Patterson–Sullivan distributions for rank one symmetric spaces of the noncompact type

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Abstract

There is a remarkable relation between two kinds of phase space distributions associated to eigenfunctions of the Laplacian of a compact hyperbolic manifold: It was observed in [1] that for compact hyperbolic surfaces $X_\Gamma = \Gamma\backslash\mathbb{H}$ Wigner distributions $\int_{S^*X_\Gamma} a\,dW_{ir_j} = \langle \mathrm{Op}(a)\varphi_{ir_j}, \varphi_{ir_j}\rangle_{L^2(X_\Gamma)}$ and Patterson–Sullivan distributions PS_{ir_j} are asymptotically equivalent as $r_j \to \infty$. We generalize the definitions of these distributions to all rank one symmetric spaces of noncompact type and introduce off-diagonal elements PS_{λ_j,λ_k}. Further, we give explicit relations between off-diagonal Patterson–Sullivan distributions and off-diagonal Wigner distributions and describe the asymptotic relation between these distributions.

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Contents

1 Introduction

In this paper we generalize an interesting link between two kinds of phase space distributions which was observed in [1] for hyperbolic surfaces to all rank one

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Riemannian symmetric spaces of the noncompact type. The distributions of interest arise in the study of quantum ergodicity. To put our results in a general context, we follow [20] to briefly recall some relevant notions of the framework of quantum ergodicity.

If (X, g) is an n-dimensional compact Riemannian manifold with Laplace operator Δ, then $L^2(X) = \oplus_{\lambda_k} \mathcal{H}_{\lambda_k}$, where $\Delta = -\lambda_k$ on the eigenspaces $\mathcal{H}_{\lambda_k}$ and $\dim(H_{\lambda_k}) < \infty$. We fix ordered orthonormal bases $\{\varphi_{k_i} : 1 \leq i \leq \dim(H_{\lambda_k})\}$ for each $\mathcal{H}_{\lambda_k}$ to obtain a sequence $\{\varphi_{k_i} : k = 1, 2, 3, \ldots, 1 \leq i \leq \dim(H_{\lambda_k})\}$ of orthonormal eigenfunctions. Given a calculus of pseudodifferential operators on X, i.e. an assignment $\mathrm{Op} : C^\infty(S^*X) \to B(L^2(X))$ of bounded operators $\mathrm{Op}(a)$ to smooth zero order symbols a, satisfying the usual requirements [21], we associate to a given eigenfunction φ_k a distribution W_k, called the *Wigner distribution for* φ_k, defined by $W_k(a) := \langle \mathrm{Op}(a)\varphi_k, \varphi_k \rangle_{L^2(X)}$. A distribution $\mu \in \mathcal{D}'(S^*X)$ is called *weak***-limit point* of the $\left\{W_{k_j}\right\}$ if there is a subsequence $\mathcal{S} \subseteq \{\lambda_{k_i}\}$ such that $\lim_{\mathcal{S}} W_{k_i}(a) = \mu(a)$ for all a. One of the problems in the framework of quantum ergodicity is the question: What are the weak*-limit points of the W_{k_i}? All such limit distributions are invariant measures for the geodesic flow on S^*X. It is not known which limit points arise and how they depend on the choice of the $\{\varphi_{k_i}\}$.

It was observed in [1] that for compact hyperbolic surfaces $X_\Gamma = \Gamma\backslash\mathbb{H}$ Wigner distributions are asymptotically equivalent (and hence equivalent for the study of quantum ergodicity) to *Patterson–Sullivan distributions* $\widehat{PS}_{ir_k}$, which are also associated to the sequence $\{\varphi_k\}$ of eigenfunctions. An interesting property of these Patterson–Sullivan distributions is that they are invariant under the geodesic flow, so one might hope that the study of these invariant distributions combined with the relations to Wigner distributions yield more insight into the questions of quantum ergodicity for symmetric spaces.

Before we state our results we have to make a few remarks about the special ΨDO-calculus we use in this paper: S. Zelditch ([19]) introduced a natural quantization for G/K, when $G = PSU(1, 1)$, $K = PSO(2)$. It is in fact possible to generalize this calculus two all rank one symmetric spaces $X := G/K$, where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. The basic definitions and properties of this calculus are given in Section 4. Full details with all computations concerning this calculus will appear in [12]. An advantage of this calculus is its G-equivariance: Fix a co-compact and torsion free discrete subgroup Γ of G and let SX denote the unit tangent bundle of $X = G/K$. If $a \in C^\infty(SX)$ is Γ-invariant (under the natural action of G on SX, see Section 4), then it yields a pseudodifferential operator on the quotient $X_\Gamma := \Gamma\backslash G/K$.

Our setting is as follows: Let $X = G/K$ denote a general rank one symmetric space of the noncompact type, where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $G = KAN$ be a corresponding Iwasawa decomposition of G and let M denote the centralizer of A in K. The geodesic boundary of X can be identified with the flag manifold $B := K/M$. Let $o := K \in G/K$ denote the *origin* of the symmetric space X. Further, let Δ, resp. Δ_Γ, denote the Laplace operator of X, resp. X_Γ. We consider the following automorphic eigenvalue problem on $X = G/K$:

$$\begin{aligned} \Delta\varphi &= -c\,\varphi \\ \varphi(\gamma z) &= \varphi(z) \text{ for all } \gamma \in \Gamma \text{ and for all } z \in X. \end{aligned}$$

In other words, we study the eigenfunctions of the Laplacian on the compact manifold $X_\Gamma = \Gamma\backslash X$. If the eigenfunctions φ are real-valued, the eigenvalues $-c \in \mathbb{R}$, $-c \leq -\langle\rho,\rho\rangle$, of Δ are of the form $-c := -c_\lambda := -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)$, where $\lambda \in \mathfrak{a}^*$, the real dual of the Lie algebra $\mathfrak{a}$ of A, and where $\langle\, , \rangle$ denotes the inner product on $\mathfrak{a}^*$ induced by the Killing form (see Section 2). We fix a complete $L^2(X_\Gamma)$-orthonormal basis $\{\varphi_{\lambda_j}\}$ of real-valued and Γ-invariant eigenfunctions, where the eigenvalues are repeated according to their multiplicity. We hence obtain a corresponding sequence of eigenvalue parameters $\lambda_j \in \mathfrak{a}^*$. Then

$$\begin{aligned} \Delta\varphi_{\lambda_j} &= -(\langle\lambda_j,\lambda_j\rangle + \langle\rho,\rho\rangle)\varphi_{\lambda_j} \text{ for all } j, \\ \varphi_{\lambda_j}(\gamma z) &= \varphi_{\lambda_j}(z) \text{ for all } \gamma \in \Gamma,\ z \in X,\ j \in \mathbb{N}_0. \end{aligned}$$

If Y is a manifold, u a distribution or hyperfunction on Y and φ a test function, then we denote the pairing $\langle\varphi,u\rangle_Y$ by $\int_Y \varphi(y)u(dy)$.

For each eigenfunction φ_{λ_j} (with exponential growth, see Section 3) of the negative Laplacian $-\Delta$ with corresponding eigenvalue $c_j = \langle\lambda_j,\lambda_j\rangle + \langle\rho,\rho\rangle$ there is a unique distribution boundary value (also described in Section 3) $T_{\lambda_j} \in \mathcal{D}'(B)$ such that

$$\varphi_{\lambda_j}(x) = \int_B e^{(i\lambda_j+\rho)\langle x,b\rangle} T_{\lambda_j}(db).$$

Here $\langle x,b\rangle$ denotes the horocyclic bracket defined in (2.13) below. Given $a \in C^\infty(SX)$, the *Wigner distributions* are defined by

$$W_{\lambda_j,\lambda_k}(a) := \langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_\Gamma)}.$$

In the special case when $j = k$, we write $W_{\lambda_j}(a) := W_{\lambda_j,\lambda_j}(a)$.

Let $B^{(2)} = (B \times B) \setminus \Delta = \{(b,b') \in B \times B : b \neq b'\}$ denote the set of pairs of distinct boundary points (Δ denotes the diagonal of $B \times B$). We will describe the geodesic boundary in Section 2. Each geodesic of X has a unique forward limit point and a unique backward limit point in B. In particular, we identify $B^{(2)}$ with the space of geodesics. We will see in Section 3 that in the case of Γ-invariant eigenfunctions the boundary values T_{λ_j} satisfy the following equivariance property:

$$T_{\lambda_j}(d\gamma b) = e^{-(i\lambda_j+\rho)\langle\gamma o,\gamma b\rangle} T_{\lambda_j}(db), \quad \gamma \in \Gamma. \tag{1.1}$$

It is then possible to introduce (see Section 5 for details) functions d_{λ_j} on $B^{(2)}$ and a Radon transform $\mathcal{R} : C_c^\infty(SX) \to C_c^\infty(B^{(2)})$ such that the expression

$$\langle a, PS_{\lambda_j}\rangle_{SX} := \int_{B^{(2)}} d_{\lambda_j}(b,b')\, \mathcal{R}(a)(b,b')\, T_{\lambda_j}(db)\, T_{\lambda_j}(db') \tag{1.2}$$

defines a Γ-invariant distribution on SX. We call these distributions the *Patterson–Sullivan distributions* associated to the $\{\varphi_{\lambda_j}\}$. The PS_{λ_j} are invariant under the geodesic flow and under time reversal (see Section 5 for details). The weight functions d_{λ_j} will be called *intermediate values* because of (5.5), which generalizes the intermediate value formula (5.1) for hyperbolic surfaces,

Let $H : KAN \to \mathfrak{a}$ denote the Iwasawa projection (see Section 2) and let w denote the non-trivial Weyl group element (see Section 2). Given $j \in \mathbb{N}_0$, define

$$L_{\lambda_j} a(g) := \int_N e^{-(i\lambda_j+\rho)(H(nw))} a(gn)dn, \quad a \in C(G), \tag{1.3}$$

whenever the integral exists.

Following [1] we use a cutoff $\chi \in C_c^\infty(X)$, which is a smooth replacement for the characteristic function of a fundamental domain $\mathcal{F}$ for Γ (cf. Section 5). A concrete relation between the W_{λ_j} and the PS_{λ_j} is given by the operators L_{λ_j} and it generalizes the "exact formula" in Theorem 1.1 of [1]:

Theorem 1.1. *Let $a \in C^\infty(SX_\Gamma)$. Then*

$$\langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_j}\rangle_{L^2(X_\Gamma)} = \langle L_{\lambda_j}(\chi a), PS_{\lambda_j}\rangle_{SX}. \tag{1.4}$$

Still following [1], we also define normalized Patterson–Sullivan distributions

$$\widehat{PS}_{\lambda_j} := \frac{1}{\langle 1, PS_{\lambda_j}\rangle_{SX_\Gamma}} PS_{\lambda_j}, \tag{1.5}$$

which satisfy the same normalization condition $\langle 1, \widehat{PS}_{\lambda_j}\rangle_{SX_\Gamma} = 1$ as the W_{λ_j} on the quotient SX_Γ.

As was pointed out in the introduction of [1] it is of interest to also have analogous results for off-diagonal matrix entries. To this end we introduce (in Section 6) off-diagonal Patterson–Sullivan distributions PS_{λ_j,λ_k} such that $PS_{\lambda_j,\lambda_j} = PS_{\lambda_j}$ for all $j \in \mathbb{N}_0$. We then prove the off-diagonal analog of Theorem 1.1:

Theorem 1.2. *Let $a \in C^\infty(SX_\Gamma)$. Then*

$$\langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_\Gamma)} = \langle L_{\lambda_k}(\chi a), PS_{\lambda_j,\lambda_k}\rangle. \tag{1.6}$$

Theorem 1.1 is an immediate consequence of Theorem 1.2, but we intentionally separated the definitions and the results. One reason is that the definitions are based on quite different ideas and that the PS_{λ_j} have nicer invariance properties than the PS_{λ_j,λ_k}. Another reason is that the normalization of the PS_{λ_j} motivates the normalization of the PS_{λ_j,λ_k} (see Definition 6.8).

