Gröbner bases for p-group algebras

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Abstract

Experiment shows that the reverse length-lexicographical word ordering consistently yields far smaller Gröbner bases for modular p-group algebras than the length-lexicographical ordering. For the so-called Jennings word ordering, based on a special power-conjugate group presentation, the associated monomial algebra is a group invariant. The package Present finds Gröbner bases for these three orderings.

Note added 9 October 2009

These notes document some experiments I did in the late 1990s to compare the usefulness of local and well orderings for noncommutative Gröbner basis calculations in modular group algebras of p-groups. They are not (or rather no longer) intended for publication. They are provided in the hope that the tables in section 5 may be of interest.

Part of the material here formed the basis for Chapter 1 of my book[1]. The system Present is no longer being maintained, but parts of it live on in the p-group cohomology package[2] for Sage.

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[1] D. J. Green, *Gröbner Bases and the Computation of Group Cohomology*, Lecture Notes in Math. vol. 1828, Springer-Verlag, 2003, xii+138pp.

[2] S. A. King and D. J. Green, *p-Group Cohomlogy Package* for the Sage computer algebra system. http://sage.math.washington.edu/home/SimonKing/Cohomology/

1 Introduction

Let G be a finite p-group. One way to do computational homological algebra over the modular group algebra $\mathbb{F}_pG$ is to realise it as a finitely presented associative algebra, together with a Gröbner basis for the relations ideal. There are then several methods available for computing minimal projective resolutions [Feustel et al. 1993], [E. Green et al. 1998], [D. Green 1999].

The algebra generators are required to lie in the Jacobson radical. Usually one chooses generators g_i for the group, and then takes the $a_i = g_i - 1$ as generators for $\mathbb{F}_pG$. See [Farkas et al. 1993] or [D. Green 1997] for an introduction to noncommutative Gröbner bases.

The choice of word ordering is of prime importance in the design of efficient Gröbner basis methods. This paper compares the usefulness in our context of three word orderings, and announces a package Present for computing Gröbner bases with respect to these orderings. They are:

- The **length-lexicographical** ordering (LL): the standard ordering for homological algebra over associative algebras.
- The **reverse length-lexicographical** ordering (RLL) allows one to read off the radical layers. Empirically, it always yields a small Gröbner basis.
- The **Jennings** ordering (see Definition 3.5) is defined on special presentations, which are constructed using the Jennings series and involve some redundant generators. Broadly speaking, it is for p-group algebras what power-conjugate presentations are for p-groups. All Gröbner bases have the same size, and in fact the associated monomial algebra is an invariant of the group's order.

The criteria for comparing orderings are:

- Smallest size of Gröbner basis, taken over all choices of generators.
- How easy is it to realise a small Gröbner basis? That is, how sensitive is size of Gröbner basis to choice of generators?
- How many properties of the Gröbner basis (e.g., maximum length of a reduced word) do not depend on choice of generators?

In Sections 2 and 3 we look at the orderings one by one. For each ordering, a method is presented for computing the Gröbner basis for the relations ideal. These methods were implemented in the package Present. Data structures and other aspects of the implementation are discussed in Section 4. Results obtained using Present are presented in Section 5, where we draw conclusions about the comparative usefulness of the orderings.

The package Present is written partly in Magma code and partly in C. See the end of the paper for details of how to obtain it.

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2 Gröbner bases

First we recall the basics about Gröbner bases in free associative algebras. Let M be the free associative monoid with 1 on a finite set A. That is, the elements of M are words in the alphabet A. Assume we have an ordering on M which is admissible in the following sense (far weaker than that of [Feustel et al. 1993]):

Definition 2.1 Let M be a free monoid. An ordering on M is called admissible if $uv_1w \leq uv_2w$ whenever u, v_1, v_2, w are elements of M with $v_1 \leq v_2$.

Now let A be a set of algebra generators for $\mathbb{F}_pG$, all lying in the Jacobson radical $J(\mathbb{F}_pG)$. Then M is the set of monomials in the free associative $\mathbb{F}_p$-algebra on A. Write I for the relations ideal: the kernel of projection from this free algebra to $\mathbb{F}_pG$. A word $w \in M$ is called a tip if it is tha largest word in the support of some element of I; if not, it is called a nontip. That is, the nontips are the words whose images in $\mathbb{F}_pG$ are linearly independent of the images of their predecessors. It follows immediately that the images of the nontips are linearly independent in $\mathbb{F}_pG$.

