

REFLECTION GROUPOIDS OF RANK TWO AND CLUSTER ALGEBRAS OF TYPE A

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ABSTRACT. We extend the classification of finite Weyl groupoids of rank two. Then we generalize these Weyl groupoids to ‘reflection groupoids’ by admitting non-integral entries of the Cartan matrices. This leads to the unexpected observation that the spectrum of the cluster algebra of type A_{n-3} completely describes the set of finite reflection groupoids of rank two with $2n$ objects.

1. INTRODUCTION

In the last years, the classification of pointed Hopf algebras has grown to a very fruitful subject. Amongst others, the Weyl groupoid was invented [9] [10], a structure which plays a similar role as the Weyl group for semisimple Lie algebras. A Weyl groupoid is a groupoid which is defined using a family of generalized Cartan matrices. The abstract notion of the Weyl groupoid is perhaps somewhat too general from the viewpoint of pointed Hopf algebras, but nevertheless its system of axioms is good enough to provide a rich general theory and to admit a classification at least of the finite case (see [3], [2]), thus it has proven to be very useful anyway.

Recent work on finite rank three Weyl groupoids has revealed a connection to combinatorics, because a Weyl groupoid yields a simplicial arrangement in the same manner a Coxeter group does. Since the classification of simplicial arrangements in the real projective plane is still an open problem - Grünbaum conjectures that he has a complete list [7] - it appears now to be a natural question to generalize Weyl groupoids to the case in which the Cartan entries are not necessarily integers: This would at least yield one more case, the Coxeter group of type H_3 .

However, the classification algorithm for rank three in [2] requires a good knowledge on finite rank two Weyl groupoids. So to find an explanation of simplicial arrangements in terms of Weyl groupoids, it is necessary to understand at least finite reflection groupoids of rank two first. These are certain groupoids generated by reflections given

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by a vast generalization of Cartan matrices, see Section 2 for a precise definition.

In general, we are mostly interested in Weyl groupoids which admit a finite root system in the sense of [4, Def. 2.2]. A Weyl groupoid is called *universal* if for each object a the group $\text{End}(a)$ is trivial. It is *irreducible* if none of the generalized Cartan matrices are decomposable. In [3], a construction is given to obtain all finite universal Weyl groupoids of rank two which admit a finite root system. We refine this result by explaining the combinatorics:

Theorem 1.1 (Theorem 3.4). *There is a natural bijection between the set of isomorphism classes of connected irreducible universal Weyl groupoids of rank two with $2n$ objects which admit a finite root system, and the triangulations of a convex n -gon by non-intersecting diagonals, up to the symmetry of the dihedral group \mathbb{D}_n .*

Further, we explicitly give the corresponding root systems in Proposition 3.7 in terms of sequences closely related to the well-known Farey series. We conclude in Corollary 3.8 that any positive root is either simple or the sum of two positive roots (at the same object). We also describe the quotients of universal coverings in Proposition 3.12.

For the classification of connected finite reflection groupoids of rank two we proceed in the most natural way. We view the entries of the defining ‘Cartan matrices’ as indeterminates, translate the axioms for finiteness into polynomials and consider the resulting variety. Using an induction on the number of objects with a similar rule as in [3], we obtain a surprising explanation for the theory in rank two, in which matrix mutation appears in a completely natural way.

Theorem 1.2 (Theorem 4.3). *Let $n \in \mathbb{N}_{\geq 3}$ and let \mathcal{W} be a connected finite reflection groupoid of rank two with $2n$ objects. The set of connected finite reflection groupoids of rank two with the same objects and the same object change diagram as \mathcal{W} is a variety isomorphic to the spectrum of the cluster algebra of type A_{n-3} , where the edges of the n -gon q_1, \dots, q_n are specialized to 1.*

For the definition of the object change diagram see Section 2.

This note is organized as follows. In Section 2 we recall the definition of the main structures as in [4], except that the Cartan entries may come from an arbitrary ring here. In Section 3 we explain the combinatorics of finite Weyl groupoids of rank two. In the last section, we shortly describe the Grassmannian $\text{Gr}(2, n)$ following [5, Lecture 3] and exhibit the connection to the reflection groupoids of rank two.

2. REFLECTION GROUPOIDS OF RANK TWO

Let K be a ring. We first introduce and recall some definitions and notations needed to formulate the definition of a finite reflection groupoid of rank two. Remark that the only reason why we restrict to the case of rank two is that we have no canonical choice for a system of axioms in the higher rank yet.

Definition 2.1. Let I be a set with $|I| = 2$. Let A be a non-empty set, $\rho_i : A \rightarrow A$ a map for all $i \in I$, and $C^a = (c_{jk}^a)_{j,k \in I}$ a matrix in $K^{I \times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$$

is called a K -Cartan scheme if

- (M1) $c_{ii}^a = 2$ for all $a \in A, i \in I$,
- (C1) $\rho_i^2 = \text{id}$ for all $i \in I$,
- (C2) $c_{ij}^a = c_{ij}^{\rho_i(a)}$ for all $a \in A$ and $i, j \in I$.

The notion K -Cartan scheme has its origins in [4] where Cartan schemes are defined using a family of generalized Cartan matrices. More precisely, a Cartan scheme is a \mathbb{Z} -Cartan scheme for which all matrices C^a , where $a \in A$, are generalized Cartan matrices in the sense of [11, Sect. 1.1]. In Definition 2.1 the matrices C^a are not necessarily generalized Cartan matrices, even if $K = \mathbb{Z}$. The reason for the generality in our definition is twofold. First, if we considered additional axioms on the matrices C^a then the interpretation of Theorem 4.3 would be more complicated than Theorem 1.2. Second, let $K = \mathbb{Z}$ and assume that any real root (see below) associated to any $a \in A$ is either positive or negative. Then for all $a \in A$ the matrix C^a is a generalized Cartan matrix, see [4, Lemma 2.5].

Two K -Cartan schemes are termed *equivalent* if there exist bijections between their sets I respectively A which satisfy the natural compatibility conditions, see [4, Def. 2.1] for details.

For all $i \in I$ and $a \in A$ define $\sigma_i^a \in \text{Aut}(K^I)$ by

$$(2.1) \quad \sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \quad \text{for all } j \in I.$$

Then for all $i \in I$ and $a \in A$ the linear map σ_i^a is a reflection in the sense of [1, Ch. V, §2.2].

To \mathcal{C} belongs a category $\mathcal{W}(\mathcal{C})$ such that $\text{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps σ_i^a which are by definition in $\text{Hom}(a, \rho_i(a))$ with $i \in I, a \in A$.

