PATCHING SUBFIELDS OF DIVISION ALGEBRAS

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ABSTRACT. Given a field F, one may ask which finite groups are Galois groups of field extensions E/F such that E is a maximal subfield of a division algebra with center F. This question was originally posed by Schacher, who gave partial results in the case $F = \mathbb{Q}$. Using patching, we give a complete characterization of such groups in the case that F is the function field of a curve over a complete discretely valued field with algebraically closed residue field of characteristic zero, as well as results in related cases.

1. INTRODUCTION

In this manuscript we consider a problem, posed by Schacher in [Sch68], that relates inverse Galois theory to division algebras. Given a field F, Schacher asked which finite groups G are *admissible* over F, meaning that there is a G-Galois field extension E/F with the property that E is a maximal subfield of an F-division algebra D. Like the original inverse Galois problem, this problem is generally open; but unlike the original problem, the set of groups that can arise in this manner is often known to be quite restricted (even for $F = \mathbb{Q}$).

This problem is a natural one because of the relationship between maximal subfields of division algebras and crossed product algebras over a given field. Crossed product algebras can be described explicitly, and are well understood. In those terms, the above problem can be rephrased as asking for the set of groups G for which there exists an F-division algebra that is a crossed product with respect to G.

Past work on admissibility has concentrated on the case of global fields. In [Sch68], Schacher gave a criterion that is necessary for admissibility of a group over the field $\mathbb{Q}$, and which he conjectured is also sufficient:

Conjecture [Sch68]
Let G be a finite group. Then G is admissible over $\mathbb{Q}$ if and only if every Sylow subgroup of G is metacyclic.

Although still open in general, many particular groups and types of groups satisfying this criterion have been shown in fact to be admissible over $\mathbb{Q}$; see for example [Son83, SS92, CS81, FF90, FV87, Fei93, Fei02, Fei04]. Also, Corollary 10.3 of [Sch68] shows that admissible groups over a global field of characteristic p have metacyclic Sylow subgroups at the primes other than p.

The main theorem of our paper is the following result (see Theorem 4.5):

Theorem
Let K be a field that is complete with respect to a discrete valuation and whose residue field k is algebraically closed. Let F be a finitely generated field extension of K of transcendence degree one. Then a finite group G with $\mathrm{char}(k) \nmid |G|$ is admissible over F if and only if every Sylow subgroup of G is abelian of rank at most 2.

The forward direction of this theorem (Proposition 3.5) is analogous to Schacher's results. As in [Sch68], a key ingredient is the equality of period and index in the Brauer group. The converse direction to our theorem is proven using patching methods from [HH07], an approach that is not available in the case of global fields, and which makes possible a variety of results for function fields as in the above theorem. In fact, the equality of period and index used in the forward direction can also be proven by such methods (see [HHK08]).

In the equal characteristic zero situation, the base field K is quasi-finite (i.e. perfect with absolute Galois group $\widehat{\mathbb{Z}}$; see [Ser79], XIII.2), and F is thus analogous to a global field, viz. to a function field over a finite field. In that situation, our main theorem provides a necessary and sufficient condition for an arbitrary finite group to be admissible over F.

This manuscript is organized as follows. Section 2 provides background and introduces the notion of an element of the Brauer group being "determined by ramification". This notion is used in Section 3 to obtain a criterion (Theorem 3.3) that is then applied to prove the forward direction of our main theorem. That section concludes with two corollaries on admissibility for rational function fields. Finally, Section 4 recalls ideas from [HH07] concerning patching, and uses them to prove the converse direction of our main result, Theorem 4.5.

2. Brauer Groups and Ramification

The notions of admissibility and crossed product algebras can be understood in terms of Brauer groups of fields, and we review that relationship in this section. In the case of discretely valued fields, we also describe properties of the ramification map, which associates a cyclic field extension of the residue field to each Brauer class. This ramification map is an important tool in studying the Brauer group, and Proposition 2.3 below will play a key role in the next section.

First, we recall some standard facts about central simple algebras and (central) division algebras; for more detail see [Pie82], [Sal99], [Jac96] or [GS06]. The *degree* of a central simple F-algebra A is the square root of its F-dimension, and its *(Schur) index* is the degree of the division algebra D such that $A \cong \mathrm{Mat}_r(D)$ for some $r \geq 1$. Equivalently, the index is the degree of a minimal *splitting field* for A, i.e. a field extension E/F such that A splits over E in the sense that $A \otimes_F E$ is a matrix algebra over F. In fact the index divides the degree of any splitting field ([Pie82], Lemma 13.4). The *Brauer group* $\mathrm{Br}(F)$ of F consists of the *Brauer equivalence classes* of central simple F-algebras, with the operation of tensor product. Here two algebras are declared equivalent if they have isomorphic underlying division algebras. The order of the class $[A]$ of A in $\mathrm{Br}(F)$ is called its *period* (or *exponent*), and per $A \mid \mathrm{ind}\, A$ ([Pie82], Proposition 14.4b(ii)). We will also write $\mathrm{per}(a)$ for the order of an element a in an arbitrary abelian group.

The Brauer group of F may be explicitly identified with the second Galois cohomology group $H^2(F, \mathbb{G}_m)$. Concretely, if E/F is a G-Galois extension and $c : G \times G \to E^\times$ is a 2-cocycle with respect to the standard action of G on $E^\times$, there is an associated central simple F-algebra $\Delta(E, G, c)$. As an E-vector space, this algebra has a basis u_σ in bijection with the elements σ of the group G. Multiplication in this algebra is given by the formulas

$$u_\sigma u_\tau := c(\sigma, \tau) u_{\sigma\tau} \qquad u_\sigma x := \sigma(x) u_\sigma$$

for $\sigma, \tau \in G$, $x \in E$. We say that an F-algebra is a *crossed product* if it is isomorphic to an algebra of this form. This construction gives rise

([Sal99], Corollary 7.8) to an isomorphism

$$H^2(F, \mathbb{G}_m) \to \mathrm{Br}(F),$$

implying in particular that every central simple algebra is Brauer equivalent to a crossed product algebra.

If A is a central simple F-algebra of degree n, then a commutative separable F-subalgebra of A is maximal among all such subalgebras if and only if its dimension over F is n ([Gro68a], Proposition 3.2). In the case that E is a subfield, we will refer to E as a *maximal subfield*, following [Sch68]. Note that "maximal subfield" therefore means not merely that E is maximal as a subfield, but in fact that it is maximal as a commutative separable subalgebra. In particular, if $A = \mathrm{Mat}_2(\mathbb{C})$, then A has no maximal subfields, since $\mathbb{C}$ has no proper algebraic extensions. In the case that A is a division algebra, however, any commutative separable subalgebra must actually be a field, and so the two notions of maximality agree. (In [Pie82], §13.1, the term "strictly maximal" is used for what we call "maximal"; and the term "maximal" is used there in the weaker sense.)