Lemma 2.2 *With the above assumptions, the nontips form a basis for* $\mathbb{F}_pG$.

Proof. By Nakayama's Lemma, $J^N(\mathbb{F}_pG)$ is zero for some N. All algebra generators lie in J, and so only finitely many words are nonzero in $\mathbb{F}_pG$. These span $\mathbb{F}_pG$, and contain the nontips as a spanning subset. ∎

By admissibility, the tips form an ideal in the free monoid M. This ideal has a unique smallest generating set. It consists of the *minimal tips*, a tip being called minimal if and only if all proper subwords are nontips. Define

$$\mathcal{G} := \{w - \nu(w) \mid w \text{ a minimal tip}\},$$

where $\nu(w)$ is the linear combination of nontips which in $\mathbb{F}_pG$ equals w. Then $\mathcal{G}$ and the length N words together generate I. So, in the presence of the constraint $A^N \subseteq I$, the set $\mathcal{G}$ is a Gröbner basis for I: the unique completely reduced Gröbner basis for this ordering.

Remark 2.3 For the length-lexicographical and Jennings orderings, the constraint $A^N \subseteq I$ is not necessary.

2.1 The length-lexicographical ordering

Denote by $\ell(w)$ the length of a word $w \in M$. Putting an ordering on the set A of algebra generators induces a lexicographical ordering $\leq_{\text{lex}}$ on M. The length-lexicographical ordering $\leq_{\text{LL}}$ on M is then defined as follows:

$$\begin{aligned} w_1 \leq_{\text{LL}} w_2 \text{ if } & \ell(w_1) < \ell(w_2), \\ \text{or } & \text{same length, and } w_1 \leq_{\text{lex}} w_2. \end{aligned}$$

This ordering is admissible. Moreover, each word $w \in M$ has only finitely many predecessors under $\leq_{\text{LL}}$. We do not use the Buchberger algorithm to compute the Gröbner basis, because for p-group algebras it takes an unreasonable amount of time to stop. Rather, we exploit the fact that $\mathbb{F}_p G$ is finite-dimensional, which means that it is feasible to list all the nontips.

Computing a Gröbner basis Construct G in Magma using a faithful permutation representation. Choose minimal generators $g_1, \ldots, g_r$ for G, and set $a_i = g_i - 1$. Take the a_i as the set A of algebra generators. Record the matrix for the right multiplication action of each a_i, with basis the group elements.

Using these matrices, each word is constructed as a linear combination of the group elements, starting with the smallest word and proceeding in LL-order. Gaussian elimination allows us to distinguish the tips from the nontips. We stop when we have found all the nontips: their number is known in advance. The list of minimal tips is deduced from the list of nontips by word manipulation. Change of basis gives us the matrix for the multiplication action of each generator with respect to the basis of nontips, and we can read off $\nu(w)$ for each minimal tip w from these matrices.

2.2 The reverse length-lexicographical ordering

Again defined on the free monoid M on an ordered set A, this admissible ordering is the opposite of LL. Namely,

$$w_1 \leq_{\text{RLL}} w_2 \quad \text{if and only if} \quad w_1 \geq_{\text{LL}} w_2 \,.$$

Note that RLL is not a well-ordering.

Lemma 2.4 *The* RLL*-nontips of length at least r constitute a basis for $J^r(\mathbb{F}_p G)$. Hence the length r* RLL*-nontips are a basis for a complement of J^{r+1} in J^r.*

Proof. The words of length at least r span J^r. If a linear combination of length r words lies in J^{r+1}, then the largest length r word involved is a tip. ∎

Magma can compute the smallest N such that J^N is zero (using the Jennings series, the dimensions of all radical layers can be computed). So we could obtain

the nontips by running through the words of length at most N in RLL order. But this could take forever. For example, if G is a Sylow 2-subgroup of the sporadic finite simple group Co_3, then $N = 23$ and A has size 4. Hence the number of words of length N is 2^{46}.

