ON THE COMPLEMENT OF THE DENSE ORBIT FOR A QUIVER OF TYPE $\mathbb{A}$

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ABSTRACT. Let $\mathbb{A}_t$ be the directed quiver of type $\mathbb{A}$ with t vertices. For each dimension vector d there is a dense orbit in the corresponding representation space. The principal aim of this note is to use just rank conditions to define the irreducible components in the complement of the dense orbit. Then we compare this result with already existing ones by Knight and Zelevinsky, and by Ringel. Moreover, we compare with the fan associated to the quiver $\mathbb{A}$ and derive a new formula for the number of orbits using nilpotent classes. In the complement of the dense orbit we determine the irreducible components and their codimension. Finally, we consider several particular examples.

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1. Introduction

The principal aim of this note is to describe the complement of the generic orbit in the representation space of a directed quiver of type $\mathbb{A}_t$ with vertices $\{1, 2, \ldots, t\}$ and arrows $\alpha_i : i+1 \longrightarrow i$. For a dimension vector $d = (d_1, \ldots, d_t)$ and a representation $A = (A_{i,i+1}) = (A_{1,2}, A_{2,3}, \ldots, A_{t-1,t})$ with $A_{i,i+1} : V_{i+1} \longrightarrow V_i$ we define

$$r_{i,j} := \min\{d_l \mid i \leq l \leq j\} \text{ and } A_{(i,j)} := A_{i,i+1}A_{i+1,i+2}\ldots A_{j-1,j}.$$

Let Y be defined as the complement in

$$\mathcal{R}(Q,d) = \bigoplus_{i=1}^{t-1} \mathrm{Hom}(k^{d_i}, k^{d_{i+1}}) \text{ with action of } \mathrm{Gl}(d) = \prod_{i=1}^{t} \mathrm{Gl}(d_i)$$

of the generic orbit

$$\mathcal{O}(d) := \{A = (A_{1,2}, \ldots, A_{i,i+1}, \ldots, A_{t-1,t}) \in \mathcal{R}(Q,d) \mid \mathrm{rk}\, A_{(i,j)} = r_{i,j}\}.$$

In the complement Y we define closed (not neccesarily irreducible) varieties

$$Y_{i,j} := \{(A_{i,i+1})_{i=1}^{t-1} \in \mathcal{R}(Q,d) \mid \mathrm{rk}\, A_{(i,j)} \leq r_{i,j} - 1\}.$$

We claim in our main result that all irreducible components of Y are among the $Y_{i,j}$ and at most $t-1$ of the $Y_{i,j}$ occure as irreducible components in Y.

For the formulation of the main result we need to define a set of pairs

$$J(d) := \{(i,j) \mid 1 \leq i < j \leq t \text{ and } \text{ for all } i < l < j, d_l > \max\{d_i, d_j\}\}.$$

and a subset $I(d)$ consisting of alle elements in $J(d)$ satisfying one of the following properties (where we define $d_0 = d_{t+1} = 0$)
(i) $d_i = d_j$;
(ii) $d_i < d_j$ and we define a to be the minimal index $a > j$ with $d_a < d_i$. Then $d_l \geq d_j$ for all $j < l < a$.
(iii) $d_i > d_j$ and we define b to be the maximal index $b < i$ with $d_b < d_j$. Then $d_l \geq d_i$ for all $b < l < i$.

If $(i,j) \in J(d)$ then we show that $Y_{i,j}$ is irreducible (Theorem 1.1) and there exists a unique representation $M(i,j)$ whose orbit is dense in $Y_{i,j}$. For any irreducible component of Y there exists a representation M whose orbit is dense in this component. Such a representation M is called *almost generic*.

Then we prove the following theorem.

Theorem 1.1. *Assume $d_i > 0$ for each $1 = 1, \ldots, t$.*
(i)

$$Y = \cup_{(i,j) \in I(d)} Y_{i,j}$$

is the decomposition of Y into pairwise different irreducible components.
(ii) Each component is the closure of an orbit corresponding to an almost generic representation $M(i,j)$.
(iii) For any $(i,j) \in J(d)$ the codimension of $Y_{i,j}$ is $|d_j - d_i| + 1$.
(iv) The irreducible components $Y_{i,j}$ in Y of codimension 1 are in bijection with the pairs (i,j) with $d_i = d_j$ and $d_l > d_i$ for all $i < l < j$.

In fact we prove the following stronger results. First of all $Y_{i,j}$ is irreducible precisely when (i,j) is in $J(d)$ (Prop 3.1, Cor. 3.4). Then Y obviously decomposes into the union of all possible $Y_{i,j}$ (Lemma 3.3). Next we show that any $Y_{k,l}$ for $(k,l) \notin I(d)$ is already contained in a union of some other $Y_{(i,j)}$ (Prop 3.2). Moreover, we interprete our result in terms of multisegments and nilpotent classes in Section 4.

Note that the techniques are similar to the ones in [BH], our case corresponds to $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)'$ therein, however, the index sets are different, no case follows from the other. With some technical modifications on the index sets one can also handle the case $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$ for the remaining values of l in a similar way (Section 6).

The paper is organized as follows. In Section 2 we only collect the details we need for the proof of the main result in Section 3. Then we proceed in Section 4 with some further descriptions related to tilting modules and trees, the structure of the fan associated to tilting modules and other combinatorial descriptions. The associated simplicial complex of the fan coincides with the simplicial complex considered by Riedtman and Schofield ([RS]). Then, in Section 5 we consider several examples that are of interest: convex and concave dimension vectors, pure and generic dimension vectors, and symmetric ones. In the last section we compare with the results in [BH] and mention some generalizations without proofs.

We always work over an infinite field k, the results here do not depend on the ground field. For finite fields, one needs to modify the definition of a dense orbit slightly: an orbit is dense, if it is dense over the algebraic closure. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we denote by $C(\lambda)$ the corresponding nilpotent class defined by

$$C(\lambda) = \{A \in \mathrm{End}(V) \mid \dim A^l(V) = \max\{j \mid \lambda_j \geq l\}\}.$$

All varieties are considered over the algebraic closure and might be reducible. Also the action of the group should be understood over the algebraic closure. We will always identify isomorphism classes of representations of $\mathbb{A}_t$ (with directed orientation) with so-called multisegments defined below. With $\sharp[i,j]$ we denote the number $j - i + 1$ of integers in the interval $[i,j]$.

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2. Description of the Orbits

In this section we recall some of the various descriptions of the isomorphism classes of representations of $\mathbb{A}_t$ with the directed orientation that we need in the proof. Moreover, we recall some well-known facts from the classification of tilting modules and compute the extension groups. We proceed with these descriptions in Section 4. Further related results can be found in [KZ] and in the classical papers [AF] and [AFK].