To each object $a \in A$, one can associate a set of *real roots*

$$R^a = \{\varphi(\alpha_i) \mid i \in I, \varphi \in \text{Hom}(b, a), b \in A\}.$$

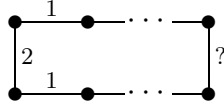


FIGURE 1. Cycle diagram

In the special case $K = \mathbb{Z}$ we have a set $R_+^a = R^a \cap \mathbb{N}_0^I$ of *positive roots*. We write $\mathcal{R}^{re}(\mathcal{C}) = (\mathcal{C}, (R^a)_{a \in A})$.

Remark 2.2. Let \mathcal{C} be a \mathbb{Z} -Cartan scheme and assume that for all $a \in A$ the matrix C^a is a generalized Cartan matrix. Then \mathcal{C} is a *Cartan scheme*. Although for the purposes of this paper we assumed that $|I| = 2$, in the general theory [4] this assumption is not necessary.

We say that \mathcal{C} is *irreducible* if the generalized Cartan matrix C^a is indecomposable for any $a \in A$. The groupoid $\mathcal{W}(\mathcal{C})$ is called the *Weyl groupoid of \mathcal{C}* (compare [4]). The Weyl groupoid $\mathcal{W}(\mathcal{C})$ is *finite* if the number of objects in each connected component of $\mathcal{W}(\mathcal{C})$ is finite and if $|\text{Hom}(a, b)| < \infty$ for all $a, b \in A$. We say that \mathcal{C} is *connected* if $\mathcal{W}(\mathcal{C})$ is connected. Further, \mathcal{C} is *simply connected* if $|\text{End}(a)| = 1$ for all $a \in A$.

For the existence of a root system of type \mathcal{C} in the sense of [4, Def. 2.2] it is necessary that the real roots are contained in $\mathbb{N}_0^I \cup -\mathbb{N}_0^I$. This is not always the case even if the rank of \mathcal{C} is two: A counterexample can be found in the proof of [4, Thm. 6.1]. By [4, Prop. 2.12], if there is a finite root system of type \mathcal{C} then all roots are real. Therefore there exists a finite root system of type \mathcal{C} if and only if $\mathcal{R}^{re}(\mathcal{C})$ is a finite root system of type \mathcal{C} .

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a K -Cartan scheme. We will omit the source of a morphism in the notation if it is clear from the context. The number $|I|$ is called the *rank* of \mathcal{C} . It is always two in this article.

Let Γ be a non directed graph, such that the vertices of Γ correspond to the elements of A . Assume that for all $i \in I$ and $a \in A$ with $\rho_i(a) \neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i , and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of \mathcal{C} .

Assume that \mathcal{C} is connected. Then the object change diagram of \mathcal{C} is either a chain or a cycle. In the second case the cardinality of A is even, see Figure 1. Let now $n \in \mathbb{N}$ and assume that the object change diagram of \mathcal{C} is a cycle with $2n$ vertices.

Let $i, j \in I$ and $a \in A$. In analogy to [6, 1.1.6], for all $m \in \mathbb{N}$ let

$$\text{Prod}(m; a, i, j) = \dots \sigma_i \sigma_j \sigma_i^a$$

where the number of factors is m . Let $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 2.3. For a K -Cartan scheme \mathcal{C} (of rank two, i.e. $|I| = 2$) we call $\mathcal{W}(\mathcal{C})$ a *finite reflection groupoid* if the object change diagram of \mathcal{C} is a cycle and

$$(2.2) \quad \text{Prod}(n; a, i, j) = -\tau^n$$

for all $a \in A$ and $i, j \in I$ with $i \neq j$, where $n = |A|/2$.

Although we have no roots here, axiom (2.2) is reasonable since it generalizes the fact that a “longest word” should map all positive roots to negative ones.

For the remaining part of this section assume that \mathcal{C} is a K -Cartan scheme such that $\mathcal{W}(\mathcal{C})$ is a finite reflection groupoid with $|A| = 2n$ objects. For simplicity assume that $I = \{1, 2\}$.

By (C2), the K -Cartan scheme \mathcal{C} is locally of the form

$$\dots \xrightarrow{1} \begin{pmatrix} 2 & -c_1 \\ -c_2 & 2 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 2 & -c_3 \\ -c_2 & 2 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 2 & -c_3 \\ -c_4 & 2 \end{pmatrix} \xrightarrow{2} \dots$$

for some $c_1, c_2, \dots \in K$. Hence giving the sequence of Cartan entries c_1, c_2, \dots and the label of the first morphism in this sequence completely determines the K -Cartan scheme (and the reflection groupoid). More precisely (compare [3, Prop. 6.5]):

Lemma 2.4. *Let $a \in A$ and $i, j \in I$ with $I = \{i, j\}$. Let $a_1, a_2, \dots, a_{2n} \in A$ and $c_1, c_2, \dots, c_{2n} \in K$ such that*

$$\begin{aligned} a_1 &= a, & a_2 &= \rho_i(a_1), & a_3 &= \rho_j(a_2), & a_4 &= \rho_i(a_3), & a_5 &= \rho_j(a_4), & \dots \\ c_1 &= -c_{ij}^{a_1}, & c_2 &= -c_{ji}^{a_2}, & c_3 &= -c_{ij}^{a_3}, & c_4 &= -c_{ji}^{a_4}, & c_5 &= -c_{ij}^{a_5}, & \dots \end{aligned}$$

Then $c_{n+r} = c_r$ for all $r \in \{1, 2, \dots, n\}$.

Proof. By Equation (2.2),

$$\begin{aligned} (\sigma \text{Prod}(n-1; \rho_i(a), j, i)) &= \text{Prod}(n; \rho_i(a), j, i) \\ &= -\tau^n \\ &= \text{Prod}(n; a, i, j) \\ &= (\text{Prod}(n-1; \rho_i(a), j, i) \sigma_i^a) \end{aligned}$$

in $\text{Aut}(K^I)$, where σ is the first map in $\text{Prod}(n; \rho_i(a), j, i)$. Hence $\sigma = \sigma_i^a$ or $\tau\sigma\tau = \sigma_i^a$ depending on whether n is even or odd. But in both cases the c_1 and c_{n+1} defining the reflections are equal. Starting at the other objects gives $c_{n+r} = c_r$ for all $r \in \{1, 2, \dots, n\}$. \square

Let

$$(2.3) \quad \Phi : I \times A \rightarrow K^n, \quad (i, a) \mapsto (c_1, c_2, \dots, c_n),$$

where $c_1, c_2, \dots, c_n \in K$ are depending on $i \in I$ and $a \in A$ as in Lemma 2.4.

