be possible.

The setting is as follows. Let q be a prime power, let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a Moufang twin building arising from a group with an $\mathbb{F}_{q^2}$-locally split root group datum considered as a vertex-colored simplicial complex, and let (W, S) be the type of Δ, i.e., the associated Coxeter system. We say that Δ is *defined over* $\mathbb{F}_{q^2}$.

Definition 6.1. A *flip* θ is a simplicial involutory permutation of $\Delta_+ \cup \Delta_-$ such that

(1) $\theta(\Delta_+) = \Delta_-$ (and thus also $\theta(\Delta_-) = \Delta_+$),
(2) $\delta_\varepsilon(c, d) = \delta_{\bar\varepsilon}(c^\theta, d^\theta)$ for $(\varepsilon, \bar\varepsilon) \in \{(+,-), (-,+), (*,*)\}$, and
(3) for every panel $\operatorname{st} p$ of Δ_+ there exists a chamber $c \in \operatorname{st} p$ such that $\operatorname{coproj}_{\operatorname{st} p} c^\theta \neq c$.

The theory of flips of (twin) buildings is currently developing rapidly. In particular, there does not yet exist a uniform notion of a flip in the literature. We refer to [Hor08] for a concise account on different types of flips and their properties. Flips as defined above were introduced in [DM07].

For a chamber $c \in \Delta_+$ we denote by $\delta_\theta(c)$, the *θ-codistance of* c, the element $\delta_*(c, c^\theta)$ of W. Note that, if $R \leq \Delta_+$ is a spherical residue parallel to its image R^θ, then (R, R^θ, δ'_*) is a twin building in a natural way. In this case $\delta'_*(c, c^\theta)$ equals $\delta_*(c, c^\theta)$ for $c \in R$ if and only if R and R^θ are opposite in Δ. A *Phan chamber* with respect to a flip θ is a chamber c with $\delta_\theta(c) = 1$. The *numerical θ-codistance of c* is defined as $l_\theta(c) := l(\delta_\theta(c))$.

We start with some folklore, which we include for the sake of completeness of our arguments.

Lemma 6.2. *Let θ be a flip. Let c be a chamber and $w = \delta_\theta(c)$. If $s \in S$ is such that $l(sw) < l(w)$, then there exists a chamber d that is s-adjacent to c with $l_\theta(d) < l_\theta(c)$. In particular, the flip θ admits Phan chambers.*

Proof. Consider the s-panel $\operatorname{st} p$ of c. Note that w, being a θ-codistance, is an involution, so that $sw = ws$. A chamber $d \geq p$ can have θ-codistance w, sw or sws where w is attained and either $l(sws) = l(w) - 2$ or $sws = w$. So $\delta_*(\operatorname{st} p, \operatorname{st} p^\theta) = w$. By (3) of Definition 6.1 there is a chamber d with $\operatorname{coproj}_{\operatorname{st} p} d^\theta \neq d$, so that in particular $\delta_\theta(d) \neq w$. Hence $l_\theta(d) < l_\theta(c)$. The last statement follows by induction. □

The following fact essentially states that every flip of a building of type A_n, B_n, or C_n defined over a field is represented by a hermitian form. Its proof is a straightforward application of classical building theory, using SL_2 theory, the fundamental theorem of projective geometry (e.g., [Art57, Theorem 2.26]) and [Tit74, Theorem 8.6, (II)].

Proposition 6.3. *Let V be a vector space over a field $\mathbb{F}$ equipped with a building geometry of type A_n, B_n, or C_n and let θ be a flip. There is an involutory $\mathbb{F}$-automorphism σ and a σ-hermitian form ω on V such that U is opposite U^θ if and only if U is ω-non-degenerate.*

For later reference we also record the following special case.

Lemma 6.4. *Let $\mathbb{F}$ be a finite field and let V be a two-dimensional $\mathbb{F}$-vector space. Let θ be an involutory automorphism of the Moufang set arising from the geometry $\mathbb{P}(V)$ of type A_1.*

If there exists $a \in V$ with $\langle a\rangle^\theta \neq \langle a\rangle$, then there is an involutory $\mathbb{F}$-automorphism σ and a σ-hermitian form ω such that a point of $\mathbb{P}(V)$ is θ-fixed if and only if it is ω-isotropic. Furthermore, there are $q+1$ points that are θ-fixed, if $\sigma \neq \mathrm{id}$ and $\mathbb{F} = \mathbb{F}_{q^2}$, there are two or no θ-fixed points, if $\theta = \mathrm{id}$ and $\operatorname{char}\mathbb{F} \neq 2$, and there is one θ-fixed point, if $\theta = \mathrm{id}$ and $\operatorname{char}\mathbb{F} = 2$.