Finally, we generalize the "asymptotic formula" in Theorem 1.1 of [1] to off-diagonal elements:

Theorem 1.3. *Let $a \in C^\infty(SX_\Gamma)$. Assume that λ_{j_n}, $\lambda_{k_n} \to \infty$ are sequences of spectral parameters such that $|\lambda_{j_n} - \lambda_{k_n}| \leq \tau$ for some $\tau > 0$. Then*

$$\langle \mathrm{Op}(a)\varphi_{\lambda_{j_n}}, \varphi_{\lambda_{k_n}}\rangle_{L^2(X_\Gamma)} = \langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle_{SX_\Gamma} + O(1/|\lambda_{k_n}|) \quad (n \to \infty).$$

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2 Preliminaries

In this section we collect a number of geometric definitions and facts needed to formulate our main results.

Semisimple Lie Groups

Let G be a non-compact connected semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G, and $\langle\, ,\, \rangle$ the *Killing form* of $\mathfrak{g}$. Let θ be a *Cartan involution* of $\mathfrak{g}$ such that the form $(X, Y) \mapsto -\langle X, \theta Y\rangle$ is positive definite on $\mathfrak{g} \times \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of θ and K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. We choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ its dual and $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$. At this point we do *not yet* make the assumption that the rank of $X = G/K$, i.e. $\dim \mathfrak{a}$, is one. Later, however, it will be indispensable. Let $A = \exp \mathfrak{a}$ denote the corresponding analytic subgroup of G and let log denote the inverse of the map $\exp : \mathfrak{a} \to A$.

Given $\lambda \in \mathfrak{a}^*$, put $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \ \forall\, H \in \mathfrak{a}\}$. If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a *(restricted) root* and $m_\lambda = \dim(\mathfrak{g}_\lambda)$ is called its *multiplicity.* Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$ and if $\mathfrak{s}$ is any subspace of $\mathfrak{g}$ let $\mathfrak{s}_{\mathbb{C}}$ denote the complex subspace of $\mathfrak{g}_{\mathbb{C}}$ spanned by $\mathfrak{s}$.

For $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$ we put $\langle \lambda, \mu\rangle := \langle H_\lambda, H_\mu\rangle$. Since $\langle\, ,\, \rangle$ is positive definite on $\mathfrak{p} \times \mathfrak{p}$ we put $|\lambda| := \langle \lambda, \lambda\rangle^{1/2}$ for $\lambda \in \mathfrak{a}^*$ and $|X| := \langle X, X\rangle^{1/2}$ for $X \in \mathfrak{p}$. The $\mathbb{C}$-bilinear extension of $\langle\, ,\, \rangle$ to $\mathfrak{a}^*_{\mathbb{C}}$ will be denoted by the same symbol.

Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. The components of $\mathfrak{a}'$ are called Weyl chambers. We fix a Weyl chamber $\mathfrak{a}^+$ and call a root α positive (> 0) if it is positive on $\mathfrak{a}^+$. Let $\mathfrak{a}^*_+$ denote the corresponding Weyl chamber in $\mathfrak{a}^*$, that is the preimage of $\mathfrak{a}^+$ under the mapping $\lambda \mapsto H_\lambda$. Let Σ denote the set of restricted roots, Σ^+ the set of positive roots and $\Sigma^- := -\Sigma^+$ the set of negative roots.

Let $\Sigma_0 = \left\{\alpha \in \Sigma : \frac{1}{2}\alpha \notin \Sigma\right\}$, and put $\Sigma_0^+ = \Sigma^+ \cap \Sigma_0$, $\Sigma_0^- = \Sigma^- \cap \Sigma_0$. We set $\rho := 2^{-1}\Sigma_{\alpha\in\Sigma^+} m_\alpha \alpha$ and let N denote the analytic subgroup of G with Lie algebra $\mathfrak{n} := \Sigma_{\alpha>0}\mathfrak{g}_\alpha$. Then $\overline{\mathfrak{n}} = \theta(\mathfrak{n}) = \Sigma_{\alpha<0}\mathfrak{g}_\alpha$. The involutive automorphism θ of $\mathfrak{g}$ extends to an analytic involutive automorphism of G, also denoted by θ, whose differential at the identity $e \in G$ is the original θ. It thus makes sense to define $\overline{N} = \theta N$. The Lie algebra of $\overline{N}$ is $\theta(\mathfrak{n})$.

Let M denote the centralizer of A in K and let M' denote the normalizer of A in K. Let W denote the (finite) *Weyl group* M'/M. The group W acts as a group of linear transformations of $\mathfrak{a}$ and also on $\mathfrak{a}^*_{\mathbb{C}}$ by $(s\lambda)(H) := \lambda(s^{-1}H)$ for $s \in W$, $H \in \mathfrak{a}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Let r denote the order of W and let $w_1, \ldots, w_r$ be a complete set of representatives in M'. Let $A^+ := \exp(\mathfrak{a}^+)$, $B := K/M$, $P := MAN$. Then we have the decompositions

(1) $G = KAN$ (Iwasawa decomposition),
(2) $G = \bigcup_{j=1}^r Pw_jP$ (Bruhat decomposition).

Here (1) means that each $g \in G$ can be uniquely written in the form

$$g = k(g) \exp H(g) n(g), \tag{2.1}$$

where $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$. The functions k, H, n are called *Iwasawa projections.* In (2), the union is disjoint. Let w^* denote the Weyl group element mapping $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Exactly one of the summands in (2), namely Pw^*P, is open in G. Thus the set $\overline{N}MAN$ is open in G.

We call $\dim(\mathfrak{a})$ the real rank of G and the rank of the symmetric space $X = G/K$. Let $o := K \in G/K$ denote the *origin* of X. If X has rank one, the Weyl group has only two elements. In this case we denote the nontrivial Weyl group element by w and pick an element $m' \in M'$ such that $m'M = w \in W$.

By abuse of notation we write w for m'. Then we have the important formula

$$waw^{-1} = a^{-1} \ \ \forall a \in A. \tag{2.2}$$

If Y is a manifold satisfying the second countability axiom we write $\mathcal{D}(Y)$ for the space of C^∞ functions on Y of compact support. $\mathcal{D}'(Y)$ denotes the dual space of distributions on Y. The space $\mathcal{E}(Y)$ denotes the space of C^∞ functions on Y and $\mathcal{E}'(Y)$ denotes the dual space of distributions on Y of compact support.

Normalization of Measures

We briefly recall some normalizations of the measures on the homogeneous spaces we work with. We follow [8]. The Killing form induces euclidean measures on A, $\mathfrak{a}$ and $\mathfrak{a}^*$. For $l = \dim(A)$ we multiply these measures by $(2\pi)^{-l/2}$ and obtain invariant measures da, dH and $d\lambda$ on $A, \mathfrak{a}$ and $\mathfrak{a}^*$. This normalization has the advantage that the euclidean Fourier transform of A is inverted without a multiplicative constant. We normalize the Haar measures dk and dm on the compact groups K and M such that the total measure is 1. If U is a Lie group and P a closed subgroup, with left invariant measures du and dp, the U-invariant measure $du_P = d(uP)$ on U/P (if it exists) will be normalized by

$$\int_U f(u)du = \int_{U/P} \left(\int_P f(up)dp \right) du_P. \tag{2.3}$$

This measure exists in particular if P is a compact subgroup of U. In particular, we have a K-invariant measure $dk_M = d(kM)$ on K/M of total measure 1. We also have a G-invariant measure $dx = dg_K = d(gK)$ on $X = G/K$. By uniqueness, dx is a constant multiple of the measure on X induced by the Riemannian structure on X given by the Killing form. The Haar measures dn and $d\overline{n}$ on the nilpotent groups N and $\overline{N}$ are normalized such that

$$\theta(dn) = d\overline{n}, \qquad \int_{\overline{N}} e^{-2\rho(H(\overline{n}))} d\overline{n} = 1. \tag{2.4}$$

The Haar measure on G can ([8], Ch. I, §5) then be normalized such that

$$\begin{aligned} \int_G f(g)dg &= \int_{KAN} f(kan) e^{2\rho(\log a)}\, dk\, da\, dn & (2.5)\\ &= \int_{NAK} f(nak) e^{-2\rho(\log a)}\, dn\, da\, dk & (2.6) \end{aligned}$$

for all $f \in C_c(G)$. Let $f_1 \in C_c(AN)$, $f_2 \in C_c(G)$, $a \in A$. Then ([8], pp. 182)

$$\int_N f_1(na)\, dn = e^{2\rho(\log(a))} \int_N f_1(an)\, dn \tag{2.7}$$

and

$$\int_G f_2(g)\, dg = \int_{KNA} f_2(kna)\, dk\, dn\, da = \int_{ANK} f_2(ank)\, da\, dn\, dk. \tag{2.8}$$

Let $f_3 \in C_c(X)$. It follows from (2.8) that

$$\int_X f_3(x)\, dx = \int_{AN} f_3(an \cdot o)\, da\, dn. \tag{2.9}$$

For any (restricted) root α we write $\alpha_0 := \alpha/\langle\alpha,\alpha\rangle$. We will need *Harish-Chandra's e-functions* ([7], p. 163, the rank of X is arbitrary)

$$e_s(\lambda) = \prod_{\alpha\in\Sigma_s^+} \Gamma\left(\frac{m_\alpha}{4} + \frac{1}{2} + \frac{\langle i\lambda,\alpha_0\rangle}{2}\right) \Gamma\left(\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{\langle i\lambda,\alpha_0\rangle}{2}\right), \quad (2.10)$$

where $s \in W$, $\Sigma_s^+ = \Sigma_0^+ \cap s^{-1}\Sigma_0^-$ and where Γ denotes the classical Gamma-function. Let X have rank one. Now, the set Σ of (restricted) roots contains at most two positive elements: α and possibly 2α. We adopt the usual convention that $m_{2\alpha} = 0$ if 2α is not a root. *Harish-Chandra's c-function* ([8], Ch. IV, §6) is the meromorphic function

$$c(\lambda) \;=\; c_0 \, \frac{2^{-\langle i\lambda,\alpha_0\rangle}\,\Gamma\left(\langle i\lambda,\alpha_0\rangle\right)}{\Gamma\left(\frac{m_\alpha}{4} + \frac{1}{2} + \frac{\langle i\lambda,\alpha_0\rangle}{2}\right)\Gamma\left(\frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2} + \frac{\langle i\lambda,\alpha_0\rangle}{2}\right)}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad (2.11)$$

where $c_0 = 2^{\frac{1}{2}m_\alpha + m_{2\alpha}}\,\Gamma(\frac{m_\alpha}{2} + \frac{m_{2\alpha}}{2} + \frac{1}{2})$.

Geodesics, Boundary and the Unit Tangent Bundle

G acts on G/P via $g \cdot xP = gxP$ and $K/M \to G/P$, $kM \mapsto kP$ is a diffeomorphism ([9], p. 407) inverted by $gP \mapsto k(g)M$, where $g = k(g)\exp H(g)n(g)$. Hence this map intertwines the G-action on G/P with the action on K/M defined by $g \cdot kM = k(gk)M$. These spaces are thus equivalent for the study of $B = K/M = G/P$. Although the following remarks are basically trivial, we write them down for later reference: With respect to the actions described above, the stabilizer of $M \in K/M$ is the subgroup $P = MAN$. The action of the groups AN and P on G/K are transitive. For the remainder of this section, let $X = G/K$ be of rank one. As above, let $w \in M'$ denote a representative of the nontrivial Weyl group element.

