Invariant Berezin integration on homogeneous supermanifolds

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Abstract. Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ a closed subsupergroup. We study the unimodularity of the homogeneous supermanifold $\mathcal{G}/\mathcal{H}$, *i.e.* the existence of $\mathcal{G}$-invariant sections of its Berezinian line bundle. To that end, we express this line bundle as a $\mathcal{G}$-equivariant associated bundle of the principal $\mathcal{H}$-bundle $\mathcal{G} \to \mathcal{G}/\mathcal{H}$. We also study the fibre integration of Berezinians on oriented fibre bundles. As an application, we prove a formula of 'Fubini' type: $\int_{\mathcal{G}} f = (-1)^{\dim \mathfrak{h}_1 \cdot \dim \mathfrak{g}/\mathfrak{h}} \int_{\mathcal{G}/\mathcal{H}} \int_{\mathcal{H}} f$, for all $f \in \Gamma_c(G, \mathcal{O}_{\mathcal{G}})$.

Moreover, we derive analogues of integral formulae for the transformation under local isomorphisms $\mathcal{G}/\mathcal{H} \to \mathcal{S}/\mathcal{T}$, and under the products of Lie subsupergroups $\mathcal{M} \cdot \mathcal{H} \subset \mathcal{U}$. The classical counterparts of these formulae have numerous applications in harmonic analysis.

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1. Introduction

Let G be a Lie group and H a closed subgroup. The homogeneous space G/H is called *unimodular* if there exists a non-zero G-invariant volume form. It is an important prerequisite to the study of harmonic and global analysis on the space G/H to find necessary and sufficient conditions for its unimodularity.

Classically, it is well-known that the unimodularity of G/H is equivalent to the condition that $\det \mathrm{Ad}_{\mathfrak{g}} = \det \mathrm{Ad}_{\mathfrak{h}}$ on H. Put differently, G/H is unimodular if and only if $\bigwedge^{top}(\mathfrak{g}/\mathfrak{h})$ is a trivial H-module, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively.

When G/H is unimodular, one has, for a suitable normalisation of invariant volume forms, the 'Fubini' type formula

$$\int_G f(g)\,dg = \int_{G/H} \int_H f(gh)\,dh\,d\dot{g} \quad \text{for all} \quad f \in \mathcal{C}_c(G)\ . \tag{1}$$

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Moreover, there is a well-known lemma which allows one to compute the behaviour of invariant-volume forms under local diffeomorphisms of homogeneous spaces for different groups. As an important application, if the Lie group U is as a manifold the direct product of two subgroups M and H, then for a suitable normalisation of measures,

$$\int_U f(u)\,du = \int_{M\times H} f(mh)\frac{\det \mathrm{Ad}_{\mathfrak{h}}(h)}{\det \mathrm{Ad}_{\mathfrak{u}}(h)}\,dm\,dh \quad \text{for all} \quad f\in\mathcal{C}_c(U)\ . \tag{2}$$

This formula has manifold applications: It applies, *e.g.*, in the context of Riemannian symmetric spaces, to the Bruhat and Iwasawa decompositions. It plays a role in the proof of the Harish-Chandra isomorphism for Riemannian symmetric spaces, and has many applications in the representation theory of semi-simple Lie groups.

From the point of view of the geometry and analysis of Lie supergroups and their homogeneous superspaces, it is desirable to have generalisations of all of these facts to the supermathematical context. The basic problem is to characterise unimodularity; here, the Berezinian bundle takes the place of the determinant bundle (whose sections are the volume forms).

We prove that for any Lie supergoup $\mathcal{G}$ and any closed subsupergroup $\mathcal{H}$, the Berezinian bundle $\mathrm{Ber}(\mathcal{G}/\mathcal{H})$ is, as a $\mathcal{G}$-equivariant vector bundle, isomorphic to the associated bundle $\mathcal{G}\times^{\mathcal{H}}\mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*)$ where $\mathfrak{g}$ and $\mathfrak{h}$ are, respectively, the Lie superalgebras of $\mathcal{G}$ and $\mathcal{H}$ (Corollary 4.12). From this fact, our main theorem (Theorem 4.13) follows: $\mathcal{G}/\mathcal{H}$ supports a non-zero $\mathcal{G}$-invariant Berezinian form if and only if $\mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*)$ is a trivial $\mathcal{H}$-module. Along the way, we discuss all the basic machinery of associated bundles: principal bundles, quotients by free and proper actions, equivariant vector bundles.

The formula (1) is best understood in the context of fibre integration. We introduce a general fibre integration map for oriented fibre bundles, and show that it satisfies a 'Fubini' type formula (Proposition 5.7). We then apply this to homogeneous principal bundles $\mathcal{G}\to\mathcal{G}/\mathcal{H}$ (Proposition 5.10 and Corollary 5.12). Finally, in Proposition 5.16, we generalise the formula (2).

Our investigation of invariant Berezin integration is motivated by an ongoing joint project with M.R. Zirnbauer (Köln), concerning the harmonic analysis on a certain class of reductive symmetric superspaces.

Zirnbauer [Zir91] has employed harmonic superanalysis on the super Poincaré disk in an application to a problem in mesoscopic physics (the determination of the mean conductivity for a quasi-one dimensional metallic system). These methods fit into a general framework of Riemannian symmetric spaces embedded as subspaces into infinite series of complex symmetric superspaces [Zir96]. In this generality, the harmonic analysis on symmetric superspaces has as yet not been developed.

In [AHZ10], we have established a generalisation of Chevalley's restriction theorem to the context of reductive symmetric superpairs. In a series of forthcoming papers, we will employ the results on invariant Berezin integration established in this paper to a generalisation of the Harish-Chandra isomorphism, and to the study of spherical superfunctions on symmetric superspaces.

Let us fix our notation for what follows. The graded parts of a super vector space V will be written V_0 and V_1, respectively; Π will denote the grading inverting functor. Supermanifolds will in general by denoted $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$, $\mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}})$. We will assume, as is common, that the underlying manifolds of supermanifolds are Hausdorff and second countable (where the latter assumption will be used in Section 5). Morphisms of supermanifolds will be denoted by $\varphi = (f, f^*)$ where $f : X \to Y$ and $f^* : \mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$. When appropriate, we will write $\varphi : X \to Y$ and $\varphi^* : \mathcal{O}_{\mathcal{X}} \to \varphi_* \mathcal{O}_{\mathcal{Y}}$, thereby slightly abusing the notation. We will say that a morphism of supermanifolds is injective, surjective, bijective, open or closed if so is the underlying map of topological spaces. Sometimes we will write $h \in \mathcal{O}_{\mathcal{X}}$; by this notation we mean that $h \in \mathcal{O}_{\mathcal{X}}(U)$ for some fixed but unspecified open subset $U \subset X$. Finally, for supermanifolds $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X}(\mathcal{Y}) = \mathrm{Hom}(\mathcal{Y}, \mathcal{X})$ be the set of morphisms $\mathcal{Y} \to \mathcal{X}$. This is also called the set of $\mathcal{Y}$-points of $\mathcal{X}$ (by the usual notion that a point is the same as a morphism $* \to \mathcal{X}$).

2. Quotients and actions

In this section, we discuss a generalisation of Godement's theorem on quotient manifolds to the context of supermanifolds, due to Almorox [Alm87]. As an application, we show that the quotient of a supermanifold by a free and proper Lie supergroup action is again a supermanifold.

2.1. Quotient supermanifolds.

Definition 2.1. Consider a morphism of supermanifolds $\varphi : \mathcal{X} \to \mathcal{Y}$ given by the map $f : X \to Y$ and the sheaf morphism $f^* : \mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$. We say that φ is an *open (closed) embedding* if f is an open (closed) embedding, and f^* is an isomorphism (epimorphism); and that it is a *subsupermanifold* if it factors as the composition of a closed and an open embedding. When the morphism φ is understood, then in the latter case, we will sometimes also refer to its domain $\mathcal{X}$ as a subsupermanifold of $\mathcal{Y}$.

Recall $T_x\mathcal{X} = \mathrm{Der}(\mathcal{O}_{\mathcal{X},x}, \mathbb{R})$. The morphism φ induces tangent maps $T_x\varphi : T_x\mathcal{X} \to T_{f(x)}\mathcal{Y}$ as follows. Given $v \in T_x\mathcal{X}$ and $h \in \mathcal{O}_{\mathcal{Y},f(x)}$, we have a germ $f^*h \in \mathcal{O}_{\mathcal{X},x}$, and set $[T_x\varphi(v)]h = v(f^*h)$. If $T_x\varphi$ is surjective (injective) for all $x \in X$, then φ is called a *submersion* (an *immersion*).

More generally, if $\psi = (g, g^*) : \mathcal{Z} \to \mathcal{Y}$ is another morphism, then φ and ψ are *transversal* whenever for any $x \in X$, $y \in Y$ and $z \in Z$ such that $f(x) = g(z) = y$, one has $T_y\mathcal{Y} = T_x\varphi(T_x\mathcal{X}) + T_z\psi(T_z\mathcal{Z})$. If φ, ψ are transversal, then the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ exists in the category of supermanifolds, and is a submanifold of $\mathcal{X} \times \mathcal{Z}$ [BBHRP98, Proposition 2.9].

Remark 2.2. Concerning the definition of submersions, we point out that φ being a submersion implies that $\varphi^* : \mathcal{O}_{\mathcal{Y}} \to \varphi_* \mathcal{O}_{\mathcal{X}}$ is a monomorphism of sheaves [Kos77, Proposition 2.16.2]. The converse it obvious, since the graded dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ equals the graded dimension of $T_x\mathcal{X}$, for any $x \in X$. Here, $\mathfrak{m}_x$ denotes the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. Thus, submersions might have been defined in terms of the structure sheaves.

Definition 2.3. Let $\mathcal{X}$ be a supermanifold and $\iota : \mathcal{R} \to \mathcal{X} \times \mathcal{X}$ a subsupermanifold. Let $\delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ be the diagonal morphism and $p_j : \mathcal{R} \to \mathcal{X}$ be the projections induced by ι. We call $\mathcal{R}$ an *equivalence relation* if:

1. There is given a morphism $\varrho : \mathcal{X} \to \mathcal{R}$ such that $\iota \circ \varrho = \delta$,
2. There is given a morphism $\tau : \mathcal{R} \times_{\mathcal{X}} \mathcal{R} \to \mathcal{R}$ such that

$$p_i \circ \tau = p_i \circ \pi_i \quad \text{and} \quad \tau \circ (\tau \times \mathrm{id}) = \tau \circ (\mathrm{id} \times \tau)$$

   where $\pi_1, \pi_2 : \mathcal{R} \times_{\mathcal{X}} \mathcal{R} \to \mathcal{R}$ are the projections onto the first and second factor, respectively.

The fibre product $\mathcal{R} \times_{\mathcal{X}} \mathcal{R}$ exists for the following reason: We have the identity $\iota = (p_1, p_2)$, so that $\operatorname{rk} T_r p_1 + \operatorname{rk} T_r p_2 = \dim_r \mathcal{R}$ for all $r \in R$. By (i), ϱ is an immersion, so $\dim_x \mathcal{X} \leqslant \dim_{\varrho(x)} \mathcal{R}$ for $x \in X$. If $p_1(r) = p_2(r)$, then $r = \varrho(x)$ for some $x \in X$, and then p_1, p_2 are transversal at r.

In (ii), the equation for τ is to be understood on $(\mathcal{R} \times_{\mathcal{X}} \mathcal{R}) \times_{\mathcal{X}} \mathcal{R} \cong \mathcal{R} \times_{\mathcal{X}} (\mathcal{R} \times_{\mathcal{X}} \mathcal{R})$. The latter canonical isomorphism obtains since products in any category are commutative, and the fibre product is the product in the category of supermanifolds over $\mathcal{X}$.

Remark 2.4. If we were to drop the assumption that $\mathcal{R}$ be a subsupermanifold of $\mathcal{X} \times \mathcal{X}$, then the above axioms would be those of a category with space of objects $\mathcal{X}$. This will be useful in understanding the treatment, below, of the equivalence relations defined by supergroup actions—these are reminiscent of the 'action groupoids' known from the theory of Lie groupoids.

If $\mathcal{R}$ is an equivalence relation, then there exists a morphism $\sigma : \mathcal{R} \to \mathcal{R}$ such that $p_1 \circ \sigma = p_2$ and $p_2 \circ \sigma = p_1$ [Alm87, Lemma 2.2].

Definition 2.5. Let $\mathcal{X}$ be a supermanifold and $\iota : \mathcal{R} \to \mathcal{X} \times \mathcal{X}$ an equivalence relation. If $\varphi : \mathcal{X} \to \mathcal{Y}$ is a morphism of supermanifolds, then φ is called a *quotient* by $\mathcal{R}$ if φ is a submersion and the coequaliser of $p_1, p_2 \colon \mathcal{R} \rightrightarrows \mathcal{X}$. If moreover, p_1, p_2 is the kernel pair of φ (*i.e.* ι induces an isomorphism $\mathcal{R} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$), then $\mathcal{R}$ is called *effective* and $\mathcal{Y}$ an *effective quotient*.

If a quotient of $\mathcal{X}$ by $\mathcal{R}$ exists, then it is unique up to canonical isomorphism. In this situation, we write $\mathcal{Y} = \mathcal{X}/\mathcal{R}$. Note that since coequalisers are epimorphisms and the Yoneda embedding preserves colimits, a quotient morphism φ is necessarily surjective on the base spaces (consider $\operatorname{Hom}(*, \varphi)$).

We have the following generalisation to supermanifolds of a theorem which in the context of smooth manifolds is attributed to Godement.

Theorem 2.6. *Let $\mathcal{X}$ be a supermanifold and $\mathcal{R}$ an equivalence relation on $\mathcal{X}$. Then $\mathcal{X}$ admits a quotient by $\mathcal{R}$ if and only if $\mathcal{R}$ is closed as a subsupermanifold of $\mathcal{X} \times \mathcal{X}$, and the projections $p_i : \mathcal{R} \to \mathcal{X}$ are submersions. Whenever a quotient exists, it is effective.*

Proof. According to [Alm87, Theorem 2.6], the condition stated in the theorem is necessary and sufficient for the existence of a supermanifold $\mathcal{Y}$ and a submersion $\varphi : \mathcal{X} \to \mathcal{Y}$ such that the underlying map $\varphi : X \to Y$ is the canonical projection with respect to the equivalence relation R on X underlying $\mathcal{R}$.