If $A = \Delta(E, G, c)$, then E must be a maximal subfield of A, since $\deg A = [E : F]$. Conversely, if E is a maximal subfield of A, and E is a G-Galois extension of F, then E is a splitting field for A ([Pie82], Theorem 13.3) and hence $A = \Delta(E, G, c)$ for some 2-cocycle c ([Sal99], Corollary 7.3). Thus G is admissible over F if and only if there is a crossed product F-division algebra with respect to G.

Now consider a field F together with a discrete valuation v, completion F_v, and residue field k_v at v. Let $\mathrm{Br}(F)'$ (resp. $H^1(k_v, \mathbb{Q}/\mathbb{Z})'$) denote the subgroup of elements whose period is prime to $\mathrm{char}(k_v)$ (so in particular, $\mathrm{Br}(F)' = \mathrm{Br}(F)$ etc. if $\mathrm{char}(k_v) = 0$). Recall from [Sal99], Chapter 10, that there is a *ramification map* $\mathrm{ram}_v : \mathrm{Br}(F)' \to H^1(k_v, \mathbb{Q}/\mathbb{Z})'$, which factors through the corresponding map on $\mathrm{Br}(F_v)'$. More generally, if Ω is a set of discrete valuations on F, then we may consider the intersection of the above subgroups of $\mathrm{Br}(F)$, as v ranges over the elements of Ω. Below, the choice of Ω will be clear from the context, and we will simply write $\mathrm{Br}(F)'$ for the intersection. In that situation, for each $v \in \Omega$ the map $\mathrm{ram}_v : \mathrm{Br}(F)' \to H^1(k_v, \mathbb{Q}/\mathbb{Z})'$ is defined on this intersection.

Definition 2.1
Let F be a field, Ω a set of discrete valuations on F, and $\alpha \in \mathrm{Br}(F)'$. We say that α is *determined by ramification* (with respect to Ω) if there is some $v \in \Omega$ such that

$$\mathrm{per}(\alpha) = \mathrm{per}(\mathrm{ram}_v \, \alpha).$$

Note that if ram_v is injective for some $v \in \Omega$, then every class in $\mathrm{Br}(F)'$ is determined by ramification. This rather special situation is generalized in Proposition 2.3 below, which will play an important role in Section 3. First we make a definition and introduce some notation.

Definition 2.2
Let F be a field and Ω a set of discrete valuations on F. We define the *unramified Brauer group* of F (with respect to Ω), denoted by $\mathrm{Br}_u(F)'$, to be the kernel of

$$\mathrm{Br}(F)' \xrightarrow{\;\prod \mathrm{ram}_v\;} \prod_{v \in \Omega} H^1(k_v, \mathbb{Q}/\mathbb{Z})'.$$

For a prime p and an integer n, let n_p denote the largest power of p that divides n. A Brauer class $\alpha \in \mathrm{Br}(F)$ may be expressed as $\alpha = \alpha_p + (\alpha - \alpha_p)$ for some unique class α_p (viz. a prime-to-p multiple of α) satisfying $\mathrm{per}(\alpha_p) = (\mathrm{per}(\alpha))_p$ and $(\mathrm{per}(\alpha - \alpha_p))_p = 1$. Moreover, this class satisfies $\mathrm{ind}(\alpha_p) = \mathrm{ind}(\alpha)_p$ and $(\mathrm{ind}(\alpha - \alpha_p))_p = 1$. (see [Pie82], §14.4, Primary Decomposition Theorem and its proof). Given a division algebra D, let D_p be the division algebra in the class $[D]_p$; i.e. $[D]_p = [D_p]$.

Proposition 2.3
Let F be a field, Ω a nonempty set of discrete valuations on F with $\mathrm{Br}_u(F)' = 0$, and suppose $\alpha \in \mathrm{Br}(F)$. Then for every prime p not equal to any residue characteristic of Ω, α_p is determined by ramification.

Proof. By definition, $\alpha_p \in \mathrm{Br}(F)'$ for p as in the statement. By the hypothesis on the unramified Brauer group, the map

$$\mathrm{Br}(F)' \xrightarrow{\;\prod \mathrm{ram}_v\;} \prod_{v \in \Omega} H^1(k_v, \mathbb{Q}/\mathbb{Z})'$$

is injective. It follows that the image of α_p under the map $\prod \mathrm{ram}_v$ has order equal to $\mathrm{per}(\alpha_p) = \mathrm{per}(\alpha)_p$, a power of p. Since the least common multiple of p-powers is their maximum, the order of the image is the maximum value of $\mathrm{per}(\mathrm{ram}_v(\alpha_p))$, for $v \in \Omega$. Hence $\mathrm{per}(\alpha_p) = \mathrm{per}(\mathrm{ram}_v(\alpha_p))$ for some $v \in \Omega$. So α_p is determined by ramification. $\square$

3. Admissibility Criteria

In this section, we give a criterion for a group to be admissible. It is based on an auxiliary lemma together with an analysis of the ramification map defined in Section 2. In Theorem 3.3, we give a ramification condition on a crossed product algebra that implies that the Sylow subgroups of the associated group are metacyclic, or even abelian of

rank at most two. We then describe a situation in which this condition is met (Proposition 3.5), thereby providing the forward direction of the main result, Theorem 4.5. Finally, we state some corollaries for (retract) rational varieties.

We begin with the auxiliary lemma based on Kummer theory.

Lemma 3.1

Let E/F be a finite Galois extension of complete discretely valued fields, with Galois group G and with cyclic residue field extension ℓ/k of characteristic not dividing $|G|$. Then:

(a) *The Galois group G is metacyclic.*
(b) *Let e be the ramification index of E over F. Then G is abelian (of rank at most 2) if and only if F contains a primitive e-th root of unity.*

Proof. (a) Let E_0 be the maximal unramified extension of F contained in E. Thus E/E_0 is Galois and totally ramified ([Ser79], Corollary III.5.3), and it is tamely ramified since $\mathrm{char}(k) \nmid [E : E_0]$. Hence its Galois group is equal to its inertia group, which is cyclic ([Ser79], Corollaries 2 and 4 of IV.2). Note that ℓ is the residue field of E_0.

Since the residue field extension ℓ/k is cyclic, Proposition I.7.20 and Corollary II.3.4 of [Ser79] imply that the unramified extension E_0/F is also cyclic. Thus E/F is metacyclic.