Lemma 2.1. *$P = MAN$ acts transitively on $G/P \setminus \{P\}$.*

Proof. This follows from the Bruhat decomposition

$$G = P \cup PwP \quad \text{(disjoint union)}.$$

In fact, let $gP \in G/P \setminus \{P\}$. Then $g \notin P$, so g is of the form p_1wp_2, where $p_1, p_2 \in P$. Hence $gP = p_1wP = p_1 \cdot wP$ and we have proven that each $gP \neq P \in G/P$ lies in the P-orbit of wP. □

Remark 2.2. Let H_0 denote the unique unit vector (with respect to the norm on $\mathfrak{a}$ induced by the Killing form) in $\mathfrak{a}^+$. It is well known ([8]) that $K\cdot H_0 = S(\mathfrak{p})$, i.e., the group K acts transitively on the set $S(\mathfrak{p})$ of unit vectors in the tangent space $T_oX = \mathfrak{p}$. The subgroup $AN = NA$ of G acts transitively on G/K, so $G = NAK$ acts transitively on the the unit tangent bundle SX of $X = G/K$. The group M is the stabilizer in K of $H_0 \in S(\mathfrak{p})$. Hence the unit tangent bundle of X can be identified G-equivariantly with the homogeneous space G/M. We will from now on write $SX = G/M$ (for X of real rank one). The geodesic flow on G/M reads as the action of A by right translations on G/M.

As in the introduction, consider the space $B \times B$ and its diagonal Δ. We let $B^{(2)} = (B \times B) \setminus \Delta$ denote the set of distinct boundary points. We may now describe the space of geodesics and the geodesic connections in the rank one case. We describe the map that assigns to a geodesic its forward and backward limit points.

We call $\gamma_{H_0}(t) = e^{tH_0} \cdot o$ the *standard geodesic*. If we write $B = K/M$, the forward limit point b_∞ of the standard geodesic identifies with $M \in K/M$ (that is $P \in G/P$) and (since $\mathrm{Ad}_G(w)$ is $-\mathrm{id}$ on $\mathfrak{a}$) its backward limit point $b_{-\infty}$ identifies with $wM \in K/M$ (that is $wP \in G/P$). Since $wM \neq M$ in K/M, the point (M, wM) is an element of $B^{(2)}$ and the standard geodesic is the unique (up to parameter translation and time reversal) geodesic of X that joins the boundary points M and wM at infinity. We also write $a_t := e^{tH_0} \in A$. Then the standard geodesic is the curve $t \mapsto a_t \cdot o$.

We consider the action of G on $B^{(2)}$ given by

$$G \times B^{(2)} \to B^{(2)}, \quad g \cdot (b_1, b_2) = (g \cdot b_1, g \cdot b_2). \tag{2.12}$$

Lemma 2.3. *G acts transitively on $B^{(2)}$. The stabilizer of $(b_\infty, b_{-\infty}) \in B^{(2)}$ is the subgroup MA of G.*

Proposition 2.4. *$B^{(2)} = G/MA$ as homogeneous spaces.*

Proof. Let $b_1 \neq b_2$ be points in B. Since K acts transitively on B, we find a $k \in K$ such that $k \cdot b_1 = b_\infty$. Since P acts transitively on $B \setminus \{b_\infty\}$, we also find a $p \in P$ such that $p \cdot k \cdot b_2 = b_{-\infty}$. Let $g = pk$. Then $g \cdot (b_1, b_2) = (b_\infty, b_{-\infty})$.

It remains to show $g \cdot (b_\infty, b_{-\infty}) = (b_\infty, b_{-\infty})$ if and only if $g \in MA$. Let $ma \in MA$. Then $ma \cdot b_\infty = b_\infty$ and $ma \cdot b_{-\infty} = m \cdot a \cdot wP = w\tilde{m}\tilde{a}P = wP$, since M' normalizes A and M. Hence MA acts trivially on $(b_\infty, b_{-\infty}) \in B^{(2)}$. For the converse assume $g \cdot (b_\infty, b_{-\infty}) = (b_\infty, b_{-\infty})$. Then $g \cdot b_\infty$, so $g = man \in MAN$.

It suffices to prove $n = e$. For $b \in B$ let G_b denote the subgroup of G fixing b. Then $n \in G_{b_\infty} \cap G_{b_{-\infty}} = MAN \cap wMANw^{-1}$, so $n \in wMANw^{-1} = MA \cdot wNw^{-1} = MA\overline{N}$. Hence there exists an element $\overline{n}' \in \overline{N}$ such that $n\overline{n}' = ma \in N\overline{N} \cap MA = \{e\}$ (cf. [9], Ch. VI, Exercise B2. See also [7], Lemma 1.6 on page 79.). But $N \cap \overline{N} = \{e\}$, since $\mathfrak{g}$ is the direct vector space sum of the root-spaces $\mathfrak{g}_\alpha$. Hence $n = e$ as desired, ☐

Definition 2.5. We will from now on always write $g(b, b')MA \in G/MA$ for the unique coset corresponding to $(b, b') \in B^{(2)}$. The representative $g(b, b')$ is uniquely determined modulo MA.

Remark 2.2 yields a G-equivariant identification $G/M = SX$. Another identification is $G/M = X \times B$: It is clear that with respect to the diagonal action of G on $X \times B$ the group M is the stabilizer of $(K, P) \in G/K \times G/P$. Using the Iwasawa decomposition (see [12] for details) we also see that G acts transitively on the space $X \times B$. This induces the following G-equivariant identification $SX = X \times B$: If $(z, b) \in X \times B$, then let $v(z, b)$ denote the unit vector in SX tangential to the geodesic through z with forward endpoint b. This geodesic exists since X has rank one (see [5], [11] and also [12] for details).

Horocycle bracket and Iwasawa Projection

In this subsection, we describe the so-called *horocycle bracket* on $X \times B$, because we need some formulae corresponding to this inner product. For details on the geometric interpretation of this horocycle bracket see [7], Ch. II (and [12]).

Let $\langle\cdot,\cdot\rangle : X \times B \to \mathfrak{a}, (x,b) \mapsto \langle x,b\rangle$ be defined by

$$\langle x,b\rangle = \langle gK, kM\rangle = -H(g^{-1}k). \tag{2.13}$$

Each (x,b) is of the form (gK,kM) and it is easy to see that (2.13) is well-defined. We remark that the use of $\langle\, ,\, \rangle$, whether we mean the Killing form or the horocycle bracket, will always be clear from the context.

Proposition 2.6. *$\langle\cdot,\cdot\rangle$ is invariant under the diagonal action of K on $X \times B$.*

Recall that $g \in G$ acts on K/M by $g \cdot kM = k(gk)M$ (Iwasawa projection).

Lemma 2.7. *Let $g_1, g_2 \in G$, $k \in K$. Then $H(g_1g_2k) = H(g_1k(g_2k)) + H(g_2k)$.*

Proof. Decompose $g_2k = \tilde{k}\tilde{a}\tilde{n}$ and $g_1\tilde{k} = k'a'n'$. Then

$$H(g_1g_2k) = H(k'a'n'\tilde{a}\tilde{n}) = H(a'n'\tilde{a}).$$

Since A normalizes N this equals $\log(a') + \log(\tilde{a})$. □

Lemma 2.8. *Let $x = hK \in G/K$, $b = kM \in K/M$, $g \in G$. Then*

$$\langle g\cdot x, g\cdot b\rangle = \langle x,b\rangle + \langle g\cdot o, g\cdot b\rangle\,. \tag{2.14}$$

Proof. By definition, $\langle g\cdot x, g\cdot b\rangle = -H(h^{-1}g^{-1}k(gk))$. Then by Lemma 2.7 applied to $g_1 = h^{-1}g^{-1}$ and $g_2 = g$ this equals

$$-H(h^{-1}g^{-1}gk) + H(gk) = -H(h^{-1}k) + H(gk).$$

For $h = e$ we obtain $\langle g\cdot o, g\cdot b\rangle = -H(k) + H(gk) = H(gk)$. Hence

$$\langle g\cdot x, g\cdot b\rangle - \langle g\cdot o, g\cdot b\rangle \quad = \quad [-H(h^{-1}k) + H(gk)] - [-H(k) + H(gk)],$$

which equals $-H(h^{-1}k) = \langle hK, kM\rangle = \langle x,b\rangle$. □

Lemma 2.9. *Let $\gamma, g \in G$ and $w \in K$. Then*
(i) $\langle g\cdot o, g\cdot M\rangle = H(g)$ and $\langle g\cdot o, g\cdot wM\rangle = H(gw)$.
(ii) $H(\gamma g) = H(g) + \langle \gamma\cdot o, \gamma g\cdot M\rangle$ and $H(\gamma g w) = H(gw) + \langle\gamma\cdot o, \gamma g\cdot wM\rangle$.

Proof. *(i)* is a direct computation. The second part of *(ii)* follows from the first part applied to gw instead of g. For this assertion, let $z = g\cdot o$. Then by *(i)*

$$H(\gamma g) = \langle \gamma g\cdot o, \gamma g\cdot M\rangle = \langle \gamma\cdot z, \gamma g\cdot M\rangle$$

and by (2.14) this equals $\langle z, g\cdot M\rangle + \langle\gamma\cdot o, \gamma g\cdot M\rangle = H(g) + \langle\gamma\cdot o, \gamma g\cdot M\rangle$. □

3 Helgason Boundary Values

In this section we recall the Poisson transform, which plays a key role in the proofs of our results, and use it to prove the estimate (3.16) which will allow us to define the Patterson–Sullivan distributions. Even though part of what we describe here could be done in greater generality we restrict ourselves to the case of rank one spaces.

Eigenfunctions and Poisson Transform

We fix a co-compact, torsion free discrete subgroup Γ of G and choose a G-invariant measure ν on $\Gamma\backslash G$ such that

$$\int_G f(x)dx = \int_{\Gamma\backslash G} \left(\sum_\gamma f(\gamma x) \right) d\nu(\Gamma x)$$

for $f \in C_c(G)$. We will denote the Hilbert space $L^2(\Gamma\backslash G, \nu)$ simply by $L^2(\Gamma\backslash G)$. The G-invariance of ν implies that the equation

$$(R_\Gamma(g)f)(\Gamma x) = f(\Gamma x g)$$

($g, x \in G$, $f \in L^2(\Gamma\backslash G)$) defines a unitary representation R_Γ of G on $L^2(\Gamma\backslash G)$, which is called the *right-regular representation* of G on $\Gamma\backslash G$.

As before, let Δ denote the Laplace operator of X. The eigenspaces corresponding to eigenvalues $-c \leq -\langle \rho, \rho \rangle$ of Δ are ([6], Theorem 7.1) the spaces

$$\mathcal{E}_\lambda(X) = \{ f \in \mathcal{E}(X) : \Delta f = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle) f \},$$

where $\lambda \in \mathfrak{a}^*$ and where $\langle\, ,\, \rangle$ denotes the inner product on $\mathfrak{a}^*$ induced by the Killing form as described in Section 2. We fix a Γ-invariant eigenfunction $\varphi \in \mathcal{E}_\lambda(X)$ and assume that φ is normalized with respect to the $L^2(X_\Gamma)$-norm. Then $\Delta\varphi = -(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)\varphi$.