Assume the condition is satisfied, *i.e.* that $\mathcal{R}$ is closed and the p_i are submersions. Then there exists a submersion $\varphi : \mathcal{X} \to \mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}})$ where $Y = X/R$ and $\varphi : X \to Y$ is the canonical projection. Then φ is as a morphism of topological spaces the coequaliser of $p_1, p_2 \colon R \rightrightarrows X$, and the latter are also a kernel pair of φ on the level of spaces. By [Alm87, proof of Theorem 2.6],

$$\mathcal{O}_{\mathcal{Y}}(U) = \left\{ f \in \mathcal{O}_{\mathcal{X}}(\varphi^{-1}(U)) \;\middle|\; p_1^* f = p_2^* f \right\}$$

and φ^* is defined by $\varphi^* f = f$. By this definition, $\varphi^* \colon \mathcal{O}_{\mathcal{Y}} \to \varphi_* \mathcal{O}_{\mathcal{X}}$ is as a sheaf morphism the equaliser of $p_1^*, p_2^* \colon \mathcal{O}_{\mathcal{X}} \rightrightarrows \mathcal{O}_{\mathcal{R}}$, and the latter form a cokernel pair for φ^*. Hence, φ is an effective quotient morphism. The converse implication is obvious by Almorox's theorem. ■

Although we shall not use this fact in the sequel, any equivalence relation on a supermanifold in the sense defined above automatically satisfies the assumptions of the theorem in the *odd* variables. To state this more precisely, we make the following definition.

Definition 2.7. A morphism $\varphi = (f, f^*) : \mathcal{X} \to \mathcal{Y}$ is called an *even (odd) submersion* if for all $x \in X$, the tangent map $T_x\varphi$ induces by restriction a surjection $(T_x\mathcal{X})_i \to (T_{f(x)}\mathcal{Y})_i$ where $i = 0, 1$, respectively.

Proposition 2.8 ([BBHRP98, Theorem 4.3])**.** *For any equivalence relation $\mathcal{R}$ on a supermanifold $\mathcal{X}$, the projections $p_i : \mathcal{R} \to \mathcal{X}$ are odd submersions.*

Corollary 2.9. *A supermanifold $\mathcal{X}$ admits a quotient by a given equivalence relation $\mathcal{R}$ if and only if $\mathcal{R}$ is closed as a subsupermanifold of $\mathcal{X} \times \mathcal{X}$, and the projections $p_i : \mathcal{R} \to \mathcal{X}$ are even submersions.*

2.2. Actions of Lie supergroups.

Definition 2.10. Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{X}$ a left $\mathcal{G}$-space. Let $\alpha : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ denote the action. Then α is *free* if $(\alpha, \mathrm{pr}_2) \colon \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is an embedding, *transitive* if (α, pr_2) is a surjective submersion, and *proper* if so is the morphism (α, pr_2). Here, a morphism of supermanifolds is called *proper* if the underlying map of topological spaces is proper, *i.e.* closed and with quasi-compact fibres. These definitions can be easily modified for the case of right actions.

Definition 2.11. Let $\mathcal{X}$, $\mathcal{Y}$ be supermanifolds with actions $\alpha : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ and $\beta : \mathcal{G} \times \mathcal{Y} \to \mathcal{Y}$ of a Lie supergroup $\mathcal{G}$. A morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ is $\mathcal{G}$*-equivariant* if the following diagram commutes:

$$\begin{tikzcd} \mathcal{G}\times\mathcal{X} \arrow[r, "\mathrm{id}\times\varphi"] \arrow[d, "\alpha"'] & \mathcal{G}\times\mathcal{Y} \arrow[d, "\beta"] \\ \mathcal{X} \arrow[r, "\varphi"'] & \mathcal{Y} \end{tikzcd}$$

We say that $\mathcal{X}$ admits a *quotient* by $\mathcal{G}$ if there exists a supermanifold $\mathcal{Y}$ and a submersion $\pi:\mathcal{X}\to\mathcal{Y}$ which is equivariant for the trivial $\mathcal{G}$-action on $\mathcal{Y}$, and such that the following universal property obtains: For any supermanifold $\mathcal{Z}$, and for any morphism $\varphi:\mathcal{X}\to\mathcal{Z}$ which is equivariant for the trivial $\mathcal{G}$-action, there exists a unique morphism $\psi:\mathcal{Y}\to\mathcal{Z}$ making the following diagram commutative:

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathcal{Z} \\ & \mathcal{Y} \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

If it exists, π is unique up to canonical isomorphism, and we write $\mathcal{Y}=\mathcal{X}/\mathcal{G}$.

Theorem 2.12. *Let $\mathcal{X}$ be a supermanifold and $\mathcal{G}$ a Lie supergroup acting freely and properly on $\mathcal{X}$. Then $\mathcal{X}$ admits a quotient by $\mathcal{G}$.*

Proof. Let α denote the action. Then $\iota=(\alpha,\mathrm{pr}_2):\mathcal{R}=\mathcal{G}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is a proper and hence closed embedding. In other words, $\mathcal{R}$ (or ι) is a closed subsupermanifold of $\mathcal{X}\times\mathcal{X}$.

We need to see that $\mathcal{R}$ is an equivalence relation with submersive projections. First, note for the projections $p_i:\mathcal{R}\to\mathcal{X}$ that $p_2=\mathrm{pr}_2$ whereas $p_1=\alpha$. The required morphism $\varrho:\mathcal{X}\cong *\times\mathcal{X}\to\mathcal{R}$ is given by $\varrho=\eta\times\mathrm{id}$ where $\eta:*\to\mathcal{G}$ is the unit. Next, we may define $\sigma:\mathcal{R}\to\mathcal{R}$ by $\sigma=(i\circ\mathrm{pr}_1,\alpha)$ where $i:\mathcal{G}\to\mathcal{G}$ is inversion. Then

$$p_1\circ\sigma=\alpha\circ(i\circ\mathrm{pr}_1,\alpha)=\mathrm{pr}_2=p_2\ .$$

Moreover, $\sigma^2=\mathrm{id}$ since $i^2=\mathrm{id}$, so that $p_2\circ\sigma=p_1$, and σ is an isomorphism. The morphism $p_2=\mathrm{pr}_2$ is manifestly a submersion. Since σ is an isomorphism, so is p_1.

Next, we consider the fibred product $\mathcal{R}\times_{\mathcal{X}}\mathcal{R}$. We define a morphism

$$\phi:\mathcal{R}^{(2)}=\mathcal{G}\times\mathcal{G}\times\mathcal{X}\to\mathcal{R}\times\mathcal{R}\quad\text{by}\quad\phi=\big(\mathrm{pr}_1,(\alpha\circ(\mathrm{pr}_2\times\mathrm{pr}_3)),\mathrm{pr}_2\times\mathrm{pr}_3\big)\ .$$

With the projections $\pi_i:\mathcal{R}^{(2)}\to\mathcal{R}$, defined by $\pi_1=(\mathrm{pr}_1,\alpha\circ(\mathrm{pr}_2\times\mathrm{pr}_3))$ and $\pi_2=\mathrm{pr}_2\times\mathrm{pr}_3$, one checks readily that $\mathcal{R}^{(2)}$ satisfies the universal property of the fibred product of p_1,p_2. Thus, we may write $\mathcal{R}^{(2)}=\mathcal{R}\times_{\mathcal{X}}\mathcal{R}$. Let $\psi=\mathrm{pr}_1\times\mathrm{pr}_3\times\mathrm{pr}_2:\mathcal{R}\times\mathcal{R}=\mathcal{G}\times\mathcal{X}\times\mathcal{G}\times\mathcal{X}\to\mathcal{G}^2\times\mathcal{X}$. Then $\psi\circ\phi=\mathrm{id}$, and it follows that ϕ is a closed embedding, *i.e.* a closed subsupermanifold.

Now, we may define $\tau:\mathcal{R}^{(2)}\to\mathcal{R}$ by $\tau=(m\circ(\mathrm{pr}_1\times\mathrm{pr}_2),\mathrm{pr}_3)$ where the morphism $m:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is multiplication. Then

$$p_1\circ\tau=\alpha\circ(m\circ(\mathrm{pr}_1\times\mathrm{pr}_2),\mathrm{pr}_3)=\alpha\circ(\mathrm{pr}_1,\alpha\circ(\mathrm{pr}_2\times\mathrm{pr}_3))=p_1\circ\pi_1$$

and $p_2 \circ \tau = \mathrm{pr}_3 = p_2 \circ \pi_2$. Thus, $\mathcal{R}$ is indeed an equivalence relation, and the assumptions of Theorem 2.6 are satisfied, so that the quotient supermanifold $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{R}$ exists.

It remains to check that this quotient supermanifold satisfies the universal property of the quotient by a Lie supergroup. To that end, let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a morphism which is $\mathcal{G}$-equivariant with respect to the trivial $\mathcal{G}$-action on $\mathcal{Y}$. Then the following diagram commutes:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{X} \arrow[d, "\varphi"] \\ \mathcal{X} \arrow[r, "\varphi"'] & \mathcal{Y} \end{tikzcd}$$

Now, by the universal property of coequalisers, there manifestly exists a morphism $\psi : \mathcal{X}/\mathcal{R} \to \mathcal{Y}$ such that $\varphi = \psi \circ \pi$. This proves the claim. ∎

3. Principal and associated bundles

3.1. Principal bundles.

Definition 3.1. Given a morphism $p : \mathcal{X} \to \mathcal{B}$, the tuple $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{F})$ is called a *fibre bundle with fibre* $\mathcal{F}$ if there exist an open cover $(\mathcal{U}_\alpha)$ of $\mathcal{B}$ and isomorphisms $\tau_\alpha : \mathcal{X}|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathcal{F}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{X}|_{\mathcal{U}_\alpha} \arrow[rr, "\tau_\alpha"] \arrow[dr, "p"'] & & \mathcal{U}_\alpha \times \mathcal{F} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathcal{U}_\alpha & \end{tikzcd}$$

Here, if $U_\alpha \subset B$ is the underlying manifold of $\mathcal{U}_\alpha$, $X|_{\mathcal{U}_\alpha}$ denotes the open subsupermanifold of $\mathcal{X}$ with base $p^{-1}(U_\alpha)$. The τ_α are called *local trivialisations*. We shall call $\mathcal{X}$ the *total space*, $\mathcal{B}$ the *base space*, $\mathcal{F}$ the *fibre* and p the *bundle projection* of $\mathcal{E}$.

If $\mathcal{G}$ is a Lie supergroup and $\mathcal{X}$, $\mathcal{B}$ carry left $\mathcal{G}$-actions $\alpha_{\mathcal{X}}$, $\alpha_{\mathcal{B}}$ (say) such that p is $\mathcal{G}$-equivariant, then $\mathcal{E}$ is called a *$\mathcal{G}$-equivariant fibre bundle.*

Given a second fibre bundle $\mathcal{E}' = (\mathcal{X}', \mathcal{B}', p', \mathcal{F}')$, a morphism of fibre bundles $\mathcal{E} \to \mathcal{E}'$ is a pair of morphisms $\varphi : \mathcal{X} \to \mathcal{X}'$, $f : \mathcal{B} \to \mathcal{B}'$ such that $f \circ p = p' \circ \varphi$. We denote the automorphisms of $\mathcal{E}$ by $\mathrm{Aut}(\mathcal{E})$ or $\mathrm{Aut}_{\mathcal{F}}(\mathcal{X})$.

Definition 3.2. Let $\mathcal{X}$ be an arbitrary supermanifold, endowed with a left action $\alpha : \mathcal{H} \times \mathcal{X} \to \mathcal{X}$ of a Lie supergroup $\mathcal{H}$. For any morphism $\gamma : \mathcal{Y} \to \mathcal{H}$, define a morphism $\alpha_\gamma : \mathcal{Y} \times \mathcal{X} \to \mathcal{Y} \times \mathcal{X}$ by $\alpha_\gamma = (\mathrm{pr}_1, \alpha \circ (\gamma \times \mathrm{id}))$. Then α_γ is an automorphism of the trivial bundle $\mathrm{pr}_1 : \mathcal{Y} \times \mathcal{X} \to \mathcal{Y}$. The action α is called *effective* if $\gamma \mapsto \alpha_\gamma : \mathcal{H}(\mathcal{Y}) = \mathrm{Hom}(\mathcal{Y}, \mathcal{H}) \to \mathrm{Aut}_{\mathcal{Y}}(\mathcal{Y} \times \mathcal{X})$ is injective for any supermanifold $\mathcal{Y}$.

Definition 3.3. Let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{F})$ be a fibre bundle and α an effective left action of $\mathcal{H}$ on $\mathcal{F}$. For any open $U \subset B$, let $\tau_{\mathcal{X}}(U)$ be the set of trivialisations of

$\mathcal{X}$ over the open subsupermanifold $\mathcal{U} \subset \mathcal{B}$ corresponding to U. If $\tau \in \tau_{\mathcal{X}}(U)$ and $\gamma \in \mathcal{H}(\mathcal{U})$, then $\gamma.\tau = \alpha_\gamma \circ \tau$ defines an effective left action of the group $\mathcal{H}(\mathcal{U})$ on $\tau_{\mathcal{X}}(U)$ [Sch84, 6.4].

An $\mathcal{H}$*-structure* is a subsheaf of sets $\mathcal{A} \subset \tau_{\mathcal{X}}$ such that for every $x \in B$, there exists an open neighbourhood U such that $\mathcal{A}(U) \neq \varnothing$, and $\mathcal{H}(U)$ acts transitively on $\mathcal{A}(U)$ whenever this set is non-void. *I.e.*, for any two $\tau, \tau' \in \tau_{\mathcal{X}}(U)$, there exists a $\gamma \in \mathcal{H}(\mathcal{U})$ (unique by effectiveness) such that $\tau = \gamma.\tau'$. We also say that $\mathcal{H}$ is the *structure group* of $\mathcal{E}$, and that the elements of $\mathcal{A}(U)$ are *bundle charts*. Occasionally, we will also refer to τ^{-1}, where $\tau \in \mathcal{A}(U)$, as a bundle chart.

Assume that $\mathcal{E}$ is at the same time a fibre bundle with $\mathcal{H}$-structure $\mathcal{A}$ and a $\mathcal{G}$-equivariant fibre bundle. Denote the $\mathcal{H}$-action in the fibre by α, and the $\mathcal{G}$-actions on $\mathcal{X}$ and $\mathcal{B}$ by $\tilde{\beta}$ and β, respectively. We say that $\mathcal{E}$ is a $\mathcal{G}$*-equivariant fibre bundle with* $\mathcal{H}$*-structure* if for each $b \in B$, there are open subsupermanifolds $\mathcal{U}, \mathcal{W}_i$ of $\mathcal{B}$ and $\mathcal{V}_i$ of $\mathcal{G}$ such that $b \in U$, $(\mathcal{V}_i \times \mathcal{W}_i)$ is an open cover of $\beta^{-1}(\mathcal{U})$, and there exist bundle charts $\tau \in \mathcal{A}(U)$, $\tau_i \in \mathcal{A}(W_i)$, and $\gamma_i \in \mathcal{H}(\mathcal{V}_i \times \mathcal{W}_i)$ such that for all i, the following diagram commutes

$$\begin{tikzcd}
\mathcal{V}_i \times \mathcal{X}|_{\mathcal{W}_i} \arrow[r, "\mathrm{id}\times\tau_i"] \arrow[d, "\tilde{\beta}"'] & \mathcal{V}_i \times \mathcal{W}_i \times \mathcal{F} \arrow[r, "(\mathrm{id},\gamma_i)\times\mathrm{id}"] & \mathcal{V}_i \times \mathcal{W}_i \times \mathcal{H} \times \mathcal{F} \arrow[dl, "\beta\times\alpha"] \\
\mathcal{X}|_{\mathcal{U}} \arrow[r, "\tau"'] & \mathcal{U} \times \mathcal{F} &
\end{tikzcd} \tag{3}$$

Equivalently, the morphism $\mathcal{G} \times \mathcal{X} \to \beta^* \mathcal{X}$ induced by $\tilde{\beta}$ is one of $\mathcal{H}$-fibre bundles, in the sense of [Sch84, 6.6].