(b) We first show that E_0 always contains a primitive e-th root of unity. Since the tamely ramified extension E/E_0 is totally ramified, its degree is equal to its ramification index e. By [Has02], Chapter 16, p. 249, E/E_0 is a radical extension $E = E_0(z)$ for some $z \in E$ with $f := z^e \in E_0$. The conjugates of z over E_0 are just the multiples of z by the primitive e-th roots of unity in a fixed algebraic closure of E. But E/E_0 is Galois; so these conjugates and hence these roots of unity lie in E. Thus there is a primitive e-th root of unity in the residue field of E, or equivalently of E_0 (since E/E_0 is totally ramified). By Hensel's Lemma, there is a primitive e-th root of unity ζ in E_0.

By the first part of the proof, the extensions E/E_0 and E_0/F are cyclic. Let σ be a generator of $\mathrm{Gal}(E/E_0)$ for which $\sigma(z) = \zeta z$, and let $\tau \in G$ be a lift of a generator of $\mathrm{Gal}(E_0/F)$. Thus G is abelian if and only if σ and τ commute; and it suffices to show that this latter condition is equivalent to the assertion that $\zeta \in F$.

Since E_0/F is Galois, $\langle \sigma \rangle$ is normal, and thus $\tau^{-1}\sigma\tau = \sigma^i$ for some i prime to e with $1 \le i < e$. Moreover $\tau(z)^e = \tau(f) \in E_0$, so $\sigma(\tau(z)) = \zeta^j\tau(z)$ for some non-negative integer $j < e$. Hence $r := \tau(z)/z^j$ is fixed by σ; so $r \in E_0$.

Let v denote the discrete valuation on F, normalized so that the valuation of a uniformizer is 1. We may extend v to a discrete valuation on E, taking values in $\frac{1}{e}\mathbb{Z}$. Since τ maps the maximal ideal of the valuation ring of E to itself, it follows that $v(\tau(h)) = v(h)$ for $h \in E$. Also, v takes integral values on E_0, because E_0 is unramified over F.

In particular, $v(f), v(r) \in \mathbb{Z}$. We have $v(f) = v(z^e) = v(\tau(z)^e) = v(z^{je}r^e) = v(f^j r^e) = jv(f) + ev(r) \in jv(f) + e\mathbb{Z}$. But $v(f)$ is prime to e since E/E_0 is totally ramified. So $j = 1$, i.e. $\sigma(\tau(z)) = \zeta\tau(z)$. It follows that $\tau(\zeta)^i\tau(z) = \tau(\zeta^i z) = \tau\sigma^i(z) = \sigma\tau(z) = \zeta\tau(z)$, where the third equality uses that $\tau^{-1}\sigma\tau = \sigma^i$. That is, $\tau(\zeta)^i = \zeta$. Thus $\zeta \in F$ if and only if $i = 1$; i.e. if and only if $\sigma\tau = \tau\sigma$. $\qquad\square$
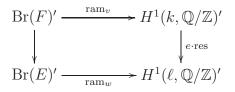
The next proposition makes it possible to verify the cyclicity of the residue field extension required in the previous lemma, in certain cases when the image of a Brauer class under the ramification map becomes trivial upon an extension E/F. It uses the following connection between 1-cocycles and cyclic extensions:

Consider a field k, with fixed separable closure k^{s} and absolute Galois group $G_k := \mathrm{Gal}(k^{\mathrm{s}}/k)$. An element $\psi \in H^1(k, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ with $\mathrm{per}(\psi) = n$ defines an n-cyclic field extension $k' := (k^{\mathrm{s}})^{\ker\psi}$ of k. Let ℓ be an algebraic field extension of k, viewed as a subfield of an algebraic closure of k containing k^{s}. Then there is a natural inclusion of G_ℓ into G_k and hence a restriction map $\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(G_\ell, \mathbb{Q}/\mathbb{Z})$. By Galois theory the image of ψ under this map defines the compositum $k'\ell/\ell$ as a cyclic extension of ℓ.

Proposition 3.2
Let $(E, w)/(F, v)$ be an extension of discretely valued fields and let ℓ/k denote the residue field extension. Let $\alpha \in \mathrm{Br}(F)'$ (with respect to $\{v\}$), and suppose that $\mathrm{per}(\mathrm{ram}_v(\alpha)) = [E : F]$. Let $\alpha_E \in \mathrm{Br}(E)'$ be the element induced by α. If $\mathrm{ram}_w(\alpha_E) = 0$ then ℓ is cyclic over k and w is the only discrete valuation on E extending v.

Proof. Let $\chi := \mathrm{ram}_v(\alpha) \in H^1(k, \mathbb{Q}/\mathbb{Z})'$ and let $e := e_{w/v}$ be the ramification index of w over v. By [Sal99], Theorem 10.4, there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Br}(F)' & \xrightarrow{\ \mathrm{ram}_v\ } & H^1(k, \mathbb{Q}/\mathbb{Z})' \\
\downarrow & & \downarrow{\scriptstyle e\cdot\mathrm{res}} \\
\mathrm{Br}(E)' & \xrightarrow[\ \mathrm{ram}_w\ ]{} & H^1(\ell, \mathbb{Q}/\mathbb{Z})'
\end{array}
$$

where the vertical map on the right is the composition of the restriction map with multiplication by e.

Hence $e \operatorname{res}(\chi) = \operatorname{ram}_w(\alpha_E) = 0$. Let k' be the cyclic extension of k defined by $e\chi$. Then since $\operatorname{res}(e\chi) = e \operatorname{res}(\chi) = 0$, $k'\ell/\ell$ is a trivial extension, i.e. $k' \subseteq \ell$. Moreover, $[k' : k] = \operatorname{per}(e\chi) \geq \operatorname{per}(\chi)/e$, and thus

$$[E : F] = \operatorname{per}(\chi) \leq [k' : k] \cdot e \leq [\ell : k] \cdot e \leq [E : F].$$

So the chain of inequalities is actually a chain of equalities and hence $\ell = k'$ is cyclic and $[E : F] = [\ell : k] \cdot e$. Hence w is the only discrete valuation on E lying over v. $\qquad \square$

Recall the notation n_p and α_p introduced before Proposition 2.3.

Theorem 3.3
Let F be a field, Ω a set of discrete valuations on F, and p a prime unequal to all of the residue characteristics of Ω. Let D be a crossed product division algebra over F with respect to a finite group G. If $[D]_p \in \operatorname{Br}(F)'$ is determined by ramification and $(\operatorname{per} D)_p = (\operatorname{ind} D)_p$, then every p-Sylow subgroup P of G is metacyclic. If moreover F contains a primitive $|P|$-th root of unity, then P is abelian of rank at most 2.