Proof. Let $b \in V$ be such that $\langle b\rangle = \langle a\rangle^\theta$. Let X_1 denote the Moufang set of $\mathbb{P}(V)$ with $\infty = \langle a\rangle$ and $0 = \langle b\rangle$ and X_2 denote the Moufang set of $\mathbb{P}(V)$ with $\infty = \langle b\rangle$ and $0 = \langle a\rangle$. Then θ can be seen as an isomorphism $X_1 \to X_2$ that fixes 0 and ∞. So by [Hua49, Theorem 1] (see also [Kno05, Theorem 3.3.1]) there is an $\mathbb{F}$-automorphism σ such that

$$\langle \alpha a + \beta b\rangle^\theta = \langle \beta^\sigma \mu a + \alpha^\sigma \xi b\rangle \ . \tag{6.1}$$

Consider the σ-hermitian map ω with $\omega(a, a) = -\mu$, $\omega(b, b) = \xi$ and $\omega(a, b) = 0$. We have $\omega(\alpha a + \beta b) = -\mu\alpha\alpha^\sigma + \xi\beta\beta^\sigma$ which vanishes if and only if

$$\mu\alpha\alpha^\sigma = \xi\beta\beta^\sigma \ . \tag{6.2}$$

Comparing this with (6.1) (and recalling that θ interchanges $\langle a\rangle$ and $\langle b\rangle$) one sees that this is precisely the condition for $\langle \alpha a + \beta b\rangle$ to be θ-stable. The number of θ-fixed points can be obtained by counting the number of solutions of (6.2). (Whether there are two or no solutions in the case $\operatorname{char}\mathbb{F} \neq 2$, $\sigma = \mathrm{id}$ depends on whether μ/ξ is a square.) □

We now turn to some elementary abstract building theory in order to link flip-flop systems with generalized Phan geometries. We start with the definition of a flip-flop system.

Definition 6.5. Given a flip of $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and a residue R of Δ_+ we denote by R_θ the complex of chambers c of R for which $l_\theta(c)$ is minimal. This is called the *flip-flop system* of R.

If R and R^θ are parallel, such that (R, R^θ, δ'_*) is a twin building with a flip $\theta|_{R\cup R^\theta}$, then R_θ consists of the Phan chambers with respect to δ'_*. If R and R^θ are not parallel, the situation is described by the following proposition. The precise description of the relative links in Proposition 6.6 below is due to Bernhard Mühlherr.

Proposition 6.6. *Let R be a spherical residue of Δ_+ and let $Q = \operatorname{coproj}_R R^\theta$. Then the map $\theta' = \operatorname{coproj}_R \circ\theta$ is a flip on the spherical twin building associated to Q.*

Moreover a chamber c of R is in R_θ if and only if the following conditions are satisfied:

(1) *c is opposite Q in R, and*
(2) *$x = proj_Q(c)$ is opposite $x^{\theta'}$ in Q.*

Proof. First let us see that θ' is a flip on Q. Since θ is an isometry, we have $(\operatorname{coproj}_R c^\theta)^\theta = \operatorname{coproj}_{R^\theta} c$, and $\operatorname{coproj}_R \operatorname{coproj}_{R^\theta} c = \operatorname{proj}_Q c$. So θ' is involutory on Q.

Since Q is spherical, in order to verify property (2) it suffices to show that $\delta_+^Q(c,d) = \delta_-^Q(c^{\theta'}, d^{\theta'})$ for $c, d \in Q$, i.e. that $\delta_+(c,d) = w_0\delta_+(c^{\theta'}, d^{\theta'})w_0$ where w_0 is the longest word in the Coxeter group of Q. Now

$$\begin{aligned}\delta_+(c,d) &= \delta_*(c, \operatorname{coproj}_{R^\theta} c)\delta_*(\operatorname{coproj}_{R^\theta} c, d)\\ &= \delta_*(c, \operatorname{coproj}_{R^\theta} c)\delta_-(\operatorname{coproj}_{R^\theta} c, \operatorname{coproj}_{R^\theta} d)\delta_*(\operatorname{coproj}_{R^\theta} d, d)\ .\end{aligned}$$

Here $\delta_*(c, \operatorname{coproj}_{R^\theta} c)$ is the longest word of the Coxeter group of Q, because for any $e \in Q$ we have

$$l_*(c,e) = l_*(c, \operatorname{coproj}_Q c) - l_-(\operatorname{coproj}_Q c, e).$$

So for $l_*(c, \operatorname{coproj}_Q c)$ to be maximal, $l_-(\operatorname{coproj}_Q c, e)$ has to be maximal. Hence $\delta_*(c, \operatorname{coproj}_{R^\theta} c)$ and $\delta_*(\operatorname{coproj}_{R^\theta} d, d)$ both are the longest word in the Coxeter group of Q^θ which is the same as that of Q.

The last condition for θ' to be a flip is, that for every panel $\operatorname{st} p$ of Q there is a chamber $c \in \operatorname{st} p$ such that $\operatorname{proj}_{\operatorname{st} p} c^{\theta'} \neq c$. But this follows immediately from the fact that $\operatorname{proj}_p \operatorname{coproj}_R = \operatorname{coproj}_P$.