2.1. **Multisegements.** A multisegment M consists of a union of intervalls $[i,j]$ with $1 \leq i \leq j \leq t$, written as $M = \oplus [i,j]^{a_{i,j}}$ (since a multisegments represents an isomorphism class of representations we write 'direct sum' instead of 'union'). The dimension vector of such a multisegemt is defined as

$$\underline{\dim} M = (d(M)_1, \ldots, d(M)_t); \quad d(M)_l := \sum_{(i,j) \mid i \leq l \leq j} a_{i,j}.$$

There are natural bijections between the multisegments of dimensions vector d, the isomorphism classes of representations of $\mathbb{A}_t$, and the orbits of the $\mathrm{Gl}(d)$–action on $\mathcal{R}(Q,d)$. Moreover, for any dimension vector d there exists a unique multisegment $M(d)$ corresponding to the dense orbit. This multisegment can be constructed recursively as follows: Define $a_{1,t}$ to be the minimum of the entries d_i in d. Then we consider $d^1 := d - a_{1,t}(1, \ldots, 1)$ and consider the longest interval $[i,j]$ in d^1 with minimal i. Then

$$d^2 := d^1 - a_{i,j}\underline{\dim}[i,j] = d^1 - a_{i,j}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots 0)$$

is nonnegative for some maximal $a_{i,j}$ and we proceed with d^2 instead of d^1 in the same way. Eventually, we obtain a multisegment $M(d)$ with at most t different direct summands. A second way to obtain this multisegment is described in in [H1], Section 8 (this is a similar, but not the same, construction as in [BHRR]) as follows: consider the unique diagram with d_i vertices in the ith column and connect each vertex in the ith column and the kth row with the vertex in the $(i+1)$th column and the kth row (if it exists). Roughly one connects all neighboured vertices in the same row. The connected components of this diagram are the direct summands and this diagram represents the multisegment $M(d)$ (see Section 5 for examples).

Definition. A dimension vector d is *generic* if $M(d)$ contains precisely t pairwise different segments. A dimension vector is *pure* if $d_1 = d_t$, $d_l \geq d_1 = d_t$ for each $1 \leq l \leq t$ and this condition holds recursively for each connected component in the support of $d(\geq a) := (\max\{d_1 - a, 0\}, \ldots, \max\{d_t - a, 0\})$. Examples can be found in Section 5, see 5.1 and 5.4.

2.2. **Extensions and homomorphisms.** The category of finite dimensional representations of $\mathbb{A}_t$ is a hereditary category and the Euler characteristic $\langle -, - \rangle = \dim \mathrm{Hom}(-,-) - \dim \mathrm{Ext}(-,-)$, respectively the Hom- and Ext-spaces are

$$\langle [i,j],[k,l] \rangle \text{ is just } \sharp([i,j] \cap [k,l]) - \sharp([i+1,j+1] \cap [k,l])$$

$$\mathrm{Hom}([i,j],[k,l]) = \begin{cases} k & \text{if } k \leq i \leq l \leq j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}([i,j],[k,l]) = \begin{cases} k & \text{if } i < k \leq j+1 < l+1 \\ 0 & \text{otherwise} \end{cases}$$

All this follows from direct calculations using a projective or an injective resolution

$$0 \longrightarrow [j+1,t] \longrightarrow [i,t] \longrightarrow [i,j] \qquad \text{or} \qquad [i,j] \longrightarrow [1,j] \longrightarrow [1,i+1] \longrightarrow 0.$$

Proposition 2.1. *a) A multisegment M has no selfextension precisely when for each pair of direct summands $[i,j]$ and $[k,l]$ of M one of the following conditions hold*
(i) $[i,j] \subseteq [k,l]$, (ii) $j < k-1$, (iii) $[k,l] \subseteq [i,j]$, or (iv) $l < i-1$.
b) The multisegment $M(d)$ has no selfextension and any other multisegment M of dimension vector d satisfies $\mathrm{Ext}(M,M) \neq 0$.
c) A multisegement M satisfies $\mathrm{Ext}(M,M) = k$ precisely when it contains two segments $[i,j]$ and $[k,l]$ with $j \geq k-1$, $i < k$, and $j < l$ as a direct summand and the complement of $[i,j]$ equals $M(d')$, where $d' = \underline{\dim} M - \underline{\dim}[i,j]$ and the complement of $[k,l]$ equals $M(d'')$, where $d'' = \underline{\dim} M - \underline{\dim}[k,l]$.
d) A multisegmet $M = \oplus [i,j]^{a_{i,j}}$ is almost generic precisely when the direct sum of the pairwise non-isomorphic direct summands $N = \oplus_{(i,j) | a_{i,j} > 0} [i,j]$ satisfies $\mathrm{Ext}^1(N,N) = k$ and one of the direct summands with non-trivial Ext–group occurs with multiplicity one in M.

Proof. a) and the first claim of b) is a direct consequence from the formula for the extension groups above. The uniqueness in b) follows either directly from the construction, or since $\mathcal{R}(Q,d)$ is irreducible (it can contain at most one dense orbit). To prove c) one uses that Ext^1 is additive, thus there is at most one non–vanishing extension group. Finally, to prove d) we note that for $d = \underline{\dim} N$ we have $\dim \mathrm{End}(N,N) = \dim \mathrm{End}(M(d), M(d)) + 1$ by a simple computation of the Euler characteristic $\langle M(d), M(d) \rangle = \langle N, N \rangle$. Thus, the stabilizer of the orbit of $M(d)$ and N differ by one and then the dimension of the orbits also differ by one. The closure of the orbit of $M(d)$ obviously contains the orbit of N. Now assume M is a multisegmet as in the claim and it is neither generic nor almost generic. Take two direct summands $[a,b]$ and $[c,d]$ of M with $\mathrm{Ext}([a,b],[c,d]) \neq 0$. We define a new multisegment M' of the same dimension vector by deleting $[a,b]$ and $[c,d]$ and replacing it by $[c,b] \oplus [a,d]$ (if $c = b+1$ we replace it just by $[a,d]$). Then M is in the closure of the orbit of M' and M is almost generic precisely when M' is already generic. This in turn is equivalent to the second condition in d), proving one direction of the claim.

Now assume N satisfies $\mathrm{Ext}(N,N) = k$. Then the closure of the orbit of N is of codimension one in the space of all representations of dimensions vector $\underline{\dim} N$. Thus it is some irreducible component in the complement of the dense orbit. Now we add the remaining segments to N so that we obtain $M = M' \oplus N$. The multisegment M' is then also a direct summand of $M(d)$, since on can get $M(d)$ from $N \oplus M'$ by extending only two segments in N.

Assume M contains two indecomposable direct summands $[a,b]$ and $[c,d]$, both occuring with multiplicity at least two and $\mathrm{Ext}([a,b],[c,d]) \neq 0$, then we can again (using extensions) construct an orbit that is not generic and contains M in ist closure. Consequently, such an M is not almost generic. $\square$

The proof also follows directly from Zwara's result [Z] that the partial order of the Ext-degeneration and the partial order for the geometric degeneration coincide. In the proof above we only used the trivial direction.