Proof. By hypothesis there is a discrete valuation $v \in \Omega$ such that $\operatorname{per}([D]_p) = \operatorname{per}(\operatorname{ram}_v[D]_p)$. Since D is a crossed product with respect to G, there is a maximal subfield L of D that is G-Galois over F. Moreover D is split by L ([Sal99], Corollary 7.3), i.e. $D_L := D \otimes_F L$ is split.

Let P be a p-Sylow subgroup of G, and let $w_1, \ldots, w_n$ denote the extensions of v to the fixed field L^P. Then $[L^P : F] = \sum_{i=1}^{n} e_{w_i/v} f_{w_i/v}$ where $e_{w_i/v}$ (resp. $f_{w_i/v}$) is the ramification index (resp. residue degree) of w_i over v. Since p does not divide $[L^P : F]$, there exists an index j for which $p \nmid e_{w_j/v} f_{w_j/v}$. Let $w := w_j$, and let u be any discrete valuation on L extending w.

Let β be the class $[D \otimes_F L^P]$ in $\operatorname{Br}(L^P)$. In order to invoke Proposition 3.2, we will show that the following three conditions are satisfied:

(i) $\beta \in \operatorname{Br}(L^P)'$ (with respect to $\{w\}$);
(ii) $\operatorname{per}(\operatorname{ram}_w \beta) = |P|$;
(iii) $\operatorname{ram}_u(\beta_L) = 0$.

For (i), the algebra D is split by L; hence the class $\beta = [D \otimes_F L^P]$ is split by L, which is of degree $|P|$ over L^P. Consequently, $\operatorname{per} \beta \mid \operatorname{ind} \beta \mid$

$|P|$. Hence per β is a power of p, which is not the residue characteristic of w. Thus $\beta \in \mathrm{Br}(L^P)'$ as we wanted to show.

For (ii), consider the commutative diagram ([Sal99], Theorem 10.4)

$$
\begin{array}{ccc}
\mathrm{Br}(F)' & \xrightarrow{\ \mathrm{ram}_v\ } & H^1(k, \mathbb{Q}/\mathbb{Z})' \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle e_{w/v}\cdot\mathrm{res}} \\
\mathrm{Br}(L^P)' & \xrightarrow[\ \mathrm{ram}_w\ ]{} & H^1(\ell, \mathbb{Q}/\mathbb{Z})'
\end{array}
$$

where k and ℓ denote the respective residue fields. The vertical right hand map in the diagram is the composition of multiplication by $e_{w/v}$ and the restriction map. Since $e_{w/v}$ is prime to p, multiplication by $e_{w/v}$ is injective on p-power torsion. Also, the degree of ℓ/k is prime to p, and the composition of restriction and corestriction is multiplication by that degree. So the restriction map is also injective on p-power torsion. Hence the vertical right hand map in the above diagram is injective on p-power torsion.

Now $[D_p] = [D]_p = r[D]$ for some $r \in \mathbb{N}$ that is prime to p, and so $[D_p \otimes_F L^P] = r\beta$. Thus $[D]_p$ is sent to $r\beta$ under the left hand vertical map ϕ. Since per β is a power of p, so is per$(\mathrm{ram}_w \beta)$, and hence

$$
\mathrm{per}(\mathrm{ram}_w \beta) = \mathrm{per}(\mathrm{ram}_w r\beta) = \mathrm{per}(\mathrm{ram}_w(\phi([D]_p)) = \mathrm{per}(\mathrm{ram}_v([D]_p).
$$

Recall that D is a crossed product with respect to G, so $\mathrm{ind}(D)_p = |G|_p = |P|$, and this quantity equals per$(D)_p$ by hypothesis. Thus, using again that $[D]_p$ is determined by ramification at v, per$(\mathrm{ram}_w \beta) = \mathrm{per}(\mathrm{ram}_v[D]_p) = (\mathrm{per}\, D)_p = |P|$, showing (ii).

Finally, (iii) is immediate since $\mathrm{ram}_u(\beta_L) = \mathrm{ram}_u([D \otimes_F L]) = 0$ because D splits over L.

Now Proposition 3.2 (applied to the P-Galois extension L/L^P) implies that the residue field extension ℓ/k is cyclic and that u is the unique discrete valuation on L extending w on L^P. Hence the extension $L_u/(L^P)_w$ of the corresponding completions also has Galois group P, and the conclusions of the theorem follow immediately from Lemma 3.1. $\qquad\square$

Corollary 3.4
Let F be a field and let Ω be a nonempty set of discrete valuations on F with $\mathrm{Br}_u(F)' = 0$. Let p be a prime unequal to all of the residue characteristics of Ω, and assume that per $\alpha_p = \mathrm{ind}\,\alpha_p$ for every $\alpha \in \mathrm{Br}(F)$. If G is admissible over F, then every Sylow p-subgroup P is metacyclic. If moreover F contains a primitive $|P|$-th root of unity, then P is abelian of rank at most two.

Proof. Since G is admissible over F, there is a crossed product division algebra D over F with respect to G. By Proposition 2.3 and the assumption on $\mathrm{Br}_u(F)'$, it follows that $[D]_p \in \mathrm{Br}(F)'$ is determined by ramification. By hypothesis, $(\mathrm{per}[D])_p = \mathrm{per}([D]_p) = \mathrm{ind}([D]_p) = (\mathrm{ind}[D])_p$. So the conclusion follows from Theorem 3.3.   □

Using this corollary, we obtain the following proposition, which provides a necessary condition for admissibility and implies the forward direction of our main result, Theorem 4.5 below.

Proposition 3.5
Let K be a complete discretely valued field with algebraically closed residue field k, and let F be a finitely generated field extension of K of transcendence degree one. Let G be a finite group that is admissible over F. Then for every $p \neq \mathrm{char}(k)$, the Sylow p-subgroups of G are each abelian of rank at most two.

Proof. Let $p \neq \mathrm{char}(k)$ be a prime that divides the order of G, and let P be a Sylow p-subgroup of G. Let T denote the valuation ring of K, and choose a regular connected projective T-curve $\widehat{X}$ with function field F. (Given F, such an $\widehat{X}$ always exists by resolution of singularities; cf. [Abh69] or [Lip75].) Let Ω be the set of discrete valuations corresponding to the codimension 1 points of $\widehat{X}$. Thus p is not a residue characteristic of any valuation v in Ω. With respect to Ω we have $\mathrm{Br}_u(F)' = 0$, by results in [COP02] and [Gro68b]. (Namely, $\mathrm{Br}(\widehat{X}) = 0$ by [COP02], Corollary 1.10(b) (Corollary 1.9(b) in the preprint), and then $\mathrm{Br}_u(F)' = \mathrm{Br}(\widehat{X})' = 0$ by Proposition 2.3 of [Gro68b].)