Now we want to prove the characterization of R_θ. Let $c \in R$ be arbitrary. Then

$$\begin{aligned}\delta_*(c, c^\theta) &= \delta_*(c, \operatorname{coproj}_{R^\theta} c)\delta_-(\operatorname{coproj}_{R^\theta} c, c^\theta)\\ &= \delta_+(c, \operatorname{coproj}_R \operatorname{coproj}_{R^\theta} c)\delta_*(\operatorname{coproj}_R \operatorname{coproj}_{R^\theta} c, \operatorname{coproj}_{R^\theta} c)\delta_-(\operatorname{coproj}_{R^\theta} c, c^\theta)\end{aligned}$$

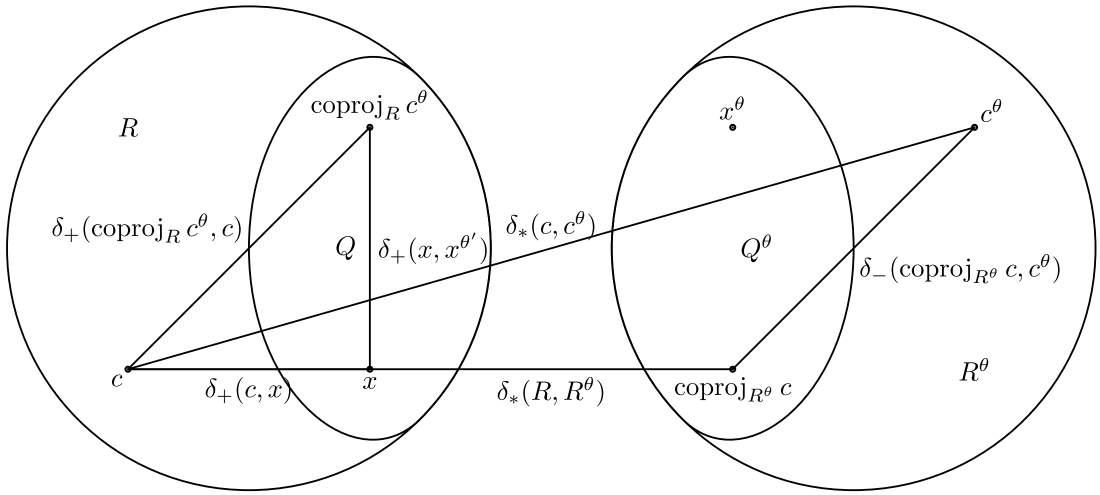

FIGURE 1. $l_*(c, c^\theta) = l_*(R, R^\theta) - 2l_+(c, x) - l_+(x, x^{\theta'})$

(see Figure 1). Here $\delta_*(\operatorname{coproj}_R \operatorname{coproj}_{R^\theta} c, \operatorname{coproj}_{R^\theta} c)$ is independent of the chamber c and we denote it just by $\delta_*(R, R^\theta)$. Note that $\operatorname{coproj}_R \operatorname{coproj}_{R^\theta} c = \operatorname{proj}_Q c =: x$. We have $\delta_-(\operatorname{coproj}_{R^\theta} c, c^\theta) = \delta_+(\operatorname{coproj}_R c^\theta, c)$ where $\operatorname{coproj}_R c^\theta = \operatorname{coproj}_Q c^\theta = x^{\theta'}$. And we can write $\delta_+(\operatorname{coproj}_R c^\theta, c)$ as $\delta_+(c, x)\delta_+(x, x^{\theta'})$. Putting this together we get that

$$l_*(c, c^\theta) = l_*(R, R^\theta) - 2l_+(c, x) - l_+(x, x^{\theta'}) \ .$$

We see that this number is minimal if and only if $l_+(c, x)$ and $l_+(x, x^{\theta'})$ are both maximal among chambers $x \in Q$ and $c \in R$. For one can first choose x so as to maximize $l_+(x, x^{\theta'})$ and then c opposite x inside R. And the longest word of Q can be obtained as $l_+(x, x^{\theta'})$ by Lemma 6.2. $\square$

Lemma 6.7. *Let Δ be a twin building of irreducible type defined over a finite field $\mathbb{F}_{q^2}$ and let θ be a flip of Δ. The involutory $\mathbb{F}_{q^2}$-automorphism σ whose existence is guaranteed by Proposition 6.3 is the same for all spherical residues R of types A_n, B_n, or C_n that are parallel to their image under θ.*

Proof. Clearly, it suffices to distinguish between the cases $\sigma = \mathrm{id}$ and $\sigma \neq \mathrm{id}$. Note that it furthermore suffices to consider panels, because if R is a residue of higher rank, we may take any Phan chamber c of (R, R^θ, δ'_*), take a panel P of c and consider P and P^θ instead of R and R^θ. It is clear, that the $\mathbb{F}_{q^2}$-automorphism for θ on P and on R is the same.

So let c be a chamber, let $s \in S$, and let R be the s-panel of c. Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be the twin root datum associated to Δ (see [AB08, Section 8.5]). The flip induces an automorphism of the Kac-Moody group G which we also denote by θ (see [Hor08]). By [CM06, Corollary A], since the type is irreducible, θ splits as a product of an inner, a diagonal, a graph, a field and a sign automorphism. In particular, it acts with the same field automorphism on all subgroups it stabilizes. Let L be a Levi factor of the stabilizer of the s-residue R of c and of R^θ (cf. [CM06, Proposition 3.6]). Then $\theta|_L$ is an automorphism of L, since θ stabilizes $R \cup R^\theta$. Note that a chamber $d \in R$ is in R_θ if and only if the root containing d is not θ-fixed.