Let $\mathcal{A}(B)$ denote the vector space of analytic functions on $B = K/M$, topologized as in [6], Section 5. The *analytic functionals* are (loc. cit.) the functionals in the dual space $\mathcal{A}'(B)$ of $\mathcal{A}(B)$. Fix $\lambda \in \mathfrak{a}^*$ and recall the following fundamental result ([7], p. 507):

Theorem 3.1. *The Poisson–Helgason transform $P_\lambda : \mathcal{A}'(B) \to \mathcal{E}_\lambda(X)$ given by*

$$P_\lambda(T)(z) := \int_B e^{(i\lambda+\rho)\langle z,b \rangle} T(db) \tag{3.1}$$

is a bijection of the dual space $\mathcal{A}'(B)$ onto the eigenspace $\mathcal{E}_\lambda(X)$.

For an eigenfunction $f \in \mathcal{E}_\lambda(X)$ of the Laplacian we call the unique functional T_f with $f = P_\lambda(T_f)$, given by Theorem 3.1, the *boundary values* of f. We will now consider a special class of these eigenfunctions that have distributional boundary values: Let d_X denote the distance function on X and define the space $\mathcal{E}^*(X)$ of smooth functions of exponential growth by

$$\mathcal{E}^*(X) := \left\{ f \in \mathcal{E}(X) \mid \exists C > 0 : |f(x)| \leq C e^{C d_X(o,x)} \ \ \forall x \in X \right\}. \tag{3.2}$$

We put $\mathcal{E}^*_\lambda(X) := \mathcal{E}^*(X) \cap \mathcal{E}_\lambda(X)$ and recall (2.10). Then (cf. [7], p. 508):

Theorem 3.2. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ be such that $e_w(\lambda) \neq 0$. Then $P_\lambda(\mathcal{D}'(B)) = \mathcal{E}^*_\lambda(X)$.*

G acts on B, hence on $\mathcal{D}'(B)$ by push-forward: Given $T \in \mathcal{D}'(B)$, a test function $f \in \mathcal{E}(B)$ and $g \in G$, the action is $(gT)(f) = T(f \circ g^{-1})$. When we denote the pairing between distributions and test functions by an integral, we also write $T(d\gamma b)$ for $(\gamma T)(db)$. Consider a Γ-invariant eigenfunction φ

with boundary values T_φ: Then $\varphi(\gamma z) = \varphi(z)$ for all γ and z implies (recall $\langle g \cdot x, g \cdot b\rangle = \langle x, b\rangle + \langle g \cdot o, g \cdot b\rangle$ from equation (2.14))

$$\begin{aligned}\varphi(z) &= \int_B e^{(i\lambda+\rho)\langle \gamma z, b\rangle} T_\varphi(db) = \int_B e^{(i\lambda+\rho)\langle \gamma z, \gamma b\rangle} T_\varphi(d\gamma b) \\ &= \int_B e^{(i\lambda+\rho)\langle z, b\rangle} e^{(i\lambda+\rho)\langle \gamma o, \gamma b\rangle} T_\varphi(d\gamma b).\end{aligned}$$

By uniqueness of the Poisson–Helgason transform (Theorem 3.1) we obtain

$$T_\varphi(d\gamma b) = e^{-(i\lambda+\rho)\langle \gamma o, \gamma b\rangle} T_\varphi(db). \tag{3.3}$$

Spherical Principal Series

We recall some facts concerning the *principal series* representations of G. Following [7] and [18], let $\lambda \in \mathfrak{a}$ and consider the representation $\sigma_\lambda(man) = e^{(i\lambda+\rho)\log(a)}$ of $P = MAN$ on $\mathbb{C}$. We denote the *induced representation* on G by $\pi_\lambda = \mathrm{Ind}_P^G(\sigma_\lambda)$. The *induced picture* of this representation is constructed as follows: A dense subspace of the representation space is

$$H_\lambda^\infty := \left\{ f \in C^\infty(G) : f(gman) = e^{-(i\lambda+\rho)\log(a)} f(g) \right\}$$

with inner product

$$(f_1, f_2) = \int_{K/M} f_1(k)\overline{f_2(k)}\, dk = \langle f_{1|K}, f_{2|K}\rangle_{L^2(K/M)}$$

and corresponding norm $\|f\|^2 = \int_{K/M} |f(k)|^2\, dk$. The group action of G is given by $(\pi_\lambda(g)f)(x) = f(g^{-1}x)$. The actual Hilbert space, which we denote by H_λ, and the representation on H_λ, which we also denote by π_λ, is obtained by completion (cf. [18], Ch. 9). The representations π_λ $(\lambda \in \mathfrak{a})$ form the *spherical principal series* of G. (π_λ, H_λ) is a unitary ([7], p. 528) and irreducible (loc. cit. p. 530) Hilbert space representation.

Given $f \in C^\infty(K/M)$ we may extend it to a function on G by $\tilde{f}(g) = e^{-(i\lambda+\rho)H(g)} f(k(g))$. A direct computation shows that $\tilde{f} \in H_\lambda^\infty$. On the other hand, if $f \in H_\lambda^\infty$, then the restriction $f_{|K}$ of f to K is an element of $C^\infty(K/M)$. Moreover, if $f \in C^\infty(K/M)$ and if $\tilde{f}$ is as above, then $\tilde{f}_{|K} = f$. The mapping $f \mapsto \tilde{f}$ described above is isometric with respect to the $L^2(K/M)$-norm. We may hence identify $C^\infty(K/M) \cong H_\lambda^\infty$. The advantage is that the representation space is independent of λ. The group action on $C^\infty(K/M)$ is realized by

$$(\pi_\lambda(g)f)(kM) = f(k(g^{-1}k)M) e^{-(i\lambda+\rho)H(g^{-1}k)}. \tag{3.4}$$

This is called the *compact picture* of the (spherical) principal series. Notice that for $g \in K$ the group action (3.4) simplifies to the left-regular representation of the compact group K on K/M.

Let $\lambda \in \mathfrak{a}$. It follows from

$$(\pi_\lambda(g)1)(k) = e^{-(i\lambda+\rho)H(g^{-1}k)} = e^{(i\lambda+\rho)\langle gK, kM\rangle} \tag{3.5}$$

that the Poisson transform $P_\lambda(T) : G/K \to \mathbb{C}$ of $T \in \mathcal{D}'(B)$ is given by

$$P_\lambda(T)(gK) = T(\pi_\lambda(g) \cdot 1). \tag{3.6}$$

Let φ denote a Γ-invariant eigenfunction of the Laplace operator with boundary values $T_\varphi \in \mathcal{D}'(B)$ such that $\varphi = P_\lambda(T_\varphi)$. Let $\widetilde{\pi}_\lambda$ denote the dual representation on $\mathcal{D}'(B)$ corresponding to π_λ. Since φ is invariant, it follows from (3.6) and the uniqueness of the boundary values that T_φ is invariant under the actions $\widetilde{\pi}_\lambda(\gamma)$, $\gamma \in \Gamma$.

Regularity of Distribution Boundary Values

In this subsection we prove a regularity statement for distribution boundary values corresponding to Laplace eigenfunctions with eigenvalue parameter $\lambda \in \mathfrak{a}^*$ on a compact quotient X_Γ. These estimates may not be the sharpest possible, but they are sufficient for our purposes.

Let $T_\varphi \in \mathcal{D}'(K/M)$ be the (unique) preimage (under the Poisson transform) of a normalized $L^2(X_\Gamma)$-eigenfunction φ (with exponential growth). Under the identification $H_\lambda^\infty \cong C^\infty(K/M)$ we view T_φ as a functional on H_λ^∞: For $f \in H_\lambda^\infty$ let $T_\varphi(f)$ be defined by $T_\varphi(f_{|K})$. Then T_φ is a continuous linear functional on H_λ^∞, invariant under $\widetilde{\pi}_\lambda(\gamma)$. As proven in [3], Theorem A.1.4, if f is a smooth vector for the principal series representation, then $f \in H_\lambda^\infty$ is a smooth function on G. We consider the mapping

$$\Phi_\varphi : H_\lambda^\infty \to C^\infty(\Gamma\backslash G), \qquad \Phi_\varphi(f)(\Gamma g) = T_\varphi(\pi_\lambda(g)f).$$

Lemma 3.3. *Φ_φ is an isometry w.r.t. the norms of $L^2(K/M)$ and $L^2(\Gamma\backslash G)$.*

Proof. The operator Φ_φ is equivariant with respect to the actions π_λ on H_λ^∞ and the right regular representation of G on $L^2(\Gamma\backslash G)$. We pull-back the $L^2(\Gamma\backslash G)$ inner product onto the $(\mathfrak{g}, K)$-module $H_{\lambda,K}^\infty$ of K-finite and smooth vectors (which is dense in H_λ^∞, [17], p. 81):

$$\langle f_1, f_2\rangle_2 := \langle \Phi_\varphi(f_1), \Phi_\varphi(f_2)\rangle_{L^2(\Gamma\backslash G)}.$$

Let $f_1 \in H_{\lambda,K}^\infty$. Then $A_{f_1} : H_{\lambda,K}^\infty \to \mathbb{C}$, $f_2 \mapsto \langle f_1, f_2\rangle_2$ is a conjugate-linear, K-finite functional on the $(\mathfrak{g}, K)$-module $H_{\lambda,K}^\infty$. This module is irreducible and admissible, since H_λ is unitary and irreducible ([17], theorems 3.4.10 and 3.4.11). As A_{f_1} is K-finite it is nonzero on at most finitely many K-isotypic components. It follows that there is a linear map $A : H_{\lambda,K}^\infty \to H_{\lambda,K}^\infty$ such that for each $f_1 \in H_{\lambda,K}^\infty$ the functional A_{f_1} equals $f_2 \mapsto \langle Af_1, f_2\rangle_{L^2(K/M)}$. The equivariance of Φ_φ and the unitarity of π_λ imply that A is $(\mathfrak{g}, K)$-equivariant. Using Schur's lemma for irreducible $(\mathfrak{g}, K)$-modules ([17], p. 80), we deduce that A is a constant multiple of the identity and hence $\langle\cdot,\cdot\rangle_2$ is a constant multiple of the original $L^2(K/M)$-inner product on $H_{\lambda,K}^\infty$. This constant is 1: First, $\Phi_\varphi(1) = P_\lambda(T_\varphi) = \varphi$ is the K-invariant lift of φ to $L^2(\Gamma\backslash G)$. Then $\|\Phi_\varphi(1)\|_{L^2(\Gamma\backslash G)} = 1 = \|1\|_{L^2(K/M)}$. □

Let (y_j) and (x_j) be bases for $\mathfrak{k}$ and $\mathfrak{p}$, respectively, such that $\langle y_j, y_i\rangle = -\delta_{ij}$, $\langle x_j, x_i\rangle = \delta_{ij}$, where $\langle\, ,\, \rangle$ denotes the Killing form. The Casimir operator of $\mathfrak{k}$ is $\Omega_\mathfrak{k} = \sum_i y_i^2$ and the Casimir operator of $\mathfrak{g}$ is

$$\Omega_\mathfrak{g} = -\sum_j x_j^2 + \Omega_\mathfrak{k} \in \mathcal{Z}(\mathfrak{g}),$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

It follows from $T_\varphi(f) = \Phi_\varphi(f)(\Gamma e)$ that

$$|T_\varphi(f)| \leq \|\Phi_\varphi(f)\|_\infty. \tag{3.7}$$

We may now estimate this by a convenient Sobolev norm on $L^2(\Gamma\backslash G)$. Let $\widetilde{\Delta}$ denote the Laplace operator of $\Gamma\backslash G$. Then we have

$$\widetilde{\Delta} = -\Omega_{\mathfrak{g}} + 2\Omega_{\mathfrak{k}},$$

where $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{k}}$ are the Casimir operators on G and K, respectively.