Since $p \neq \mathrm{char}(k)$, it follows by Theorem 5.5 of [HHK08] (or by [Lie08], Theorem 5.3) that $\mathrm{per}\,\alpha_p = \mathrm{ind}\,\alpha_p$ for every $\alpha \in \mathrm{Br}(F)$. Again since $p \neq \mathrm{char}(k)$, the algebraically closed residue field k contains a primitive $|P|$-th root of unity; thus by Hensel's Lemma, so does K and hence F. Thus the hypotheses of Corollary 3.4 hold, and therefore each Sylow p-subgroup of G is abelian of rank at most 2.   □

We conclude this section with two corollaries (which are not used in the remainder of the paper). They use Theorem 3.3 to give an obstruction for a function field to be rational, or even retract rational (which is more general; see [Sal99], p.77).

Corollary 3.6
Let F be a retract rational field extension of an algebraically closed field C. Suppose that there is an F-division algebra D with $\mathrm{per}\,D =$

ind D, such that D is a crossed product for a group G with $\mathrm{char}(C) \nmid |G|$. Then all Sylow subgroups of G are abelian of rank at most 2.

Proof. Let p be a prime that divides the order of G and let P be a Sylow p-subgroup. Let Ω be the set of discrete valuations on F that are trivial on C. By Proposition 11.8 of [Sal99], $\mathrm{Br}_u(F)'$ is isomorphic to $\mathrm{Br}(C)'$, the subgroup of $\mathrm{Br}(C)$ consisting of elements of order not divisible by $\mathrm{char}(C)$. (Note that $\Omega = R_F$ in the notation in [Sal99].) But $\mathrm{Br}(C)$ and hence $\mathrm{Br}(C)'$ is trivial, because C is algebraically closed. Thus by Proposition 2.3, $[D]_p \in \mathrm{Br}(F)'$ is determined by ramification. Since C is algebraically closed of characteristic unequal to p, it contains a primitive p^r-th root of unity for all $r \in \mathbb{N}$. Hence so does F. But $\mathrm{per}\, D = \mathrm{ind}\, D$ by assumption, so Theorem 3.3 implies the assertion. $\qquad\square$

Remark 3.7

This should be compared to a result of Saltman ([Sal90]), stating that if G is a finite group with at least one Sylow subgroup that is not abelian of rank at most 2, then the center of a *generic* crossed product algebra with group G is not rational.

In the case of a rational function field in two variables over an algebraically closed field, the condition on period and index is satisfied by [dJ04]; so from Corollary 3.6 we obtain the following

Corollary 3.8

Suppose $F = C(x, y)$ where C is algebraically closed. If G is admissible over F with $\mathrm{char}(C) \nmid |G|$ then every Sylow subgroup of G is abelian of rank at most 2.

4. ADMISSIBILITY AND PATCHING

To prove the converse direction of our main theorem, we first recall the method of patching over fields as introduced in [HH07]. Throughout this section, T is a complete discrete valuation ring with uniformizer t, fraction field K, and residue field k. We consider a finitely generated field extension F/K of transcendence degree one. Let $\widehat{X}$ be a regular connected projective T-curve with function field F such that the reduced irreducible components of its closed fiber X are regular. (Given F, such an $\widehat{X}$ always exists by resolution of singularities; cf. [Abh69] or [Lip75].) Let $f : \widehat{X} \to \mathbb{P}^1_T$ be a finite morphism such that the inverse

image S of $\infty \in \mathbb{P}^1_k$ contains all the points of X at which distinct irreducible components meet. (Such a morphism exists by Proposition 6.6 of [HH07].) We will call $(\widehat{X}, S)$ a *regular T-model* of F.

Following Section 6 of [HH07], for each point $Q \in S$ as above we let R_Q be the local ring of $\widehat{X}$ at Q, and we let $\widehat{R}_Q$ be its completion at the maximal ideal corresponding to the point Q. Also, for each connected component U of $X \smallsetminus S$ we let R_U be the subring of F consisting of the rational functions that are regular at the points of U, and we let $\widehat{R}_U$ denote its t-adic completion. If $Q \in S$ lies in the closure $\bar{U}$ of a component U, then there is a unique branch $\wp$ of X at Q lying on $\bar{U}$ (since $\bar{U}$ is regular). Here $\wp$ is a height one prime of $\widehat{R}_Q$ that contains t, and we may identify it with the pair (U, Q). We write $\widehat{R}_\wp$ for the completion of the discrete valuation ring obtained by localizing $\widehat{R}_Q$ at its prime ideal $\wp$. Thus $\widehat{R}_Q$ is naturally contained in $\widehat{R}_\wp$.

In the above situation, with $\wp = (U, Q)$, there is also a natural inclusion $\widehat{R}_U \hookrightarrow \widehat{R}_\wp$. To see this, first observe that the localizations of R_U and of R_Q at the generic point of $\bar{U}$ are the same; and this localization is naturally contained in the t-adically complete ring $\widehat{R}_\wp$. Thus so is R_U and hence its t-adic completion $\widehat{R}_U$.

The inclusions of $\widehat{R}_U$ and of $\widehat{R}_Q$ into $\widehat{R}_\wp$, for $\wp = (U, Q)$, induce inclusions of the corresponding fraction fields F_U and F_Q into the fraction field $F_\wp$ of $\widehat{R}_\wp$. Let I be the index set consisting of all $U, Q, \wp$ described above. Via the above inclusions, the collection of all F_ξ, for $\xi \in I$, then forms an inverse system with respect to the ordering given by setting $U \succ \wp$ and $Q \succ \wp$ if $\wp = (U, Q)$.

Under the above hypotheses, suppose that for every field extension L of F, we are given a category $\mathcal{A}(L)$ of algebraic structures over L (i.e. finite dimensional L-vector spaces with additional structure, e.g. associative L-algebras), along with base-change functors $\mathcal{A}(L) \to \mathcal{A}(L')$ when $L \subseteq L'$. An $\mathcal{A}$-*patching problem* for $(\widehat{X}, S)$ consists of an object V_ξ in $\mathcal{A}(F_\xi)$ for each $\xi \in I$, together with isomorphisms $\phi_{U,\wp} : V_U \otimes_{F_U} F_\wp \to V_\wp$ and $\phi_{Q,\wp} : V_Q \otimes_{F_Q} F_\wp \to V_\wp$ in $\mathcal{A}(F_\wp)$. These patching problems form a category, denoted by $\mathrm{PP}_\mathcal{A}(\widehat{X}, S)$, and there is a base change functor $\mathcal{A}(F) \to \mathrm{PP}_\mathcal{A}(\widehat{X}, S)$. (Note that the above definition of patching problem is equivalent to that given in Section 2 of [HH07] for vector spaces, since the restriction of $\phi_{U,\wp}$ to $V_U \subset V_U \otimes_{F_U} F_\wp$ is an F_U-linear map that induces $\phi_{U,\wp}$ upon tensoring, and similarly for $\phi_{Q,\wp}$.)