Computing a Gröbner basis A length r word is a tip if and only if it lies in the space spanned by its length r RLL-predecessors and by $J^{r+1}(\mathbb{F}_pG)$. Using Gaussian elimination and the matrices for the algebra generators, we can calculate a basis for J^{r+1} from one for J^r. So we start with $r = 0$ and work through to $r = N$. All length r nontips are words of the form $w.a$ with $a \in A$ and w a length $r - 1$ nontip. We work through these words in RLL-order. Once we have the nontips, we proceed as for LL.

3 The Jennings ordering

For $r \geq 1$, define $F_r(G)$ to be the rth dimension subgroup of the finite p-group G. That is,

$$F_r(G) = \{g \in G \mid g - 1 \in J^r(\mathbb{F}_pG)\}\,.$$

Clearly $G = F_1(G) \geq F_2(G) \geq \cdots$. The F_r do eventually reach the trivial group, and they form a central series, the Jennings series for G. This series need not be strictly decreasing, however. These and more facts about the Jennings series are demonstrated in Section 3.14 of [Benson 1991]

Definition 3.1 Let G be a p-group of order p^n. Elements $g_1, \ldots, g_n$ of G are Jennings pc-generators for G if the first m_1 elements g_i are minimal generators for $F_1(G)/F_2(G)$, the next m_2 are minimal generators for F_2/F_3, and so on.

Jennings pc-generators do yield a polycyclic presentation for G, as we now see.

Lemma 3.2 *Let $g_1, \ldots, g_n$ be Jennings pc-generators for G. For each $1 \leq i \leq n$, we have $g_i^p \in \langle g_{i+1}, \ldots, g_n\rangle$; and $[g_i, g_j] \in \langle g_{j+1} \ldots, g_n\rangle$ for $1 \leq i < j \leq n$.*

Proof. It is known that $[F_r, F_s] \leq F_{r+s}$ and that pth powers of elements of F_r lie in F_{pr}. ∎

Example 3.3 Let $G = \langle a, b, c, \phi\rangle$ be the following semidirect product group of order 32: the subgroup $\langle a, b, c\rangle$ is normal, and elementary abelian of order eight. The order of ϕ is four, and the effect of conjugation on the left by ϕ is

$$a \longmapsto b \longmapsto c \longmapsto abc\,.$$

Then $F_2 = \langle ab, ac, \phi^2\rangle$ and $F_3 = \langle ac\rangle$ are elementary abelian, and so a, ϕ, ab, ϕ^2, ac are Jennings pc-generators for G.

The last three generators are Jennings pc-generators for F_2. Another family of Jennings pc-generators for F_2 is ac, ab, ϕ^2. This family cannot be extended to Jennings pc-generators for G, since $[\phi, ab] = ac$. Hence to find Jennings pc-generators for G, it is not enough to take minimal generators for G and add on Jennings pc-generators for F_2.

Definition 3.4 Let $g_1, \dots, g_n$ be Jennings pc-generators for G. Then $a_1, \dots, a_n$ are Jennings generators for the group algebra $\mathbb{F}_p G$, where $a_i = g_i - 1$. We say a_i has dimension r if $a_i \in J^r(\mathbb{F}_p G) \setminus J^{r+1}(\mathbb{F}_p G)$, or equivalently $g_i \in F_r \setminus F_{r+1}$.

Order the Jennings generators as follows: $a_1 < a_2 < \cdots < a_n$. For a word w in these generators, define its dimension $\dim(w)$ in the obvious additive way:

$$\dim(a_{i_1} \dots a_{i_r}) = \sum_{j=1}^{r} \dim(a_{i_j}) \,.$$

Definition 3.5 The Jennings ordering is the following ordering on the monoid of words in the Jennings generators:

$$\begin{aligned} w_1 \leq_{\mathrm{J}} w_2 \text{ if } & \dim(w_1) > \dim(w_2), \\ \text{or } & \text{dimensions equal and } \ell(w_1) < \ell(w_2), \\ \text{or } & \text{same dimension and length, and } w_1 \geq_{\mathrm{lex}} w_2. \end{aligned}$$

Observe that the Jennings ordering is admissible; and that $\dim(w) \geq \ell(w)$ for each word w. The next result tells us that it is straightforward to calculate the Gröbner basis for the Jennings ordering.