Definition 3.4. Let $s \in \mathbb{R}$. The Sobolev space $W^{2,s}(\Gamma\backslash G)$ is (cf. [14], p. 22) the space of functions f on $\Gamma\backslash G$ satisfying $(1+\widetilde{\Delta})^{s/2}(f) \in L^2(\Gamma\backslash G)$ with norm

$$\|f\|_{W^{2,s}(\Gamma\backslash G)} = \|(1+\widetilde{\Delta})^{s/2}(f)\|_{L^2(\Gamma\backslash G)}.$$

Let $m = \dim(\Gamma\backslash G) = \dim(G)$, and let $s > m/2$. The Sobolev imbedding theorem for the compact space $\Gamma\backslash G$ ([14], p. 19) states that the identity $W^{2,s}(\Gamma\backslash G) \hookrightarrow C^0(\Gamma\backslash G)$ is a continuous inclusion ($C^0(\Gamma\backslash G)$ is equipped with the usual sup-norm $\|\cdot\|_\infty$). It follows that there exists a $C > 0$ such that

$$\|\Phi_\varphi(f)\|_\infty \leq C\|\Phi_\varphi(f)\|_{W^{2,s}(\Gamma\backslash G)} \qquad \forall f \in C^\infty(K/M). \tag{3.8}$$

Now we derive the announced regularity estimate for the boundary values: First, by increasing the Sobolev order, we may assume $s/2 \in \mathbb{N}$, so

$$(1+\widetilde{\Delta})^{s/2} = (1-\Omega_{\mathfrak{g}}+2\Omega_{\mathfrak{k}})^{s/2} \in \mathcal{U}(\mathfrak{g}).$$

Hence $(1+\widetilde{\Delta})^{s/2}$ commutes with each G-equivariant mapping. Let $f \in H_\lambda^\infty$. Then

$$\begin{aligned} \|\Phi_\varphi(f)\|_{W^{2,s}(\Gamma\backslash G)} &= \left\|(1+\widetilde{\Delta})^{s/2}\Phi_\varphi(f)\right\|_{L^2(\Gamma\backslash G)} \\ &= \left\|\Phi_\varphi((1-\Omega_{\mathfrak{g}}+2\Omega_{\mathfrak{k}})^{s/2}(f))\right\|_{L^2(\Gamma\backslash G)} \\ &= \left\|(1-\Omega_{\mathfrak{g}}+2\Omega_{\mathfrak{k}})^{s/2}(f)\right\|_{L^2(K/M)}. \end{aligned} \tag{3.9}$$

Recall $\pi_\lambda(\Omega_{\mathfrak{k}}) = \Delta_{K/M}$ and $\Omega_{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$. Then (3.9) equals

$$\begin{aligned} &\left\|\sum_{k=0}^{s/2}\binom{s/2}{k}(1+2\Delta_{K/M})^k(-\Omega_{\mathfrak{g}})^{s/2-k}(f)\right\|_{L^2(K/M)} \\ &\leq \sum_{k=0}^{s/2}\binom{s/2}{k}\left\|(1+2\Delta_{K/M})^k(-\Omega_{\mathfrak{g}})^{s/2-k}(f)\right\|_{L^2(K/M)}. \end{aligned} \tag{3.10}$$

Assume $f \in H_{\lambda,K}^\infty$ and recall that $\Omega_{\mathfrak{g}}$ acts on the irreducible $\mathcal{U}(\mathfrak{g})$-module $H_{\lambda,K}^\infty$ by multiplication with the scalar $-(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)$ (cf. [18], p. 163), that is

$$\Omega_{\mathfrak{g}|H_{\lambda,K}^\infty} = -\left(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\right)\mathrm{id}_{H_{\lambda,K}^\infty}.$$

Then (3.10) equals

$$\sum_{k=0}^{s/2} \binom{s/2}{k} \left\| (1+2\Delta_{K/M})^k (|\lambda|^2+|\rho|^2)^{s/2-k}(f) \right\|_{L^2(K/M)}. \tag{3.11}$$

But $\left(|\lambda|^2+|\rho|^2\right)^{-k} \leq 1+|\rho|^{-s} =: C'$ $(0 \leq k \leq s/2)$, so the term in (3.11) is bounded by

$$C' \left(|\lambda|^2+|\rho|^2\right)^{s/2} \sum_{k=0}^{s/2} \binom{s/2}{k} \left\| (1+2\Delta_{K/M})^k (f) \right\|_{L^2(K/M)}. \tag{3.12}$$

Since $H^\infty_{\lambda.K}$ is dense in H^∞_λ, this bound holds for all $f \in H^\infty_\lambda$. Using (3.7)-(3.12) we get

$$|T_\varphi(f)| \leq C' \left(|\lambda|^2+|\rho|^2\right)^{s/2} \sum_{k=0}^{s/2} \binom{s/2}{k} \left\| (1+2\Delta_{K/M})^k (f) \right\|_{L^2(K/M)}. \tag{3.13}$$

for all $f \in H^\infty_\lambda$ and hence for all $f \in C^\infty(K/M)$. We estimate (3.13) by a continuous $C^\infty(K/M)$-seminorm $\|\cdot\|'$ (independent of φ) and obtain:

Proposition 3.5. *Let $2s > \dim(G)$ such that $s/2 \in \mathbb{N}$. There exists a continuous $C^\infty(B)$-seminorm $\|\cdot\|'$, such that*

$$|T_\varphi(f)| \leq (1+|\lambda|)^s \|f\|' \quad \forall\, f \in C^\infty(K/M) \tag{3.14}$$

for the distribution boundary values T_φ corresponding to a real-valued and $L^2(X_\Gamma)$-normalized eigenfunction φ of Δ_Γ with eigenvalue $-(|\lambda|^2+|\rho|^2)$.

Each $f \in C^\infty(B) \otimes C^\infty(B)$ has the form $f = \sum_{i,j} c_{i,j} f_i \otimes f_j$. We define a cross-norm $\|\cdot\|''$ on $C^\infty(B) \otimes C^\infty(B)$ by

$$\|f\|'' = \inf \left\{ \sum_{i,j} |c_{i,j}| \|f_i\|' \|f_j\|' : f = \sum_{i,j} c_{i,j} f_i \otimes f_j \right\}.$$

This norm induces a continuous seminorm on the projective tensor product $C^\infty(B) \widehat{\otimes}_\pi C^\infty(B)$ (cf. [13], p. 435). Let ψ denote another normalized eigenfunction with distribution boundary values $T_\psi \in \mathcal{D}'(B)$ and eigenvalue parameter $\mu \in \mathfrak{a}^*$. Given $f = \sum_{i,j} c_{i,j} f_i \otimes f_j \in C^\infty(B) \otimes C^\infty(B)$ we obtain

$$\begin{aligned} |(T_\varphi \otimes T_\psi)(f)| &\leq \sum_{i,j} |c_{i,j}| \cdot |T_\varphi(f_i)| \cdot |T_\psi(f_j)| \\ &\leq (1+|\lambda|)^s (1+|\mu|)^s \sum_{i,j} |c_{i,j}| \cdot \|f_i\|' \cdot \|f_j\|', \end{aligned} \tag{3.15}$$

which implies (by taking the infimum)

$$|(T_\varphi \otimes T_\psi)(f)| \leq (1+|\lambda|)^s (1+|\mu|)^s \|f\|'' \tag{3.16}$$

for all $f \in C^\infty(B) \otimes C^\infty(B)$. But $C^\infty(B \times B) \cong C^\infty(B) \widehat{\otimes}_\pi C^\infty(B)$ (cf. [13], p. 530) implies that (3.16) holds for all $f \in C^\infty(B \times B)$.

4 Non-Euclidean Pseudodifferential Operators

We use a special G-equivariant ΨDO-calculus that generalizes the *Zelditch quantization* from ([19]). In this section we state some basic definitions and results we need. Full details will appear in [12]. For the moment, we may drop the rank one assumption. Fix a co-compact and torsion free discrete subgroup Γ of G. Using the identification $X \times B = G/M$ we identify functions $a(z,\lambda,b) = a(gK,\lambda,g\cdot M)$ on $X \times \mathfrak{a}^* \times B$ with functions $a(gM,\lambda)$ on $G/M \times \mathfrak{a}^*$. Let $n = \dim G$ and $\{X_1,...,X_n\}$ be a basis for $\mathfrak{g}$ (the elements are acting on functions on G/M as left-invariant differential operators). A ΨDO of order 0 is a properly supported operator $A: C_c^\infty(X) \to C_c^\infty(X)$ defined by

$$Au(z) = \int_{\mathfrak{a}_+^*} \int_B e^{(i\lambda+\rho)\langle z,b\rangle} a(z,\lambda,b) \tilde{u}(\lambda,b)\, db\, đ\lambda, \tag{4.1}$$

where:

(i) $\tilde{u}(\lambda,b) = \int_X u(x) e^{(-i\lambda+\rho)\langle x,b\rangle} dx$ is Helgason's non-euclidean Fourier transform of u ([7], p. 223).

(ii) $đ\lambda = \frac{1}{|W|}|c(\lambda)|^{-2} d\lambda$, where $|W|$ is the order of the Weyl group.

We call $a(z,\lambda,b)$ the *complete symbol* of A, which is equivalently given by

$$\left(Ae_{\lambda,b}\right)(z) = a(z,\lambda,b) e_{\lambda,b}(z), \tag{4.2}$$

where for $\lambda \in \mathfrak{a}^*$ and $b \in B$ the functions $e_{\lambda,b} : X \to \mathbb{C},\ z \mapsto e^{(i\lambda+\rho)\langle z,b\rangle}$ are called *non-Euclidean plane waves.*

Let now X have rank one and denote by $|\cdot|$ the norm on $\mathfrak{a}^*$ induced by the Killing form. We identify $\mathfrak{a} = \mathbb{R} = \mathfrak{a}^*$: Define $\lambda_0 \in \mathfrak{a}_+^*$ by $\lambda_0(X) = \langle X, H_0\rangle$ ($X \in \mathfrak{a}$). We always assume that $a(z,\lambda,b)$ is a classical symbol of order 0, i.e. it has an asymptotic expansion of homogeneous symbols of decreasing order:

$$a(z,\lambda,b) \sim \sum_{j=0}^{\infty} \lambda^{-j} a_{-j}(z,b). \tag{4.3}$$

Asymptotics here means that $a(z,b,\lambda) - \sum_{j=0}^{R} a_j(z,b)\lambda^{-j+m} \in S^{m-R-1}$, where $a \in C^\infty(X \times \mathfrak{a}^* \times B) = C^\infty(G/M \times \mathfrak{a}^*)$ is a *symbol of order* $m \in \mathbb{R}$ ($a \in S^m$) if for all $\beta \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and for each compact subset $C \subset G/M$ it satisfies

$$\|\partial_\lambda^\beta X_1^{\alpha_1} \cdots X_n^{\alpha_n} a(gM,\lambda)\| \leq C_\beta(C)(1+|\lambda|)^{m-\beta}. \tag{4.4}$$

We call $\sigma_A := a_0$ the *principal symbol* of $\mathrm{Op}(a) = A$. Theorems 1.1, 1.2, 1.3 only concern principal symbols, so we often assume that a is independent of λ.

By S_Γ^m we denote symbols of order m which are invariant under the diagonal action of Γ on $X \times B$:

$$a(\gamma\cdot z,\lambda,\gamma\cdot b) = a(z,\lambda,b), \quad \gamma \in \Gamma. \tag{4.5}$$

Let L_Γ^m be the space of operators associated with such symbols. If $(T_g u)(z) = u(g\cdot z)$ denotes the translation of functions on X we find (see [12] for details):

Proposition 4.1. *Let* $a \in S^0$. *Then* $\mathrm{Op}(a) : L^2(X) \to L^2(X)$ *is continuous. Moreover,* $A \in L_\Gamma^m$ *if and only if* A *commutes with each* T_γ, $\gamma \in \Gamma$.