If an object $V \in \mathcal{A}(F)$ induces a given patching problem up to isomorphism, we will say that V is a *solution* to that patching problem,

or that it is *obtained by patching* the objects V_ξ. We similarly speak of obtaining a morphism over F by patching morphisms in $\mathrm{PP}_\mathcal{A}(\widehat{X}, S)$.

The next result is given by [HH07], Theorem 7.1(i,v,vi), in the context of Theorem 6.4 of that paper.

Theorem 4.1
Let K be a complete discretely valued field with valuation ring T, and let F/K be a finitely generated field extension of transcendence degree one. Let $(\widehat{X}, S)$ be a regular T-model of F. For a field extension L of F, let $\mathcal{A}(L)$ denote any of the following categories:

(1) the category of finite dimensional associative L-algebras,
(2) the category of G-Galois L-algebras for some fixed finite group G, with G-equivariant morphisms, or
(3) the category of central simple L-algebras with algebra homomorphisms.

Then the base change functor $\mathcal{A}(F) \to \mathrm{PP}_\mathcal{A}(\widehat{X}, S)$ is an equivalence of categories. In particular, every $\mathcal{A}$-patching problem has a unique solution.

Given a finite group G, a subgroup $H \subseteq G$, and an H-Galois field extension L/F, there is an induced G-Galois F-algebra $E = \mathrm{Ind}_H^G L$ given by a direct sum of copies of L indexed by the left cosets of H in G; e.g. see [HH07], Section 7.2. In particular, if $H = 1$, then E is a split extension of F; i.e., $E \cong F^{\oplus |G|}$.

Following Schacher [Sch68], we say that a finite field extension L/F is *adequate* if there exists an F-central division algebra D such that L is isomorphic to a maximal subfield of D as an F-algebra. In particular, if L/F is an adequate Galois extension with group G, then G is admissible over F. Note that F is adequate over itself.

The following lemma shows that the objects in Theorem 4.1(2) and (3) may in a sense be patched in pairs. Note that the condition on the indices $(G : H_Q)$ implies that the subgroups H_Q generate G, which would suffice to obtain a G-Galois field extension by patching (cf. [Ha87], Proposition 2.2, or the proof of [HH07], Theorem 7.3). In our set-up, however, a stronger condition than generation is needed, due to the restriction on admissible groups given in Proposition 3.5.

Lemma 4.2
Let G be a finite group, and let F and $(\widehat{X}, S)$ be as in Theorem 4.1. Suppose that for each $Q \in S$ we are given a subgroup $H_Q \subseteq G$ and an H_Q-Galois adequate field extension L_Q/F_Q such that $L_Q \otimes_{F_Q} F_\wp$ is

a split extension $F_\wp^{\oplus|H_Q|}$ of $F_\wp$ for each branch $\wp$ at Q. Assume that the greatest common divisor of the indices $(G : H_Q)$ is equal to 1. Then there exists an adequate G-Galois field extension E/F such that $E \otimes_F F_Q \cong E_Q := \operatorname{Ind}_{H_Q}^G L_Q$ for all $Q \in S$.

Proof. We first describe our setup for patching. Let I be the set of all indices $U, Q, \wp$, as in the introduction to this section. If ξ is either a branch $\wp$ of the closed fiber X at a point $Q \in S$ or else a component U of $X \smallsetminus S$, let $L_\xi = F_\xi$ and $H_\xi = 1$ and let $E_\xi = \operatorname{Ind}_{H_\xi}^G L_\xi \cong F_\xi^{\oplus|G|}$, a split extension of F_ξ. Thus for every $\xi \in I$, the F_ξ-algebra $E_\xi = \operatorname{Ind}_{H_\xi}^G L_\xi$ is a G-Galois F_ξ-algebra, in the sense of [DI71], Section III.1. Let $n = |G|$ and let $n_\xi = (G : H_\xi)$ for $\xi \in I$. Since each field extension L_ξ/F_ξ is adequate, we may choose division algebras D_ξ that contain the fields L_ξ as maximal subfields. Thus D_ξ is a crossed product F_ξ-algebra with respect to H_ξ and L_ξ; and D_ξ is split over L_ξ ([Sal99], Corollary 7.3). Let $A_\xi = \operatorname{Mat}_{n_\xi}(D_\xi)$. In particular, if ξ is a branch $\wp$ or a component U then $n_\xi = n$, $D_\xi = F_\xi$, and $A_\xi = \operatorname{Mat}_n(F_\xi)$.

For every $\xi \in I$, we claim that E_ξ embeds as a maximal commutative separable F_ξ-subalgebra of A_ξ. To see this, note that since the central simple L_ξ-algebra $D_\xi \otimes_{F_\xi} L_\xi$ is split, so is the L_ξ-algebra $D_\xi \otimes_{F_\xi} E_\xi$, in the sense of being a direct sum of matrix algebras over L_ξ. It then follows from [DI71] (Theorem II.5.5 together with Proposition V.1.2 and Corollary I.1.11) that E_ξ is a maximal commutative separable F_ξ-subalgebra of some central simple F_ξ-algebra B_ξ that is Brauer equivalent to D_ξ. Such an algebra is a matrix algebra over D_ξ, necessarily of degree n over F_ξ; so A_ξ and B_ξ are each matrix algebras of the same degree, and hence are isomorphic. This shows the existence of an F_ξ-algebra embedding $\iota_\xi : E_\xi \to A_\xi$, proving the claim.

The proof now proceeds in three steps. First, we want to use patching to obtain a G-Galois (commutative separable) F-algebra E. Observe that for a branch $\wp = (U, Q)$, we have isomorphisms $L_U \otimes_{F_U} F_\wp \to \operatorname{Ind}_1^{H_U} F_\wp$ and $L_Q \otimes_{F_Q} F_\wp \to \operatorname{Ind}_1^{H_Q} F_\wp$; these yield isomorphisms $\phi_{U,\wp} : E_U \otimes_{F_U} F_\wp \to E_\wp$ and $\phi_{Q,\wp} : E_Q \otimes_{F_Q} F_\wp \to E_\wp$. With respect to the various maps $\phi_{U,\wp}, \phi_{Q,\wp}$, the G-Galois F_ξ-algebras E_ξ (for $\xi \in I$) may be patched, by Theorem 4.1(2), to obtain a G-Galois F-algebra E. That is, $E \otimes_F F_\xi \cong E_\xi$ for all $\xi \in I$, compatibly with the above isomorphisms.