Proposition 3.6 *Let $a_1, \dots, a_n$ be Jennings generators for $\mathbb{F}_p G$. The minimal tips are the words a_i^p and $a_j a_k$ with $j < k$. The nontips are the words $a_n^{e_n} a_{n-1}^{e_{n-1}} \dots a_1^{e_1}$ with $0 \leq e_i \leq p-1$. A nontip w lies in $J^r \setminus J^{r+1}$ if and only $\dim(w) = r$.*

Proof. We just have to show that a_i^p and $a_j a_k$ $(j < k)$ are tips. For then all nontips are of the form claimed, and the number of nontips equals the number of claimed nontips. So the nontips and hence the minimal tips are as claimed. The distribution in radical layers then follows from Jennings' theorem (Theorem 3.14.6 in [Benson 1991]).

If $\dim(a_i) = r$ then a_i^p lies in $\mathbb{F}_p F_{pr}$, which is generated by the a_j with $\dim(a_j) \geq pr$. Hence a_i^p is a linear combination of words in these a_j. Each such word either has dimension greater than pr or has length one. So a_i^p is a tip, since it has length p.

Similarly, if $\dim(a_j) = s$ and $\dim(a_k) = t$, then $c = [g_j, g_k]$ lies in F_{s+t}. Set $\gamma = c - 1$, so that $\gamma \in \mathbb{F}_p F_{s+t}$. Arguing as above, γ is a linear combination of words smaller than $a_j a_k$ or $a_k a_j$. Since $g_j g_k = c g_k g_j$, we have

$$a_j a_k = a_k a_j + \gamma + \gamma a_k + \gamma a_j + \gamma a_k a_j \,,$$

and the right hand side is a linear combination of words smaller than $a_j a_k$. ∎

Example 3.7 Let G be the cyclic group of order four. Then Jennings pc-generators for G are g_1, g_2 with $g_1^2 = g_2$, $g_2^2 = 1$ and $[g_1, g_2] = 1$. The corresponding Jennings generators for $\mathbb{F}_p G$ are $a_1 = g_1 - 1$ and $a_2 = g_2 - 1$. The minimal tips are a_1^2, $a_1 a_2$ and a_2^2; the nontips are $a_2 a_1, a_2, a_1, 1$ in ascending order; and the Gröbner basis is

$$a_1^2 + a_2\,, \qquad a_1 a_2 + a_2 a_1\,, \qquad a_2^2\,.$$

Remark 3.8 Jennings pc-generators are usually not minimal group generators. This means that there are relations involving length one words. However, the Jennings ordering does guarantee that all tips have length at least two.

Computing a Gröbner basis Get Magma to compute the Jennings series. Use this to pick Jennings pc-generators for G. The nontips and the minimal tips are known by Proposition 3.6. Proceed as for LL.

4 The implementation

In this section we provide an overview of the package Present, and describe the data structures used. Some components of Present are written in Magma code, others are written in C. The C components use M. Ringe's C MeatAxe to handle vectors and matrices over finite fields.

Groups must be constructed in Magma as permutation groups. The function `regularPermutationAction` is provided to convert pc-groups (and hence matrix groups too) into permutation groups using the regular permutation action.

4.1 Selecting minimal generators

To choose minimal generators for a p-group G in pc-presentation, Present first asks Magma for the Frattini subgroup $\Phi(G)$, and sets $H = \Phi(G)$. Then it selects an element $g \in G - H$, adds this to the list of generators, and replaces H by $\langle g, H \rangle$. This is repeated until H is equal to G. At this point the elements g constitute a minimal generating set.

There remains the question of which element of $G - H$ to select at each stage. Two generator selection methods were investigated:

- The most obvious method: pick $g \in G - H$ completely at random. Minimal generators constructed in this way are called *arbitrary* minimal generators.
- Calculate the exponent of each element of $G - H$, and pick g at random from amongst those with the smallest exponent. Such minimal generators are called *smallest exponent* minimal generators.

The merits of these two methods are compared in Section 5. By default, Present uses the smallest exponent method for LL, and the arbitrary method for RLL.

4.2 The nontips tree

The group generators are passed to the C programs as a list of permutations. For the Jennings ordering, generator dimensions are passed too. The Gröbner basis is now determined by the method for the chosen ordering given above.

The nontips are words, and all proper subwords of a nontip are again nontips. Hence the most natural way to store the nontips is as a tree, in which the children of a word w are the nontips of the form $w.a$ with $a \in A$, the set of algebra generators. On the other hand, the nontips constitute an ordered basis, and so we also want to be able to address them as an array.