3. CLASSIFICATION OF FINITE RANK TWO WEYL GROUPOIDS

In [3] we gave an inductive method to construct all finite Weyl groupoids of rank two which admit a root system. Here, we extend these results and give a natural bijection to triangulations of convex polygons. We describe in Proposition 3.7 the set of positive roots in terms of ordered sequences of vectors in \mathbb{N}_0^I . This sequences are closely related to the well-known Farey series, see the discussion in Remark 3.6. As a corollary we obtain that any non-simple positive root is a sum of two positive roots.

3.1. Connected simply connected Cartan schemes. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a connected simply connected Cartan scheme (see Rem. 2.2) with $I = \{1, 2\}$ and assume that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite root system (of type \mathcal{C}). By [3, Section 6] the object change diagram of \mathcal{C} is a cycle with an even number of vertices. For all $x \in \mathbb{Z}$ let

$$(3.1) \quad \eta(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}.$$

Following [3, Def. 5.1] let \mathcal{A} denote the set of finite sequences (c_1, \dots, c_n) of integers such that $n \geq 1$ and $\eta(c_1) \cdots \eta(c_n) = -\text{id}$. Let \mathcal{A}^+ be the subset of \mathcal{A} formed by those $(c_1, \dots, c_n) \in \mathcal{A}$, for which $c_i \geq 1$ for all $i \in \{1, \dots, n\}$ and the entries in the first column of $\eta(c_1) \cdots \eta(c_i)$ are non negative for all $i < n$.

In [3, Prop. 6.5] it was shown that Lemma 2.4 holds for \mathcal{C} , and that $(c_1, \dots, c_n) \in \mathcal{A}^+$ (using the notation of the lemma).

By [3, Prop. 5.3] (2),(3) the dihedral group \mathbb{D}_n of $2n$ elements, where $n \in \mathbb{N}$, acts on sequences of length n in \mathcal{A}^+ by cyclic permutation of the entries and by reflections. This action gives rise to an equivalence relation \sim on \mathcal{A}^+ by taking the orbits of the action as equivalence classes.

Let $n, m \in \mathbb{N}$ with $m \geq n$, and let $c = (c_1, \dots, c_n), d = (d_1, \dots, d_m) \in \mathcal{A}^+$. We write $c \approx' d$ if and only if

- $m = n, c \sim d$ or
- $m = n + 1, d = (c_1 + 1, 1, c_2 + 1, c_3, c_4, \dots, c_n)$.

The following is [3, Def. 5.4]: Let $c, d \in \mathcal{A}^+$. Write $c \approx d$ if and only if there exists $k \in \mathbb{N}$ and a sequence $c = e_1, e_2, \dots, e_k = d$ of elements of \mathcal{A}^+ , such that $e_i \approx' e_{i+1}$ or $e_{i+1} \approx' e_i$ for all $i \in \{1, 2, \dots, k-1\}$.

Theorem 3.1. [3, Thm. 5.5] *The only element of \mathcal{A}^+/\approx is $(1, 1, 1)$.*

Summarizing this, the set \mathcal{A}^+ may also be defined as follows:

Definition 3.2. Define η -sequences recursively in the following way:

- (1) $(1, 1, 1)$ is an η -sequence.
- (2) If (c_1, \dots, c_n) is an η -sequence, then $(c_2, c_3, \dots, c_{n-1}, c_n, c_1)$ and $(c_n, c_{n-1}, \dots, c_2, c_1)$ are η -sequences.
- (3) If (c_1, \dots, c_n) is an η -sequence, then $(c_1 + 1, 1, c_2 + 1, c_3, \dots, c_n)$ is an η -sequence.
- (4) Every η -sequence is obtained recursively by (1),(2),(3).

Then \mathcal{A}^+ is the set of η -sequences.

Recall the map Φ from Equation (2.3).

Theorem 3.3. [3, Thm. 6.6] *Let $n \in \mathbb{N}$ and $c = (c_1, c_2, \dots, c_n) \in \mathcal{A}^+$. Then there is a unique (up to equivalence) connected simply connected Cartan scheme of rank two such that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is an irreducible root system and $c \in \text{Im } \Phi$.*

Together with [3, Prop. 6.5] this gives that the connected simply connected Cartan schemes of rank two, for which $\mathcal{R}^{\text{re}}(\mathcal{C})$ is an irreducible root system, are those with cycle diagram as object change diagram, and where the Cartan entries yield an η -sequence. Since the symmetry group of such object change diagrams is the dihedral group, we obtain exactly one equivalence class of connected simply connected Cartan schemes for each element of \mathcal{A}^+/\sim .

Let \mathcal{T}_n be the set of triangulations of a convex n -gon by non-intersecting diagonals for a fixed labeling of the vertices by $1, \dots, n$, and let $\mathcal{T} = \bigcup_{n=3}^{\infty} \mathcal{T}_n$.

Let $\Psi : \mathcal{A}^+ \rightarrow \mathcal{T}$ be the map that maps an η -sequence (c_1, \dots, c_n) to the triangulation of the n -gon which has c_i triangles attached to the vertex i : The sequence $(1, 1, 1)$ maps to the triangle. Inserting a ‘1’ in the η -sequence as in Definition 3.2(3) corresponds to attaching a new triangle at the edge $(1, 2)$. Definition 3.2(2) just corresponds to the natural action of the dihedral group on the polygon. Thus Ψ is well-defined.

We can now deduce the result of this subsection:

Theorem 3.4. *The map Ψ is bijective.*

Proof. Since an element of \mathcal{T} uniquely defines an η -sequence, Ψ is injective. On the other hand, each triangulation of the n -gon by non-intersecting diagonals may be obtained by starting with a triangle and by successively attaching triangles at the edges (this is sometimes called a flexagon or a planar 2-tree), so Ψ is surjective. \square

We end this subsection by determining the roots of the Cartan schemes of rank two for which $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite irreducible root system.

Definition 3.5. Define \mathcal{F} -sequences as finite sequences of length ≥ 3 with entries in \mathbb{N}_0^2 given by the following recursion.

- (1) $((0, 1), (1, 1), (1, 0))$ is an \mathcal{F} -sequence.
- (2) If (v_1, \dots, v_n) is an \mathcal{F} -sequence, then

$$(v_1, \dots, v_i, v_i + v_{i+1}, v_{i+1}, \dots, v_n)$$

are \mathcal{F} -sequences for $i = 1, \dots, n - 1$.

- (3) Every \mathcal{F} -sequence is obtained recursively by (1) and (2).