The action of L on R is equivalent to the action of $\mathrm{SL}_2(\mathbb{F})$ on a projective line. In [DMGH, Section 4] the automorphisms of $\mathrm{SL}_2(\mathbb{F})$ are described. It is also described that $\mathrm{SL}_2(\mathbb{F})_\theta$ preserves a σ'-hermitian form where σ' is the $\mathbb{F}$-automorphism of θ. From this it is clear that the σ' is of order 2 if and only if the σ in Lemma 6.4 is of order two. □

Theorem 6.8. *Let Δ be a twin building of irreducible type defined over a finite field $\mathbb{F}_{q^2}$ and let θ be a flip of Δ. If R is a spherical residue of Δ_+ of type A_n, B_n, or C_n, then R_θ is isomorphic to a generalized Phan geometry of the respective type defined over $\mathbb{F}_{q^2}$.*

Generalized Phan geometries of type A_n have been introduced in [DGM]. They consists of the proper, non-trivial subspaces of a vector space V of dimension $n+1$ that are transversal to (F,ω) where F is a flag of V and ω is a family of σ-hermitian forms compatible with F. This theorem corresponds to [DGM, Fact 5.1] and extends it to residues of types B_n and C_n.

Proof. The spherical residue R is isomorphic to

(1) the geometry of proper, non-trivial subspaces of a vector space V of dimension $N = n+1$ over $\mathbb{F}_{q^2}$, if R is of type A_n;
(2) the geometry of non-trivial, totally isotropic subspaces of vector space V of dimension $N = 2n+1$ over $\mathbb{F}_{q^2}$ that is equipped with a non-degenerate bilinear form of Witt index n, if R is of type B_n; or
(3) the geometry of non-trivial, totally isotropic subspaces of vector space V of dimension $N = 2n$ over $\mathbb{F}_{q^2}$ that is equipped with a non-degenerate alternating form, if R is of type C_n.

The residue $Q = \mathrm{coproj}_R R^\theta$ consists of chambers that contain a certain flag $F = (0 = V_0 \lneq \ldots \lneq V_l = V)$. If R is of type B_n or C_n, this flag can be chosen to satisfy $F = F^\perp$.

Clearly, a chamber $W_0 \lneq \ldots \lneq W_N$ is opposite $W_0^\theta \lneq \ldots \lneq W_N^\theta$ in Q if and only if W_j/V_i is opposite W_j^θ/V_i in V_{i+1}/V_i for $V_i \lneq W_j \lneq V_{i+1}$. By Proposition 6.3 there are forms ω'_{i+1} on V_{i+1}/V_i such that U/V_i is opposite U^θ/V_i if and only if U is ω_{i+1}-non-degenerate for $V_i \leq U \leq V_{i+1}$. These induce forms ω_{i+1} on V_{i+1} by $\omega_{i+1}(v,v') = \omega'_{i+1}(v+V_i, v'+V_i)$. By Lemma 6.7 the $\mathbb{F}$-automorphisms σ_i such that ω_i is σ_i-hermitian are all the same.

A chamber $C = (U_0 \lneq \ldots \lneq U_N)$ of V is opposite Q if and only if every U_j is transversal to F. The projection of C to Q is the chamber consisting of the spaces $\langle U_i, V_{k_{U_i}-1}\rangle \cap V_{k_{U_i}} = \langle U_i \cap V_{k_{U_i}}, U_i\rangle$.

Proposition 6.6 gives two conditions that have to be satisfied for C to be in R_θ. The first is, that C is opposite Q, that is $U_j \pitchfork V_i$ for every i,j. This is the case if and only if $U_j \pitchfork V_{k_{U_j}}$ for every j. The second is that the projection of C to Q is mapped to an opposite chamber in Q under θ'. If we translate this using our discussion above and setting $i = k_{U_j}$, we obtain that $\langle U_j \cap V_i, V_{i-1}\rangle$ has to be ω_i-non-degenerate. □

Corollary 6.9. *Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be the affine twin building associated to $G = \mathrm{SL}_{n+1}(\mathbb{F}_{q^2}[t,t^{-1}])$ or to $G = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t,t^{-1}])$. Let θ be a flip of Δ. If R is an irreducible spherical residue of Δ_+, then R_θ is isomorphic to a generalized Phan geometry of type A_m or C_m defined over $\mathbb{F}_{q^2}$.*

Proof. The residues of Δ_+ are spherical buildings defined over $\mathbb{F}$. Since all irreducible residues of $\tilde{A}_n$ and $\tilde{C}_n$ are of type A_m or C_m, the result follows from the theorem. □

7. Applications

7.1. **Finiteness length of the unitary form of a Kac-Moody group.** Let σ be the involutory automorphism of the ring $\mathbb{F}_{q^2}[t, t^{-1}]$ that acts as the non-trivial involution on $\mathbb{F}_{q^2}$ and exchanges t and t^{-1}. Let $G = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t, t^{-1}])$, let θ be the automorphism of G that maps g to $(g^\sigma)^{-T}$, and let $K := G_\theta := \{g \in G \mid g^\theta = g\}$.