2.3. **Rank conditions.** To any representation A of Q one can associate the ranks of the compositions of the corresponding matrices. Consider $A = (A_{i,i+1}) \in \mathcal{R}(Q,d)$. Then we define the *rank triangle*

$$r(A) = (r_{i,j}(A))_{1 \le i < j \le t}, \text{ with } r_{i,j}(A) = \mathrm{rk}\, A_{i,i+1} \cdot \ldots \cdot A_{j-1,j} = \mathrm{rk}\, A_{(i,j)}.$$

Moreover, it is convenient to define the *extended rank triangle* with $r_{i,i} := d_i$ and to define $r_{i,j} = 0$ whenever $i \le 0$ or $j > t$. Obviously, we must have $r_{i,j}(A) \le r_{i,j} := \min \{d_l \mid i \le l \le j\}$ and (using generic matrices) the set

$$\mathcal{O}(d) := \{A \in \mathcal{R}(Q,d) \mid r_{i,j}(A) = r_{i,j}\}$$

is open and dense in $\mathcal{R}(Q,d)$. In fact, the set $\mathcal{O}(d)$ consists of all representations isomorphic to $M(d)$, since $r_{i,j}(M(d)) = r_{i,j}$ by construction.

We fix a dimension vector d and consider any triangle $s = (s_{i,j})$ of non-negative integers $s_{i,j}$ satisfying $s_{i,j} \le r_{i,j}$. Then

$$X_s^0 := \{A \in \mathcal{R}(Q,d) \mid r_{i,j}(A) = s_{i,j}\} \subseteq X_s := \{A \in \mathcal{R}(Q,d) \mid r_{i,j}(A) \le s_{i,j}\}$$

defines an open (possibly empty) subvariety X_s^0 in a closed, non-empty algebraic subvariety X_s (not necessarily irreducible) of $\mathcal{R}(Q,d)$. The rank triangles are partially ordered by $s \le u$ iff $u - s$ has only non-negative entries. It turns out that some of the X_s are irreducible (we determine which ones) and the rank conditions are very useful for determining the components in the orbit closures. Moreover, one can reconstruct the multisegment M from the rank condition s, where the orbit of M is dense in X_s with s minimal: A direct sum $[i,j]^a$ is a direct summand of M (with maximal possible a) if and only if $a = r_{i,j} - r_{i+1,j} - r_{i,j+1} + r_{i+1,j+1}$. Consequently, X_s^0 is empty, if some $r_{i,j} - r_{i+1,j} - r_{i,j+1} + r_{i+1,j+1}$ is negative. Otherwise X_s^0 is dense in X_s.

Conversely, given a multisegment M we can easily determine its rank vector $r(M) = (r(M)_{i,j})$ as follows

$$r(M)_{i,j} = \sharp \{[k,l] \in M \mid k \le i \le j \le l\}.$$

In the particular case of a segment $[k,l]$, we obtain just the characteristic function of a triangle as the rank triangle

$$r([k,l])_{i,j} = \begin{cases} 1 \text{ if } [i,j] \subseteq [k,l] \\ 0 \text{ else }. \end{cases}$$

Proposition 2.2. *a) If $s \leq u$ then $X_s \subseteq X_u$. In particular, X_r contains each X_s and X_0 (consisting of the zero matrix) is contained in each X_u.*
b) The variety X_s is irreducible precisely when it is the closure of one $Gl(d)$–orbit.
c) X_s^0 is non-empty precisely when s is a sum of functions of the form $r([i,j])$ and this is equivalent to $s_{i,j} - s_{i+1,j} - s_{i,j+1} + s_{i+1,j+1} \geq 0$ for all pairs (i,j).

Proof. Assertion a) is obvious, since $\operatorname{rk} A_{(i,j)} \leq a$ implies $\operatorname{rk} A_{(i,j)} \leq b$ for any $b > a$.

To prove b) we decompose X_s in a disjoint union of $\mathrm{Gl}(d)$–orbits. This is possible, since X_s is $\mathrm{Gl}(d)$–invariant. Thus we obtain a set of multisegments $\mathcal{M}_s$ with

$$X_s = \bigsqcup_{M \in \mathcal{M}_s} \mathrm{Gl}(d)M.$$

Consequently, X_s is the union of a finite number of orbit closures $\overline{\mathrm{Gl}(d)M}$ for a finite number of multisegments M. We can assume this set is minimal. Thus X_s is irreducible precisely when $X_s = \overline{\mathrm{Gl}(d)M}$ for some maximal M in $\mathcal{M}_s$.

For c), note that X_s^0 is nonempty, precisley when there exists a multisegment M with $s = \operatorname{rk} M$. This is also equivalent to $s = \sum_{(i,j)} a_{(i,j)} \operatorname{rk}[i,j]$ is the sum of rank functions of segments. To prove the last characterization we note that for $s = \sum_{(i,j)} a_{(i,j)} \operatorname{rk}[i,j]$ we obtain $s_{i,j} - s_{i+1,j} - s_{i,j+1} + s_{i+1,j+1} = a_{(i,j)} \geq 0$. Conversely, if $s_{i,j} - s_{i+1,j} - s_{i,j+1} + s_{i+1,j+1} \geq 0$ then we define $a_{(i,j)} = s_{i,j} - s_{i+1,j} - s_{i,j+1} + s_{i+1,j+1}$. □

3. Proof of the main theorem

We start this section by showing that some of the $Y_{i,j}$ are irreducible and compute their dimension. Then we show that all $Y_{i,j}$ for (i,j) not in $I(d)$ are already contained in some union of other ones. This allows a reduction to the case $Y_{i,j}$ for $(i,j) \in I(d)$. Finally we show that Y is already contained in the union of all $Y_{i,j}$.

3.1. **Irreducible varieties.**

Proposition 3.1. *Assume $(i,j) \in J(d)$, then $Y_{i,j}$ is irreducible of codimension $|d_j - d_i| + 1$ in $\mathcal{R}(Q,d)$.*

Proof. We consider the projection of a representation of Q to the quiver Q' with vertices $i, i+1, \ldots, j-1, j$ and its subvarieties $Y_{i,j}$ in $\mathcal{R}(Q,d)$ and $Y'_{i,j}$ in $\mathcal{R}(Q',d')$ defined by $\operatorname{rk} A_{(i,j)} < r_{i,j}$. Then $\mathcal{R}(Q,d)$ is a direct product of $\mathcal{R}(Q',d')$ with some affine space and $Y_{i,j}$ is a product of $Y'_{i,j}$ with some affine space. Thus $Y_{i,j}$ is irreducible precisely when $Y'_{i,j}$ is irreducible. Consequently, it is sufficient to prove the claim for $Y_{1,t}$ in $\mathcal{R}(Q,d)$.