Remark 3.6. Notice the resemblance to Farey series (see [8, III]): The Farey series \mathcal{F}_n of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n . A characteristic property is: If $\frac{h}{k}$, $\frac{h''}{k''}$, and $\frac{h'}{k'}$ are three successive terms of \mathcal{F}_n , then $\frac{h'}{k'} = \frac{h+h'}{k+k'}$.

Except $(1, 0)$, we may view an entry (a, b) of an \mathcal{F} -sequence as the rational number $\frac{a}{b}$. This maps an \mathcal{F} -sequence to an ascending sequence of rational numbers. Axiom (2) corresponds to the above characteristic property of a Farey series. On positive roots (including $(1, 0)$), we define

$$(a, b) \leq_{\mathbb{Q}} (c, d) \iff ad \leq cb.$$

Proposition 3.7. *To each \mathcal{F} -sequence f there is up to equivalence a unique connected simply connected Cartan scheme \mathcal{C} such that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite root system and the set of entries of f is R_+^a for some $a \in A$.*

Conversely, let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan scheme of rank two. Assume that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite irreducible root system. Let $a \in A$. Then sorting the elements of $R^a \cap \mathbb{N}_0^2$ with respect to $\leq_{\mathbb{Q}}$ in ascending order yields an \mathcal{F} -sequence.

Proof. For $c \in \mathcal{A}^+$ of length n , consider the Cartan scheme \mathcal{C} with objects a_1, \dots, a_{2n} such that $\Phi(2, a_1) = c$ and $\rho_2(a_1) = a_2$. The numbers c_1, \dots, c_n are the Cartan entries used for certain reflections $s_i \in \text{Hom}(a_{i+1}, a_i)$, $1 \leq i \leq 2n$, where $a_{2n+1} = a_1$. By the proof of [3, Thm. 6.6], the real positive roots at a_1 are

$$\{\beta_\nu := s_1 s_2 \cdots s_\nu(\alpha_{2-(\nu \bmod 2)}) \mid \nu = 0, \dots, n - 1\},$$

or equivalently

$$\{\tilde{\beta}_\nu := s_{2n}^{-1} s_{2n-1}^{-1} \cdots s_{2n+1-\nu}^{-1}(\alpha_{(\nu \bmod 2)+1}) \mid \nu = 0, \dots, n-1\}.$$

For $0 \leq \mu < n$, the equation $s_1 \cdots s_{\mu+1}(-\tau^n) = s_{2n}^{-1} \cdots s_{n+\mu+2}^{-1}$ implies that $\beta_\mu = \tilde{\beta}_{n-1-\mu}$, thus the set of real positive roots is also

$$\{\beta_0, \dots, \beta_\nu, \tilde{\beta}_{n-2-\nu}, \dots, \tilde{\beta}_0\}$$

for any fixed $0 \leq \nu < n$. If using Definition 3.2(2),(3) we include a ‘1’ in the sequence at position ν (and of course also at position $\nu + n$), i.e.

$$(\dots, c_\nu, c_{\nu+1}, \dots) \mapsto (\dots, c_\nu + 1, 1, c_{\nu+1} + 1, \dots),$$

this will not affect the roots $\tilde{\beta}_{n-2-\nu}, \dots, \tilde{\beta}_0, \beta_0, \dots, \beta_{\nu-1}$ coming from $a_{n+3+\nu}, \dots, a_{2n}, a_1, a_1, \dots, a_\nu$. Let $i, j \in I$ be such that $\rho_i(a_\nu) = a_{\nu+1}$ and $i \neq j$. If $\tilde{s}_\nu, t, \tilde{s}_{\nu+1}$ are the new morphisms corresponding to $c_\nu + 1, 1, c_{\nu+1} + 1$ respectively, then

$$s_1 \cdots s_{\nu-1} \tilde{s}_\nu(\alpha_j) = s_1 \cdots s_\nu(\alpha_j) + s_1 \cdots s_{\nu-1}(\alpha_i) = \beta_\nu + \beta_{\nu-1},$$

$$s_1 \cdots s_{\nu-1} \tilde{s}_\nu t(\alpha_i) = \beta_\nu.$$

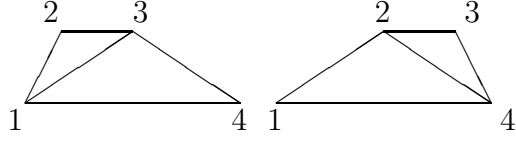
These will be the positive roots coming from the two objects replacing $a_{\nu+1}$. Hence $\beta_\nu + \beta_{\nu-1}$ is the new root after the transformation. Since the positive roots for the η -sequence $(1, 1, 1)$ are $((0, 1), (1, 1), (1, 0))$, this proves that \mathcal{F} -sequences are sets of positive roots.

Conversely, a set of positive roots uniquely determines an η -sequence by [3, Prop. 6.5], and this η -sequence corresponds to an \mathcal{F} -sequence by the above construction. \square

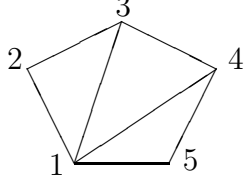
Corollary 3.8. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan scheme of rank two. If $\mathcal{R}^{re}(\mathcal{C}) = (\mathcal{C}, (R^a)_{a \in A})$ is a finite root system of type \mathcal{C} then, for all $a \in A$ and $\alpha \in R_+^a$, either α is simple or it is the sum of two positive roots in R_+^a .*

Example 3.9. We determine the sets R_+^a for all Cartan schemes \mathcal{C} where $\mathcal{R}^{re}(\mathcal{C})$ is a finite irreducible root system with at most 5 positive roots.

- (1) With three positive roots: The η -sequence is $(1, 1, 1)$, it corresponds to the universal covering of the Weyl group of type A_2 (see Definition 3.10). The set of positive roots is $\{(0, 1), (1, 1), (1, 0)\}$.
- (2) With four positive roots: There are two positions for including a ‘1’, one obtains $(2, 1, 2, 1)$ or $(1, 2, 1, 2)$ which are equivalent with respect to \sim . The sets of positive roots are (type B_2 and C_2): $\{(0, 1), (1, 2), (1, 1), (1, 0)\}, \{(0, 1), (1, 1), (2, 1), (1, 0)\}$.



- (3) With five positive roots: There is only one triangulation of a convex pentagon up to operation of the dihedral group.