For later use and reference, we record the following fundamental fact [Sch84, Propositions 5.3, 6.5].

Proposition 3.4. *Let $\mathcal{B}$, $\mathcal{F}$ be supermanifolds, $\mathcal{H}$ a Lie supergroup acting effectively on $\mathcal{F}$, $(\mathcal{U}_i)$ an open cover of $\mathcal{B}$, and $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$. Assume given $\varphi_{ij} \in \mathcal{H}(\mathcal{U}_{ij})$ such that $\varphi_{ij} \cdot \varphi_{jk} = \varphi_{ik}$ on $\mathcal{U}_{ijk} = \mathcal{U}_{ij} \cap \mathcal{U}_{jk}$ and $\varphi_{ii} = 1$. Here, $\varphi_{ij} \cdot \varphi_{jk} = m \circ (\varphi_{ij} \times \varphi_{jk})$ and 1 denotes the unique morphism $\mathcal{U}_{ii} = \mathcal{U}_i \to \mathcal{H}$ which factors through the unit $* \to \mathcal{H}$.*

There exists an $\mathcal{H}$-fibre bundle $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{F})$ with bundle charts τ_i over $\mathcal{U}_i$ such that $\varphi_{ij}.\tau_j = \tau_i$ on $\mathcal{U}_{ij}$, and $\mathcal{E}$ is unique up to unique isomorphism. We say that $\mathcal{E}$ is determined by the cocyle (φ_{ij}).

Definition 3.5. Let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{H})$ be a fibre bundle with fibre and structure group $\mathcal{H}$ where $\mathcal{H}$ a Lie supergroup which acts from the left on itself via left multiplication. (This action is effective [Sch84, Lemma 6.3].) Then $\mathcal{E}$ is called a *principal* $\mathcal{H}$*-bundle.*

Proposition 3.6. *Let $\mathcal{X}$ be a supermanifold and $\mathcal{H}$ a Lie supergroup. If $\mathcal{H}$ acts freely and properly from the right on $\mathcal{X}$ via $\alpha : \mathcal{X} \times \mathcal{H} \to \mathcal{X}$, then $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{H})$ is a principal $\mathcal{H}$-bundle, where $p : \mathcal{X} \to \mathcal{B}$ is the quotient morphism. Conversely, given any morphism $p : \mathcal{X} \to \mathcal{B}$ such that the tuple $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, \mathcal{H})$ is a principal*

$\mathcal{H}$-bundle, the supermanifold $\mathcal{X}$ allows for a free and proper $\mathcal{H}$-action such that p is the quotient morphism.

Proof. Assume that $\mathcal{E}$ be a principal $\mathcal{H}$-bundle. For each local trivialisation $\tau \in \tau_{\mathcal{X}}(U)$, $\mathcal{X}|_{\mathcal{U}} \cong \mathcal{U} \times \mathcal{H}$ is endowed with a right $\mathcal{H}$-action which is induced by the canonical right action of $\mathcal{H}$ on itself. Since the latter commutes with the canonical left action, we obtain a right action of $\mathcal{H}$ on $\mathcal{X}$ [Sch84, Proposition 6.18]. It is clear from its definition that p satisfies the universal property of the quotient morphism when restricted to $\mathcal{X}|_{\mathcal{U}}$, and hence globally.

On the level of ordinary manifolds, $E = (X, B, p, H)$ is a principal H-bundle, so the action of H on X is free and proper. To check that the morphism $(\mathrm{pr}_1, \alpha) : \mathcal{X} \times \mathcal{H} \to \mathcal{X} \times \mathcal{X}$ is an embedding, it suffices to prove that it is an immersion. We may assume that $\mathcal{X} = \mathcal{B} \times \mathcal{H}$, since this is a local property. But the multiplication morphism $m : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is a free $\mathcal{H}$-action; indeed, (pr_1, m) is an isomorphism with inverse $(\mathrm{pr}_1, m \circ (i \times \mathrm{id}))$ where i denotes inversion. This implies that (pr_1, α) is a proper embedding, so that α is free and proper.

Conversely, assume that α be a free and proper $\mathcal{H}$-action. Then the quotient $\mathcal{X}/\mathcal{H}$ exists, and we assume that $p : \mathcal{X} \to \mathcal{B}$ be the quotient morphism. Since p is a surjective submersion, $\mathcal{B}$ has an open cover $(\mathcal{U}_\alpha)$ such that there are morphisms $s_\alpha : \mathcal{U}_\alpha \to \mathcal{X}$ such that $p \circ s_\alpha = \mathrm{id}_{\mathcal{U}_\alpha}$. It follows that the s_α are open embeddings. Define $\phi_\alpha : \mathcal{U}_\alpha \times \mathcal{H} \to \mathcal{X}|_{\mathcal{U}_\alpha}$ by $\phi_\alpha = \alpha \circ (s_\alpha \times \mathrm{id})$. This is well-defined because $p \circ \alpha = p \circ \mathrm{pr}_1$.

We have $p \circ \phi_\alpha = p \circ s_\alpha \circ \mathrm{pr}_1 = \mathrm{pr}_1$. It is clear that ϕ_α is bijective on the level of ordinary spaces, and it is an immersion. Moreover, since $\mathcal{B}$ is an effective quotient by Theorem 2.6, we have $\mathcal{X} \times_{\mathcal{B}} \mathcal{X} \cong \mathcal{R} \cong \mathcal{X} \times \mathcal{H}$ (*cf.* Theorem 2.12 and proof) and thus locally $2 \dim \mathcal{X} - \dim \mathcal{B} = \dim \mathcal{X} + \dim \mathcal{H}$; hence, $\dim \mathcal{X} = \dim \mathcal{B} + \dim \mathcal{H}$. Thus, ϕ_α is an isomorphism by [Kos77, Theorem 2.16 and Corollary]. It follows that the ϕ_α define local trivialisations of $\mathcal{E}$, and they are manifestly $\mathcal{H}$-equivariant. That the ϕ_α define an $\mathcal{H}$-structure by considering the left action of $\mathcal{H}$ on itself follows from [Sch84, Lemma 6.2 and Proposition 6.5]. ∎

3.2. Associated bundles.

Definition 3.7. Let $\mathcal{X}, \mathcal{Y}$ be supermanifolds and $\mathcal{H}$ a Lie supergroup, and assume given a right action $\alpha : \mathcal{X} \times \mathcal{H} \to \mathcal{X}$ and a left action $\beta : \mathcal{H} \times \mathcal{Y} \to \mathcal{Y}$ where α is free and proper. Consider the diagonal right action $\gamma : \mathcal{X} \times \mathcal{Y} \times \mathcal{H} \to \mathcal{X} \times \mathcal{Y}$, defined by $\gamma = (\alpha \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta \circ (i \circ \mathrm{pr}_3, \mathrm{pr}_2))$.

It is clear that $(\gamma, \mathrm{pr}_1, \mathrm{pr}_2)$ is a proper injection on the level of ordinary spaces, and since $T_{(x,h)}\alpha$ is injective for any $(x, h) \in X \times H$, it follows that γ is an immersion. Hence, γ is a free and proper action, and the quotient

$$\mathcal{X} \times^{\mathcal{H}} \mathcal{Y} = (\mathcal{X} \times \mathcal{Y})/\mathcal{H}$$

exists. By the universal property, the projection $\mathrm{pr}_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ descends to a submersion $p : \mathcal{X} \times^{\mathcal{H}} \mathcal{Y} \to \mathcal{X}/\mathcal{H}$ which we will call the *induced projection.*

Assume given a local trivialisation of $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{H}$, $\tau : \mathcal{X}|_{\mathcal{U}} \to \mathcal{U} \times \mathcal{H}$ (say). Then $(\mathcal{U} \times \mathcal{H}) \times^{\mathcal{H}} \mathcal{Y} \cong \mathcal{U} \times \mathcal{Y}$, and this defines a local trivialisation of p.

Proposition 3.8. *Given a principal $\mathcal{H}$-bundle $\pi : \mathcal{X} \to \mathcal{B}$ and a left $\mathcal{H}$-supermanifold $\mathcal{Y}$, the tuple $\mathcal{E} = (\mathcal{X} \times^{\mathcal{H}} \mathcal{Y}, \mathcal{B}, p, \mathcal{Y})$ where $p : \mathcal{X} \times^{\mathcal{H}} \mathcal{Y} \to \mathcal{B}$ is the induced projection, is a fibre bundle, called the bundle with fibre $\mathcal{Y}$* associated with *$\mathcal{X}$. If the action of $\mathcal{H}$ on $\mathcal{Y}$ is effective, then $\mathcal{X} \times^{\mathcal{H}} \mathcal{Y}$ has an $\mathcal{H}$-structure.*

Proof. It only remains to specify the $\mathcal{H}$-structure if the action on $\mathcal{Y}$ is effective. With $\tau \in \tau_{\mathcal{X}}(U)$, we associate $\tilde{\tau} \in \tau_{\mathcal{X} \times^{\mathcal{H}} V}(U)$, the morphism induced on the quotient by the composite

$$\tau' : \mathcal{X}|_{\mathcal{U}} \times \mathcal{Y} \xrightarrow{\tau \times \mathrm{id}} \mathcal{U} \times \mathcal{H} \times \mathcal{Y} \xrightarrow{\mathrm{id} \times \alpha} \mathcal{U} \times \mathcal{Y} \tag{4}$$

where α is the action of $\mathcal{H}$ on $\mathcal{Y}$. This defines a subsheaf of $\tau_{\mathcal{X} \times^{\mathcal{H}} \mathcal{Y}}(U)$.

Recall the action of $\mathcal{H}(U)$ on local trivialisations from Definition 3.3. Let $\sigma, \tau \in \tau_{\mathcal{X}}(U)$, there exists $\gamma \in \mathcal{H}(U)$ such that $\sigma = \gamma.\tau = m_\gamma \circ \tau$ where m is the multiplication of $\mathcal{H}$. We compute

$$\begin{aligned} \gamma.\tau' &= (\mathrm{pr}_1, \alpha \circ (\gamma \times \mathrm{id})) \circ (\mathrm{id} \times \alpha) \circ (\tau \times \mathrm{id}) \\ &= (\mathrm{id} \times \alpha) \circ (\mathrm{pr}_1, m \circ (\gamma \circ \mathrm{pr}_1, \mathrm{pr}_2), \mathrm{pr}_3) \circ (\tau \times \mathrm{id}) = \sigma' \ . \end{aligned}$$

It follows that $\gamma.\tilde{\tau} = \tilde{\sigma}$. ∎

Definition 3.9. A fibre bundle $\mathcal{E}$ whose fibre is a *linear* supermanifold V and which is endowed with a $\mathcal{GL}(V)$-structure is called a *vector bundle*. One might also define vector bundles as locally free $\mathcal{O}_{\mathcal{B}}$-module sheaves (where $\mathcal{B}$ is the base space of $\mathcal{E}$). These notions are equivalent by [Sch84, Proposition 7.33].

Let $\mathcal{G}$ be a Lie supergroup. If $\mathcal{E}$ is a vector bundle endowed with $\mathcal{G}$-actions $\tilde{\alpha}$, α on $\mathcal{X}$ and $\mathcal{B}$, respectively, then $\mathcal{E}$ is called a *$\mathcal{G}$-equivariant vector bundle* if it is a $\mathcal{G}$-equivariant fibre bundle with $\mathcal{GL}(V)$-structure in the sense of Definition 3.3.

We shall also need the notion of a morphism of vector bundles. To that end, let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, V)$ and $\mathcal{E}' = (\mathcal{X}', \mathcal{B}, p', V')$ be vector bundles over the same base. A *morphism of vector bundles* $\mathcal{E} \to \mathcal{E}'$ is given by a bundle morphism $f : \mathcal{X} \to \mathcal{X}'$ such that for each $b \in B$, there are an open neighbourhood $U \subset B$ of x, vector bundle charts $\sigma \in \mathcal{A}_{\mathcal{X}}(U)$, $\tau \in \mathcal{A}_{\mathcal{X}'}(U)$, and a morphism $\varphi : \mathcal{U} \to \mathrm{Hom}(V, V')_0$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{X}|_{\mathcal{U}} \arrow[rr, "f"] \arrow[d, "\tau"'] & & \mathcal{X}'|_{\mathcal{U}} \arrow[d, "\sigma"] \\ \mathcal{U} \times V \arrow[dr, "(\mathrm{id}, \varphi) \times \mathrm{id}"'] & & \mathcal{U} \times V' \\ & \mathcal{U} \times \mathrm{Hom}(V, V')_0 \times V \arrow[ur, "\mathrm{id} \times \varepsilon"'] & \end{tikzcd} \tag{5}$$

where $\varepsilon : \mathrm{Hom}(V, V')_0 \times V \to V'$ is the natural ('evaluation') morphism.

If $\mathcal{E}'$ is a vector bundle over a different base $\mathcal{B}'$ (say), then a *morphism of vector bundles* $\mathcal{E} \to \mathcal{E}'$ is a pair of morphisms $\varphi : \mathcal{X} \to \mathcal{X}'$, $\phi : \mathcal{B} \to \mathcal{B}'$ such that $p' \circ \varphi = \phi \circ p$ and φ induces a vector bundle morphism $\mathcal{X} \to \phi^* \mathcal{X}'$ (where the pullback is defined in [Sch84, Propositions 5.6, 7.21]).

Proposition 3.10. *Let $\mathcal{X}$ be a principal $\mathcal{H}$-bundle and V be endowed with a linear $\mathcal{H}$-action. Then the associated bundle with fibre V is a vector bundle.*

Proof. The action of $\mathcal{H}$ on V factors through a morphism of supergroups $\mathcal{H} \to \mathcal{GL}(V)$. The same proof as that of the existence of $\mathcal{H}$-structures in Proposition 3.8 shows that $\mathcal{X}$ has a $\mathcal{GL}(V)$-structure. (It is easy to check by the definition that the canonical action of $\mathcal{GL}(V)$ on V is effective.) ∎

4. Homogeneous superspaces

In what follows, let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ a closed sub-supergroup.

4.1. The tangent bundle of $\mathcal{G}/\mathcal{H}$ as an associated bundle.

4.1. We recall some basic facts related to tangent morphisms. Let $\mathcal{X}$ be a supermanifold. The tangent bundle $T\mathcal{X} \to \mathcal{X}$ is the vector bundle which is associated with the locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathrm{Der}(\mathcal{O}_{\mathcal{X}})$.