We now assume $(i,j) = (1,t)$ and $d_i > d_1, d_t$ for any $1 < i < t$.

Now we consider a multisegment M consisting of $[1,t-1] \oplus [2,t]$ and $M(e)$ for $e = d - (1,2,2,\ldots,2,2,1)$. A computation of the ranks $r_{i,j}(M)$ yields $r_{1,t}(M) = r_{1,t} - 1$ and $r_{i,j}(M) = r_{i,j}$ for all $(i,j) \neq (1,t)$. Thus the equation $s_{1,t} = r_{1,t} - 1$ and $s_{i,j} = r_{i,j}$ for $(i,j) \neq (1,t)$ defines an orbit X_s and X_s is the closure of this orbit containing M. Consequently it is irreducible, and it coincides with $Y_{1,t}$.

Finally, we need to compute the codimension of the orbit closure $Y_{1,t}$. For this we compute the dimension of the stabilizer of $M(d)$ and of M constructed above. To make the computation easier, we delete the common direct summands that contribute with the same dimension to the stabilizer and assume without loss of generality $d_1 \geq d_t$. Then we need to compute

$$\begin{aligned}\dim \mathrm{End}([2,t-1] \oplus [1,t]^a \oplus [1,t-1]^b) &= a^2+b^2+ab+b+1\\ \dim \mathrm{End}([2,t] \oplus [1,t]^{a-1} \oplus [1,t-1]^{b+1}) &= a^2+b^2+ab+2b+2.\end{aligned}$$

Back to $M(d)$, we decompose it into $M(d) = [2,t-1] \oplus [1,t]^a \oplus [1,t-1]^b \oplus M'$ with maximal a and b. Then M is $[2,t] \oplus [1,t]^{a-1} \oplus [1,t-1]^{b+1} \oplus M'$ and

$$\begin{aligned}&-(b+1) = \dim \mathrm{End}(M(d)) - \dim \mathrm{End}(M) =\\ &\dim \mathrm{End}([2,t-1] \oplus [1,t]^a \oplus [1,t-1]^b) - \dim \mathrm{End}([2,t] \oplus [1,t]^{a-1} \oplus [1,t-1]^{b+1}).\end{aligned}$$

Consequently, the codimension of the orbit of M equals $b+1 = d_1 - d_t + 1$ and this equals the codimension of $Y_{1,t}$. Finally, note that under the reduction from arbitrary $Y_{i,j}$ to $Y_{1,t}$ the codimension does not change. □

3.2. **The reduction process.**

Proposition 3.2. *a) Assume $(i,j) \notin J(d)$ then there exists some l with $i < l < j$ and $d_l \leq d_k$ for all $i < k < j$. In particular, $d_l \leq max\{d_i, d_j\}$. In this case we have an inclusion $Y_{i,j} \subseteq Y_{i,l} \cup Y_{l,j}$.*
b) If $(i,j) \in J(d) \setminus I(d)$ with $d_i \leq d_j$ then there exists an l with $l > j$ and $d_i \leq d_l < d_j$. In this case we obtain $Y_{i,j} \subset Y_{i,l}$.
c) If $(i,j) \in J(d) \setminus I(d)$ with $d_i \geq d_j$ then there exists an l with $l < i$ and $d_j \leq d_l < d_i$. In this case we obtain $Y_{i,j} \subset Y_{l,j}$.

Proof. Without loss of generality we may assume $d_i \leq d_j$ in the proof.
a) Consider the maps $A_{(i,j)} : V_i \longrightarrow V_j$, $A_{(i,l)} : V_i \longrightarrow V_l$, and $A_{(l,j)} : V_l \longrightarrow V_j$. We consider two cases.
$d_i \geq d_l$:
Assume $\mathrm{rk}\, A_{(i,j)} < d_l$, then $\mathrm{rk}\, A_{(i,l)} < d_l$ or $\mathrm{rk}\, A_{(l,j)} < d_l$ since $A_{(i,j)} : V_i \longrightarrow V_l \longrightarrow V_j$ factors through V_l with $\dim V_l \leq \dim V_i, \dim V_j$.
$d_i < d_l$:
Assume $\mathrm{rk}\, A_{(i,j)} < d_i$, then $\mathrm{rk}\, A_{(i,l)} < d_i$
b) Consider the maps $A_{(i,j)} : V_i \longrightarrow V_j$ and $A_{(i,l)} : V_i \longrightarrow V_l$. Since $(i,j) \in J(d) \setminus I(d)$ there exist some $l > j$ with $d_i \leq d_l < d_j$ and $(i,l) \in J(d)$. Then from $\mathrm{rk}\, A_{(i,j)} < d_i$ follows $\mathrm{rk}\, A_{(i,l)} < d_i$.
c) This case is opposite to case b). □

Lemma 3.3.

$$Y = \bigcup_{1 \leq i < j \leq t} Y_{i,j} = \bigcup_{(i,j) \in J(d)} Y_{i,j} = \bigcup_{(i,j) \in I(d)} Y_{i,j}.$$

Proof. The dense orbit is defined by the condition $\mathrm{rk}\, A_{(i,j)} = r_{i,j}$. Thus, the complement satisfies $\mathrm{rk}\, A_{(i,j)} < r_{i,j}$ for at least one pair (i,j) with $r_{i,j} > 0$. Since $r_{i,j} = \min\{d_i \mid 1 \leq i \leq t\} > 0$ we finish the proof of the first equality.
To prove the second one we use the proposition above. From Proposition 3.2 a) we obtain $\bigcup_{1 \leq i < j \leq t} Y_{i,j} \subseteq \bigcup_{(i,j) \in J(d)} Y_{i,j}$ and from part b) and c) $\bigcup_{(i,j) \in J(d)} Y_{i,j} \subseteq \bigcup_{(i,j) \in I(d)} Y_{i,j}$. □

Corollary 3.4. *The variety $Y_{i,j}$ is irreducible precisely when $(i,j) \in J(d)$.*