The corresponding sets of positive roots are

$$\begin{aligned} & \{(0, 1), (1, 3), (1, 2), (1, 1), (1, 0)\} \text{ (assoc. to the given triangulation),} \\ & \{(0, 1), (1, 1), (2, 1), (3, 1), (1, 0)\}, \{(0, 1), (1, 2), (2, 3), (1, 1), (1, 0)\}, \\ & \{(0, 1), (1, 2), (1, 1), (2, 1), (1, 0)\}, \{(0, 1), (1, 1), (3, 2), (2, 1), (1, 0)\}. \end{aligned}$$

3.2. Finite rank two Weyl groupoids. After considering simply connected Cartan schemes, we may now investigate the quotients of these universal coverings. Coverings of Cartan schemes have been treated in [3, Sect. 3]. We recall the concept from [3, Def. 3.1]:

Definition 3.10. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ and $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i \in I}, (C'^a)_{a \in A'})$ be connected Cartan schemes. Let $\pi : A' \rightarrow A$ be a map such that $C^{\pi(a)} = C'^a$ for all $a \in A'$ and the diagrams

$$(3.2) \quad \begin{array}{ccc} A' & \xrightarrow{\rho'_i} & A' \\ \pi \downarrow & & \downarrow \pi \\ A & \xrightarrow{\rho_i} & A \end{array}$$

commute for all $i \in I$. We say that $\pi : \mathcal{C}' \rightarrow \mathcal{C}$ is a *covering*, and that \mathcal{C}' is a *covering of \mathcal{C}* .

By [3, Prop. 3.4] and [3, Cor. 3.6], coverings behave similarly as for example in topology: If \mathcal{C} is a Cartan scheme and a an object, then for each subgroup $U \leq \text{End}(a)$ there exists up to equivalence a unique Cartan scheme \mathcal{C}' , a covering $\pi : \mathcal{C}' \rightarrow \mathcal{C}$ and an object b of \mathcal{C}' such that $\text{End}(b)$ and U are isomorphic via π (see the definition of the covariant functor F_π in [3, Sect. 3]). In particular if U is the trivial subgroup, then we obtain the universal covering.

Let $n \in \mathbb{N}$ and let \mathcal{C} be a connected simply connected Cartan scheme of rank two with finite root system and $2n$ objects. By Definition 3.10,

to obtain all coverings $\mathcal{C} \rightarrow \mathcal{C}'$, we have to determine the pairs of objects a, b of \mathcal{C} for which $\Phi(i, a) = \Phi(i, b)$ for some (equivalently all) $i \in I$.

Consider the action of the dihedral group \mathbb{D}_{2n} on the object change diagram of \mathcal{C} and write $\mathbb{D}_{2n} = Z_{2n} \rtimes Z_2$, where Z_k is the cyclic group with k elements; let τ be the generator of the Z_2 part on the right and μ a generator of Z_{2n} rotating the object change diagram by one generating morphism. Let H be the following subgroup of \mathbb{D}_{2n} :

$$(3.3) \quad H = \{\mu^{2k}, \tau\mu^{2k+1} \in \mathbb{D}_{2n} \mid k \in \mathbb{Z}\}.$$

The group $\text{End}(a')$ for an object a' of \mathcal{C}' will at most be a subgroup of H . Indeed, let $i \in I$ and $a, b \in A$ with $\Phi(i, a) = \Phi(i, b)$. Then either the distance between a and b in the object change diagram is even, then we just rotate the object change diagram; or the distance is odd, then we rotate and change direction.

We determine first the periodicity of an η -sequence. If c is a sequence, let c^r denote the r times repeated sequence c .

Proposition 3.11. *Let c be an η -sequence. Write $c = (a_1, \dots, a_n)^r$ for suitable $r, n, a_1, \dots, a_n \in \mathbb{N}$. Then $r \in \{1, 2, 3\}$.*

Proof. Observe first that by Definition 3.2 an η -sequence always contains a 1. If $n = 1$, then $c = (1, 1, 1)$ again by Definition 3.2, so $r = 3$.

Assume that $n = 2$. Then $c = (a, 1)^r$ for some $a \in \mathbb{N}$ and $a \neq 1$ since two consecutive 1's may never appear by construction unless the sequence is $(1, 1, 1)$. If $r = 2k$ then

$$c = (a, 1, a, 1)^k \approx (a-1, a-1, 1)^k \sim (a-1, 1, a-1)^k.$$

Thus if $a-1 = 1$ then $r = 2$ and $c = (2, 1, 2, 1)$. Otherwise $a > 2$ and $(a-1, 1, a-1)^k \approx (a-2, a-2)^k$ which is a contradiction. If $r = 2k+1$ for some $k \geq 1$, then $c = (a, 1, a, 1)^k(a, 1) \approx (a-1, a-1, 1)^k(a, 1)$ and therefore $a > 2$. The sequence $(a-1, a-1, 1)^k(a, 1)$ is equivalent to

$$(a-1, 1, a-1)^{k-1}(a-1, 1, a, 1, a-1) \approx (a-2)^{2k+1}$$

and hence $a = 3, k = 1$, i.e. $c = (3, 1, 3, 1, 3, 1)$ and $r = 3$.

Assume now that $n > 2$. Then without loss of generality $a_2 = 1$ and $c \approx (a_1-1, a_3-1, a_4, \dots, a_n)^r$ has length $r(n-1)$. Hence we conclude by induction on n that $r \in \{1, 2, 3\}$. \square

Call an η -sequence $c = (a_1, \dots, a_n)$ of “type 1” if there exists k such that $(a_k, \dots, a_n, a_1, \dots, a_{k-1}) = (a_k, a_{k-1}, \dots, a_1, a_n, \dots, a_{k+1})$. Otherwise, we say that c is of “type 2”. We obtain:

Proposition 3.12. *Let \mathcal{C} be a connected simply connected Cartan scheme of rank two. Assume that $\mathcal{R}^{\text{ec}}(\mathcal{C})$ is a finite irreducible root system. Let $a \in A$ and $c = \Phi(1, a)$. Let $\pi : \mathcal{C} \rightarrow \mathcal{C}'$ be the covering*

for which $\text{End}(\pi(a))$ is maximal. Write $c = d^r$ for a sequence d and $r \in \mathbb{N}$ such that r is maximal. Let $m = |A|/2r$ be the length of d . Then $\text{End}(\pi(a))$ is given by the following subgroup of H (see Equation (3.3)):

m even:			
c	$r = 1$	$r = 2$	$r = 3$
type 1	$Z_2 \times Z_2$	\mathbb{D}_4	\mathbb{D}_6
type 2	$Z_2 = \langle \mu^n \rangle$	$Z_4 = \langle \mu^{\frac{n}{2}} \rangle$	$Z_6 = \langle \mu^{\frac{n}{3}} \rangle$
m odd:			
c	$r = 1$	$r = 2$	$r = 3$
type 1	Z_2	$Z_2 \times Z_2$	$Z_3 \times Z_2$
type 2	1	$Z_2 = \langle \mu^n \rangle$	$Z_3 = \langle \mu^{\frac{2n}{3}} \rangle$