Define $\Omega^1_{\mathcal{X}}$ by $\Pi\Omega^1_{\mathcal{X}} = \mathrm{Der}(\mathcal{O}_{\mathcal{X}})^*$. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism, then by [Sch84, Proposition 8.12], there is a unique sheaf morphism $f^* : \Omega^1_{\mathcal{Y}} \to f_*\Omega^1_{\mathcal{X}}$ such that $f^*(d\varphi) = df^*\varphi$ for all $\varphi \in \mathcal{O}_{\mathcal{Y}}$. Thus, there is a vector bundle morphism $Tf : T\mathcal{X} \to T\mathcal{Y}$ where $(Tf)^*\Pi(\omega) = \Pi(f^*\omega)$.

Put differently, the set of local sections of $T\mathcal{X}$ is $\mathrm{Der}(\mathcal{O}_{\mathcal{X}})$. Then we say that $v \in \mathrm{Der}(\mathcal{O}_{\mathcal{Y}})$ is *f-related* to $u \in \mathrm{Der}(\mathcal{O}_{\mathcal{X}})$ if and only if $f^*\omega(u) = \omega(v)$ for all $\omega \in \Omega^1_{\mathcal{Y}}$, if and only if $d(f^*\varphi)(u) = d\varphi(v)$ for all $\varphi \in \mathcal{O}_{\mathcal{Y}}$. Recall from [Sch84, 7.27] that we may interpret sections of $T\mathcal{X}$ as morphisms $\mathcal{X} \to T\mathcal{X} \oplus \Pi T\mathcal{X}$ which are right inverses of the bundle projection $T\mathcal{X} \oplus \Pi T\mathcal{X} \to \mathcal{X}$ (we have to add the tangent bundle with the opposite parity in order to treat even and odd sections). If we do so, then v is f-related to u if and only if $v \circ f = (Tf \oplus \Pi Tf) \circ u$. This also determines Tf.

The assignment $\mathcal{X} \mapsto T\mathcal{X}$, $f \mapsto Tf$ defines a product-preserving functor from supermanifolds to vector bundles [Sch84, 8.16-17], the *tangent functor*. By functoriality, if $\mathcal{G}$ is a Lie supergroup, so is $T\mathcal{G}$, and $\mathcal{G}$ is a closed subsupergroup via the zero section; if $\mathcal{X}$ is a space with a $\mathcal{G}$-action, then $T\mathcal{X}$ has a $T\mathcal{G}$-action, and in particular, $T\mathcal{X}$ is a $\mathcal{G}$-equivariant vector bundle.

Proposition 4.2. *The tangent bundle $T\mathcal{G}$ is $\mathcal{G}$-equivariantly trivial. More precisely, the composite*

$$\mathcal{G} \times \mathfrak{g} \xrightarrow{0\times\iota} T\mathcal{G} \times T\mathcal{G} \xrightarrow{Tm} T\mathcal{G}$$

is a $\mathcal{G}$-equivariant vector bundle isomorphism. Here, $0 : \mathcal{G} \to T\mathcal{G}$ is the zero section, and $\iota : \mathfrak{g} = T_1\mathcal{G} \to T\mathcal{G}$ is the inclusion of the fibre at the unit.

Proof. Let ϕ denote the morphism defined in the assertion. The zero section exhibits $\mathcal{G}$ as a closed subsupergroup of $T\mathcal{G}$, and by restriction of Tm to from $T\mathcal{G}^2$ to $\mathcal{G} \times T\mathcal{G}$, we obtain the natural left $\mathcal{G}$-action on $T\mathcal{G}$. The action of $\mathcal{G}$ on $\mathcal{G} \times \mathfrak{g}$ is simply $m \times \mathrm{id}$. From these definitions, it is clear that Tm, and hence ϕ, is $\mathcal{G}$-equivariant.

Let $p : T\mathcal{G} \to \mathcal{G}$ be the bundle projection. Now, consider the morphism $\psi = (Ti \circ \mathrm{pr}_1, Tm) \circ (Ti \times \mathrm{id}) \circ (0 \circ p, \mathrm{id}) : T\mathcal{G} \to T\mathcal{G}$. We compute

$$\begin{aligned}\psi \circ \phi &= (Ti \circ \mathrm{pr}_1, Tm) \circ (\mathrm{id} \times Tm) \circ (Ti \times \mathrm{id} \times \mathrm{id}) \circ (\delta \times \mathrm{id}) \circ (0 \times \iota) \\ &= (\mathrm{pr}_1, Tm \circ (Tm \times \mathrm{id}) \circ (Ti \times \mathrm{id} \times \mathrm{id}) \circ (\delta \times \mathrm{id})) \circ (0 \times \iota) \\ &= 0 \times \iota \ .\end{aligned}$$

It follows that ϕ is an injective immersion. Moreover,

$$p \circ \phi = m \circ (p \times p) \circ (0 \times \iota) = m \circ (\mathrm{id} \times \eta) = \mathrm{pr}_1$$

where we also write η for the unique morphism $\mathfrak{g} \to \mathcal{G}$ which factors through $\eta : * \to \mathcal{G}$. Thus, ϕ is a vector bundle morphism along the identity. We conclude that ϕ is an isomorphism. ■

4.3. Let $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ be the quotient morphism and denote the unit tangent spaces by $\mathfrak{g} = T_1\mathcal{G}$, $\mathfrak{h} = T_1\mathcal{H}$. Let $\ker T\pi$ be the kernel (in the category of vector bundles over $\mathcal{G}$) of the canonical map $T\mathcal{G} \to \pi^*T(\mathcal{G}/\mathcal{H})$. Any bundle chart $\tau : T\mathcal{G}|_{\mathcal{U}} \to \mathcal{U} \times T_x\mathcal{G}$ of $T\mathcal{G}$ restricts to a bundle chart $(\ker T\pi)|_{\mathcal{U}} \to \mathcal{U} \times \ker T_x\pi$.

Lemma 4.4. *The subbundle $\ker T\pi \subset T\mathcal{G}$ identifies with $\mathcal{G} \times \mathfrak{h}$. Therefore, $T\pi$ induces a $\mathcal{G}$-equivariant vector bundle morphism*

$$\mathcal{G} \times \mathfrak{g}/\mathfrak{h} = T\mathcal{G}/\ker T\pi \to T(\mathcal{G}/\mathcal{H}) \ .$$

Proof. Let $\varphi \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$. Then $f = \pi^*\varphi \in \mathcal{O}_{\mathcal{G}}$ is characterised by $m^*f = \mathrm{pr}_1^* f$ for $m, \mathrm{pr}_1 : \mathcal{G} \times \mathcal{H} \to \mathcal{G}$. If v is ϕ-related to a local section u of $G \times \mathfrak{h}$, then

$$df(v) = dm^*f((0 \times \iota)(u)) = d\,\mathrm{pr}_1^* f((0 \times \iota)(u)) = 0 \ .$$

It follows that v is a local section of $\ker T\pi$. By equality of dimension ϕ induces a $\mathcal{G}$-equivariant isomorphism $\mathcal{G} \times \mathfrak{h} \cong \ker T\pi$. ■

4.5. On $T\mathcal{G}$, consider the right action α of $\mathcal{H}$ induced by the tangent multiplication $Tm : T\mathcal{G} \times T\mathcal{H} \to T\mathcal{G}$. Thus, $\alpha = Tm \circ (\mathrm{id} \times 0) : T\mathcal{G} \times \mathcal{H} \to T\mathcal{G}$. Recall also the adjoint action $\mathrm{Ad} : \mathcal{G} \times \mathfrak{g} \to \mathfrak{g}$. It is given as the restriction of the composite

$$T\mathcal{G}^2 \xrightarrow{\delta \times \mathrm{id}} T\mathcal{G}^3 \xrightarrow{\mathrm{id} \times Ti \times \mathrm{id}} T\mathcal{G}^3 \xrightarrow{(23)} T\mathcal{G}^3 \xrightarrow{Tm^{(2)}} T\mathcal{G}$$

where $m^{(2)} = m \circ (m \times \mathrm{id})$ and (23) interchanges the second and third factor.

Lemma 4.6. *Let γ be the diagonal right $\mathcal{H}$-action on $\mathcal{G} \times \mathfrak{g}$. The isomorphism $\phi : \mathcal{G} \times \mathfrak{g} \to T\mathcal{G}$ is $\mathcal{H}$-equivariant.*

Proof. Let $\delta^{(2)} = (\delta \times \mathrm{id}) \circ \delta$ and $\mathrm{id}^{(2)} = \mathrm{id} \times \mathrm{id}$. Let $\tilde{\gamma}$ be the morphism

$$T\mathcal{G}^3 \xrightarrow{\mathrm{id}^{(2)} \times \delta^{(2)}} T\mathcal{G}^5 \xrightarrow{(34)\circ(23)} T\mathcal{G}^5 \xrightarrow{\mathrm{id}^{(2)} \times Ti \times \mathrm{id}^{(2)}} T\mathcal{G}^5 \xrightarrow{Tm \times Tm^{(2)}} T\mathcal{G}^2$$

Then γ satisfies $(0 \times \iota) \circ \gamma = \tilde{\gamma} \circ (0 \times \iota \times 0)$, and this determines the morphism γ uniquely. But $\phi = Tm \circ (0 \times \iota)$ and $Tm \circ \gamma' = Tm^{(2)}$. This proves the assertion. ■

Proposition 4.7. *The morphism $\mathcal{G}\times\mathfrak{g}/\mathfrak{h}=T\mathcal{G}/\ker T\pi\to T(\mathcal{G}/\mathcal{H})$ induced by $T\pi$ is the quotient morphism for the induced right $\mathcal{H}$-action. In particular, $T(\mathcal{G}/\mathcal{H})$ is $\mathcal{G}$-equivariantly isomorphic, as a vector bundle, to $\mathcal{G}\times^{\mathcal{H}}\mathfrak{g}/\mathfrak{h}$.*

Proof. For the trivial $\mathcal{H}$-action on $\mathcal{G}/\mathcal{H}$, π is $\mathcal{H}$-equivariant. Applying the tangent functor, $T\pi$ is $T\mathcal{H}$-equivariant for the trivial $T\mathcal{H}$-action on $T(\mathcal{G}/\mathcal{H})$. In particular, $T\pi$ is $\mathcal{H}$-equivariant, and hence, so is the morphism induced by $T\pi$, namely $\widetilde{T\pi}:T\mathcal{G}/\ker T\pi\to T(\mathcal{G}/\mathcal{H})$.

Hence, there exists a vector bundle morphism $\varphi:\mathcal{G}\times^{\mathcal{H}}\mathfrak{g}/\mathfrak{h}\to T(\mathcal{G}/\mathcal{H})$ such that $\varphi\circ p=\widetilde{T\pi}\circ\tilde{\phi}$ where $\tilde{\phi}:\mathcal{G}\times\mathfrak{g}/\mathfrak{h}\to T\mathcal{G}/\ker T\pi$ is the isomorphism induced by ϕ, and $p:\mathcal{G}\times\mathfrak{g}/\mathfrak{h}\to\mathcal{G}\times^{\mathcal{H}}\mathfrak{g}/\mathfrak{h}$ is the quotient morphism. We know that p and $\widetilde{T\pi}$ are along π, and $\tilde{\phi}$ is along id. On the other hand, φ is a submersion since so is $\widetilde{T\pi}\circ\tilde{\phi}$. Hence, φ is an isomorphism, and by definition, it is $\mathcal{G}$-equivariant. ■

4.2. Berezinians.

4.8. Let V be a finite-dimensional super-vector space. Recall that $\mathcal{GL}(V)$ is the open subsupermanifold of the linear supermanifold $\mathrm{End}(V)_0$ corresponding to the open subset $\mathrm{GL}(V)\subset\mathrm{End}(V)_0$. Moreover, for any supermanifold $\mathcal{Y}$, the $\mathcal{Y}$-points of $\mathcal{GL}(V)$ are $\mathrm{Aut}_{\mathcal{O}(\mathcal{Y})}(\mathcal{O}(\mathcal{Y})\otimes V)$, and this induces the Lie supergroup structure on $\mathcal{GL}(V)$.

The standard action of $\mathcal{GL}(V)$ on V is defined by the actions

$$\alpha(\mathcal{Y}):\mathrm{Aut}_{\mathcal{O}(\mathcal{Y})}(\mathcal{O}(\mathcal{Y})\otimes V)\times(\mathcal{O}(\mathcal{Y})\otimes V)_0\to(\mathcal{O}(\mathcal{Y})\otimes V)_0$$

which are natural in $\mathcal{Y}$. One defines

$$\alpha(\mathcal{Y})^*:\mathrm{Aut}_{\mathcal{O}(\mathcal{Y})}(\mathcal{O}(\mathcal{Y})\otimes V)\times(\mathcal{O}(\mathcal{Y})\otimes V^*)_0\to(\mathcal{O}(\mathcal{Y})\otimes V^*)_0$$

by

$$\alpha(\mathcal{Y})^*(\gamma,f\otimes u^*)(g\otimes v)=(f\otimes u^*)\big(\alpha(\mathcal{Y})(\gamma^{-1},g\otimes v)\big)\ .$$

This is again natural in $\mathcal{Y}$, and defines the contragredient action of $\mathcal{GL}(V)$ on V^*. Moreover, we recall that there is a natural Lie supergroup isomorphism $\mathcal{GL}(V)\to\mathcal{GL}(\Pi V)$ given on the level of $\mathcal{Y}$-points by $\gamma\mapsto\Pi\gamma\Pi$. In particular, the identity $\Pi:V\to\Pi V$ is $\mathcal{GL}(V)$-equivariant. Using $\mathcal{Y}$-points, one shows equally easily that the canonical isomorphism $\Pi V\otimes V^*\cong\mathrm{Hom}(V,\Pi V)$ is $\mathcal{GL}(V)$-equivariant.

4.9. Let V be a finite-dimensional super-vector space, and consider the (super-)symmetric algebra $S(\Pi V\oplus V^*)$. The identity $\Pi:V\to\Pi V$ may be considered as an element of $\Pi V\otimes V^*$ and thus embedded as an odd element of $S(\Pi V\oplus V^*)$. Since Π is odd, $\Pi\cdot\Pi=0$ in $S(\Pi V\oplus V^*)$, and multiplication by Π is a differential on this vector space. One defines $\Pi^p(\mathrm{Ber}(V))$, where $p=\dim V_0$, to be the homology of this differential. One can show that $\mathrm{Ber}(V)$ is a super-vector space of total dimension one, and parity $q=\dim V_1$. Thus, $\dim\mathrm{Ber}(V)=1|0$ if q is even, and $\mathrm{Ber}(V)=0|1$ if q is odd. Since Π is $\mathcal{GL}(V)$-equivariant, $\mathrm{Ber}(V)$ carries a linear

$\mathcal{GL}(V)$-action. We shall call $\mathrm{Ber}(V)$ the *Berezinian module* and the corresponding action the *Berezinian action*. In fact, this definition can be analogously performed whenever V is a graded free and finitely generated R-module, where R is any commutative ring. We will use this fact in one instance below, and to stress the base ring, we will then write $\mathrm{Ber}_R(V)$.