Proof. Thanks to Proposition 3.1, we only need to prove that $Y_{i,j}$ is not irreducible for (i,j) not in $J(d)$. Take (i,j) not in $J(d)$, thus there exists an l with $i < l < j$ and $d_l \leq \max\{d_i, d_j\}$ is minimal. Then by Proposition 3.2 a) we have $Y_{i,j} \subseteq Y_{i,l} \cup Y_{l,j}$. Assume first that $d_l \leq \min\{d_i, d_j\}$. We claim that $Y_{i,j} = Y_{i,l} \cup Y_{l,j}$

and this is a proper decomposition, none contains the other. To see the equality, consider any element A in $Y_{i,l} \cup Y_{l,j}$. Then $\mathrm{rk}(A)_{i,l} < r_{i,l}$ or $\mathrm{rk}(A)_{l,j} < r_{l,j}$. From each of the inequalities follows $\mathrm{rk}(A)_{i,j} < r_{i,j}$. On the other hand, there exists a representation with $\mathrm{rk}(A)_{i,l} = r_{i,l}$ and $\mathrm{rk}(A)_{l,j} < r_{l,j}$ and vice versa, proving also the last claim.
In the second case $\max d_i, d_j \geq d_l > \min\{d_i, d_j\}$. We construct two different subvarieties that contain $Y_{i,j}$ and none contains the other. To simplify the arguments, we assume without loss of generality $d_i \geq d_l > d_j$ and, using the first case, d_l is the minimal entry of d between d_i and d_j. The first variety is just the orbit closure $Y_{l,j}$, the second one is defined by $r_{i,l}(A) < r_{i,l}$ and $r_{i,j}(A) < r_{i,j}$. Using multisegments (or rank conditions) one can show that we obtain at least two irreducible components in this way ($Y_{l,j}$ is irreducible, the other variety need not to be). Anyway, we obtain at least two irreducible components. □

4. Further descriptions

In this section we proceed with the various descriptions of the irreducible components and the tilting modules started in Section 2. In particular, we use trees and fans to describe the irreducible components and we relate our description to the nilpotent class representations defined in [H2].

4.1. **Trees and tilting modules.** Let T be a 3–regular tree with one root and $t+1$ leaves, where the leaves are enumerated by $0, 1, 2, \ldots, t-1, t$. We denote the set of those trees with $\mathcal{T}_t$. With $\mathcal{T}_t^1$ we denote all trees that have precisely on vertex with four neighbours, all other vertices have three neighbours and admit one root and $t+1$ leaves. There is a natural map from $\mathcal{T}_t^1$ to the set of unordered pairs $\mathcal{P}^2(\mathcal{T}_t)$ of of trees in $\mathcal{T}_t$ by "resolving" the vertex with four neighbours and replacing it by two 3–regular vertices (see Figure 1).

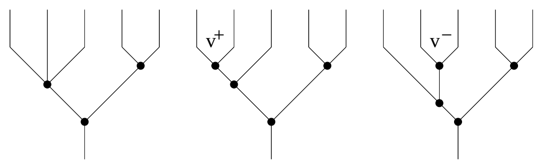

Figure 1. a tree in $\mathcal{T}_4^1$ and the two associated 3–regular trees in $\mathcal{T}_4$.

We always draw a tree in the plane and fix the numbering of the leaves $0, \ldots, t$ from left to right. Two trees are considered to be equal, if the abstract graphs are isomorphic and the numbering of the leaves is preserved under the isomorphism. Then each vertex v defines the set of leaves (in fact an interval) above the vertex $\{i_T(v)-1, i_T(v), \ldots, j_T(v)\}$. This way, each vertex v defines a segment $[i(v), j(v)]$. To any tree in $\mathcal{T}$ or $\mathcal{T}^1$ we can associate multisegments as follows. Assume $T \in \mathcal{T}_t$ and denote by T_0 the vertices in T, then we define

$$M_T = \bigoplus_{v \in T_0} [i_T(v), j_T(v)]$$

to be the union of the multisegments $[i_T(v), j_T(v)]$ above of v. If $S \in \mathcal{T}^1$ and T^+ and T^- are the two associated 3–regular trees with unique vertex $v^+ \in T^+$ and $v^- \in T^-$ (these are the only vertices defining a segment that is not obtained from the other tree), we define

$$\begin{aligned} M_S &= \textstyle\bigoplus_{v \in T_0^+} ([i_{T^+}(v), j_{T^+}(v)]) \oplus [i_{T^-}(v^-), j_{T^-}(v^-)] \\ &= \textstyle\bigoplus_{v \in T_0^-} ([i_{T^-}(v), j_{T^-}(v)]) \oplus [i_{T^+}(v^+), j_{T^+}(v^+)], \text{ and} \\ \underline{M}_S &= \textstyle\bigoplus_{v \in T_0} [i_T(v), j_T(v)] \end{aligned}$$

The module M_T for $T \in \mathcal{T}^1$ has $t+1$ pairwise nonisomorphic direct summands and the module $\underline{M}_S$ has $t-1$ pairwise nonisomorphic direct summands.

Theorem 4.1. *a) If T is a 3–regular tree, then $M_T = M(d(T))$ for $d(T) = \underline{\dim} M_T$. In particular,* $\operatorname{Ext}(M_T, M_T) = 0$ *for any tree T in $\mathcal{T}_t$.*
b) If S is in $\mathcal{T}^1$ then $\operatorname{Ext}(M_S, M_S) = k$. *In particular, M_S is almost generic.*
c) If d is generic, then there exists a unique $T \in \mathcal{T}_t$ so that $M(d)$ and M_T have the same indecomposable direct summands. Thus $M(d)$ is a direct summand of several copies of M_T.
d) If M defines the open dense subset in an irreducible component $Y_{i,j}$ of Y, then there exists some tree $S \in \mathcal{T}_t^1$ so that M is a direct summand of copies of M_S.
e) For each multisegment M with $Ext(M,M) = k$ there exists some $S \in \mathcal{T}^1$ with $[i_{T^-}(v^-), j_{T^-}(v^-)]$ and $[i_{T^+}(v^+), j_{T^+}(v^+)]$ as direct summand of M.

Proof. Using Prop. 2.1 a) one sees immediately that the segments in M_T satisfy the vanishing condition for the extension groups. Thus, the only non-vanishing extension group in M_S are in the complement of $\underline{M}_S$, that consists of two segments. This proves a) and b) (see also the proof in [H1]).

Part c) also follows from the arguments in loc. cit.: Each multisegment with non-vanishing extension group can be completed to one with t non-isomorphic direct summands. Finally, any multisegment with precisely t indecomposable summands, all pairwise non-isomorphic and vanishing extension group is isomorphic to M_T for some T in $\mathcal{T}_t$ and the segments determine T uniquely.
Using the description of an almost generic multisegment M in Prop. 2.1 d) we find two segments $[a,b]$ and $[c,d]$ in M with non-vanishing extension group. Moreover, we can assume that M has $t+1$ non-isomorphic direct summands (otherwise we add further ones to M). Deleting all summands of the form $[a,b]$ defines a tree T^+, deleting the other direct summand $[c,d]$ defines a different tree T^- by c). By construction, both trees come from a common S in $\mathcal{T}_t^1$ so that M_S and M contain the same indecomposable direct summands up to isomorphism. This proves d). The two summands in e) are just $[a,b]$, respectively $[c,d]$. □