Proof. Remember that if the η -sequence c is of length n then we have $2n$ objects and the Cartan entries used for the $2n$ generating morphisms are the entries of c^2 . So by Proposition 3.2, what we are looking for are the elements σ of H fixing the sequence $c^2 = d^{2r}$ where $r \in \{1, 2, 3\}$. \square

Remark 3.13. To construct all Cartan schemes for which \mathcal{C} is a covering, proceed in the following way: Choose a subgroup U of the group given in Proposition 3.12, and view it as a subgroup of \mathbb{D}_{2n} as explained above Proposition 3.11, so U acts on the set A of objects of \mathcal{C} . Define A' to be the set of orbits of U on A , and $\pi : A \rightarrow A'$ maps an object to its orbit. Then set $C'^{\pi(a)} = C^a$ and ρ'_i are given by the commutative diagram in Definition 3.10. The quadruple $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i \in I}, (C'^a)_{a \in A'})$ is then a Cartan scheme with finite root system and $\mathcal{C} \rightarrow \mathcal{C}'$ is a covering as defined in Definition 3.10.

4. FINITE REFLECTION GROUPOIDS AND THE GRASSMANNIANS

Let $n \in \mathbb{N}_{\geq 3}$. Recall from [5] that the cluster algebra of type A_n is constructed in the following way: Start with a convex n -gon and choose a triangulation by non-intersecting diagonals. Label these diagonals by x_1, \dots, x_{n-3} , the edges by q_1, \dots, q_n and the vertices of the n -gon by $1, \dots, n$. The labels for the remaining diagonals are obtained by flipping diagonals; two adjacent triangles (x, a, b) and (x, c, d) yield a relation for the other diagonal y (see Figure 2):

$$xy = ac + bd.$$

The cluster algebra of type A_{n-3} is the subalgebra of the rational function field in $x_1, \dots, x_{n-3}, q_1, \dots, q_n$ generated by all the labels in the n -gon. In a simplified version, which will be our point of view in this text, the variables q_1, \dots, q_n are specialized to 1.

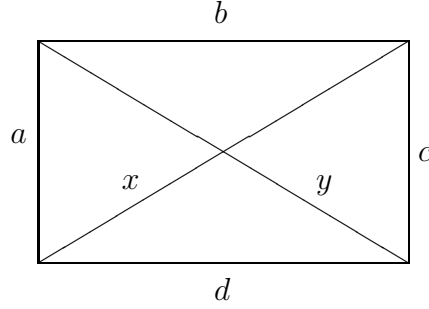


FIGURE 2. Diagonal flip

Another description is by the Grassmannian $\text{Gr}(2, n)$ (see [5, Lecture 3]): Take a matrix $z = (z_{ij}) \in K^{2 \times n}$. For $1 \leq k < l \leq n$, let

$$P_{k,l} = \det \begin{pmatrix} z_{1k} & z_{1l} \\ z_{2k} & z_{2l} \end{pmatrix}.$$

These are the Plücker coordinates of the row span of z as an element of the Grassmannian $\text{Gr}(2, n)$. They satisfy the relations

$$(4.1) \quad P_{i,k}P_{j,l} = P_{i,j}P_{k,l} + P_{i,l}P_{j,k}$$

for all $i, j, k, l \in \{1, \dots, n\}$ with $i < j < k < l$. By assigning the value of $P_{i,j}$ to the variable x_k associated to the segment (i, j) of the n -gon, it follows by induction that the rational function associated to every diagonal is equal to the corresponding $P_{k,l}$.

Let $\mathcal{O}(\text{Gr}(2, n))$ be the homogeneous coordinate ring of $\text{Gr}(2, n)$. Equivalently, it is the commutative K -algebra generated by elements $P_{i,j}$ with $1 \leq i < j \leq n$ and the Plücker relations above. Define $P_{i,j} = P_{j,i}$ whenever $1 \leq j < i \leq n$, and consider the indices of the Plücker coordinates as elements of the cyclic group Z_n . Let

$$\mathcal{G}(2, n) = \mathcal{O}(\text{Gr}(2, n))/I, \quad I = (P_{i,i+1} - 1 \mid i = 1, \dots, n).$$

Lemma 4.1. *The algebra $\mathcal{G}(2, n)$ is generated by the elements*

$$P_{1,3}, P_{2,4}, \dots, P_{n-2,n}.$$

Proof. Let \mathcal{G} be the subalgebra of $\mathcal{G}(2, n)$ generated by the elements $P_{i,i+2}$ with $1 \leq i \leq n-2$. Let $i, j \in \{1, \dots, n\}$ with $i+2 < j$. Then $P_{i,i+1}P_{i+2,j} + P_{i,j}P_{i+1,i+2} = P_{i,i+2}P_{i+1,j}$, that is

$$P_{i,j} = P_{i,i+2}P_{i+1,j} - P_{i+2,j}.$$

Hence $P_{i,j} \in \mathcal{G}$ for all $i, j \in \{1, \dots, n\}$ with $i+1 < j$ by induction on $j-i$. This proves the claim. \square

Let \mathcal{C} be a connected simply connected Cartan scheme of rank two such that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite irreducible root system. Assume that $n = |A|/2$ and that $I = \{1, 2\}$. Recall that

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta(x) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$$

for all $x \in \mathbb{Z}$. Then

$$\sigma_1^a = \begin{pmatrix} -1 & -c_{12}^a \\ 0 & 1 \end{pmatrix} = \eta(-c_{12}^a)\tau,$$

$$\sigma_2^a = \begin{pmatrix} 1 & 0 \\ -c_{21}^a & -1 \end{pmatrix} = \tau\eta(-c_{21}^a)$$

for all $a \in A$, where the matrices are given with respect to the standard basis of \mathbb{Z}^2 . Since $\tau^2 = \text{id}$,

$$\begin{aligned} \text{Prod}(n; a, 1, 2)\tau &= \dots \sigma_1 \sigma_2 \sigma_1^a \tau \\ &= \dots \sigma_1 \tau \tau \sigma_2 \sigma_1^a \tau \\ &= \tau^{n-1} \eta(c_n) \dots \eta(c_2) \eta(c_1) = -\tau^{n-1} \end{aligned}$$

for all $a \in A$, where the last equation follows from [3, Prop. 6.5]. This argument shows that Axiom (2.2) is valid for \mathcal{C} :

Corollary 4.2. *Let \mathcal{C} be a connected simply connected Cartan scheme of rank two. Assume that $\mathcal{R}^{\text{re}}(\mathcal{C})$ is a finite irreducible root system of type \mathcal{C} . Then $\mathcal{W}(\mathcal{C})$ is a finite reflection groupoid.*

We now formulate our main result on reflection groupoids in purely algebraic terms. Recall that $n \in \mathbb{N}_{\geq 3}$.