The aim of this section is to show how the topological finiteness length of K can be established by using the Main Theorem.

Theorem 7.1. *The group K is of type* F_{n-1} *but not of type* FP_n *provided* $4^{n-1}(q+1) < q^2$.

Remark 7.2.
(1) The group K is an arithmetic subgroup of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}(t))$, cf. [DGM, Remark 5.6], of local rank n. So Theorem 7.1 supports the rank conjecture, cf. [Beh98, p. 80].
(2) The topological finiteness length of the group $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t])$, which is another arithmetic subgroup of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}(t))$ of local rank n, has been established in [Abr96]. Since the class of generalized Phan geometries contains the class of geometries opposite a chamber, the proof of Theorem 7.1 below reproduces that finiteness result.

Our main tool for establishing the finiteness length of K is Brown's Criterion, see [Bro87]. We give a version that is particularly well suited for our purpose, it can be found in [Abr96] as Lemma 14.

Proposition 7.3 (Brown's criterion)**.** *Let X be a CW complex. Let Γ act on X by homeomorphisms that permute the cells. Assume that there exists an $n \geq 1$ such that the following conditions are satisfied*

(1) *X is n-connected.*
(2) *If σ is a cell of dimension $d \leq n$, then the stabilizer Γ_σ is of type F_{n-d}.*
(3) *$X = \bigcup_{j \in \mathbb{N}} X_j$ with Γ-invariant subcomplexes X_j of X which are finite complexes modulo Γ.*
(4) *$X_{j+1} = X_j \cup \bigcup_{a \in A_j} S_{a,j}$ with contractible subcomplexes $S_{a,j} \subseteq X_{j+1}$ satisfying*
    (a) *$S_{a,j} \cap S_{b,j} \subseteq X_j$ for all j and $a \neq b \in I_j$,*
    (b) *$S_{a,j} \cap X_j$ is $(n-1)$-spherical for all j and all $a \in I_j$,*
    (c) *There exist infinitely many j such that $\tilde{H}_{n-1}(S_{a,j} \cap X_j) \neq 0$ for at least one $a \in I_j$.*

Then Γ is of type F_{n-1} *and not of type* FP_n.

Recall that the group $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2}[t, t^{-1}])$ admits a twin BN-pair that gives rise to a twin building $(\Delta_+, \Delta_-, \delta_*)$ (see [AB08, Section 6.12] for the A_n case). The group G acts on the twin building by isometries, that is, $\delta_\varepsilon(c, d) = \delta_\varepsilon(gc, gd)$ for $\varepsilon \in \{+, -, *\}$ and $g \in G$.

We use the following lemma, which is taken from [Hor08].

Lemma 7.4. *The automorphism θ of G induces a flip of $(\Delta_+, \Delta_-, \delta_*)$.*

In our case, where the residue field is finite and θ induces the non-trivial field involution on the residue field, the flip θ also satisfies the following ascending version of Lemma 6.2. (Recall from the preceding section that the θ-codistance δ_θ of c is $\delta_*(c, c^\theta) \in W$ and that the numerical θ-codistance of c is the integer $l(\delta_\theta(c))$.)

Lemma 7.5. *For every panel* $\operatorname{st} p$ *of* Δ_+ *there exists a chamber* $c \geq p$ *such that* $\operatorname{coproj}_{\operatorname{st} p} c^\theta = c$*. Thus if* c *is a chamber,* $\operatorname{st} p$ *is its* s*-panel,* $w = \delta_\theta(c)$*, and* $s \in S$ *is such that* $l(sw) > l(w)$*, then there is a chamber* $d \geq p$ *such that* $l_\theta(d) > l_\theta(c)$*.*

Proof. If $\operatorname{coproj}_{\operatorname{st} p} \operatorname{st} p^\theta$ consists of only one chamber, there is nothing to show – this chamber is as desired. So we may assume that $\operatorname{coproj}_{\operatorname{st} p} \operatorname{st} p^\theta = \operatorname{st} p$. Then $\theta' := \operatorname{coproj}_{\operatorname{st} p} \circ \theta$ is a flip on $\operatorname{st} p$. Lemma 6.4 shows that there is a chamber that is θ'-fixed, i.e. satisfies $\operatorname{coproj}_{\operatorname{st} p} c^\theta = c$. The second assertion is shown as in the proof of Lemma 6.2. $\square$

Note that $(kc)^\theta = k^\theta c^\theta = kc^\theta$, so that $\delta_\theta(kc) = \delta_\theta(c)$ for $k \in K$. In particular K acts on the subcomplexes $\Delta_j = \overline{\{c \in \mathcal{C}(\Delta_+) \mid l_\theta(c) \leq j\}}$ of Δ_+. We claim that the filtration $(X_j)_{j\in\mathbb{N}} = (|\Delta_j|)_{j\in\mathbb{N}}$ and the space $X = |\Delta_+|$ satisfy the conditions of Brown's Criterion (Proposition 7.3).