4.2. **Nilpotent class representations.** There is an obvious formula for the number $N(d)$ of orbits in $\mathcal{R}(Q,d)$. We just count the number of multisegments $\oplus[i,j]^{a_{i,j}}$ defined by a non-strict triangle $a = (a_{(i,j)})_{1 \le i \le j \le t}$

$$N(d) = \sharp\{a = (a_{i,j})_{1 \le i \le j \le t} \mid a_{i,j} \in \mathbb{Z}_{\ge 0}, \quad \sum_{i,j} a_{i,j} \underline{\dim}[i,j] = d\}.$$

This function is also called *Kostants partition function* for type $\mathbb{A}$. It is for large d not efficiently computable, thus an easier formula is desirable. For we define numbers $NA(\lambda,\mu)$ for any two partitions λ of $b > 0$ and μ of $c > 0$

$$\mathrm{NA}(\lambda,\mu) = \begin{cases} \prod_{l=1}^{\infty}(\sharp\{i \mid \lambda_i = \mu_i = l\} + 1) & \text{if } |\lambda_i - \mu_i| \le 1 \text{ for all } i \\ 0 & \text{if } |\lambda_i - \mu_i| \ge 2 \text{ for some } i. \end{cases}$$

Proposition 4.2. *The number of multisegments coincides with the sum, taken over all sequences of partitions $(\lambda^1, \ldots, \lambda^t)$ with λ^i a partition of d_i, $\lambda^1 = (1)^{d_1}$ and $\lambda^t = (1)^{d_t}$ are both trivial, of the product of the numbers* $\mathrm{NA}(\lambda^i, \lambda^{i+1})$

$$N(d) = \sum_{(\lambda^1,\ldots,\lambda^t)} \prod_{i=1}^{t-1} \mathrm{NA}(\lambda^i, \lambda^{i+1}).$$

Proof. We only mention the idea of the proof, the details can be found in [H2], Section 4.2. First we consider the preprojective algebra Π_t of $\mathbb{A}_t$ and the cyclic quiver $\widetilde{\mathbb{A}}_1$ with two vertices together with the natural projection maps

$$\times_{i=1}^{t-1}\mathcal{R}(\widetilde{\mathbb{A}}_1,(d_i,d_{i+1})) \xleftarrow{p_1} \mathcal{R}(\Pi_t,d) \xrightarrow{p_2} \times_{i=1}^{t}\mathcal{R}(k[T]/T^{d_i}).$$

If we denote an element in $\mathcal{R}(\Pi_t,d)$ by (A,B), then it satisfies $B_1A_1 = A_1B_1 - B_2A_2 = \ldots = A_{t-1}B_{t-1} - B_tA_t = A_tB_t = 0$, where $A_i : V_i \longrightarrow V_{i+1}$ and $B_i : V_{i+1} \longrightarrow V_i$. The projections are defined by

$$p_1 : (A,B) \mapsto ((A_1,B_1),\ldots,(A_{t-1},B_{t-1})) \text{ and}$$

$$p_2 : (A,B) \mapsto (B_1A_1, A_1B_1,\ldots,A_{t-1}B_{t-1}).$$

In particular, each element (A,B) defines a sequence of partitions $(\lambda^1,\ldots,\lambda^t)$ (defined by the partition of the nilpotent class of $B_1A_1, A_1B_1,\ldots,A_{t-1}B_{t-1}$). By definition, λ^1 and λ^t are always the trivial ones corresponding to the zero matrix. It is known (see [KZ] or [P]) that $\mathcal{R}(\Pi_t,d)$ is equidimensional and the irreducible components are in bijection with the $\mathrm{Gl}(d)$–orbits on $\mathcal{R}(\mathbb{A}_t,d)$. Thus $N(d)$ is just the number of irreducible components in $\mathcal{R}(\Pi,d)$. Now we determine the irreducible components in a different way using the projection map above. First note, that $\mathrm{NA}(\mu,\lambda)$ is the number of irreducible components of

$$\mathcal{R}(\widetilde{\mathbb{A}}_1,(\mu,\lambda)) = \{(A,B) \mid AB \in C(\mu), BA \in C(\lambda)).$$

If one fixes the sequence of partitions $(\lambda^1,\ldots,\lambda^t)$, then one can compute the number of irreducible components in $\mathcal{R}(\Pi_t,d)$ as the sum of the products $\mathrm{NA}(\lambda^1,\lambda^2)\cdot\ldots\cdot \mathrm{NA}(\lambda^i,\lambda^{i+1})$ taken over all such sequences of partitions with λ^1 and λ^t trivial. $\square$

The advantage of the formula above is twofold. First, it is independent of the orientation of the quiver. We can, for any orientation of the quiver of type $\mathbb{A}_t$ define such a sequence of partitions. Secondly, the formula in Prop. 4.2 is much more efficient than just a simple counting.

Note that for the generic representation M for a a quiver of type $\mathbb{A}_t$ with an arbitrary orientation the corresponding sequence of partitions is just the trivial one (all λ^i are zero).

4.3. **The fan and the volume.** The sets of trees $\mathcal{T}_t$ and $\mathcal{T}_t^1$ define a graph Γ_t that is the dual graph of the simplicial complex of tilting modules defined in [RS]. This simplicial complex has a natural realisation as a fan Σ in the positive quadrant $K_{\mathbb{R}}^+$ of the real Grothendieck group K_0, where $K_{\mathbb{R}}^+ := \mathbb{R}^t_{\geq 0} \subset \mathbb{R}^t \simeq K_0 \otimes \mathbb{R}$. This fan is described in [H1]. From the fan, one can again determine the irreducible compenents in a simple way.

We start to define the graph $\Gamma = \Gamma_t$. The vertices Γ_0 are just the trees in $\mathcal{T}_t$. The set of edges is $\mathcal{T}_t^1$. The end points of the edge S consists of the two resolutions T^+ and T^- of S.

Then we recall the definition of the fan Σ. For a precise definition of a fan, some first properties and applications we refer to [F]. Note first, that a fan Σ is a finite collection of rational, convex, strongly convex, polyhedral cones that satisfy two conditions:
F1) each face of a cone in Σ is in Σ and
F2) the intersection of two cones in Σ is a face of both.

Note that we only need finite fans, for tame and wild quivers one needs to allow also infinite ones. For each $T \in \mathcal{T}_t$ we define a cone $\sigma_T \subset K_{\mathbb{R}}^+$ as the cone spanned by the dimension vectors of the indecomposable direct summands of M_T

$$\sigma_T := \{\sum \mathbb{R}_{\geq 0}\underline{\dim}[i,j] \mid [i,j] \text{ is a direct summand of } M_T\}.$$

Two cones σ_{T^+} and σ_{T^-} have a common facet, precisely when there exists a tree $S \in \mathcal{T}_t^1$ with corresponding trees $S^+ = T^+$ and $S^- = T^-$. The fan Σ consists of all cones, generated by dimension vectors of indecomposable direct summands of a rigid multisegment M (that is $\mathrm{Ext}(M, M) = 0$). Already the cones σ_T determine the fan Σ consisting of all the cones σ that are faces of a cone σ_T (including the cones σ_T themself). We recall the main result from [H1] together with some easy consequences.