Next, we will also patch the algebras A_ξ, compatibly with the inclusions ι_ξ. Let $\wp = (U, Q)$ be a branch at a point $Q \in S$, and let $\xi = U$ or Q. Since $L_\xi \otimes_{F_\xi} F_\wp \cong F_\wp^{\oplus|H_\xi|}$, there is an embedding $L_\xi \hookrightarrow F_\wp$ of

L_ξ-algebras by projecting onto the direct summand corresponding to the identity element in H_ξ. With respect to this embedding, we have

$$D_\xi \otimes_{F_\xi} F_\wp \cong (D_\xi \otimes_{F_\xi} L_\xi) \otimes_{L_\xi} F_\wp \cong \mathrm{Mat}_{|H_\xi|}(L_\xi) \otimes_{L_\xi} F_\wp \cong \mathrm{Mat}_{|H_\xi|}(F_\wp),$$

since $D_\xi \otimes_{F_\xi} L_\xi$ is a split central simple L_ξ-algebra. So there is an isomorphism $\widetilde{\psi}_{\xi,\wp} : A_\xi \otimes_{F_\xi} F_\wp \to \mathrm{Mat}_n(F_\wp) = A_\wp$ of $F_\wp$-algebras. By [Jac96], Theorem 2.2.3(2), the $F_\wp$-algebra embedding $\widetilde{\psi}_{\xi,\wp} \circ (\iota_\xi \otimes_{F_\xi} F_\wp) \circ \phi_{\xi,\wp}^{-1} : E_\wp \to A_\wp$ extends to an inner automorphism $\alpha_{\xi,\wp}$ of $A_\wp$. Then $\alpha_{\xi,\wp}^{-1}\widetilde{\psi}_{\xi,\wp} \circ (\iota_\xi \otimes_{F_\xi} F_\wp) = \phi_{\xi,\wp} : E_\xi \otimes_{F_\xi} F_\wp \to E_\wp \subseteq A_\wp$. Let $\psi_{\xi,\wp} = \alpha_{\xi,\wp}^{-1}\widetilde{\psi}_{\xi,\wp} : A_\xi \otimes_{F_\xi} F_\wp \to A_\wp$. Hence if ξ is a point of X or a component of $X \smallsetminus S$, there is the following commutative diagram:

$$
\begin{array}{ccc}
A_\xi \otimes_{F_\xi} F_\wp & \xrightarrow{\psi_{\xi,\wp}} & A_\wp \\
{\scriptstyle \iota_\xi \otimes_{F_\xi} F_\wp} \uparrow & & \uparrow {\scriptstyle \iota_\wp} \\
E_\xi \otimes_{F_\xi} F_\wp & \xrightarrow{\phi_{\xi,\wp}} & E_\wp
\end{array}
$$

By Theorem 4.1(3), we may patch the algebras A_ξ (for all $\xi \in I$) to obtain a central simple F-algebra A. Furthermore, because of the compatibility expressed in the above diagram, the morphisms ι_ξ patch to give a morphism $\iota : E \to A$. Therefore E is a commutative separable subalgebra of A, and its dimension over F equals the dimension n of each E_ξ over F_ξ. Thus E is maximal in A.

Finally, we will show that A is a division algebra, which implies that E is a field and hence is an adequate G-Galois field extension of F. For this, note that

$$n/n_\xi = |H_\xi| = [L_\xi : F_\xi] = \deg(D_\xi) = \mathrm{ind}(A_\xi) \mid \mathrm{ind}(A),$$

where the third equality holds because L_ξ is a maximal subfield of the crossed product algebra D_ξ ([Sal99], Corollary 7.3), and where the divisibility follows from [Pie82], Proposition 13.4(v). Hence

$$\deg(A) = n = \mathrm{lcm}(n/n_\xi,\ \xi \in I) \mid \mathrm{ind}(A) \mid \deg(A),$$

where the second equality follows from the hypothesis on the indices $n_\xi = (G : H_\xi)$ being relatively prime (even just for $\xi = Q \in S$), using that $n = |G|$. So $\deg(A) = n = \mathrm{ind}(A)$, and hence A is a division algebra. This finishes the proof. $\square$

Consider a two-dimensional regular local ring R, say with fraction field F, and whose maximal ideal has generators f, t. Let $w : F^\times \to \mathbb{Z}$ be the f-adic discrete valuation on F, with valuation ring $W = R_{(f)}$. The ring R/fR is also a discrete valuation ring, with fraction field

$\bar{W} = W/fW$. The corresponding discrete valuation $u : \bar{W}^\times \to \mathbb{Z}$ is equal to the $\bar{t}$-adic valuation on $\bar{W}$, where $\bar{t} \in \bar{W}$ is the image of $t \in W$ under the canonical map $\eta : W \to \bar{W}$. With $\mathbb{Z} \times \mathbb{Z}$ ordered lexicographically, there is a rank two valuation $u \circ w : F^\times \to \mathbb{Z} \times \mathbb{Z}$ defined by

$$(u \circ w)(a) = \big(w(a), u(\eta(af^{-w(a)}))\big). \qquad (*)$$

Note that $u \circ w$ is the valuation whose associated place is the composition of the places associated to the valuations u and w; this follows from the fact that both have valuation ring $\eta^{-1}(R/fR) = R + fW$. (See [Bo72], Chapter VI, Sections 3-4, for a general discussion.)

Lemma 4.3
Let R be a complete regular local ring whose maximal ideal has generators f, t; let n be a positive integer; and assume that the fraction field F of R contains a primitive n-th root of unity ζ. Let D be the associative F-algebra generated by elements Y, Z satisfying the relations

$$Y^n = \frac{f}{f-t}, \quad Z^n = \frac{f-t^2}{f-t-t^2}, \quad YZ = \zeta ZY.$$

Then D is a division algebra over F.

Proof. The F-algebra D is the symbol algebra over F with respect to the n-th root of unity ζ and the elements $a = f/(f-t)$ and $b = (f-t^2)/(f-t-t^2)$ of $F^\times$; see [Wad02], (0.3). This is a central simple F-algebra of degree n.