Theorem 4.3. *Let J be the ideal of $R = K[c_1, \dots, c_n]$ generated by the coefficients of the 2×2 -matrix*

$$(4.2) \quad \eta(c_n) \dots \eta(c_2) \eta(c_1) + \text{id}.$$

There is a unique algebra isomorphism

$$\bar{\varphi} : R/J \rightarrow \mathcal{G}(2, n), \quad c_i \mapsto P_{i, i+2}.$$

The inverse of $\bar{\varphi}$ can be defined by

$$\bar{\varphi}^{-1}(P_{i, j}) = (\eta(c_{j-2}) \dots \eta(c_{i+1}) \eta(c_i))_{11} \quad \text{for all } 1 \leq i < j \leq n,$$

where $(\cdot)_{11}$ denotes the entry in the first row and first column.

The proof of Theorem 4.3 uses several observations and lemmata which are collected here.

Lemma 4.4. *Let J be the ideal of $R = K[c_1, \dots, c_n]$ generated by the coefficients of the 2×2 -matrix in Equation (4.2). The algebra R/J is generated by the elements c_1, \dots, c_{n-2} .*

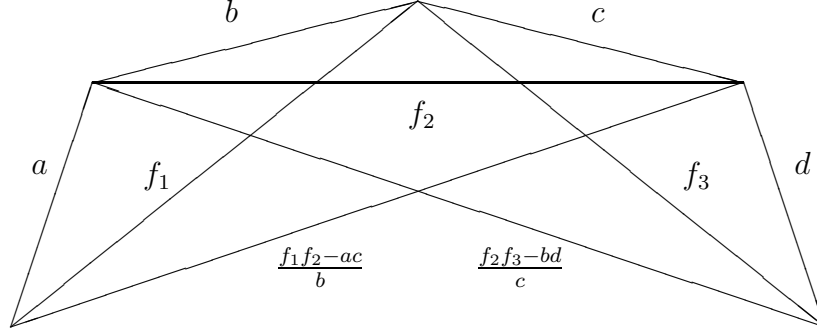


FIGURE 3. Equation (4.3)

Proof. Since $\eta(c)$ is invertible for all $c \in R$, Equation (4.2) implies that

$$\eta(c_{n-2}) \cdots \eta(c_2)\eta(c_1) = -(\eta(c_n)\eta(c_{n-1}))^{-1}.$$

Looking at the non-diagonal entries yields that c_{n-1} and c_n are polynomials in c_1, \dots, c_{n-2} . \square

For all $a, b, c \in K$ let

$$\mu(a, b, c) = \begin{pmatrix} a & -b \\ c & 0 \end{pmatrix} \in K^{2 \times 2}.$$

Explicit calculation shows that (see Figure 3 for an illustration)

$$(4.3) \quad \begin{aligned} \mu(f_1, a, b)\mu(f_2, b, c)\mu(f_3, c, d)f_2 = \\ \mu(f_1f_2 - ac, ab, bf_2)\mu(f_2f_3 - bd, cf_2, cd) \end{aligned}$$

for all $f_1, f_2, f_3, a, b, c, d \in K$. Equation (4.3) is a generalization of [3, 5.2].

Lemma 4.5. *Equation*

$$(4.4) \quad \prod_{i=1}^n \mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2}) = - \prod_{i=1}^n P_{i,i+1} \text{id}$$

holds in $\mathcal{O}(\text{Gr}(2, n))^{2 \times 2}$.

Proof. We proceed by induction on n . If $n = 3$, then Equation (4.4) becomes

$$\mu(P_{1,3}, P_{1,2}, P_{2,3})\mu(P_{1,2}, P_{2,3}, P_{1,3})\mu(P_{2,3}, P_{1,3}, P_{1,2}) = -P_{1,2}P_{2,3}P_{1,3}\text{id},$$

which can be checked by explicit calculation. Similarly, Equation (4.4) holds for $n = 4$.

Assume that $n > 4$. The subalgebra of $\mathcal{O}(\text{Gr}(2, n))$ generated by all $P_{i,j}$ with $i, j \neq 3$ is isomorphic to $\mathcal{O}(\text{Gr}(2, n-1))$, hence by induction hypothesis we have

$$(4.5) \quad \mu(P_{1,4}, P_{1,2}, P_{2,4})\mu(P_{2,5}, P_{2,4}, P_{4,5}) \prod_{i=4}^n \mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2}) \\ = -P_{1,2}P_{2,4}P_{4,5} \cdots P_{n,n+1}\text{id}.$$

By Equation (4.3) and the Grassmann-Plücker relations (4.1) we have

$$(4.6) \quad \mu(P_{1,3}, P_{1,2}, P_{2,3})\mu(P_{2,4}, P_{2,3}, P_{3,4})\mu(P_{3,5}, P_{3,4}, P_{4,5})P_{2,4} \\ = \mu(P_{1,4}, P_{1,2}, P_{2,4})\mu(P_{2,5}, P_{2,4}, P_{4,5})P_{2,3}P_{3,4}.$$

By multiplying Equation (4.5) with $P_{2,3}P_{3,4}$, inserting Equation (4.6), and dividing by $P_{2,4}$ ($\mathcal{O}(\text{Gr}(2, n))$ is an integral domain) we obtain Equation (4.4) which completes the induction. \square

Lemma 4.6. *Let S be a ring with 1. For all $x \in S$ define $\eta(x) \in S^{2 \times 2}$ similarly to Equation (3.1). Let $x, y \in S$, $A \in S^{2 \times 2}$, and $a = A_{11}$. Then $(\eta(x)A)_{21} = a$ and $(A\eta(y))_{12} = (\eta(x)A\eta(y))_{22} = -a$.*

Proof of Theorem 4.3. There exists a unique algebra map $\varphi : R \rightarrow \mathcal{G}(2, n)$ such that $\varphi(c_i) = P_{i,i+2}$ for all i . By Lemma 4.1 this map is surjective. It remains to show that $\ker \varphi = J$. First we conclude that $\bar{\varphi}$ is well-defined, that is, $J \subseteq \ker \varphi$. Indeed, specializing $P_{i,i+1}$ to 1 for all $i = 1, \dots, n$ allows us to replace the factors $\mu(P_{i,i+2}, P_{i,i+1}, P_{i+1,i+2})$ in Equation (4.4) by $\eta(P_{i,i+2})$ for all $i \in \{1, 2, \dots, n\}$. Thus $\varphi(J) = 0$, and hence $\bar{\varphi}$ is well-defined.