Theorem 4.3. *a) The cones $\sigma \in \Sigma$ are all generated by a part of a $\mathbb{Z}$–basis (they are smooth cones).*
b) The union of the cones σ_T (that is the same as the union of all cones in Σ) cover $K_{\mathbb{R}}^+$. For two cones in Σ their intersection is a face of both (and it is also in Σ). Each cone is a face of a t–dimensional cone and each t–dimensional cone equals σ_T for some $T \in \mathcal{T}_t$.
c) A dimension vector d is generic, precisely when it is in the interior of some cone σ_T. Consequently, for d generic, T is uniquely determined by d.
d) For each dimension vector d there exists a unique cone $\sigma \in \Sigma$ with $d \in \sigma$ and no face of σ does contain d. This is equivalent to saying that, d is an element of the relative interior of the cone σ. Moreover, the cone σ is generated as a cone by the dimension vectors of the indecomposable direct summands of $M(d)$. In particular, the dimension of σ is the number of pairwise non-isomorphic indecomposable direct summands of $M(d)$.
e) The dual graph of the t–dimensional cones and the $(t-1)$–dimensional cones, occuring as an intersection of two t–diemensional cones, is Γ_t.
f) Each t–dimensional cone has precisely $t-1$ neighbours, that is Γ_t is $(t-1)$–regular.

The proof can be found in [H1].

4.4. **Irreducible components and the fan Σ.** Using the fan, we can again determine the irreducible components in Y. Note that d is contained in some maximal set of cones. We denote the set of trees $T \in \mathcal{T}_t$ with $d \in \sigma_T$ by $\mathcal{T}(d)$.

Assume d is in the relative interior of a facet σ_S that is the intersection of the two t–dimensional cones σ_{T^+} and σ_{T^-}. Note that S is a tree in $\mathcal{T}_t^1$ (compare with section 4.1). Then S defines two segments $[a_S, b_S]$ and $[c_S, d_S]$ defined as the unique segments not in σ_S but one in σ_{T^+} and the other one in σ_{T^-}.

Then we obtain one component just by adding to $[a_S, b_S] \oplus [c_S, d_S]$ the unique generic complement: $M(d') \oplus [a_S, b_S] \oplus [c_S, d_S]$. In this way, each inner facet σ_S defines a unique irreducible component Y_S.

If d is generic, that is $\mathcal{T}(d)$ consists of just one tree T, then the components in Y correspond to the $t-1$ neighboured cones. In fact, each neighboured cone of σ_T has a common facet σ_S with σ_T and the component constructed above defines also a component for the dimension vector d. In this way, we obtain precisely $t-1$ components. It remains to show that they are pairwise different. Decompose $M(d) = [1,t]^a \oplus \bigoplus [i,j]^{a_{i,j}}$ into t indecomposable, pairwise nonisomorphic, direct summands. Since there is a unique cone σ_T containing d, $M(d)$ has precisley t pairwise nonisomorphic, direct summands by Prop. 4.3. Then for each segment $[k,l] \neq [1,t]$ with $a_{k,l} \neq 0$ there exists a unique S so that $[k,l]$ coincides with one of the segments, say $[a_S, b_S]$, constructed above. Then we obtain a component Y_S just by adding to $[a_S, b_S] \oplus [c_S, d_S]$ the unique generic complement. Since $[a_S, b_S]$ and $[c_S, d_S]$ determine S, we obtain the desired $t-1$ irreducible components.

Theorem 4.4. *Let d be generic with d in the interior of a t–dimensional cone $\sigma \in \Sigma$, then the irreducible components $Y_{i,j}$ of Y are in bijection with the set of* *trees T that define a cone with a common facet σ_S with σ. For such a tree, take the two unique segments $[a_S, b_S]$ and $[c_S, d_S]$ (with $a_S < c_S \le b_S + 1 < d_S + 1$) in $\sigma_T \cup \sigma$ with nontrivial extension. Then $(i, j) = (a_S, d_S)$.*

5. Examples

In general, $M(d)$ can have less than t pairwise non-isomorphic direct summands. Also, the codimension of the components and the number of components in Y can vary. We discuss several examples, where we have more precise results. This includes the pure case (all components have codimension one), the generic case (which contains all dimension vectors that do not lie on a proper face of a cone in the fan Σ), the concave case, and, eventually, the convex case.

5.1. **Generic dimension vectors.** For d generic, we have always t indecomposable direct summands and d lies in the interior of a cone σ_T in the fan Σ for some $T \in \mathcal{T}_t$.

Proposition 5.1. *Assume d is a generic dimension vector. Then Y consists of $t-1$ irreducible components, all have codimension at least 2.*

Proof. This follows directly from Theorem 4.4. □

5.2. **Concave dimension vectors.** In the concave case (that is $d_1 \ge d_2 \ge \dots d_{a-1} \ge d_a \le d_{a+1} \le \dots \le d_t$) there are also always $t-1$ components. Moreover $I(d)$ can easily be described.

Proposition 5.2. *Assume d is a concave dimension vector with $d_i > 0$.*
a) The sets $I(d)$ and $J(d)$ both coincide with $\{(1,2),(2,3),\dots(t-2,t-1),(t-1,t)\}$.
b) There are precisely $t-1$ irreducible components $Y_{i,i+1}$ for $i = 1,\dots,t-1$. The codimension of $Y_{i,j}$ equals $|d_i - d_{i+1}| + 1$.
c) The variety $Y_{i,j}$ is irreducible precisely when $j = i+1$.
d) The dimension vector d is generic precisely when for all $i < j$ with $d_i = d_j$ there exists some l with $i < l < j$ and $d_l < d_i = d_j$.
e) The components in Y of codimension one correspond to pairs $(i, i+1)$ with $d_i = d_{i+1}$.

Proof. We use our main theorem together with the methods obtained in Section 4. □

Example. We consider $d = (d_1,\dots,d_7) = (5,4,3,1,2,4,6)$ and get $I(d) = \{(1,2),(2,3),\dots,(5,6),(6,7)\}$. In this case all components have codimension at least 2 and d is generic. The corresponding multisegment $M(d)$ is $[1,7] \oplus [1,3]^2 \oplus [1,2] \oplus [1,1] \oplus [5,7] \oplus [6,7]^2 \oplus [7,7]^2$.