Consider the rank two valuation $v := u \circ w : F^\times \to \Gamma := \mathbb{Z} \times \mathbb{Z}$ as above, with respect to the generators f, t of the maximal ideal of R. Thus v is given by the expression $(*)$ above. Observe that $v(a) = (1, -1)$ and $v(b) = (0, 1)$. Thus the images of $v(a)$ and $v(b)$ in $\Gamma/n\Gamma$ generate that group, which has order n^2. Since F contains a primitive n-th root of unity, it then follows from [Wad02], Example 4.4, that D is a division algebra over F. □

The following result now provides the converse direction of our main theorem (in fact in a more general situation):

Proposition 4.4
Let G be a finite group whose Sylow subgroups are abelian of rank at most 2. Let F be a finitely generated field extension of transcendence degree one over a complete discretely valued field K, and assume that F contains a primitive $|G|$-th root of unity. Then G is admissible over F.

Proof. Let $p_1, \ldots, p_r$ be the prime numbers dividing the order of G. For each i let P_i be a Sylow p_i-subgroup of G. Let T be the valuation ring of K. As in the introduction to this section, there is a regular T-curve $\widehat{X}$ such that the reduced irreducible components of the closed fiber X are regular. Choose distinct closed points $Q_1, \ldots, Q_r \in X$ at which X is regular. Thus Q_i lies on only one (regular) irreducible component of X, and there is a unique branch $\wp_i$ of X at the point Q_i. By Proposition 6.6 of [HH07], there is a finite morphism $f : \widehat{X} \to \mathbb{P}_T^1$ whose fiber $S = f^{-1}(\infty) \subset X$ contains all the points Q_i and all the points at which two irreducible components of X meet.

As above, let I be the set of all indices $U, Q, \wp$, and consider the associated rings $\widehat{R}_\xi$ and their fraction fields F_ξ for $\xi \in I$. To prove the result, it is enough to construct P_i-Galois adequate field extensions $L_i = L_{Q_i}$ of F_{Q_i}, for $i = 1, \ldots, r$, such that $L_i \otimes_{F_{Q_i}} F_{\wp_i}$ is a split extension $F_{\wp_i}^{\oplus |P_i|}$ of $F_{\wp_i}$. Namely, if this is done, then let $H_Q = 1$ and $L_Q = F_Q$ for every point $Q \in S$ other than $Q_1, \ldots, Q_r$. Since the indices of the subgroups $H_{Q_i} := P_i$ are relatively prime for $i = 1, \ldots, r$, it follows that the indices of all the subgroups H_Q (for $Q \in S$) are relatively prime; and Lemma 4.2 will then imply the assertion.

So fix i. By hypothesis, P_i is an abelian p_i-group of rank at most 2, say $P_i = C_q \times C_{q'}$. Since $\widehat{X}$ is regular and since Q_i is a regular point of the closed fiber X, the maximal ideal of the two-dimensional regular local ring $\widehat{R}_i := \widehat{R}_{Q_i}$ is generated by two elements f_i and t_i, where $t_i \in \widehat{R}_i$ defines the reduced closed fiber of $\operatorname{Spec}(\widehat{R}_i)$. (Thus t is a power of t_i multiplied by a unit in $\widehat{R}_i$.) Consider the elements $a = f_i/(f_i - t_i)$ and $b = (f_i - t_i^2)/(f_i - t_i - t_i^2)$ of F_{Q_i}, and let $L_i = F_{Q_i}(y, z)$ be the field extension of F_{Q_i} defined by the relations $y^q = a$, $z^{q'} = b$. We will show first that L_i is a P_i-Galois adequate field extension of F_{Q_i} and then show that $L_i \otimes_{F_{Q_i}} F_{\wp_i}$ is a split extension of $F_{\wp_i}$.

By the assumption on roots of unity, Kummer theory applies. Neither a nor b is a d-th power in F_{Q_i} for any $d > 1$, so the extensions $F_{Q_i}(y)$ and $F_{Q_i}(z)$ of F_{Q_i} are Galois with group C_q and $C_{q'}$, respectively. Here $F_{Q_i}(y)$ is totally ramified over the prime (f_i) of $\operatorname{Spec}(\widehat{R}_i)$, whereas $F_{Q_i}(z)$ is unramified there. So the intersection of these two fields equals F_{Q_i}, showing that they are linearly disjoint over F_{Q_i}. Hence the Galois group of their compositum L_i/F_{Q_i} is in fact P_i as claimed.

Let $\zeta \in F$ be a primitive qq'-th root of unity; this exists since $qq' = |P_i|$ divides $|G|$. Consider the central simple F_{Q_i}-algebra D_i generated by elements Y, Z satisfying the relations $Y^{q'} = y$, $Z^q = z$, and $YZ = \zeta ZY$. Thus $Y^{qq'} = f_i/(f_i - t_i)$ and $Z^{qq'} = (f_i - t_i^2)/(f_i - t_i - t_i^2)$. By

Lemma 4.3, D_i is an F_{Q_i}-division algebra of degree qq'. Inductively, one sees that $YZ^q = \zeta^q Z^q Y$ and thus $Y^{q'}Z^q = \zeta^{qq'}Z^q Y^{q'} = Z^q Y^{q'}$. Consequently, $yz = zy$ in D_i; i.e. L_i is a subfield of D_i. But the degree of L_i/F_{Q_i} is qq', so L_i is a maximal subfield; i.e., L_i is adequate.

Finally, we show that $L_i \otimes_{F_{Q_i}} F_{\wp_i}$ is split. The above elements a and b each lie in the discrete valuation ring $\widehat{R}_{\wp_i}$, and in fact each is congruent to 1 modulo the maximal ideal (t_i) of that ring. The reductions of a and b modulo t_i are thus qq'-th powers in the residue field of $\widehat{R}_{\wp_i}$, and so Hensel's Lemma implies that a and b are each qq'-th powers in $\widehat{R}_{\wp_i}$ (using that $\widehat{R}_{\wp_i}$ is complete t-adically and hence t_i-adically). Tensoring over $\widehat{R}_{\wp_i}$ with $F_{\wp_i}$, we have that $L_i \otimes_{F_{Q_i}} F_{\wp_i}$ is a split extension of $F_{\wp_i}$, as desired. $\qquad\square$

Combining the above with Proposition 3.5 yields our main theorem:

Theorem 4.5
Let F be a finitely generated field extension of transcendence degree one over a complete discretely valued field K with algebraically closed residue field k, and let G be a finite group of order not divisible by $\mathrm{char}(k)$. Then G is admissible over F if and only if each of its Sylow subgroups is abelian of rank at most 2.

Proof. The forward direction is given by Proposition 3.5. The converse follows from Proposition 4.4, which applies because K (and hence F) contains a primitive $|G|$-th root of unity by Hensel's Lemma, since the residue field k is algebraically closed of characteristic not dividing $|G|$. $\qquad\square$

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