Recall from Section 3 that if φ is an eigenfunction of the Laplace operator with eigenvalue $-(\langle\lambda,\lambda\rangle+\langle\rho,\rho\rangle)$ ($\lambda\in\mathfrak{a}^*$) and boundary values $T\in\mathcal{D}'(B)$, then

$$\varphi(z)=\int_B e^{(i\lambda+\rho)\langle z,b\rangle}T(db). \tag{4.6}$$

Let $\{\varphi_{\lambda_j}\}$ denote the eigenfunctions of Δ_Γ with corresponding boundary values $T_{\lambda_j}\in\mathcal{D}'(B)$. Then $a\in S^0_\Gamma$ induces a bounded operator on $L^2(X_\Gamma)$ by

$$\mathrm{Op}(a)\varphi_{\lambda_j}(z)=\int_B a(z,b)e^{(i\lambda_j+\rho)\langle z,b\rangle}T_{\lambda_j}(db), \tag{4.7}$$

where we used the formula $\mathrm{Op}(a)e^{(i\lambda+\rho)\langle z,b\rangle}=a(z,b)e^{(i\lambda+\rho)\langle z,b\rangle}$ (cf. (4.2)) and pulled the operator under the integral sign in (4.6).

5 Patterson–Sullivan Distributions

In this section we introduce the central concepts we need to formulate our results: Intermediate values, the Radon transform, which really is a time average in our context, and the Patterson–Sullivan distributions.

Intermediate Values

To motivate the concept of intermediate values, consider the case where $G/K=PSU(1,1)/PSO(2)$ is the open unit disk $\mathbb{D}$ with boundary $B=\{z\in\mathbb{C}:|z|=1\}$. Let $\gamma\in G$, $b,b'\in B$. One has the *intermediate value formula* (cf. [10], p. 8)

$$|\gamma(b)-\gamma(b')|^2=|\gamma'(b)|\cdot|\gamma'(b')|\cdot|b-b'|^2. \tag{5.1}$$

It follows from [8], p. 197, that $\frac{d(\gamma\cdot b)}{db}=e^{-2\rho\langle\gamma\cdot o,\gamma\cdot b\rangle}$, where $\rho=\frac{1}{2}$. Then

$$|\gamma(b)-\gamma(b')|^2=e^{-\langle\gamma\cdot o,\gamma\cdot b\rangle}e^{-\langle\gamma\cdot o,\gamma\cdot b'\rangle}\cdot|b-b'|^2. \tag{5.2}$$

To generalize this we construct certain functions $d_\lambda: G/MA\to\mathbb{C}$, which we call *intermediate values*, and which satisfy a certain equivariance property generalizing (5.2) (cf. (5.5)). This property then leads to invariance properties of the Patterson–Sullivan distributions.

Definition 5.1. By *time reversal* we mean the involution $\iota(x,\xi)=(x,-\xi)$ on the unit cosphere bundle S^*X. Under $\Gamma\backslash G/M=S^*X_\Gamma$ the time reversal map takes the form $\Gamma g\mapsto\Gamma gw$. We say that a distribution T is *time-reversible* if $\iota^*T=T$. Recall that each $(b,b')\in B^{(2)}$ is of the form $(g\cdot M, g\cdot wM)\in B^{(2)}$, where $gMA\in G/MA$ is unique. Since $w^2\in M$, time reversal means

$$(b,b')=(g\cdot M,g\cdot wM)\mapsto(gw\cdot M,g\cdot w^2M)=(b',b),$$

which is given by $(b,b')\leftrightarrow(b',b)$.

Definition 5.2. Given $\lambda\in\mathfrak{a}^*$, we define $d_\lambda: G/MA\to\mathbb{C}$ by

$$d_\lambda(gMA):=e^{(i\lambda+\rho)(H(g)+H(gw))}. \tag{5.3}$$

Recall $w^{-1}aw=a^{-1}$ ($a\in A$), which implies that d_λ is well-defined and time reversal invariant. We call the functions d_λ *intermediate values*.

Lemma 5.3. *Let $\gamma, g \in G$. Then*

$$d_\lambda(\gamma g) = e^{(i\lambda+\rho)(\langle \gamma\cdot o, \gamma g\cdot M\rangle + \langle \gamma\cdot o, \gamma g\cdot wM\rangle)} d_\lambda(g). \tag{5.4}$$

Proof. This follows from Lemma 2.9. ☐

Note that by Lemma 2.3 we may interpret d_λ as a function on $B^{(2)}$, that is

$$d_\lambda(b,b') = d_\lambda(g\cdot M, g\cdot wM) = e^{(i\lambda+\rho)(H(g)+H(gw))}$$

for $g = g(b,b')$.

Proposition 5.4. $d_\lambda(g\cdot M, g\cdot wM) = e^{(i\lambda+\rho)(\langle g\cdot o, g\cdot M\rangle + \langle g\cdot o, g\cdot wM\rangle)}$.

Proposition 5.5. *Let $(b,b') \in B^{(2)}$ and $\gamma \in G$. Then*

$$(d_\lambda \circ \gamma)(b,b') = d_\lambda(\gamma\cdot b, \gamma\cdot b') = e^{(i\lambda+\rho)(\langle \gamma\cdot o, \gamma\cdot b\rangle + \langle \gamma\cdot o, \gamma\cdot b'\rangle)} d_\lambda(b,b'). \tag{5.5}$$

Proof. Let $g \in G$ such that $(b,b') = (g\cdot M, g\cdot wM)$. Then $d_\lambda(\gamma\cdot b, \gamma\cdot b') = d_\lambda(\gamma g)$, so the assertion follows from Lemma 5.3. ☐

Invariance Properties

As in the introduction, let $c_0 \le c_1 \le c_2 \le \ldots \to \infty$ denote the spectrum of $-\Delta_\Gamma$ and $\{\varphi_{\lambda_j}\}$ a fixed $L^2(X_\Gamma)$-orthonormal basis of real valued eigenfunctions with eigenvalues $c_j = \langle \lambda_j, \lambda_j\rangle + \langle \rho, \rho\rangle \in \mathbb{R}$. Then $\lambda_j \in \mathfrak{a}^* \cup i\mathfrak{a}^*$ and since $c_j \to \infty$ there are only finitely many $\lambda_j \in i\mathfrak{a}^*$, so we may assume $\lambda_j \in \mathfrak{a}^*$ for all $j \in \mathbb{N}_0$. We only consider eigenfunctions with exponential growth and denote the corresponding sequence of distributional boundary values by $\{T_{\lambda_j}\}$.

Definition 5.6. The *Patterson–Sullivan distribution ps_{λ_j} associated to φ_{λ_j}* is the distribution

$$ps_{\lambda_j}(db, db') := d_{\lambda_j}(b,b')\, T_{\lambda_j}(db) T_{\lambda_j}(db'). \tag{5.6}$$

on $C_c^\infty(B^{(2)})$. The same definition (5.6) extends ps_{λ_j} to a bounded linear functional on the larger space $d_\lambda(b,b')^{-1} \cdot C^\infty(B\times B)$.

Proposition 5.7. *Suppose that φ_{λ_j} is a Γ-invariant eigenfunction of the Laplacian. Let T_{λ_j} denote its boundary values. Then the distribution $ps_{\lambda_j}(db, db')$ is Γ-invariant and time reversal invariant.*

Proof. Time reversibility is obvious. Given a test function f and $\gamma \in \Gamma$ we have

$$ps_{\lambda_j}(f\circ\gamma^{-1}) = (T_{\lambda_j} \otimes T_{\lambda_j})(d_{\lambda_j} \cdot (f\circ\gamma^{-1})) = (\gamma T_{\lambda_j} \otimes \gamma T_{\lambda_j})((d_{\lambda_j}\circ\gamma)\cdot f).$$

It follows from (3.3) that

$$T_{\lambda_j}(d\gamma b) T_{\lambda_j}(d\gamma b') = e^{-(i\lambda_j+\rho)\langle \gamma\cdot o, \gamma\cdot b\rangle} e^{-(i\lambda_j+\rho)\langle \gamma\cdot o, \gamma\cdot b'\rangle} T_{\lambda_j}(db) T_{\lambda_j}(db').$$

Multiplying with (5.5) completes the proof of Γ-invariance. ☐

Recall our notation from 2.5: Let $g(b,b')MA \in G/MA$ denote the coset corresponding to $(b,b') \in B^{(2)}$.

Definition 5.8. The *Radon transform* on $SX = G/M$ is given by

$$\mathcal{R}f(b,b') := \int_A f(g(b,b')aM)da,$$

whenever the integral exists. [8], p. 91, applied to the subgroup MA, yields:

Lemma 5.9. $\mathcal{R} : C_c(SX) \to C_c(B^{(2)})$.

Definition 5.10. Let $\mathcal{F}$ denote a bounded fundamental domain for Γ in X. Following [1], pp. 380-381, we say that $\chi \in C_c^\infty(X)$ is a *smooth fundamental domain cutoff function* if it satisfies

$$\sum_{\gamma\in\Gamma} \chi(\gamma z) = 1 \quad \forall z \in X. \tag{5.7}$$

Such a function can for example be constructed by taking $\nu \in C_c^\infty(X)$, $\nu = 1$ on $\mathcal{F}$, and setting $\chi(z) = \nu(z) \cdot (\sum_{\gamma\in\Gamma} \nu(\gamma z))^{-1}$. If χ satisfies (5.7), then

$$\int_{\mathcal{F}} f\, dz = \int_X \chi f\, dz, \quad f \in C(X_\Gamma). \tag{5.8}$$

The following property of these cutoffs is proven in [1], Lemma 3.5:

Proposition 5.11. *Let $T \in \mathcal{D}'(SX)$ be a Γ-invariant distribution. Let a be a Γ-invariant smooth function on SX. Then for any $a_1, a_2 \in \mathcal{D}(SX)$ such that $\sum_{\gamma\in\Gamma} a_j(\gamma \cdot (z,b)) = a(z,b)$ $(j = 1,2)$ we have $\langle a_1, T\rangle_{SX} = \langle a_2, T\rangle_{SX}$.*

Given T and a as in Proposition 5.11 and if moreover χ_j $(j = 1,2)$ are smooth fundamental domain cutoffs, then $a_j = \chi_j a$ satisfy the assumptions of the proposition. Hence $\langle a, T\rangle_{SX_\Gamma} := \langle \chi a, T\rangle_{SX}$ defines a distribution on the quotient SX_Γ and this definition is independent of the choice of χ.

Definition 5.12. (1) The *Patterson–Sullivan distributions* PS_{λ_j} on SX are defined by

$$\langle a, PS_{\lambda_j}\rangle_{SX} := \int_{(B\times B)\setminus\Delta} (\mathcal{R}a)(b,b')\, ps_{\lambda_j}(db, db').$$

(2) On $SX_\Gamma = \Gamma\backslash SX$ we define the Patterson–Sullivan distributions by

$$\langle a, PS_{\lambda_j}\rangle_{SX_\Gamma} := \langle \chi a, PS_{\lambda_j}\rangle_{SX},$$

where χ is a smooth fundamental domain cutoff.

(3) We define normalized Patterson–Sullivan distributions

$$\widehat{PS}_{\lambda_j} = \frac{1}{\langle 1, PS_{\lambda_j}\rangle_{SX_\Gamma}} PS_{\lambda_j}, \tag{5.9}$$

which satisfy the normalization condition $\langle 1, \widehat{PS}_{\lambda_j}\rangle_{SX_\Gamma} = 1$. Note that $1 = \langle 1, W_{\lambda_j}\rangle_{SX_\Gamma}$.