In Present, the nontips are stored as a length $|G|$ array of `pathnode`s. The class `pathnode` contains

- the word w itself, its array index and its length
- an array of pointers to its children, indexed by the elements of A. (If $w.a$ is a tip, then the corresponding pointer is null.)
- a pointer to the parent word. Also, which child of the parent it is.

The nontips are stored in the array in ascending order for LL, and in descending order for RLL and the Jennings ordering. This means that the root of the tree is always located in the first entry of the array; and that the nontips tree can be built *while* the nontips are being determined.

For the second point, observe that the LL nontips are determined in ascending order. Also, the number of length r RLL nontips is known in advance, and they are determined in ascending order after all nontips of smaller length have been determined. Hence for LL and RLL, the index of each nontip is known the moment it is identified as a nontip.

Strictly, the Jennings nontips are an exception here. They are written down in lexicographical order, and then sorted into Jennings order.

4.3 The components of Present

The package Present consists of the package MakeBasis of Magma functions, and four C programs.

The Magma function `makeBasis` constructs nontips and action matrices for the LL and RLL orderings. A variant has a user-defined number of attempts at finding the smallest Gröbner basis. The function `makeJenningsBasis` constructs nontips and action matrices for the Jennings ordering. There is no need to have repeated attempts.

For each ordering, a flag allows the user to insist that the defining group generators are used. This eliminates the random component.

The C programs `makeNontips` and `makeActionMatrices` are invoked by the Magma functions to determine the nontips and the action matrices respectively. The program `writeGroebnerBasis` can then be used to write out the Gröbner basis and the minimal tips. The utility program `groupInfo` prints out relevant statistics on the groups by decoding the nontips file header.

5 Results and conclusions

The package Present was used to compare the orderings LL and RLL. The groups used in the comparison were: all 51 groups of order 32; all 267 groups of order 64; and eight Sylow p-subgroups of sporadic finite simple groups.

Define $\mathrm{LL}(G)$ to be the smallest size of a Gröbner basis for the relations ideal with respect to the LL ordering. Define $\mathrm{RLL}(G)$ similarly for RLL. Using Present we can obtain empirical approximations $\mathrm{eLL}(G)$ and $\mathrm{eRLL}(G)$ to these numbers.

By Propostion 3.6, all Gröbner bases for the Jennings ordering have the same size $\mathrm{Je}(G)$. This is $\frac{1}{2}n(n+1)$ for a group of order p^n.

5.1 Groups of order 32

There are 51 groups of order 32. The smallest Gröbner basis found in twenty attempts was recorded for each combination of: group, ordering (LL or RLL) and generator selection method (arbitrary or smallest).

For the RLL ordering, the generator selection method made no difference. For LL, smallest minimal generators yielded a smaller Gröbner basis in three cases: one less element for two groups, and three less for one group.

The comparative performance of the two orderings is shown below. The most extreme difference was for the group with Hall–Senior number 43, where eLL = 31 and eRLL = 10.

$d := \mathrm{eLL} - \mathrm{eRLL}$	$d < 0$	$d = 0$	$1 \le d \le 3$	$4 \le d \le 6$	$7 \le d \le 9$	$d \ge 10$
No. of groups	0	17	16	8	7	3

5.2 Groups of order 64

There are 267 groups of order 64. The smallest Gröbner basis found in twenty attempts was recorded for each combination of group, ordering and generator selection method.

For the RLL ordering, smallest minimal generators yielded a larger Gröbner basis in 63 cases, and a smaller Gröbner basis in one case. Each time, the difference was only one element. For LL, smallest minimal generators yielded a

		Arbitrary				Smallest exponent			
Group	Order	Min	Max	μ	σ	Min	Max	μ	σ
$\mathrm{Syl}_2(HS)$	2^9	128	338	261	50	104	127	114	5
$\mathrm{Syl}_2(M_{24})$	2^{10}	433	906	706	101	183	543	373	81
$\mathrm{Syl}_2(Co_3)$	2^{10}	502	871	690	92	212	574	405	85
$\mathrm{Syl}_3(McL)$	3^6	126	333	258	50	133	245	197	43

Table 1: Generator selection methods compared for LL. Each sample size: 30.