Let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, V)$ be a vector bundle. We define the *Berezinian bundle* as follows. Let $\mathcal{A}$ be the $\mathcal{GL}(V)$-structure of $\mathcal{E}$ and $\tau_i \in \mathcal{A}(U_i)$ bundle charts over some open cover (U_i) of B. Let $\varphi_{ij} : \mathcal{U}_{ij} \to \mathcal{GL}(V)$ be the corresponding cocyle such that $\tau_i = \varphi_{ij}.\tau_j$. By application of the Berezinian, we define a cocyle $\psi_{ij} = \mathrm{Ber}(\varphi_{ij}) : \mathcal{U}_{ij} \to \mathcal{GL}(\varepsilon)$, where $\varepsilon = 1|0$ or $\varepsilon = 0|1$ according to the parity of q. The corresponding vector bundle on $\mathcal{B}$ with fibre $\Pi^q(\mathbb{R})$ is the Berezinian $\mathrm{Ber}(\mathcal{E}) = \mathrm{Ber}_{\mathcal{B}}(\mathcal{X})$, cf. [Sch84, Propositions 5.3, 6.5]. (The definition has to be modified appropriately when $\mathcal{X}$ does not have pure dimension.)

If $\mathcal{X}$ is any supermanifold, then we write $\mathrm{Ber}(\mathcal{X}) = \mathrm{Ber}(T^*\mathcal{X})$. Here, $T^*\mathcal{X}$ is the dual bundle of $T\mathcal{X}$. Whenever $\phi : \mathcal{X} \to \mathcal{Y}$ is an isomorphism of supermanifolds, we denote $T^*\phi : T^*\mathcal{Y} \to T^*\mathcal{X}$ the super-transpose of $T\phi$. Hence, there is an induced isomorphism $\Pi(T^*(\phi^{-1})) \oplus T\phi : \Pi(T^*\mathcal{X}) \oplus T\mathcal{X} \to \Pi(T^*\mathcal{Y}) \oplus T\mathcal{Y}$ of vector bundles. This induces an isomorphism $\mathrm{Ber}(\mathcal{X}) \to \mathrm{Ber}(\mathcal{Y})$ of line bundles which we denote by $\mathrm{Ber}(\phi)$. For any local section $\omega \in \Gamma(U, \mathrm{Ber}(\mathcal{Y}))$, we define the *pullback* $\phi^*\omega \in \Gamma(\phi^{-1}(U), \mathrm{Ber}(\mathcal{X}))$ by $\mathrm{Ber}(\phi) \circ \phi^*\omega = \omega \circ \phi$. (Recall that we may consider a section of a vector bundle $\mathcal{E} \to \mathcal{X}$ as a morphism $\mathcal{X} \to \mathcal{E} \oplus \Pi(\mathcal{E})$ which is right inverse to the bundle projection.)

Proposition 4.10. *Let $\mathcal{X}$ be a principal $\mathcal{H}$-bundle, and let the super-vector space V carry a linear $\mathcal{H}$-action. Then $\mathcal{X} \times^{\mathcal{H}} \mathrm{Ber}(V) \cong \mathrm{Ber}(\mathcal{X} \times^{\mathcal{H}} V)$ as vector bundles.*

Proof. Recall the definition of the $\mathcal{GL}(V)$-structure $\mathcal{A}_V$ on the vector bundle with fibre V associated to the principal bundle $\pi : \mathcal{X} \to \mathcal{B}$: $\mathcal{A}_V(U)$, for any open $U \subset B$, consists of all the $\tilde{\tau} \in \tau_{\mathcal{X} \times^{\mathcal{H}} V}(U)$ induced by the morphisms $\tau' : \mathcal{X}|_{\mathcal{U}} \times V \to \mathcal{U} \times V$ defined in (4), where τ runs through $\tau_{\mathcal{X}}(U)$.

Fix an open cover (U_i) of B such that $\tau_{\mathcal{X}}(U_i) \neq \varnothing$ for all i, and fix local trivialisations $\tau_i \in \tau_{\mathcal{X}}(U_i)$. For all i, j, there exist unique $\varphi_{ij} \in \mathcal{GL}(V)(\mathcal{U}_{ij})$ such that $\tau_i'|_{\mathcal{X}|_{\mathcal{U}_{ij}}} = \varphi_{ij}.\tau_j'|_{\mathcal{X}|_{\mathcal{U}_{ij}}}$, by the proof of Proposition 3.8. For the sake of simplicity, we will write this as $\tau_i' = \varphi_{ij}.\tau_j'$.

If $\tau_i'' : \mathcal{X}|_{\mathcal{U}_i} \times \mathrm{Ber}(V) \to \mathcal{U} \times \mathrm{Ber}(V)$ denotes the morphism defined via (4) with $\mathrm{Ber}(V)$ as the fibre, then $\tau_i'' = \varphi_{ij}.\tau_j'' = \mathrm{Ber}(\varphi_{ij}).\tau_j''$, by the same proof. Thus, if $\tilde{\tilde{\tau}}_i \in \mathcal{A}_{\mathrm{Ber}(V)}$ is induced by τ_i'', then $\tilde{\tilde{\tau}}_i = \mathrm{Ber}(\varphi_{ij}).\tilde{\tilde{\tau}}_j$. This is manifestly the cocycle defining $\mathrm{Ber}(\mathcal{X} \times^{\mathcal{H}} V)$. ∎

4.11. Let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, p, V)$ be a $\mathcal{G}$-equivariant vector bundle. Denote by $\mathcal{A}$ the $\mathcal{GL}(V)$-structure, and by $\tilde{\alpha}$ and α the actions on $\mathcal{X}$ and $\mathcal{B}$, respectively. We use these data to turn $\mathrm{Ber}_{\mathcal{B}}(\mathcal{X})$ into a $\mathcal{G}$-equivariant vector bundle. This is a somewhat technical construction, but we will need it.

There exist an open cover $\mathcal{U}_i$ of $\mathcal{B}$, and for each i, open subsupermanifolds $\mathcal{V}_k^i$ of $\mathcal{G}$ and $\mathcal{W}_k^i$ of $\mathcal{B}$ together with bundle charts $\tau_i \in \mathcal{A}(U_i)$ and $\sigma_k^i \in \mathcal{A}(W_k^i)$, and elements $\gamma_k^i \in \mathcal{GL}(V)(\mathcal{V}_k^i \times \mathcal{W}_k^i)$ such that $\mathcal{V}_k^i \times \mathcal{W}_k^i$ form an open cover of

$\alpha^{-1}(\mathcal{U}_i)$, and the following diagrams commute:

$$\begin{tikzcd}
\mathcal{V}^i_k \times \mathcal{X}|_{\mathcal{W}^i_k} \arrow[r, "\mathrm{id}\times\sigma^i_k"] \arrow[dd, "\tilde{\alpha}"'] & \mathcal{V}^i_k \times \mathcal{W}^i_k \times V \arrow[dr, "(\mathrm{id},\gamma^i_k)\times\mathrm{id}"] & \\
& & \mathcal{V}^i_k \times \mathcal{W}^i_k \times \mathcal{GL}(V) \times V \arrow[dl, "\alpha\times\beta"] \\
\mathcal{X}|_{\mathcal{U}_i} \arrow[r, "\tau_i"'] & \mathcal{U}_i \times V &
\end{tikzcd}$$

Here, we write β for the canonical action of $\mathcal{GL}(V)$ on V.

Let $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$, $\mathcal{V}^{ij}_{k\ell} = \mathcal{V}^i_k \cap \mathcal{V}^j_\ell$ and $\mathcal{W}^{ij}_{k\ell} = \mathcal{W}^i_k \cap \mathcal{W}^j_\ell$. There exist elements $\varphi_{ij} \in \mathcal{GL}(\mathcal{U}_{ij})$ and $\psi^{ij}_{k\ell} \in \mathcal{GL}(\mathcal{W}^{ij}_{k\ell})$ such that $\varphi_{ij}.\tau_j = \tau_i$ on $\mathcal{X}|_{\mathcal{U}_{ij}}$ and $\psi^{ij}_{k\ell}.\sigma^j_\ell = \sigma^i_k$ on $\mathcal{X}|_{\mathcal{W}^{ij}_{k\ell}}$. Since the action β is faithful, this implies that $(\varphi_{ij} \circ \alpha) \cdot \gamma^j_\ell = \gamma^i_k \cdot (\psi^{ij}_{k\ell} \circ \mathrm{pr}_2)$ on $\mathcal{V}^{ij}_{k\ell} \times \mathcal{W}^{ij}_{k\ell}$ (where the product $\cdot$ of $\mathcal{GL}(V)$-valued morphisms is defined in the obvious way).

Let $\tilde{\mathcal{A}}$ denote the $\mathcal{GL}(\mathrm{Ber}(V))$-structure of $\mathrm{Ber}_{\mathcal{B}}(\mathcal{X})$. By its construction, there exist bundle charts $\tilde{\tau}_i \in \tilde{\mathcal{A}}(U_i)$ and $\tilde{\sigma}^i_k \in \tilde{\mathcal{A}}(W^i_k)$ such that we have $\mathrm{Ber}(\varphi_{ij}).\tilde{\tau}_j = \tilde{\tau}_j$ and $\mathrm{Ber}(\psi^{ij}_{k\ell}).\tilde{\sigma}^j_\ell = \tilde{\sigma}^i_k$. Since

$$(\mathrm{Ber}(\varphi_{ij}) \circ \alpha) \cdot \mathrm{Ber}(\gamma^j_\ell) = \mathrm{Ber}(\gamma^i_k) \cdot (\mathrm{Ber}(\psi^{ij}_{k\ell}) \circ \mathrm{pr}_2) \quad \text{on} \quad \mathcal{V}^{ij}_{k\ell} \times \mathcal{W}^{ij}_{k\ell}\ ,$$

there exists a unique morphism $\mathrm{Ber}(\tilde{\alpha}) : \mathcal{G} \times \mathrm{Ber}_{\mathcal{B}}(\mathcal{X}) \to \mathrm{Ber}_{\mathcal{B}}(\mathcal{X})$ such that

$$\tilde{\tau}_i \circ \mathrm{Ber}(\tilde{\alpha}) = (\alpha \times \tilde{\beta}) \circ \big((\mathrm{id}, \mathrm{Ber}(\gamma^i_k)) \times \mathrm{id}\big) \circ (\mathrm{id} \times \tilde{\sigma}^i_k) \quad \text{on} \quad \mathcal{V}^i_k \times \mathrm{Ber}_{\mathcal{B}}(\mathcal{X})|_{\mathcal{W}^i_k}$$

where $\tilde{\beta}$ denotes the canonical action of $\mathcal{GL}(\mathrm{Ber}(V))$ on $\mathrm{Ber}(V)$. It is easy if somewhat tedious to check that $\mathrm{Ber}(\tilde{\alpha})$ is an action, and hence, $\mathrm{Ber}_{\mathcal{B}}(\mathcal{X})$ really is a $\mathcal{G}$-equivariant vector bundle.

Corollary 4.12. *There exists an isomorphism* $\mathrm{Ber}(\mathcal{G}/\mathcal{H}) \cong \mathcal{G} \times^{\mathcal{H}} \mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*)$ *of* $\mathcal{G}$*-equivariant vector bundles.*

Proof. The point is to check the $\mathcal{G}$-equivariance of the vector bundle isomorphism. By the above considerations, to that end, we need to see that if $\gamma^i_k \in \mathcal{GL}((\mathfrak{g}/\mathfrak{h})^*)(\mathcal{V}^i_k \times \mathcal{W}^i_k)$ define the $\mathcal{G}$-action on $\mathcal{G} \times^{\mathcal{H}} (\mathfrak{g}/\mathfrak{h})^*$, then $\mathrm{Ber}(\gamma^i_k)$ define the $\mathcal{G}$-action on $\mathcal{G} \times^{\mathcal{H}} \mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*)$. By the construction of the vector bundle structure on the associated bundles from the proof of Proposition 3.8, the local expressions of the actions factor through a Lie supergroup morphism $\mathcal{H} \to \mathcal{GL}(V)$ (the same one in both cases), so that the claim follows immediately. ∎

We come to our first main result.

Theorem 4.13. *Let* $\mathcal{G}$ *be a Lie supergroup and* $\mathcal{H}$ *a closed subsupergroup with Lie superalgebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, respectively. The following are equivalent:*

1. *The Berezinian bundle* $\mathrm{Ber}(\mathcal{G}/\mathcal{H})$ *is* $\mathcal{G}$*-equivariantly trivial.*

2. $\mathrm{Ber}(\mathcal{G}/\mathcal{H})$ *has a non-zero* $\mathcal{G}$*-invariant global section.*

3. For the action induced by $\mathrm{Ad}^*_{\mathcal{G}}$, $(\mathfrak{g}/\mathfrak{h})^*$ *is a trivial* $\mathcal{H}$*-module.*

Here, a global section is a morphism $s:\mathcal{G}/\mathcal{H}\to\Pi^q\mathrm{Ber}(\mathcal{G}/\mathcal{H})$ *which is right inverse to the bundle projection. It is called invariant if it is equivariant as a morphism,* i.e.

$$\mathrm{Ber}(\alpha_{\mathcal{G}/\mathcal{H}})\circ(\mathrm{id}\times s)=s\circ\alpha\ .$$

Whenever the equivalent conditions are satisfied, then the non-zero $\mathcal{G}$*-invariant section of* $\mathrm{Ber}(\mathcal{G}/\mathcal{H})$ *is unique up to constant multiples.*

Proof. From what we have proved, 1 and 3 are clearly equivalent. The equivalence of 1 and 2 follows by standard procedures. ■

Definition 4.14. If the equivalent conditions of Theorem 4.13 are fulfilled, then the homogeneous supermanifold $\mathcal{G}/\mathcal{H}$ is called *unimodular*. (The notion of course depends on the choice of the Lie supergroup $\mathcal{G}$.)

Remark 4.15. Any Lie supergroup $\mathcal{G}$ is unimodular as left $\mathcal{G}$-space. Considered as the quotient $(\mathcal{G}\times\mathcal{G})/\mathcal{G}$ (for the diagonal action of $\mathcal{G}$ on $\mathcal{G}\times\mathcal{G}$) of the Lie supergroup $\mathcal{G}\times\mathcal{G}$, it need not be unimodular. In fact, if $\mathcal{G}=G$ is a Lie group, then $(G\times G)/G$ is a unimodular $G\times G$-space if and only if G is unimodular as a topological group. This justifies our terminology.

If $(\mathcal{G},\mathcal{H})$ is a symmetric pair, *i.e.* $\mathcal{H}$ is a closed subsupergroup fixed by an automorphism θ of $\mathcal{G}$ of order two, and H is open in G^θ, and $\mathfrak{g}$ carries a non-degenerate $\mathcal{G}$-invariant bilinear form for which $\mathfrak{h}$ is a non-degenerate subspace, then $\mathcal{G}/\mathcal{H}$ is unimodular as a $\mathcal{G}$-space.