We take A_j to be the set of simplices which are in Δ_{j+1} but not in Δ_j and all whose proper faces are in Δ_j. Moreover, we let $S_{a,j} = |\operatorname{st}_{\Delta_{j+1}} a|$.

Lemma 7.6. *We have* $X_{j+1} = \bigcup_{i\in I_j} X_j$*. More precisely every chamber* $c \in \Delta_{j+1}$*,* $c \notin \Delta_j$ *has a unique face that is in* A_j*.*

Proof. First let us see that, if $b \leq c$ is a face that is contained in Δ_j, then there is a facet (maximal proper face) p of c that contains b such that p is contained in Δ_j. Let $d \geq b$ be a chamber with $l_\theta(d) < l_\theta(c)$. By [Hor08], there is a chamber $c' \geq b$ that is adjacent to c with $l_\theta(c') < l_\theta(c)$. So $p = c' \cap c$ is as desired.

So $\overline{c} \cap \Delta_j$ consists of (rather, is the complex generated by) facets $p_1, \ldots, p_k$ of c. Let $v_1, \ldots, v_k$ be the vertices of c such that $v_i \notin p_i$ and let a be the simplex spanned by $v_1, \ldots, v_k$. Clearly a is in A_j. Conversely if $a' \in A_j$, $a' \leq c$, then for every facet p_j there is a vertex v_j of a' not contained in p_j (because otherwise $a' \leq p_j$). If there were another vertex, say v, then $a' \setminus v$ would not be in Δ_j, hence $a = a'$. $\square$

Lemma 7.7. *Let* $j \in \mathbb{N}$*,* $a \in A_j$ *and set* $R = \operatorname{lk}_{\Delta_+} a$*. Then* $\operatorname{lk}_{\Delta_{j+1}} a = R_\theta$ *and* $\operatorname{st}_{\Delta_{j+1}} a \cap \Delta_j = \operatorname{lk}_{\Delta_{j+1}} a * \partial a$*.*

Proof. Note that chambers $c \in \operatorname{st}_{\Delta_+} a$ satisfy $l_\theta(c) \geq j + 1$ and are in $\operatorname{st}_{\Delta_{j+1}} a$ if and only if $l_\theta(c) = j + 1$. So these are precisely the chambers that have minimal numerical θ-codistance among the chambers of $\operatorname{st}_{\Delta_{j+1}} a$. This shows the first statement.

For the second we have to see that each chamber $c \in \operatorname{st}_{\Delta_{j+1}} a$ is adjacent to a chamber of strictly shorter numerical θ-codistance along every facet that does not contain a. Let c be as above and let $\operatorname{st} p$ be a panel that does not contain a. Then $p \cap a$ is a facet of a, so it is contained in Δ_j. By the proof of the preceding lemma there is a panel of c in Δ_j, that contains $p \cap a$. This panel, of course, cannot contain a, so it is p. $\square$

Lemma 7.8 (cf. [Abe91, Lemma 2.4])**.** *Let* $j \in \mathbb{N}$ *and* $a, a' \in A_j$*. Either* $a = a'$ *or* $\operatorname{st}_{\Delta_{j+1}} a \cap \operatorname{st}_{\Delta j+1} a' \subseteq \Delta_j$*.*

Proof. By the preceding lemma if $b \in \operatorname{st}_{\Delta_{j+1}} a \cap \operatorname{st}_{\Delta j+1} a'$, but $b \notin \Delta_j$, then $b \geq a$ and $b \geq a'$. Let c be a chamber containing b. By Lemma 7.6 there is a unique face of c that is one of the a, hence $a = a'$. $\square$

Lemma 7.9. *For every $j \in \mathbb{N}$ there is a $j' \geq j$, a chamber $c \in \Delta_+$ and an $s \in S$ such that $w = \delta_\theta(c)$ satisfies that $l(w) = j'$, and $l(tw) < l(w)$ for all $t \in S \setminus \{s\}$. Moreover there is a chamber c' that is s-adjacent to c with $\delta_\theta(c') = w$.*

Proof. We start with a chamber c_0 with $l_\theta(c_0) \geq j$, which is possible by Lemma 7.5. As long as possible, we take t from $S \setminus \{s\}$ such that $l(tw_j) > l(w_j)$. By Lemma 7.5, there is a chamber c_{j+1} with $\delta_\theta(c_{j+1}) = tw_j$ or $\delta_\theta(c_{j+1}) = tw_jt$ and $l_\theta(c_{j+1}) > l_\theta(c_j)$. We let $w_{j+1} = \delta_\theta(c_{j+1})$. This process has to come to a halt because $S \setminus \{s\}$ is spherical, so at some point $w := w_j$ begins with the longest word of $S \setminus \{s\}$. The chamber $c := c_j$ is as desired, see Lemma 6.2.