In view of Proposition 5.11 the definitions made in 5.12 do not depend on χ. Consider the expression

$$PS_{\lambda_j}(a) = \langle a, PS_{\lambda_j}\rangle = \int_{B^{(2)}} d_{\lambda_j}(b,b')\, \mathcal{R}(a)(b,b')\, T_{\lambda_j}(db)\, T_{\lambda_j}(db').$$

$PS_{\lambda_j}(a)$ is defined if $d_{\lambda_j}\mathcal{R}(a) \in C^\infty(B \times B)$, which is the case for $a \in C_c^\infty(SX)$, since then $\mathcal{R}a \in C_c^\infty(B^{(2)})$, which in turn implies $d_{\lambda_j}\mathcal{R}(a) \in C_c^\infty(B^{(2)}) \subset C_c^\infty(B \times B) = C^\infty(B \times B)$.

As an immediate consequence of Proposition 5.11 we obtain:

Proposition 5.13. *Each PS_{λ_j} is a geodesic flow invariant and Γ-invariant distribution on $G/M = SX$. On the quotient SX_Γ, PS_{λ_j} still is invariant under the geodesic flow.*

Proof of Theorem 1.1

Lemma 5.14. $L_{\lambda_j} : C_c^\infty(G) \to C_c^\infty(G)$.

Proof. It is well-known (cf. [8], Ch. IV, §6, Corollary 6.6) that

$$\rho(H(\overline{n})) \geq 0 \quad \forall \overline{n} \in \overline{N}. \tag{5.10}$$

Hence the weight $|e^{-(i\lambda_j+\rho)H(nw)}| \leq C$ is bounded by a constant. The assertion follows from [8], p. 91, applied to the closed subgroup N of G. ☐

The following formula is the key tool in the proof of Theorem 1.1.

Lemma 5.15. *Let $a \in C^\infty(SX)$, $(b, b') \in B^{(2)}$. Then*

$$\int_X \chi a(z,b) e^{(i\lambda_j+\rho)(\langle z,b\rangle + \langle z,b'\rangle)}\, dz = d_{\lambda_j}(b,b')\, \mathcal{R}(L_{\lambda_j}\chi a)(b,b'). \tag{5.11}$$

In view of (6.6), (5.11) is the special case $\lambda_j = \lambda_k$ of the more general formula in Lemma 6.9 and hence we do not give a proof here. Recall that the φ_{λ_j} are real-valued. Let $a \in C^\infty(\Gamma\backslash G/M)$. Then (4.7) yields

$$\langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_j}\rangle = \int_{B^{(2)}} \left(\int_X \chi a(z,b) e^{(i\lambda_j+\rho)(\langle z,b\rangle + \langle z,b'\rangle)}\, dz \right) T_{\lambda_j}(db)\, T_{\lambda_j}(db').$$

It follows from Lemma 5.15 that $d_{\lambda_j}\mathcal{R}(L_{\lambda_j}\chi a)$ has removable singularities in each $(b,b) \in B \times B$. Hence by the same lemma $\langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_j}\rangle$ equals

$$\langle d_{\lambda_j}\mathcal{R}(L_{\lambda_j}\chi a), T_{\lambda_j} \otimes T_{\lambda_j}\rangle = \langle \mathcal{R}(L_{\lambda_j}\chi a), ps_{\lambda_j}\rangle = \langle L_{\lambda_j}(\chi a), PS_{\lambda_j}\rangle,$$

which proves Theorem 1.1.

6 Off-diagonal Patterson–Sullivan Distributions

In this section we generalize the results of Section 5 to the off-diagonal case and thus prove Theorem 1.2.

Off-diagonal Intermediate Values

The construction of PS_{λ_j,λ_k} is different from the construction of PS_{λ_j}. We will see in this section why it is impossible to define functionals ps_{λ_j,λ_k} ($\lambda_j \neq \lambda_k$).

Definition 6.1. Given $\lambda, \mu \in \mathfrak{a}$, define $d_{\lambda,\mu} : G/M \to \mathbb{C}$ by

$$d_{\lambda,\mu}(g) = e^{(i\lambda+\rho)H(g)} e^{(i\mu+\rho)H(gw)}. \tag{6.1}$$

This is well-defined, since the Iwasawa projection is M-invariant and M' normalizes M. What we really need is a geodesic flow invariant function on G/M, that is $d_{\lambda,\mu}$ *should* be invariant under the right action of A. In other words, we would wish to have $d_{\lambda,\mu}$ well-defined on G/MA. But for $g \in G$, $m \in M$ and $a \in A$ a direct computation shows

$$d_{\lambda,\mu}(gam) = d_{\lambda,\mu}(g) e^{i(\lambda-\mu)\log(a)}. \tag{6.2}$$

It follows that $d_{\lambda,\mu}$ is *not* a function on G/MA. This implies that for $\lambda \neq \mu$ we *cannot* define a more general function $d_{\lambda,\mu}(b,b')$ in analogy with (5.1). We will see in (6.4) how to circumvent this problem. Exactly as in Lemma 5.3 we have:

Lemma 6.2. *Let $\gamma, g \in G$. Then*

$$d_{\lambda,\mu}(\gamma g) = e^{(i\lambda+\rho)\langle \gamma\cdot o, \gamma g\cdot M\rangle} e^{(i\mu+\rho)\langle \gamma\cdot o, \gamma g\cdot wM\rangle} d_{\lambda,\mu}(g). \tag{6.3}$$

Invariance Properties

Let f be a function on G/M and let $\lambda, \mu \in \mathfrak{a}^*$. The *weighted Radon transform* on G is defined by

$$(\mathcal{R}_{\lambda,\mu} f)(g) := \int_A d_{\lambda,\mu}(ga) f(ga)\, da, \tag{6.4}$$

whenever the integral exists. As in Lemma 5.9 we deduce:

Remark 6.3. Let $f \in C_c^\infty(G/M)$. Then $\mathcal{R}_{\lambda,\mu}(f) \in C_c^\infty(G/MA)$ is invariant under the geodesic flow of $G/M = SX$. Hence $\mathcal{R}_{\lambda,\mu}(f)$ is defined on G/MA (see (6.2) and its subsequent remark).

Definition 6.4. As before, let $g(b,b') \in G$ be a representative for the element in G/MA that corresponds to $(b,b') \in B^{(2)}$. Given $f \in C_c^\infty(G/M)$, we define

$$\mathcal{R}_{\lambda,\mu}(f)(b,b') := \mathcal{R}_{\lambda,\mu}(f)(g(b,b')). \tag{6.5}$$

Then $\mathcal{R}_{\lambda,\mu} f \in C_c^\infty(B^{(2)})$. This definition is independent of the choice of representative $g(b,b')$, since $\mathcal{R}_{\lambda,\mu}(f)$ is invariant.

Let $f \in C_c^\infty(G/M)$. The values $|d_{\lambda,\mu}(g)|$ are independent of λ, μ and all derivatives of $d_{\lambda,\mu}$ have polynomial growth in λ, μ. It follows that given a continuous seminorm $\|\cdot\|_1$ on $C^\infty(B \times B)$ there exist $K_1 > 0$ and a continuous seminorm $\|\cdot\|_2$ on $C_c^\infty(G/M)$ such that

$$\|\mathcal{R}_{\lambda,\mu}(f)\|_1 \leq (1+|\lambda|)^{K_1}(1+|\mu|)^{K_1}\|f\|_2. \tag{6.6}$$

Definition 6.5. The *off-diagonal Patterson–Sullivan distribution associated to φ_{λ_j} and φ_{λ_k}* is the distribution on $SX = G/M$ defined by

$$PS_{\lambda_j,\lambda_k}(f) := \langle f, PS_{\lambda_j,\lambda_k}\rangle := \int_{B^{(2)}} (\mathcal{R}_{\lambda_j,\lambda_k} f)(b,b')\, T_{\lambda_j}(db)\, T_{\lambda_k}(db'). \tag{6.7}$$

Assume $\mathcal{R}_{\lambda_j,\lambda_k}(f) \in C^\infty(B \times B)$. Then $PS_{\lambda_j,\lambda_k}(f)$ is well-defined. A simple example is when $f \in C_c^\infty(SX) = C_c^\infty(G/M)$. In this case, it follows from (3.16), (6.6) and (6.7) that there exist $K > 0$ and a continuous seminorm $\|\cdot\|_2$ on $C_c^\infty(G/M)$ such that

$$|PS_{\lambda_j,\lambda_k}(f)| \leq (1+|\lambda_j|)^K(1+|\lambda_k|)^K\|f\|_2. \tag{6.8}$$

Remark 6.6. Let $(b, b') \in B^{(2)}$ and $g = g(b, b')$. Then

$$\mathcal{R}_{\lambda_j,\lambda_j}(f)(g) = \int_A d_{\lambda_j,\lambda_j}(ga) f(ga)\, da = d_{\lambda_j}(g(b,b'))(\mathcal{R}f)(b,b'), \tag{6.9}$$

which implies $PS_{\lambda_j,\lambda_j} = PS_{\lambda_j}$.

Proposition 6.7. *Suppose that* φ_{λ_j} *and* φ_{λ_k} *are* Γ*-invariant eigenfunctions. Then the distribution* PS_{λ_j,λ_k} *on* $SX = G/M$ *is* Γ*-invariant.*

Proof. Let $f \in C_c^\infty(G/M)$ and let f_γ denote the translation $f \circ \gamma^{-1}$. Then

$$\langle f_\gamma, PS_{\lambda_j,\lambda_k}\rangle = \int_{B^{(2)}} \int_A d_{\lambda_j,\lambda_k}(g(b,b')a)\, f(\gamma^{-1} g(b,b')a)\, da\, T_{\lambda_j}(db)\, T_{\lambda_k}(db'),$$

where $(b, b') = (g \cdot M, g \cdot wM)$ for $g = g(b, b')$. By (3.3) this equals

$$\int_{B^{(2)}} \int_A d_{\lambda_j,\lambda_k}(g(\gamma \cdot (b,b'))a)\, f(\gamma^{-1} g(\gamma(b,b'))a) e^{-(i\lambda_j+\rho)\langle \gamma\cdot o, \gamma\cdot b\rangle} \\ \times\ e^{-(i\lambda_k+\rho)\langle \gamma\cdot o, \gamma\cdot b'\rangle} da\, T_{\lambda_j}(db)\, T_{\lambda_k}(db'). \tag{6.10}$$

Recall that $a \in A$ acts trivially on (M, wM). Using this and (6.3) we observe

$$d_{\lambda_j,\lambda_k}(\gamma g a) = e^{(i\lambda_j+\rho)\langle \gamma\cdot o, \gamma\cdot b\rangle} e^{(i\lambda_k+\rho)\langle \gamma\cdot o, \gamma\cdot b'\rangle} d_{\lambda_j,\lambda_k}(ga).$$

We also have $g(\gamma \cdot (b, b')) = \gamma g(b, b'))$, since $(b, b') \mapsto g(b, b') \in G/MA$ is G-equivariant. Hence $\gamma^{-1} g(\gamma \cdot (b, b')) = g(b, b')$. Thus we have

$$\begin{aligned} \langle f_\gamma, PS_{\lambda_j,\lambda_k}\rangle &= \int_{B^{(2)}} \int_A d_{\lambda_j,\lambda_k}(g(b,b')a) f(g(b,b')a)\, da\, T_{\lambda_j}(db) T_{\lambda_k}(db') \\ &= \int_{B^{(2)}} \mathcal{R}_{\lambda_j,\lambda_k} f(b,b')\, T_{\lambda_j}(db) T_{\lambda_k}(db') = \langle f, PS_{\lambda_j,\lambda_k}\rangle, \end{aligned}$$

and the proposition follows. □

In view of Proposition 6.7, the definition of PS_{λ_j,λ_k} descends to $SX_\Gamma = \Gamma\backslash SX$:

Definition 6.8. (1) The off-diagonal Patterson–Sullivan distributions on SX_Γ are defined by (χ is a smooth fundamental domain cutoff)

$$\langle a, PS_{\lambda_j,\lambda_k}\rangle_{SX_\Gamma} := \langle \chi a, PS_{\lambda_j,\lambda_k}\rangle_{SX}. \tag{6.11}$$

(2) We normalize these distributions by

$$\widehat{PS}_{\lambda_j,\lambda_k} = \frac{1}{\langle 1, PS_{\lambda_k,\lambda_k}\rangle_{SX_\Gamma}} PS_{\lambda_j,\lambda_k}. \tag{6.12}$$

Proof of Theorem 1.2

The following lemma is the off-diagonal analog of Lemma 5.15.