5. Fibre integration of Berezinians

5.1. Integration along the fibre.

5.1. If $\mathcal{F}$ is a sheaf of vector spaces on a topological space X, then for $f\in\mathcal{F}(U)$, the *support* $\operatorname{supp} f\subset X$ is the set of points $x\in X$ where the germ $f_x\neq 0$. This set is closed. We let $\Gamma_c(U,\mathcal{F})$ be the set of $f\in\mathcal{F}(U)$ where $\operatorname{supp} f$ is compact. This defines a presheaf $\Gamma_c(\mathcal{F})$ on X. We call the elements *compactly supported local sections.*

If $\pi:\mathcal{X}\to\mathcal{B}$ is a fibre bundle in the category of supermanifolds, then for open subsets $U\subset B$ (!), $\Gamma_{cf}(U,\mathcal{O}_{\mathcal{X}})$ denotes the set of $h\in\mathcal{O}_{\mathcal{X}}(\pi^{-1}(U))$ such that $\pi|\operatorname{supp} h:\operatorname{supp} h\to X$ is proper. This defines a presheaf $\Gamma_{cf}(\mathcal{O}_{\mathcal{X}})$ on B, and its elements are called *compactly supported in the fibre*. One does *not* obtain a presheaf on X. Similarly, one may define for any vector bundle on $\mathcal{X}$ the local sections with compact support or compact support in the fibre (w.r.t. π).

Definition 5.2. A supermanifold $\mathcal{X}$ is called *oriented* if the underlying manifold X is. An isomorphism of oriented supermanifolds is called *orientation preserving* if the underlying isomorphism of oriented manifolds is orientation preserving.

If $\mathcal{X}$ is an oriented supermanifold, then there exists a linear morphism of presheaves on X [Man88, § 6.2]

$$\int_{\mathcal{X}} : \Gamma_c(\mathrm{Ber}(\mathcal{X})) \to \mathbb{R}_X \ ,$$

which in the case of a manifold $\mathcal{X} = X$ is the integration of volume forms. Moreover, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is an orientation preserving isomorphism of oriented supermanifolds, then [Lei80, Theorem 2.4.5]

$$\int_{\mathcal{X}} \varphi^* \omega = \int_{\mathcal{Y}} \omega \quad \text{for all} \ \ \omega \in \Gamma_c(U, \mathrm{Ber}(\mathcal{Y})) \tag{6}$$

where we recall the definition of $\varphi^*\omega$ from 4.9.

Let $\mathcal{E} = (\mathcal{X}, \mathcal{B}, \pi, \mathcal{F})$ be a fibre bundle. It is called *oriented* if $\mathcal{X}$, $\mathcal{B}$ and $\mathcal{F}$ are oriented and it is supplied with an *oriented bundle atlas*. The latter is the data of an open covering $(\mathcal{U}_i)$ of $\mathcal{B}$, and of local trivialisations $\tau_i \in \tau_{\mathcal{X}}(U_i)$ which are *orientation preserving* isomophisms $\tau_i : \mathcal{X}|_{\mathcal{U}_i} \to \mathcal{U}_i \times \mathcal{F}$ where on the right hand side, we take the product orientation.

5.3. We wish to define fibre integration of Berezinians. To that end, we have to introduce topologies on the presheaves $\mathcal{O}_{\mathcal{X}}$ and $\Gamma_c(\mathcal{O}_{\mathcal{X}})$ for any supermanifold $\mathcal{X}$, and on $\Gamma_{cf}(\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})$ for any direct product of supermanifolds.

Let $U \subset X$ be open. For any compact $K \subset U$, and any differential operator $D \in \mathcal{D}_{\mathcal{X}}(U)$, we define a seminorm $p_{K,D}$ on $\mathcal{O}_{\mathcal{X}}(U)$ as follows:

$$p_{K,D}(h) = \sup\nolimits_{x\in K} |(Dh)(x)| \ .$$

Here, the value $h(x)$ at x of $h \in \mathcal{O}_{\mathcal{X}}(U)$ is defined as the value at x of the image of h in $\mathcal{C}^\infty_X(U)$.

Lemma 5.4. *Let $\mathcal{X}$ be a supermanifold and $U \subset X$ an open subset. Then $\mathcal{O}_{\mathcal{X}}(U)$ is a nuclear and m-convex Fréchet algebra. (Where we recall that a topological algebra is called m-*convex *if its topology is the locally convex topology defined by a family of submultiplicative seminorms.)*

Proof. If $\mathcal{U}$, the open subsupermanifold of $\mathcal{X}$ with base U, is a superdomain, then $\mathcal{O}_{\mathcal{X}}(U) \cong \mathcal{C}^\infty(U) \otimes \bigwedge(\mathbb{R}^q)$ as topological algebras (where $\dim \mathcal{U} = p|q$). The right hand side is certainly an m-convex Fréchet algebra, and both tensor factors are nuclear. Thus, in this case, the statement follows from the fact that the projective tensor product of nuclear spaces is nuclear [Sch71, Chapter II, §7.5].

In general, $\mathcal{U}$ has an open cover $(\mathcal{U}_i)$ by superdomains. Then $\mathcal{O}_{\mathcal{X}}(U)$ carries the initial locally convex topology with respect to the restriction maps $\varrho^U_{U_i} : \mathcal{O}_{\mathcal{X}}(U) \to \mathcal{O}_{\mathcal{X}}(U_i)$. Thus, $\mathcal{O}_{\mathcal{X}}(U)$ is a projective limit of complete nuclear locally convex spaces and as such, complete and nuclear [Sch71, Chapter II, § 5.3; Chapter III, § 7.4, Corollary]. It follows also that it is m-convex. Since X is second countable, the topology of $\mathcal{O}_{\mathcal{X}}(U)$ is generated by a countable family of seminorms. Therefore, $\mathcal{O}_{\mathcal{X}}(U)$ is metrisable, and hence, a Fréchet space. ■

Lemma 5.5. *Let $\mathcal{X}\times\mathcal{Y}$ be supermanifolds, $U\subset X$, $V\subset V$ be open subsets. If τ is any tensor product topology, then $\mathcal{O}_{\mathcal{X}}(U)\mathbin{\hat{\otimes}_\tau}\mathcal{O}_{\mathcal{Y}}(V)\cong\mathcal{O}_{\mathcal{X}\times\mathcal{Y}}(U\times V)$ via the continuous linear extension of the map $\phi:g\otimes h\mapsto\mathrm{pr}_1^*g\cdot\mathrm{pr}_2^*h$.*

Proof. The map $\phi:\mathcal{O}_{\mathcal{X}}(U)\otimes\mathcal{O}_{\mathcal{Y}}(V)\to\mathcal{O}_{\mathcal{X}\times\mathcal{Y}}(U\times V)$ is injective and has dense image. This is easily checked for superdomains, and the general statement follows by a projective limit argument, or by considering partitions of unity.

Both tensor factors are nuclear Fréchet spaces, so the point is to show that the restriction ψ of ϕ to $\mathcal{O}_{\mathcal{X}}(U)\times\mathcal{O}_{\mathcal{Y}}(V)$ is a continuous bilinear map. We need to show that $p_{K,D}\circ\psi$ is continuous for any compact $K\subset U\times V$ and any $D\in\mathcal{D}_{\mathcal{X}\times\mathcal{Y}}(U\times V)$. Since K is compact, it has a finite cover by superdomains, and we may restrict our attention to the case where $\mathcal{U}$ and $\mathcal{V}$ are superdomains. By an easy estimate, the case of a general differential operator D is reduced to the case of one which is a polynomial in the vector fields $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial\xi_j}$, $\frac{\partial}{\partial y_k}$, $\frac{\partial}{\partial\eta_\ell}$ where (x_i,ξ^j), (y_k,η^ℓ) are coordinates for $\mathcal{U}$ and $\mathcal{V}$, respectively. But this case is entirely trivial. ∎

5.6. In what follows, we denote by $\hat{\otimes}$ the completed (and graded) *projective* tensor product of locally convex vector spaces.

Let $\Gamma_c(U,\mathcal{O}_{\mathcal{X}})$ be topologised as the inductive limit with respect to the inclusions $\Gamma_K(U,\mathcal{O}_{\mathcal{X}})\subset\Gamma_c(U,\mathcal{O}_{\mathcal{X}})$, where for any compact $K\subset U$, $\Gamma_K(U,\mathcal{O}_{\mathcal{X}})$ is the subspace of $\mathcal{O}_{\mathcal{X}}(U)$ consisting of all h with $\operatorname{supp}h\subset K$. Since U is second countable, the topology is defined by a *countable* inductive limit. Thus, $\Gamma_c(U,\mathcal{O}_{\mathcal{X}})$ is an (LF)-space, and it is nuclear [Sch71, Chapter III, § 7.4, Corollary].

Next, we fix a second supermanifold $\mathcal{Y}$. Consider the first projection $p:\mathcal{X}\times\mathcal{Y}\to\mathcal{X}$ and $\Gamma_{cf}(U,\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})$. This space is topologised as the inductive limit with respect to the inclusions $\Gamma_{cf,K}(U,\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})\subset\Gamma_{cf}(U,\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})$ where for any closed $K\subset X\times Y$ with $p|_K:K\to X$ proper, the former space is defined to be the subspace of $\mathcal{O}_{\mathcal{X}\times\mathcal{Y}}$ which consists of those h such that $\operatorname{supp}h\subset K$. This makes $\Gamma_{cf}(U,\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})$ a nuclear (LF)-space. Moreover, we have as above that $\Gamma_{cf}(U,\mathcal{O}_{\mathcal{X}\times\mathcal{Y}})=\mathcal{O}_{\mathcal{X}}(U)\mathbin{\hat{\otimes}}\Gamma_c(Y,\mathcal{O}_{\mathcal{Y}})$.

If we are given a vector bundle $\mathcal{E}$ on $\mathcal{X}$, then the sheaf of sections $\Gamma(\mathcal{E})$ is locally free over $\mathcal{O}_{\mathcal{X}}$. On trivialising open subsupermanifolds of $\mathcal{X}$, this determines a topology on $\Gamma(\mathcal{E})$ (the product topology). If $U\subset X$ is any open subset, then we let $\Gamma(U,\mathcal{E})$ be equipped with the initial locally convex topology with respect to all restrictions to trivialising open subsets (by the open mapping theorem, this is well-defined).

Finally, one defines topologies on $\Gamma_c(\mathcal{E})$ and $\Gamma_{cf}(\mathcal{E})$ (in the case of a vector bundle over a direct product of supermanifolds) in the same way as above, namely, by taking inductive limits over subsets K which are compact or closed and such that $p|_K:K\to X$ is proper, respectively. The statements for projective tensor products carry over.

Proposition 5.7. *Let $\mathcal{E}=(\mathcal{X},\mathcal{B},\pi,\mathcal{F})$ be an oriented fibre bundle in supermanifolds where $\dim\mathcal{B}=m|n$ and $\dim\mathcal{F}=p|q$. There is an even morphism*

$\pi_! : \Gamma_{cf}(\mathrm{Ber}(\mathcal{X})) \to \Gamma(\mathrm{Ber}(\mathcal{B}))$ *of graded presheaves over* B, *such that*

$$\pi_!(\pi^* h \cdot \omega) = h \cdot \pi_!(\omega) \quad \textit{and} \quad \operatorname{supp} \pi_!(\omega) \subset \pi(\operatorname{supp} \omega) \tag{7}$$

for all open $U \subset B$, $h \in \mathcal{O}_{\mathcal{B}}(U)$, $\omega \in \Gamma_{cf}(U, \mathrm{Ber}(\mathcal{X}))$, *and*

$$\int_{\mathcal{X}} \omega = (-1)^{(m+n)q} \cdot \int_{\mathcal{B}} \pi_!(\omega) \tag{8}$$

for all $\omega \in \Gamma_c(\pi^{-1}(U), \mathrm{Ber}(\mathcal{X}))$.

Moreover, if $\mathcal{E}'$ *is another fibre bundle over* $\mathcal{B}$ *and* $\varphi : \mathcal{X}' \to \mathcal{X}$ *is an oriented isomorphism of the total spaces such that* $\pi' = \pi \circ \varphi$, *then*

$$\pi'_!(\varphi^* \omega) = \pi_!(\omega) \quad \textit{for all} \;\; \omega \in \Gamma_{cf}(U, \mathrm{Ber}(\mathcal{X})) \; . \tag{9}$$

In the *proof*, we first establish the local picture.

Lemma 5.8. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be supermanifolds where* $\mathcal{Y}$ *is oriented, and set* $p = \mathrm{pr}_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$. *Let* $\mathcal{E}$ *be a vector bundle on* $\mathcal{X}$. *For any open* $U \subset X$, *we define the map* $p_! : \Gamma(U, \mathcal{E}) \otimes \Gamma_c(Y, \mathrm{Ber}(\mathcal{Y})) \to \Gamma(U, \mathcal{E})$ *by*

$$p_!(\omega_1 \otimes \omega_2) = \omega_1 \cdot \int_{\mathcal{Y}} \omega_2 \; .$$

Then $p_!$ *has an extension* $\Gamma_{cf}(U, \mathcal{E} \boxtimes \mathrm{Ber}(\mathcal{Y})) \to \Gamma(U, \mathcal{E})$ *uniquely determined by the requirement that it be continuous and linear (here,* $\boxtimes$ *denotes the graded external tensor product). Thus, we obtain an even morphism of graded presheaves on* X, $p_! : \Gamma_{cf}(\mathcal{E} \boxtimes \mathrm{Ber}(\mathcal{Y})) \to \Gamma(\mathcal{E})$, *which satisfies*

$$p_!(p^* h \cdot \omega) = h \cdot p_!(\omega) \quad \textit{and} \quad \operatorname{supp} p_!(\omega) \subset p(\operatorname{supp} \omega) \tag{10}$$

for all $h \in \mathcal{O}_{\mathcal{X}}(U)$, $\omega \in \Gamma_{cf}(U, \mathcal{E} \boxtimes \mathrm{Ber}(\mathcal{Y}))$.

Proof. From our remarks in 5.6 on projective tensor products, it follows that $\Gamma_{cf}(U, \mathcal{E} \boxtimes \mathrm{Ber}(\mathcal{Y})) = \Gamma(U, \mathcal{E}) \hat{\otimes} \Gamma_c(Y, \mathrm{Ber}(\mathcal{Y}))$. Thus, to prove unique existence of a continuous linear extension, it suffices to see that for any open subset $U \subset X$, $p_!$ restricts to a continuous linear bilinear map on $\Gamma(U, \mathcal{E}) \times \Gamma_c(Y, \mathrm{Ber}(\mathcal{Y}))$. Since $\int_{\mathcal{Y}}$ is continuous on any $\Gamma_K(Y, \mathrm{Ber}(\mathcal{Y}))$ for $K \subset Y$ compact, the assertion follows immediately.

To check (10), we note that the second statement follows from the first; and to prove the first, it suffices to consider $p_!$ on the algebraic tensor product. Here, we have

$$p_!\big(p^* h \cdot (\omega_1 \otimes \omega_2)\big) \; = \; p_!\big((h \cdot \omega_1) \otimes \omega_2\big) \; = \; h \cdot \omega_1 \cdot \int_{\mathcal{Y}} \omega_2 \; = \; h \cdot p_!(\omega_1 \otimes \omega_2) \; .$$

This proves the assertion. ∎

Proof of Proposition 5.7. First, note that $\mathrm{Ber}(\mathcal{Y} \times \mathcal{Z}) = \mathrm{Ber}(\mathcal{Y}) \boxtimes \mathrm{Ber}(\mathcal{Z})$. Indeed, for super-vector spaces V and W, it follows from the definition of $\mathrm{Ber}(V)$

that there is a canonical isomorphism $\mathrm{Ber}(V \oplus W) = \mathrm{Ber}(V) \otimes \mathrm{Ber}(W)$. Thus, Lemma 5.8 gives a morphism $p_! : \Gamma_{cf}(\mathrm{Ber}(\mathcal{Y} \times \mathcal{Z})) \to \Gamma(\mathrm{Ber}(\mathcal{Y}))$.