Now clearly $l(sw) > l(w)$, because otherwise w would be a longest word of (W, S) which does not exist. We consider the s-panel $\operatorname{st} p$ of c. Either all chambers in $\operatorname{st} p$ have θ-codistance w or sw or they all have θ-codistance w or sws. If $\operatorname{coproj}_{\operatorname{st} p}(\operatorname{st} p^\theta)$ consists of just one chamber, any other chamber will have smaller numerical θ-codistance, i.e. θ-codistance w. Since the building is thick, there is at least one which is distinct from c. We pick one and take it to be c'. If $\operatorname{coproj}_{\operatorname{st} p}(\operatorname{st} p^\theta) = \operatorname{st} p$, we may take c' to be $\operatorname{coproj}_{\operatorname{st} p} c^\theta$. $\square$

Proof. (Proof of Theorem 7.1) We verify the prerequisites of Brown's Criterion (Proposition 7.3). The first is clear by the Solomon-Tits Theorem ([Sol69, Theorem 1], see also [Bro89, Section IV.6], [AB08, Section 4.12]).

Now we want to verify condition (2). In fact we argue that the stabilizer $\operatorname{Stab}_K(a)$ is finite for any (non-empty) simplex a of Δ_+, which implies that $\operatorname{Stab}_K(a)$ is of finiteness type F_∞.

First we consider the case that $a = c$ is a chamber. To do so, we use facts from [AB08, Section 8.2]. Fix a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$ that contains c and c^θ. By Proposition 8.15 and Proposition 8.19 of [AB08] the stabilizer of $c \cup c^\theta$ is $U_{\alpha_1} \cdots U_{\alpha_m} H$ where $\alpha_1, \ldots, \alpha_m$ is a certain ordering of the twin roots α of Σ with $c, c^\theta \in \alpha$, the group U_α is the root group of α, and H is the torus. So $\operatorname{Stab}_G(c \cup c^\theta)$ is finite, since H and every of the U_α is. Hence $\operatorname{Stab}_K(c)$ is finite.

Now let $a \in \Delta_+$ be arbitrary and let $c \geq a$ be a chamber. Once we realize that $\operatorname{Stab}_{\operatorname{Stab}_K(a)}(c)$ is finite and that c has finite orbit under $\operatorname{Stab}_K(a)$, finiteness of $\operatorname{Stab}_K(a)$ follows from the orbit-stabilizer formula. That $\operatorname{Stab}_{\operatorname{Stab}_K(a)}(c)$ is finite is clear from the above argument because $\operatorname{Stab}_{\operatorname{Stab}_K(a)}(c) \leq \operatorname{Stab}_K(c)$. That the orbit of c under $\operatorname{Stab}_K(a)$ is finite follows from the fact that Δ_+ is locally finite.

We claim that K acts transitively on the chambers that have a given θ-codistance. This immediately implies (3). Let T be the torus of diagonal matrices as an algebraic group scheme defined over the residue field $\mathbb{F}_{q^2}$. Let (Σ_+, Σ_-) be the twin apartment corresponding to T. Let c' be an arbitrary chamber and let $c \in \Sigma_+$ be such that $c^\theta \in \Sigma_-$ and $\delta_\theta(c) = \delta_\theta(c')$. Let (Σ'_+, Σ'_-) be a twin apartment such that $c' \in \Sigma'_+$, ${c'}^\theta \in \Sigma'_-$. By strong transitivity there is a $g \in G$ such that $g\Sigma_+ = \Sigma'_+$ and $g\Sigma_- = \Sigma'_-$ and there is an $h \in G$ normalizing (Σ'_+, Σ'_-) such that $hgc = c'$. Then hgc^θ is in Σ'_- and has Weyl-distance $\delta_\theta(c')$ from c', so $hgc^\theta = {c'}^\theta$. Now

$$(hg)^{-\theta} hgc = (hg)^{-\theta} c' = ((hg)^{-1} {c'}^\theta)^\theta = c$$

and, similarly, $(hg)^{-\theta}hgc^\theta = c^\theta$, so $t = (hg)^{-\theta}hg \in T(\mathbb{F}_{q^2})$. Moreover $t^\theta = (hg)^{-1}(hg)^\theta = t^{-1}$.

Let $\overline{\mathbb{F}_{q^2}}$ denote the algebraic closure of $\mathbb{F}_{q^2}$ and note that $T(\overline{\mathbb{F}_{q^2}})$ is connected with respect to the Zariski topology. Let σ be the endomorphism of raising elements of $\overline{\mathbb{F}_{q^2}}$ to the qth power and note that the fixed points set of $\overline{\mathbb{F}_{q^2}}$ under σ^2 is exactly $\mathbb{F}_{q^2}$. The map $\theta\colon T(\overline{\mathbb{F}_{q^2}}) \to T(\overline{\mathbb{F}_{q^2}}), g \mapsto g^{-\sigma}$ is an endomorphism of algebraic groups, which satisfies $g^{\theta^2} = g^{\sigma^2}$, so that $s \in T(\mathbb{F}_{q^2})$ if and only if $s = s^{\theta^2}$ for $s \in T(\overline{\mathbb{F}_{q^2}})$.