We prove the injectivity of $\bar{\varphi}$ by determining the inverse map. First let

$$(4.7) \quad \psi(P_{i,j}) = (\eta(c_{j-2}) \cdots \eta(c_{i+1})\eta(c_i))_{11} \in R/J$$

for all $1 \leq i < j \leq n$. Then $\psi(P_{i,i+1}) = 1$ for all $i \in \{1, 2, \dots, n-1\}$ and $\psi(P_{i,i+2}) = c_i$ for all $i \in \{1, 2, \dots, n-2\}$. Lemma 4.6 implies that

$$(4.8) \quad \psi(P_{j,k+1}) = \sum_{r=1}^2 \eta(c_{k-1})_{1r} (\eta(c_{k-2}) \cdots \eta(c_{j+1})\eta(c_j))_{r1} \\ = c_{k-1}\psi(P_{j,k}) - \psi(P_{j,k-1})$$

for all $1 \leq j < k < n$, where the potential summand $\psi(P_{j,j})$ is considered as 0. Using Equation (4.8), one shows by induction on k that

$$(4.9) \quad \psi(P_{i,j}) = \psi(P_{i,k})\psi(P_{j,k+1}) - \psi(P_{j,k})\psi(P_{i,k+1})$$

for all $1 \leq i < j \leq k < n$.

Let $\psi : \mathcal{O}(\text{Gr}(2, n)) \rightarrow R/J$ be the unique algebra map where $\psi(P_{i,j})$ is as in Equation (4.7). We prove that ψ is well-defined. Let $i, j, k, l \in \{1, 2, \dots, n\}$ with $i < j < k < l$. Then

$$\begin{aligned} \psi(P_{i,l})\psi(P_{j,k}) &= (\eta(c_{l-2}) \cdots \eta(c_i))_{11} \psi(P_{j,k}) \\ &= \sum_{r=1}^2 (\eta(c_{l-2}) \cdots \eta(c_k))_{1r} (\eta(c_{k-1}) \cdots \eta(c_i))_{r1} \psi(P_{j,k}) \\ &= (\psi(P_{k,l})\psi(P_{i,k+1}) - \psi(P_{k+1,l})\psi(P_{i,k}))\psi(P_{j,k}), \end{aligned}$$

where the last equation follows from Lemma 4.6. Apply Equation (4.9) to the first summand of the last expression. Then

$$\begin{aligned} \psi(P_{i,l})\psi(P_{j,k}) &= \psi(P_{k,l})\psi(P_{i,k})\psi(P_{j,k+1}) - \psi(P_{k,l})\psi(P_{i,j}) \\ &\quad - \psi(P_{k+1,l})\psi(P_{i,k})\psi(P_{j,k}) \\ &= \psi(P_{i,k})\psi(P_{j,l}) - \psi(P_{k,l})\psi(P_{i,j}), \end{aligned}$$

where the second equation is obtained from the definition of $\psi(P_{j,l})$ and Lemma 4.6. The obtained formula proves that ψ is well-defined. Since $\psi(P_{i,i+1}) = 1$ for all $i \in \{1, 2, \dots, n-1\}$, it induces an algebra map $\bar{\psi} : \mathcal{G}(2, n) \rightarrow R/J$. Further, Lemma 4.4 tells that R/J is generated by the elements c_1, \dots, c_{n-2} . Since $\bar{\psi}\bar{\varphi}(c_i) = \bar{\psi}(P_{i,i+2}) = c_i$ for all $i \in \{1, 2, \dots, n-2\}$, we conclude that $\bar{\psi}\bar{\varphi} = \text{id}_{R/J}$, and hence $\bar{\varphi}$ is injective. Clearly, $\bar{\psi}$ is the inverse of $\bar{\varphi}$. \square

Remark 4.7. Alternatively, we can prove that $J = \ker \varphi$ by constructing a matrix $z = (z_{ij})_{i,j} \in (R/J)^{2 \times n}$ such that if

$$\det_z(i, j) := \det \begin{pmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{pmatrix} \quad \text{for } 1 \leq i, j \leq n,$$

then $\det_z(i, i+1) = 1$ and $\det_z(i, i+2) = c_i$ for all $i = 1, \dots, n$, where all indices are to be read modulo n and $\det_z(n-1, 1) = -c_{n-1}$, $\det_z(n, 2) = -c_n$, $\det_z(n, 1) = -1$, in the following way:

Let $\xi^{(i)} = -\eta(c_i)^{-1} \cdots \eta(c_2)^{-1} \eta(c_1)^{-1}$ and write $\bar{\xi}^{(i)} = (\xi^{(i)})^{-1}$; this is well-defined over R/J since $\det \eta(x) = 1$. For $i \equiv 1 \pmod{4}$ set

$$\begin{aligned} z_{1i} &= \xi_{11}^{(i-1)}, & z_{1i+1} &= \xi_{11}^{(i)}, & z_{1i+2} &= -\bar{\xi}_{21}^{(i)}, & z_{1i+3} &= -\bar{\xi}_{21}^{(i+1)}, \\ z_{2i} &= \xi_{12}^{(i-1)}, & z_{2i+1} &= \xi_{12}^{(i)}, & z_{2i+2} &= \bar{\xi}_{11}^{(i)}, & z_{2i+3} &= \bar{\xi}_{11}^{(i+1)}. \end{aligned}$$

(Ignore the cases for which $i, i+1, i+2$ or $i+3$ are greater than n .) Several explicit calculations give that $\det_z(i, i+1) = 1$ for $i = 1, \dots, n-1$ and $\det_z(i, i+2) = c_i$ for $i = 1, \dots, n-2$ hold even in R . For $\det_z(n-1, 1) = -c_{n-1}$, $\det_z(n, 2) = -c_n$, and $\det_z(n, 1) = -1$ we use Equation (4.2).

Recall that the determinants satisfy the Plücker relations. Hence, once the matrix z is constructed, it follows that there is an algebra map from $\mathcal{G}(2, n)$ to R/J which maps $P_{i,i+2}$ to c_i .

Remark 4.8. Equation (4.3) gives a rule how to produce reflection groupoids of rank two with explicit Cartan entries: Start with a triangle and inductively append triangles. If a new triangle is glued to an edge with label y , then the new labels a, b will satisfy $ab = y$. If all edges of the n -gon have value 1, then one has completed a reflection groupoid.

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