Let $(\mathcal{U}_i)$, $\tau_i \in \tau_{\mathcal{X}}(U_i)$ be the data of an oriented bundle atlas where we assume that U_i is relatively compact in B for any i, and set $\sigma_i = \tau_i^{-1}$. Let $\varphi_i \in \mathcal{C}^\infty(B)$ form a partition of unity on B subordinate to (U_i). We write p for the first projections $\mathcal{U}_i \times \mathcal{F} \to \mathcal{U}_i$.

For any open $U \subset B$, we define

$$\pi_!(\omega) = \textstyle\sum_i p_!(p^*\varphi_i \cdot \sigma_i^*(\omega)) \quad \text{for all } \ \omega \in \Gamma_{cf}(U, \mathrm{Ber}(\mathcal{X})) \ .$$

As usual, the issue is to see that this is well-defined independently of all choices. First, to prove independence of the partition of unity, one applies (10). Next, the independence of the definition on the choice of oriented bundle atlas follows by considering elementary tensors and applying (6). Hence, the definition is independent of all choices, and in particular, it follows that $\pi_!$ is linear and defines a morphism of presheaves. Then (7) follows from (10), and (9) is also an application of (6).

Finally, to prove (8), we may assume that $\mathcal{X} = \mathcal{B} \times \mathcal{F}$ and that π is the first projection, so we are in the situation of Lemma 5.8. By shrinking coordinate charts in the first factor, and by taking partitions of unity in the second, we may assume that the line bundles $\mathrm{Ber}(\mathcal{B})$ and $\mathrm{Ber}(\mathcal{F})$ are trivial, and that $\mathcal{B}$ and $\mathcal{F}$ are superdomains. By (7), it suffices to prove

$$\int_{\mathcal{B}\times\mathcal{F}} (\omega_1 \otimes \omega_2) = (-1)^{(m+n)q} \cdot \int_{\mathcal{B}} \omega_1 \cdot \int_{\mathcal{F}} \omega_2 \tag{$*$}$$

for all $\omega_1 \in \Gamma_c(B, \mathrm{Ber}(\mathcal{B}))$ and $\omega_2 \in \Gamma_c(F, \mathrm{Ber}(\mathcal{F}))$.

Fix coordinates $(x_1, \dots, x_m, \xi^1, \dots, \xi^n)$ of $\mathcal{B}$ and $(y_1, \dots, y_p, \eta^1, \dots, \eta^q)$ of $\mathcal{F}$. We have global sections $D(x, \xi)$ and $D(y, \eta)$ of $\mathrm{Ber}(\mathcal{B})$ and $\mathrm{Ber}(\mathcal{F})$, respectively [Man88, Chapter 3, § 4.7]. Here, we recall that

$$\Pi^m(D(x,\xi)) \equiv \Pi(dx_1)\cdots\Pi(dx_m) \cdot \tfrac{\partial}{\partial \xi^1} \cdots \tfrac{\partial}{\partial \xi^n}$$

(modulo the image of Π). Moreover, $D(x, y, \xi, \eta) = (-1)^{np} D(x, \xi) \otimes D(y, \eta)$ is a global section of $\mathrm{Ber}(\mathcal{B} \times \mathcal{F})$.

Let $g \in \mathcal{O}_{\mathcal{B}}(B)$, $h \in \mathcal{O}_{\mathcal{F}}(F)$. Write $g = \sum_\alpha \xi^\alpha g_\alpha$ and $h = \sum_\beta \eta^\beta h_\beta$ where g_α, h_β are even, and $dx = dx_1 \cdots dx_m$ and $dy = dy_1 \dots dy_q$. The Berezin integral on $\mathcal{B}$ is given by $\int_{\mathcal{B}} D(x, \xi) \cdot g = (-1)^{mn} \int_B g_{1,\dots,1}\, dx$. Thus

$$\begin{aligned}
\int_{\mathcal{B}\times\mathcal{F}} D(x,y,\xi,\eta)\,(g \otimes h) &= (-1)^{(m+p)(n+q)} \cdot \int_{B\times F} g_{1,\dots,1} h_{1,\dots,1}\, dx dy \\
&= (-1)^{(m+p)(n+q)} \cdot \int_B g_{1,\dots,1}\, dx \cdot \int_F h_{1,\dots,1}\, dy \\
&= (-1)^{mq+np} \cdot \int_{\mathcal{B}} D(x,\xi) \cdot g \cdot \int_{\mathcal{F}} D(y,\eta) \cdot h
\end{aligned}$$

Finally, note

$$(D(x,\xi)\cdot g) \otimes (D(y,\eta)\cdot h) = (-1)^{n|g|} (D(x,\xi) \otimes D(y,\eta)) \cdot (g \otimes h)$$

and $|\xi^{1,\dots,1}g_{1,\dots,1}| = q$. This proves $(*)$, and therefore, the assertion. ■

5.2. 'Fubini' formula for quotients $\mathcal{G}/\mathcal{H}$.

5.9. Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ be a closed Lie subsupergroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is a unimodular $\mathcal{G}$-space. By Theorem 4.13, there exist non-zero $\mathcal{G}$-invariant section $\omega_{\mathcal{G}}$ of $\mathrm{Ber}(\mathcal{G})$ and $\omega_{\mathcal{G}/\mathcal{H}}$ of $\mathrm{Ber}(\mathcal{G}/\mathcal{H})$, and a non-zero $\mathcal{H}$-invariant section $\omega_{\mathcal{H}}$ of $\mathrm{Ber}(\mathcal{H})$. For $f \in \Gamma_c(G, \mathcal{O}_{\mathcal{G}})$, we define

$$\int_{\mathcal{G}} f = \int_{\mathcal{G}} f \cdot \omega_{\mathcal{G}} \ ,$$

and similarly for $\mathcal{H}$ and $\mathcal{G}/\mathcal{H}$.

Proposition 5.10. *Retain the above assumptions. For a suitable normalisation of $\omega_{\mathcal{G}}$, $\omega_{\mathcal{H}}$ and $\omega_{\mathcal{G}/\mathcal{H}}$, we have, for each bundle chart $\tau : \mathcal{G}|_{\mathcal{U}} \to \mathcal{U} \times \mathcal{H}$ of the principle $\mathcal{H}$-bundle $\mathcal{G} \to \mathcal{G}/\mathcal{H}$, $\omega_{\mathcal{G}} = \tau^*(\omega_{\mathcal{G}/\mathcal{H}} \otimes \omega_{\mathcal{H}})$.*

Proof. Let $\mathcal{A}$ denote the $\mathcal{H}$-structure of $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$. For any open subset $U \subset G/H$ such that $\mathcal{A}(U) \neq \varnothing$ and any $\tau \in \mathcal{A}(U)$, we may define a local section $\omega_\tau \in \Gamma(\pi^{-1}(U), \mathrm{Ber}(\mathcal{G}))$ by $\omega_\tau = \tau^*(\omega_{\mathcal{G}/\mathcal{H}} \otimes \omega_{\mathcal{H}})$. Also, certainly, ω_τ is non-zero on $\mathcal{G}|_{\mathcal{U}}$.

Let $V \subset G/H$ be an open subset such that $\mathcal{A}(V) \neq \varnothing$ and $U \cap V \neq \varnothing$. Let $\sigma \in \mathcal{A}(V)$. There exists a unique $\varphi \in \mathcal{H}(\mathcal{U} \cap \mathcal{V})$ such that $\tau = \varphi.\sigma$. Then

$$(\omega_{\mathcal{G}/\mathcal{H}} \circ \mathrm{pr}_1) \otimes \big(\mathrm{Ber}(m_{\mathcal{H}}) \circ (\mathrm{id} \times \omega_{\mathcal{H}}) \circ (\mathrm{pr}_2, \mathrm{pr}_3)\big) = (\omega_{\mathcal{G}/\mathcal{H}} \otimes \omega_{\mathcal{H}}) \circ (\mathrm{id} \times m_{\mathcal{H}})$$

by the invariance of $\omega_{\mathcal{H}}$. It follows that on $\mathcal{U} \cap \mathcal{V}$,

$$(\omega_{\mathcal{G}/\mathcal{H}} \otimes \omega_{\mathcal{H}}) \circ (\mathrm{pr}_1, m_{\mathcal{H}} \circ (\varphi \times \mathrm{id})) = \big((\omega_{\mathcal{G}/\mathcal{H}} \circ \mathrm{pr}_1) \otimes \mathrm{Ber}(m_{\mathcal{H}})\big) \circ (\mathrm{pr}_1, \varphi \circ \mathrm{pr}_1, \omega_{\mathcal{H}} \circ \mathrm{pr}_2) \ ,$$

so

$$\omega_\tau = \sigma^*(\mathrm{pr}_1, m_{\mathcal{H}} \circ (\varphi \times \mathrm{id}))^*(\omega_{\mathcal{G}/\mathcal{H}} \otimes \omega_{\mathcal{H}}) = \omega_\sigma \ .$$

Hence, there exists a unique $\omega \in \Gamma(G, \mathrm{Ber}(\mathcal{G}))$ such that $\omega|_{\pi^{-1}(U)} = \omega_\tau$ (where $\tau \in \mathcal{A}(U)$ is arbitrary), for all open sets $U \subset G/H$.

To prove the assertion, we have to see that ω is $\mathcal{G}$-invariant. Choose open subsets $U \subset G$, $V, W \subset G/H$ such that $m_{\mathcal{G}}(U \times \pi^{-1}(V)) \subset \pi^{-1}(W)$ and $\mathcal{A}(V), \mathcal{A}(W) \neq \varnothing$. Let $\tau \in \mathcal{A}(V)$ and $\sigma \in \mathcal{A}(W)$. By the proof of Proposition 3.6, there exist sections $t : \mathcal{V} \to \mathcal{G}|_{\mathcal{V}}$ and $s : \mathcal{W} \to \mathcal{G}|_{\mathcal{W}}$ of π such that $\tau^{-1} = m_{\mathcal{G}} \circ (t \times \mathrm{id})$ and $\sigma^{-1} = m_{\mathcal{G}} \circ (s \times \mathrm{id})$.

Define $\varphi : \mathcal{U} \times \mathcal{V} \to \mathcal{G}$ by $\varphi = m_{\mathcal{G}} \circ (i_{\mathcal{G}} \circ s \circ \alpha, m_{\mathcal{G}} \circ (\mathrm{id} \times t))$. Here, α denotes the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$, and $i_{\mathcal{G}}$ denotes the inversion of $\mathcal{G}$.

We have $\pi \circ m_{\mathcal{G}} = \alpha \circ (\mathrm{id} \times \pi)$, so

$$\pi \circ s \circ \alpha = \alpha = \pi \circ m_{\mathcal{G}} \circ (\mathrm{id} \times t) \ ,$$

so that $s \circ \alpha$ and $m_{\mathcal{G}} \circ (\mathrm{id} \times t)$ induce a morphism $\tilde{\varphi} : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times_{\mathcal{G}/\mathcal{H}} \mathcal{G}$. By the effectiveness of the quotient $\mathcal{G}/\mathcal{H}$ (Theorem 2.6), $\iota = (m_{\mathcal{G}}, \mathrm{pr}_1) : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{G}$

induces by corestriction an isomorphism $\mathcal{G}\times\mathcal{H}\to\mathcal{G}\times_{\mathcal{G}/\mathcal{H}}\mathcal{G}$. Hence, there exists a morphism $\psi:\mathcal{U}\times\mathcal{V}\to\mathcal{H}$ such that $s\circ\alpha=m_{\mathcal{G}}\circ(m_{\mathcal{G}}\circ(\mathrm{id}\times t),\psi)$ (take $\psi=\mathrm{pr}_2\circ\iota^{-1}\circ\tilde{\varphi}$). An easy computation shows that $\varphi=i_{\mathcal{G}}\circ\psi$, so that the corestriction $\varphi:\mathcal{U}\times\mathcal{V}\to\mathcal{H}$ exists. Moreover, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}\times\mathcal{G}|_{\mathcal{V}} \arrow[rr, "m_{\mathcal{G}}"] & & \mathcal{G}|_{\mathcal{W}} \\
\mathcal{U}\times\mathcal{V}\times\mathcal{H} \arrow[u, "\mathrm{id}\times\tau^{-1}"] \arrow[dr, "(\mathrm{id},\varphi)\times\mathrm{id}"'] & & \mathcal{W}\times\mathcal{H} \arrow[u, "\sigma^{-1}"'] \\
& \mathcal{U}\times\mathcal{V}\times\mathcal{H}\times\mathcal{H} \arrow[ur, "\alpha\times m_{\mathcal{H}}"'] &
\end{tikzcd}$$

Indeed, if $m_{\mathcal{G}}^{(2)}=m_{\mathcal{G}}\circ(m_{\mathcal{G}}\times\mathrm{id})=m_{\mathcal{G}}\circ(\mathrm{id}\times m_{\mathcal{G}})$, then

$$\begin{aligned}
&\sigma^{-1}\circ(\alpha\times m_{\mathcal{H}})\circ((\mathrm{id},\varphi)\times\mathrm{id})\\
&\qquad=m_{\mathcal{G}}\circ(s\times\mathrm{id})\circ(\alpha\times m_{\mathcal{H}})\circ((\mathrm{id},\varphi)\times\mathrm{id})\\
&\qquad=m_{\mathcal{G}}^{(2)}\circ(s\circ\alpha\circ(\mathrm{pr}_1,\mathrm{pr}_2),i_{\mathcal{G}}\circ s\circ\alpha\circ(\mathrm{pr}_1,\mathrm{pr}_2),m_{\mathcal{G}}^{(2)}\circ(\mathrm{id}\times t\times\mathrm{id}))\\
&\qquad=m_{\mathcal{G}}^{(2)}\circ(\mathrm{id}\times t\times\mathrm{id})=m_{\mathcal{G}}\circ(\mathrm{id}\times\tau^{-1})\ .
\end{aligned}$$

By the invariance of $\omega_{\mathcal{G}/\mathcal{H}}$ and $\omega_{\mathcal{H}}$,

$$(\omega_{\mathcal{G}/\mathcal{H}}\otimes\omega_{\mathcal{H}})\circ(\alpha\times m_{\mathcal{H}})=\mathrm{Ber}(\alpha\times m_{\mathcal{H}})\circ(\mathrm{id}\times(\omega_{\mathcal{G}/\mathcal{H}}\otimes\omega_{\mathcal{H}}))$$

on $\mathcal{U}\times\mathcal{V}\times\mathcal{H}$. Hence, a similar computation as for the well-definedness of ω shows the relation $\mathrm{Ber}(m_{\mathcal{G}})\circ(\mathrm{id}\times\omega)=\omega\circ m_{\mathcal{G}}$ on $\mathcal{U}\times\mathcal{G}|_{\mathcal{V}}$. This proves the assertion. ∎

5.11. For any $f\in\Gamma_c(G,\mathcal{O}_{\mathcal{G}})$, there exists a unique $h\in\Gamma_c(G/H,\mathcal{O}_{\mathcal{G}/\mathcal{H}})$ such that $\pi_!(f\cdot\omega_{\mathcal{G}})=h\cdot\omega_{\mathcal{G}/\mathcal{H}}$. We write $f_{\mathcal{H}}=h$. Then

$$\int_{\mathcal{G}}f=(-1)^{\dim\mathfrak{h}_1\cdot\dim\mathfrak{g}/\mathfrak{h}}\cdot\int_{\mathcal{G}/\mathcal{H}}f_{\mathcal{H}}\quad\text{for all}\quad f\in\Gamma_c(G,\mathcal{O}_{\mathcal{G}})\ . \tag{11}$$

Corollary 5.12. *Retain the assumptions of Proposition 5.10. For a suitable normalisation of the Berezinians, for all bundle charts $\tau:\mathcal{U}\times\mathcal{H}\to\mathcal{G}|_{\mathcal{U}}$, and all $f\in\Gamma_c(G,\mathcal{O}_{\mathcal{G}})$, the following identity holds:*

$$f_{\mathcal{H}}|_U=p_!(\tau^*f\cdot(1\otimes\omega_{\mathcal{H}})) \tag{12}$$

Here, $p:\mathcal{U}\times\mathcal{H}\to\mathcal{U}$ is the first projection, and we denote by $p_!$ the linear map $\Gamma_{cf}(U,\mathcal{O}_{\mathcal{G}/\mathcal{H}}\otimes\mathrm{Ber}(\mathcal{H}))\to\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U)$ defined in Lemma 5.8 for the case of the trivial bundle on $\mathcal{G}/\mathcal{H}$.