Consequently $T(\overline{\mathbb{F}_{q^2}})_\theta \subseteq T(\overline{\mathbb{F}_{q^2}})_{\theta^2} = T(\mathbb{F}_{q^2})$ is finite and so by Lang's Theorem [Lan56, Corollary to Theorem 1] there is an $s \in T(\overline{\mathbb{F}_{q^2}})$ such that $s^{-\theta}s = t$. Now

$$s^{\theta^2} = (st^{-1})^\theta = s^\theta t = s \ ,$$

so $s \in T(\mathbb{F}_{q^2})$. Hence the element hgs lies in K and it maps c to c'. Therefore K acts transitively on the chambers that have a given θ-codistance.

Property (4a) follows from Lemma 7.8. As for (4b) we know by Lemma 7.7 that $S_{a,j} \cap X_j = |(\mathrm{lk}_{\Delta_+} a)_\theta| * |\partial a|$, where $\mathrm{lk}_{\Delta_+} a$ is a residue of type A_n or C_n. If a has dimension d, then by Corollary 6.9 $(\mathrm{lk}_{\Delta_+} a)_\theta$ is isomorphic to a generalized Phan geometry of type A_{n-d} or C_{n-d}. Using the Main Theorem we get that $(\mathrm{lk}_{\Delta_+} a)_\theta$ is $(n-d-1)$-spherical in the C_n case. In the A_n case apply the Main Theorem of [DGM]. Since ∂a is a $(d-1)$-sphere, $|(\mathrm{lk}_{\Delta_+} a)_\theta| * |\partial a|$ is $(n-1)$-spherical by Proposition 3.2.

The condition (4c) follows from Lemma 7.9 as follows: Let $j \in \mathbb{N}$ be arbitrary and let c, c', j' and s be as in the Lemma. Let $\mathrm{st}\, p$ be the s-panel of c. Then p is in $A_{j'}$, because for every $t \in S \setminus \{s\}$ by Lemma 6.2 there is a chamber d that is t-adjacent to c with $l_\theta(d) < l_\theta(c)$. Now $(\mathrm{lk}_{\Delta_+} p)_\theta$ contains at least two vertices, namely $c \setminus p$ and $c' \setminus p$. So $|(\mathrm{lk}_{\Delta_+} p)_\theta| * |\partial p|$ is the join of an $(n-2)$-sphere with a space that is properly 0-spherical, hence properly $(n-1)$-spherical. □

7.2. **Phan theory.**

Theorem 7.10. *Let $\mathbb{F}_{q^2}$ be a finite field of square order and let σ be the non-trivial $\mathbb{F}_{q^2}$-involution. Let X be an irreducible diagram of rank at least 3 all irreducible rank 3-residues of which are of type A_3, B_3, or C_3. Assume that $q^2 \geq 16(q+1)$. Let Δ be a twin building of type X defined over $\mathbb{F}_{q^2}$ with a flip θ that induces σ on $\mathbb{F}_{q^2}$ in the sense of Lemma 6.7.*

Then $\Delta_\theta = \overline{\{c \in \mathcal{C}(\Delta_+) \mid \delta_\theta(c) = 1\}}$ is 2-simply connected.

Proof. The geometry of Δ_+ is 2-simply connected, see [Tit81, Theorem 3]. We want to use (the proof of) [DM07, Theorem 3.14] to see that Δ_θ is 2-simply connected if and only if Δ_+ is. To do so, we have to show that the residues of rank 3 are simply connected and the residues of rank 2 are connected. But this is true by Theorem 6.8 and Theorem 4.1, respectively [DGM, Main Theorem]. □

The following result has been published in [BGHS07], [GHN07] in the case B_n and [GHS03], [GHN06] in the case C_n. For the F_4 case Gramlich, Hoffman, Mühlherr, and Shpectorov found a proof in Oberwolfach in summer 2005, which because of its length and its tedious case distinctions they did not publish. Our proof here logically depends on the same result as the original 2005 proof, such as [DM07], [Dun05], [Tit86], but is much shorter as the concept of generalized Phan geometries allowed us to get rid of all case distinctions except the distinction of the Dynkin diagrams.

Theorem 7.11 (Gramlich, Hoffman, Mühlherr, Shpectorov 2005; unpublished)**.** *Let* $n \geq 3$*, let* $q \geq 17$*, let* $X \in \{B_n, C_n, F_4\}$*, and let* K *be a group with a weak Phan system of type* X *over* $\mathbb{F}_{q^2}$*. Then* K *is a central quotient of* $\mathrm{Spin}_{2n+1}(q)$*,* $\mathrm{Sp}_{2n}(q)$*, or the simply connected version of* $F_4(q)$*, respectively.*

Proof. By Theorem 7.10, Δ_θ is 2-simply connected, so in particular it is simply connected. Tits' Lemma [Tit86, Corollaire 1] then implies that K is the universal enveloping group of the weak Phan amalgam in K, cf. [Dun05]. The amalgams that can occur have been classified in [BS04], [Gra04], [Dun05] and they are unique up to passage to quotients. □

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