		Arbitrary				Smallest exponent			
Group	Order	Min	Max	μ	σ	Min	Max	μ	σ
$\mathrm{Syl}_2(HS)$	2^9	10	13	11.0	1.1	11	13	11.9	0.9
$\mathrm{Syl}_2(M_{24})$	2^{10}	15	22	17.3	1.5	15	20	17.1	1.0
$\mathrm{Syl}_2(Co_3)$	2^{10}	13	21	15.5	1.7	13	21	16.2	2.0
$\mathrm{Syl}_3(McL)$	3^6	11	14	11.8	1.0	12	14	12.6	0.9

Table 2: Generator selection methods compared for RLL. Each sample size: 30.

smaller Gröbner basis in 97 cases, with mean difference 3.3; and a larger Gröbner basis in 69 cases, with mean difference 4.6.

The comparative performance of the two orderings is shown below. The most extreme difference was for the group with Hall–Senior number 187, where eLL = 57 and eRLL = 10.

$d := \mathrm{eLL} - \mathrm{eRLL}$	$d < 0$	$d = 0$	$1 \leq d \leq 8$	$9 \leq d \leq 16$	$17 \leq d \leq 32$	$d \geq 33$
No. of groups	0	29	112	73	50	3

5.3 Sylow p-subgroups of sporadic finite simple groups

One of the main applications driving the development of group cohomology software is calculating the cohomology rings of sporadic finite simple groups. So Sylow p-subgroups of these groups are important examples for Present. Moreover, they are a good source of larger p-groups.

The sensitivity of Gröbner basis size to generator choice was investigated by looking at four such Sylow p-subgroups. The program was run thirty times for each combination of: group, ordering (LL or RLL) and generator selection method. The results are shown in Tables 1 and 2. We give the smallest and largest sizes of Gröbner basis found, together with mean and standard deviation.

In Table 3, we compare smallest Gröbner basis sizes for all three orderings by looking at eight Sylow p-subgroups of sporadic finite simple groups. Each empirical value is based on at least twenty calculations.

	Sylow 2-subgroup of						Syl_3 of	
	M_{22}	HS	M_{24}	Co_3	Suz	Ru	McL	Suz
Order	2^7	2^9	2^{10}	2^{10}	2^{13}	2^{14}	3^6	3^7
eLL	34	104	150	236	2669	3111	126	417
eRLL	8	9	15	13	21	18	11	13
Je	28	45	55	55	91	105	21	28

Table 3: Smallest known Gröbner bases. Each sample size at least 20.

5.4 Conclusions

Gröbner bases for the length-lexicographical ordering LL are consistently much larger than for the reverse length-lexicographical ordering RLL. In Table 3, the size of the smallest LL Gröbner basis lies between 14% and 33% of the group order. This means that the LL ordering is unsuitable for p-group algebras.

By contrast, RLL consistently yields very small Gröbner bases: the size behaving very roughly as the logarithm of the group order. Moreover, RLL allows one to read off the radical layers in the group algebra. Hence the RLL ordering is well-suited for computing with p-group algebras.

The Jennings ordering also seems to be suitable for computing with p-group algebras. Finding Jennings generators is easy, and it would appear that all Jennings generating sets are equally useful. Many properties of a Gröbner basis are invariants of the group's order, including the size. Gröbner bases are not as small as for RLL, but this is compensated for by the fact that fewer multiplication operations are necessary to obtain the action of the average minimal tip.

The smallest exponent generator selection method seems to deliver a small LL Gröbner basis more often than the arbitrary method does: see Table 1. However, there are numerous exceptions amongst the groups of order 64. For the RLL ordering, all evidence suggests that the smallest exponent method should be avoided: it is no more reliable at yielding a small Gröbner basis than the arbitrary method, and sometimes misses the smallest Gröbner bases.

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Software availability The package Present is available at http://www.math.uni-wuppertal.de/~green/software.html, as is a copy of the C MeatAxe. The most recent version of the C MeatAxe is available at http://www-gap.dcs.st-and.ac.uk/~gap as a GAP 3 share package. The Magma home page is at http://www.maths.usyd.edu.au:8000/u/magma/.

[3]Subsequently published as: *Gröbner Bases and the Computation of Group Cohomology*, Lecture Notes in Math. vol. 1828, Springer-Verlag, 2003, xii+138pp.

[4]Subsequently appeared as: *Trans. Amer. Math. Soc.* **353** (2001), 2915–2939.