Proof. This follows from Proposition 5.7, the definitions in Lemma 5.8, Proposition 5.10, and standard considerations using tensor products. ∎

Remark 5.13. The formulae (11) and (12) constitute the graded version of the classical 'Fubini' formula

$$\int_G f(g)\,dg=\int_{G/H}\left(\int_H f(gh)\,dh\right)d\dot{g}\quad\text{for all}\quad f\in\mathcal{C}_c(G)$$

valid for any unimodular homogeneous G-space G/H.

5.3. Products of subsupergroups.

In this section, we will generalise the results [Hel84, Chapter I, Lemma 1.11, Proposition 1.12] to the setting of homogeneous supermanifolds.

5.14. Let V be a finite-dimensional super-vector space, $x_1, \ldots, x_n$ a homogeneous basis, and $\xi_1, \ldots, \xi_n$ its dual basis. We extend the definition of $D(x, \xi)$ for graded bases to this setting. Let $D(x) = D(x_1, \ldots, x_n) \in \mathrm{Ber}(V)$ be the element which is represented modulo the image of the differential Π on $S(\Pi V \oplus V^*)$ by

$$y_1 \ldots y_n \quad \text{where} \quad y_i = \begin{cases} \Pi(x_i) & |x_i| = 0 \\ \xi_i & |x_i| = 1 \end{cases}$$

We define a *canonical* perfect pairing $\langle\cdot,\cdot\rangle : \mathrm{Ber}(V^*) \otimes \mathrm{Ber}(V) \to \mathbb{R}$ by

$$\langle D(\xi_n, \ldots, \xi_1), D(x_1, \ldots, x_n)\rangle = 1 \ .$$

The definition is independent of the choice of bases [DP07, Lemma 1.4].

Next, recall from Proposition 4.7 that for any Lie supergroup $\mathcal{G}$ and any closed subsupergroup $\mathcal{H}$, we have a $\mathcal{G}$-equivariant vector bundle isomorphism $T(\mathcal{G}/\mathcal{H}) \cong \mathcal{G} \times^{\mathcal{H}} \mathfrak{g}/\mathfrak{h}$. In particular, for any bundle chart $\tau : \mathcal{U} \times \mathcal{H} \to \mathcal{G}|_{\mathcal{U}}$, there is a bundle chart $\mathcal{U} \times \mathfrak{g}/\mathfrak{h} \to T(\mathcal{G}/\mathcal{H})|_{\mathcal{U}}$ which we denote by $T\tau$.

Explicitly, it is given as follows. If $t : \mathcal{U} \to \mathcal{G}$ is a local section of $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ such that $\tau = m_{\mathcal{G}} \circ (t \times \mathrm{id})$, then $T\tau = \alpha_{\mathcal{G}/\mathcal{H}} \circ (t \times \eta)$ where $\alpha_{\mathcal{G}/\mathcal{H}} : \mathcal{G} \times \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{H}$ is the $\mathcal{G}$-action, and $\eta : \mathfrak{g}/\mathfrak{h} \to T(\mathcal{G}/\mathcal{H})$ is (again) the canonical inclusion of the fibre at the base point $o = H \in G/H$.

Since we also have an induced $\mathcal{G}$-equivariant vector bundle isomorphism $\mathrm{Ber}(\mathcal{G}/\mathcal{H}) \cong \mathcal{G} \times^{\mathcal{H}} \mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*)$, by Corollary 4.12, the construction of bundle charts carries over to this situation. Indeed, associated with τ, there is a bundle chart $\mathrm{Ber}(\tau) : \mathcal{U} \times \mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*) \to \mathrm{Ber}(\mathcal{G}/\mathcal{H})|_{\mathcal{U}}$, and it is given explicitly by $\mathrm{Ber}(\tau) = \mathrm{Ber}(\alpha_{\mathcal{G}/\mathcal{H}}) \circ (t \times \eta)$ where $\eta : \mathrm{Ber}((\mathfrak{g}/\mathfrak{h})^*) \to \mathrm{Ber}(\mathcal{G}/\mathcal{H})$ is the canonical inclusion of the fibre at the base point $o = H \in G/H$, and $\mathrm{Ber}(\alpha_{\mathcal{G}/\mathcal{H}})$ is the Berezinian action defined in 4.11.

Lemma 5.15. *Let $\mathcal{G}/\mathcal{H}$ and $\mathcal{S}/\mathcal{T}$ be unimodular as a $\mathcal{G}$- and as an $\mathcal{S}$-space, respectively. Fix invariant forms $\omega_{\mathcal{G}/\mathcal{H}}$, $\omega_{\mathcal{S}/\mathcal{T}}$, and base points $o = H \in G/H$ and $o' = T \in S/T$. Assume that $\mathcal{G}/\mathcal{H}$ and $\mathcal{S}/\mathcal{T}$ have common graded dimension $p|q$, $n = p + q$, that $\phi : \mathcal{G}/\mathcal{H} \to \mathcal{S}/\mathcal{T}$ is a local isomorphism onto an open subsupermanifold of $\mathcal{S}/\mathcal{T}$, and that $\phi(o) = o'$.*

For any bundle charts $\tau : \mathcal{U} \times \mathcal{H} \to \mathcal{G}|_{\mathcal{U}}$ and $\sigma : \mathcal{V} \times \mathcal{T} \to \mathcal{S}|_{\mathcal{V}}$ such that $\phi : \mathcal{U} \to \mathcal{V}$ is an open embedding, there exists $d \in \mathcal{O}_{\mathcal{G}/\mathcal{U}}(U)$ such that

$$\phi^*(\omega_{\mathcal{S}/\mathcal{T}}) = d \cdot \omega_{\mathcal{G}/\mathcal{H}} \quad \text{on} \quad \mathcal{U} \ .$$

The superfunction d may be computed as follows: There are local sections $t : \mathcal{U} \to \mathcal{G}$ and $s : \mathcal{V} \to \mathcal{H}$ such that $\tau = m_{\mathcal{G}} \circ (t \times \mathrm{id})$ and $\sigma = m_{\mathcal{S}} \circ (s \times \mathrm{id})$. Let $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be homogeneous bases of $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{s}/\mathfrak{t}$, respectively, such that

$$\langle(\omega_{\mathcal{G}/\mathcal{H}})_o, D(x_1, \ldots, x_n)\rangle = \langle(\omega_{\mathcal{S}/\mathcal{T}})_{o'}, D(y_1, \ldots, y_n)\rangle \ .$$

Consider the local expression $\varphi : \mathcal{U} \to \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{s}/\mathfrak{t})_0$ *of* $T\phi$*; according to* (5),

$$T\phi \circ T\tau = T\sigma \circ (\mathrm{id} \times \varepsilon) \circ ((\mathrm{id}, \varphi) \times \mathrm{id}) \tag{13}$$

where $T\tau$, $T\sigma$ *where defined in 5.14. Define* $\xi = (\xi_{ij}) \in \mathrm{GL}\big(p|q, \mathcal{O}(\mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{s}/\mathfrak{t}))\big)$ *by* $a(x_i) = \sum_{j=1}^n \xi_{ij}(a) y_j$ *for all* $a \in \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{s}/\mathfrak{t})$ *(recall* $V^* \subset \mathcal{O}(V)$ *for* V *any linear supermanifold). Then* $\varphi^*(\xi) = (\varphi^*(\xi_{ij})) \in \mathrm{GL}\big(p|q, \mathcal{O}(\mathcal{U})\big)$ *and*

$$d = \mathrm{Ber}(\varphi^*(\xi)) \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}(U) \ .$$

Proof. In analogy with [Sch84, 4.19], we may consider

$$\varphi \in \big(\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U) \otimes \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{s}/\mathfrak{t})\big)_0 \ .$$

Then $\varphi(x_i) \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}(U) \otimes \mathfrak{s}/\mathfrak{t}$ makes sense, is of parity $|x_i|$, and we have the equation $\varphi(x_i) = \sum_{j=1}^n \varphi^*(\xi_{ij}) y_j$. It follows that

$$D(\varphi(x_1), \ldots, \varphi(x_n)) = \mathrm{Ber}(\varphi^*(\xi_{ij})) \cdot D(y_1, \ldots, y_n)$$

as elements of $\mathrm{Ber}_{\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U)}(\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U) \otimes \mathfrak{s}/\mathfrak{t})$. Similarly, the Berezinian sections may be considered as

$$\begin{aligned} \omega_{\mathcal{G}/\mathcal{H}}|_{\mathcal{U}} &\in \mathrm{Ber}_{\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U)}(\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U) \otimes (\mathfrak{g}/\mathfrak{h})^*) \ , \\ \omega_{\mathcal{S}/\mathcal{T}}|_{\mathcal{V}} &\in \mathrm{Ber}_{\mathcal{O}_{\mathcal{S}/\mathcal{T}}(V)}(\mathcal{O}_{\mathcal{S}/\mathcal{T}}(V) \otimes (\mathfrak{s}/\mathfrak{t})^*) \ . \end{aligned}$$

For $\omega = \omega_{\mathcal{G}/\mathcal{H}}$, we have on $\mathcal{U}$:

$$\tau^*\omega = \mathrm{Ber}(\tau)^{-1} \circ \omega \circ \tau = \mathrm{id} \times \omega_o \equiv \omega_o \ ,$$

by the $\mathcal{G}$-invariance of ω and the definition of $\mathrm{Ber}(\tau)$ in 5.14. A similar equation holds for $\omega_{\mathcal{S}/\mathcal{T}}$. Hence, abbreviating $D(x) = D(x_1, \ldots, x_n)$, *etc.*, we obtain

$$\begin{aligned} \big\langle \tau^*\phi^*\omega_{\mathcal{S}/\mathcal{T}}, D(x) \big\rangle &= \big\langle \sigma^*\omega_{\mathcal{S}/\mathcal{T}}, D(\varphi(x)) \big\rangle \\ &= \mathrm{Ber}(\varphi^*(\xi_{ij})) \cdot \big\langle (\omega_{\mathcal{S}/\mathcal{T}})_{o'}, D(y) \big\rangle \\ &= \mathrm{Ber}(\varphi^*(\xi_{ij})) \cdot \big\langle (\omega_{\mathcal{G}/\mathcal{H}})_o, D(x) \big\rangle \\ &= \mathrm{Ber}(\varphi^*(\xi_{ij})) \cdot \big\langle \tau^*\omega_{\mathcal{G}/\mathcal{H}}, D(x) \big\rangle \end{aligned}$$

where the first equality follows from (13). This proves the assertion. ∎

Proposition 5.16. *Let* $\mathcal{U}$ *be a Lie supergroup and* $\mathcal{M}$, $\mathcal{H}$ *closed subsupergroups such that the map* $m = m_{\mathcal{U}} : \mathcal{M} \times \mathcal{H} \to \mathcal{U}$ *is an isomorphism onto an open subsupermanifold* $\mathcal{V}$ *of* $\mathcal{U}$*. For a suitable normalisation of Berezinian sections,*

$$\int_{\mathcal{U}} f = \int_{\mathcal{M} \times \mathcal{H}} m^* f \cdot \frac{\mathrm{pr}_2^* \mathrm{Ber}(\mathrm{Ad}_{\mathfrak{h}})}{\mathrm{pr}_2^* \mathrm{Ber}(\mathrm{Ad}_{\mathfrak{u}})}$$

for all $f \in \Gamma_c(U, \mathcal{O}_{\mathcal{U}})$ *such that* $\mathrm{supp}\, f \subset V$.

Proof. We consider $\mathcal{U}$ and $\mathcal{M}\times\mathcal{H}$ as quotients by the trivial subsupergroup. A standard calculation (for instance, using generalised points) shows that the local expression $\varphi:\mathcal{M}\times\mathcal{H}\to\mathrm{Hom}(\mathfrak{m}\oplus\mathfrak{h},\mathfrak{u})_0$ of Tm is given by

$$\varphi=\iota_{\mathfrak{h}\to\mathfrak{u}}+\mathrm{Ad}_{\mathcal{H}}^{\mathfrak{m}\to\mathfrak{u}}\circ i_{\mathcal{H}}\circ\mathrm{pr}_2$$

where $\iota_{\mathfrak{h}\to\mathfrak{u}}$ is the constant morphism $\mathcal{M}\times\mathcal{H}\to\mathrm{Hom}(\mathfrak{h},\mathfrak{u})$ whose value is the inclusion $\mathfrak{h}\to\mathfrak{u}$, and where $\mathrm{Ad}_{\mathcal{H}}^{\mathfrak{m}\to\mathfrak{u}}$ is the morphism $\mathcal{H}\to\mathrm{Hom}(\mathfrak{m},\mathfrak{u})_0$ induced by restriction and corestriction by the adjoint morphism of $\mathcal{U}$.

It follows that $\varphi^*(\xi_{ij})=(\mathrm{pr}_2^*\,\mathrm{Ber}(\mathrm{Ad}_{\mathfrak{h}}))\cdot(\mathrm{pr}_2^*\,\mathrm{Ber}(\mathrm{Ad}_{\mathfrak{u}}))^{-1}$, so that the assertion follows from Lemma 5.15 and the invariance of the Berezin integral under oriented isomorphisms. ∎

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