Lemma 6.9. *Let* $a \in C^\infty(SX_\Gamma)$*,* $(b, b') \in B^{(2)}$*. Then*

$$\int_X \chi a(z,b) e^{(i\lambda_j+\rho)\langle z,b\rangle} e^{(i\lambda_k+\rho)\langle z,b'\rangle} dz = \mathcal{R}_{\lambda_j,\lambda_k}(L_{\lambda_k}(\chi a))(b,b'). \tag{6.13}$$

Proof. Select $g \in G$ such that $(b, b') = (g \cdot M, g \cdot wM)$. The following manipulations do not depend on the choice of g. By G-invariance of dz, the left hand side of (6.13) equals

$$\int_X \chi a(g \cdot z, b) e^{(i\lambda_j+\rho)\langle g\cdot z, b\rangle} e^{(i\lambda_k+\rho)\langle g\cdot z, b'\rangle} dz. \tag{6.14}$$

Identify χa with a function on G/M: Then since $b = g \cdot o$ we have

$$\chi a(gan \cdot o, b) = \chi a(gan \cdot o, g \cdot M) = \chi a(gan \cdot o, gan \cdot M) = \chi a(ganM)$$

(recall that $P = MAN$ fixes $M \in K/M$, in particular $an \in AN$ fixes $M = b_\infty$). From the integral formula (2.9) we obtain that (6.14) equals

$$\int_{AN} \chi a(ganM) e^{(i\lambda_j+\rho)\langle gan\cdot o, g\cdot M\rangle} e^{(i\lambda_k+\rho)\langle gan\cdot o, g\cdot wM\rangle} \, dn \, da. \tag{6.15}$$

But $\langle gan \cdot o, g \cdot M\rangle = \langle gan \cdot o, gan \cdot M\rangle = H(gan) = H(ga)$ and $H(n^{-1}w) = H(nw)$ (which is equivalent to $H(\overline{n}) = H(\overline{n}^{-1})$ and thus follows from [8], p. 436 (8)). Then (2.13) and (2.14) yield

$$\langle gan \cdot o, g \cdot wM\rangle = -H(n^{-1}a^{-1}w) + H(gw),$$

which by (2.2) equals $-H(nw) + H(gaw)$. Hence (6.15) becomes

$$\begin{aligned} & \int_{AN} \chi a(ganM) e^{(i\lambda_j+\rho)H(ga)} e^{(i\lambda_k+\rho)H(gaw)} e^{-(i\lambda_k+\rho)H(nw)} dn \, da \\ = & \int_A d_{\lambda_j,\lambda_k}(gaM) \int_N \chi a(ganM) e^{-(i\lambda_k+\rho)H(nw)} dn \, da \\ = & \int_A d_{\lambda_j,\lambda_k}(gaM)(L_{\lambda_k}(\chi a))(gaM) da = \mathcal{R}_{\lambda_j,\lambda_k}(L_{\lambda_k}(\chi a))(b, b'). \end{aligned}$$

The independence of the representative $g(b, b')$ follows from the unimodularity of A and because the mapping $N \ni n \mapsto \tilde{m}^{-1} n \tilde{m} \in N$ ($\tilde{m} \in M$) preserves the measure dn (since M is compact). ☐

As in Section 5 we may now integrate (6.13) against $T_{\lambda_j}(db) T_{\lambda_k}(db')$, which completes the proof of Theorem 1.2.

7 Proof of Theorem 1.3

Given a *phase function* $\psi : \mathbb{R}^n \to \mathbb{C}$ such that $\mathrm{Im}(\psi) \geq 0$ and an *amplitude* $\alpha \in C_c^\infty(\mathbb{R}^n)$ and $\tau > 0$, consider the integral

$$I(\tau) := \int e^{i\tau\psi(x)} \alpha(x) dx.$$

It is well known ([15], p. 195) that if $\psi' \neq 0$ on the support of a, then $I(\tau) = O(\tau^{-\infty})$ as $\tau \to \infty$. Assume that $0 \in \mathbb{R}^n$ is the only critical point of ψ and let $H = \psi''(0)$ be nonsingular at 0. Also assume $\psi(0) = 0$. Then

$$\psi(x) = \langle Hx, x\rangle/2 + O(|x|^3) \text{ as } x \to 0$$

and one proves (loc. cit., p. 171) the asymptotic expansion

$$\int e^{i\tau\psi(x)}\alpha(x)dx \sim C(2\pi/\tau)^{n/2}\sum_{k=0}^{\infty}\tau^{-k}R_k a(0) \quad (\tau\to\infty), \tag{7.1}$$

where $R_k = (\langle H^{-1}D, D\rangle/2i)^k$ is a differential operator on $\mathbb{R}^n$ of order $2k$ with $D = (D_1,\ldots,D_n)$, where $D_j = -i\partial_j$ and $C = |\det H|^{-1/2}e^{\pi i\,\mathrm{sign}(H)/4}$ is a constant depending on ψ. We refer to (7.1) as the *MSP-formula* (method of stationary phase).

We now assume $\lambda_j \in \mathfrak{a}_+^*$ for all $j\in\mathbb{N}_0$, identify $\mathfrak{a}_+^* = \mathbb{R}^+$, and write $\mathfrak{a} = \mathbb{R}\cdot H_0$. If $\overline{n} := wnw^{-1}\in\overline{N}$ for $n\in N$, then

$$L_\lambda(\chi a)(g) = \int_{\overline{N}} e^{-(i\lambda)\langle H(\overline{n}),H_0\rangle}e^{-\rho\langle H(\overline{n}),H_0\rangle}\chi a(gw\overline{n}w^{-1})d\overline{n}, \quad \lambda,\rho\in\mathbb{R}.$$

Proposition 7.1. *The phase function $\psi(\overline{n}) = \langle H(\overline{n}), H_0\rangle$ has exactly one critical point, namely $\overline{n} = e$. The Hessian form at $\overline{n} = e$ is non-degenerate.*

Proof. [4], pp. 343. □

Clearly $\psi(e) = 0$ and for the amplitude $\alpha(\overline{n}) = e^{-\rho\langle H(\overline{n}),H_0\rangle}\chi a(gw\overline{n}w^{-1})$ we have $\alpha(e) = \chi a(g)$. Let $s = \dim(N) = \dim(\overline{N})$. The MSP-formula yields

$$L_\lambda(\chi a)(g) = C\cdot(2\pi/\lambda)^{s/2}\sum_n \lambda^{-n}R_{2n}(\chi a)(g), \tag{7.2}$$

where R_{2n} is a differential operator on SX of order $2n$ and R_0 is the identity. Although we consider off-diagonal elements, the proof in [1] applies with almost no change: Let K be defined as in (6.8). Theorem 1.2 implies

$$\begin{aligned}\langle \mathrm{Op}(a)\varphi_{\lambda_j},\varphi_{\lambda_k}\rangle_{SX_\Gamma} &= \langle L_{\lambda_k}(\chi a), PS_{\lambda_j,\lambda_k}\rangle_{SX}\\ &= C\,(2\pi/\lambda_k)^{s/2}\sum_{n=0}^{N}\lambda_k^{-n}\langle R_{2n}(\chi a), PS_{\lambda_j,\lambda_k}\rangle + O(\lambda_k^{-N-1+2K}).\end{aligned}$$

We choose $N > 2K$. Since R_0 is the identity, the operator $L_\lambda^{(N)} = \sum_n^N \lambda^{-n}R_{2n}$ can be inverted up to $O(\lambda^{-N-1})$, i.e. one finds differential operators $M_\lambda^{(N)} = \sum_{n=0}^N \lambda^{-n}M_{2n}$, where $M_0 = \mathrm{id}$, and $R_\lambda^{(N)}$ such that

$$L_\lambda^{(N)}M_\lambda^{(N)} = \mathrm{id} + \lambda^{-N-1}R_\lambda^{(N)}.$$

An application of Theorem 1.2 to $M_{\lambda_k}^{(N)}a$ yields

$$\begin{aligned}\langle \mathrm{Op}(M_{\lambda_k}^{(N)}a)\varphi_{\lambda_j},\varphi_{\lambda_k}\rangle_{SX_\Gamma} &= \langle L_{\lambda_k}^{(N)}\chi M_{\lambda_k}^{(N)}a, PS_{\lambda_j,\lambda_k}\rangle_{SX} + O(\lambda_k^{-N-1+2K})\\ &= \langle L_{\lambda_k}^{(N)}M_{\lambda_k}^{(N)}\chi a, PS_{\lambda_j,\lambda_k}\rangle_{SX} + O(\lambda_k^{-N-1+2K})\\ &= \langle a, PS_{\lambda_j,\lambda_k}\rangle_{SX_\Gamma} + O(\lambda_k^{-N-1+2K}).\end{aligned}$$

The second line is a consequence of Proposition 5.11. But

$$M_\lambda^{(N)}(a) = a + \lambda^{-1}\left(M_2 + \ldots + \lambda^{-N+1}M_{2N}\right)(a), \tag{7.3}$$

so the L^2-continuity of zero order pseudodifferential operators implies

$$\langle \mathrm{Op}(M_{\lambda_k}^{(N)}(a))\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_\Gamma)} = \langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{L^2(X_\Gamma)} + O(1/\lambda_k), \qquad (7.4)$$

which proves

$$C \cdot (2\pi/\lambda_k)^{s/2} \langle a, PS_{\lambda_j,\lambda_k}\rangle_{SX_\Gamma} = \langle \mathrm{Op}(a)\varphi_{\lambda_j}, \varphi_{\lambda_k}\rangle_{SX_\Gamma} + O(1/\lambda_k). \qquad (7.5)$$

We put $\langle a, PS_{\lambda_j,\lambda_k}\rangle = \langle 1, PS_{\lambda_k,\lambda_k}\rangle\langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle$ into (7.5) and obtain

$$C \cdot (2\pi/\lambda_k)^{s/2} \cdot \langle 1, PS_{\lambda_k,\lambda_k}\rangle \cdot \langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle = \langle a, W_{\lambda_j,\lambda_k}\rangle + O(1/\lambda_k). \qquad (7.6)$$

In particular, for $a = 1$, we get

$$C \cdot (2\pi/\lambda_k)^{s/2} \cdot \langle 1, PS_{\lambda_k,\lambda_k}\rangle_{SX_\Gamma} = 1 + O(1/\lambda_k). \qquad (7.7)$$

Together with (7.6) this yields

$$(1 + O(1/\lambda_k)) \cdot \langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle = \langle a, W_{\lambda_j,\lambda_k}\rangle + O(1/\lambda_k). \qquad (7.8)$$

The Wigner distributions and hence by (7.8) the $\langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle$ are uniformly bounded. It follows that the left side of (7.8) is asymptotically the same as $\langle a, \widehat{PS}_{\lambda_j,\lambda_k}\rangle$. This completes the proof of Theorem 1.3.

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