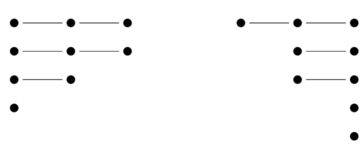

Figure 2. $M(5,4,3,1,2,4,6)$

and the almost generic multisegments (corresponding to minimal degenerations) look like (all are different and minimal)

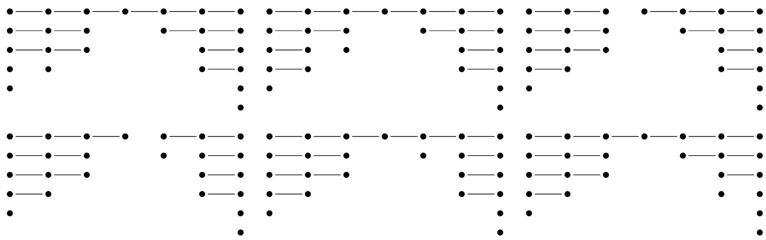

Figure 3. The minimal degenerations of $M(5,4,3,1,2,4,6)$

Thus we get the following irreducible components
$Y_{i,i+1} := \{(A_{i,i+1} \mid \operatorname{rk} A_{i,i+1} < \min\{d_i, d_{i+1}\}\}$ of codimension 2 and 3, respectively.

5.3. **Unimodular (Convex) Dimension Vectors.** For an unimodular dimension vector (or a convex one) we have the opposite inequalities $d_1 \leq d_2 \leq \ldots \leq d_{a-1} \leq d_a \geq d_{a+1} \geq \ldots \geq d_t$ In the unimodular case there can be less than $t-1$ components. To illustrate this we give two examples, one with $t-1$ components and one with $(t-1)/2$ components, which is the minimal number of components. Moreover, it is convenient to exclude the previous case of a concave dimension vector in what follows. We denote by d^+ the maximal entry in d.

Proposition 5.3. *Let d be an unimodular, sincere dimension vector that is not concave. a) $I(d) \neq J(d)$.*
b) There are precisely $t-1$ irreducible components in Y if and only if from $d_i = d_j$ follows $i = j-1$ or $j+1$.

Proof. Under the assumptions we have a maximal entry d^+ in d with $d_1 \neq d^+$ and $d_t \neq d^+$. Then $(1,2),(2,3),\ldots,(t-1,t)$ are all in $J(d)$. If they are also in $I(d)$, then $I(d)$ is just the set of all pairs $(i,i+1)$. Then we get $d_i = d_{i+1}$ for all i, a contradiction to our assumption. □

Example 1. We consider $d = (d_1,\ldots,d_7) = (1,3,5,7,6,4,2)$. The corresponding multisegment is

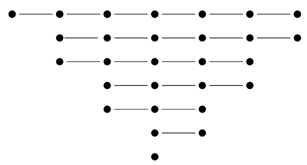

Figure 4. $M(1,3,5,7,6,4,2)$

and its minimal degenerations look like (all are different and irreducible)

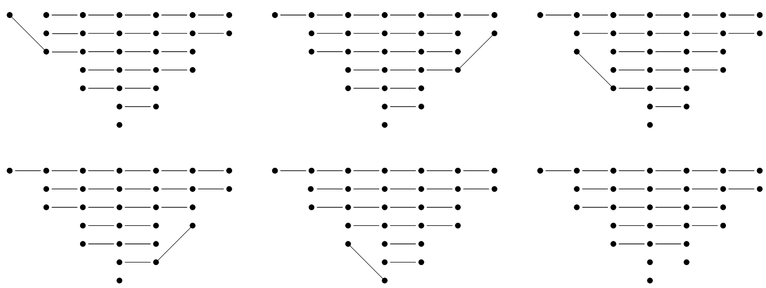

Figure 5. minimal degenerations of $M(1,3,5,7,6,4,2)$

Example 2. The other extreme case is to have only $(t-1)/2$ components. This forces that d has often the same entry and the entries in between two equal entries are all larger. Thus, in the convex case we may consider $d = (1,2,4,5,4,2,1)$ with $I(d) = \{(1,7),(2,6),(3,5)\}$ and generic multisegment $[1,7] \oplus [2,6] \oplus [3,5]^2 \oplus [4,4]$.

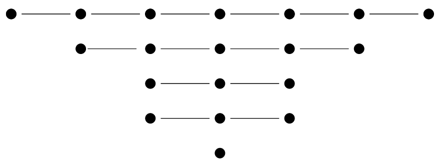

Figure 6. $M(1, 2, 4, 5, 4, 2, 1)$

and its minimal degenerations look like (all are different and irreducible)

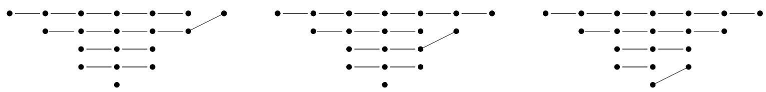

Figure 7. The minimal degenerations of $M(1, 2, 4, 5, 4, 2, 1)$

5.4. **Pure dimension vectors.** A characterization of all dimension vectors d with Y equidimensional seems to be quite technical. So we restrict the result to the codimension one case. We define d to be pure, if for alle elements (i, j) in $I(d)$ we have $d_i = d_j$. The following proposition is easy to check.

Proposition 5.4. *a) The complement Y of the dense orbit in $\mathcal{R}(Q, d)$ is equidimensional of codimension 1 precisely when d is pure.*
b) Assume d is pure, then $J(d)$ contains all pairs $(i, i+1)$ and $I(d)$ only consists of the pairs (i, j) with $d_i = d_j$ and $d_l > d_i = d_j$ for all $i < l < j$.

Example. We have already seen a pure example that is also convex in section 5.3, Example 1. So we consider a pure one that is not convex. Let d be $(1, 2, 3, 5, 3, 2, 3, 2, 1)$

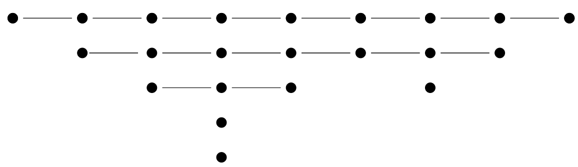

Figure 8. $M(1, 2, 3, 5, 3, 2, 3, 2, 1)$

The components are given by $I(1, 2, 3, 5, 3, 2, 3, 2, 1) = \{(1, 9), (2, 6), (6, 8), (3, 5)\}$.

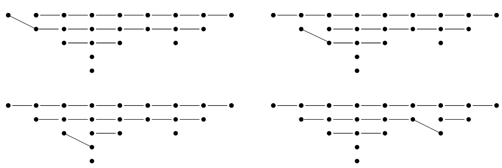

Figure 9. The minimal degenerations of $M(1, 2, 3, 5, 3, 2, 3, 2, 1)$

6. Parabolic group actions

The results in this note are inspired by the decription of the complement of the Richardson orbit (the dense orbit) for the action of a parabolic subgroup in Gl_N on its unipotent radical as considered recently in [BH]. We explain the common idea and some generalizations.

6.1. **The Richardson orbit.** Given a dimension vector d as above, then we define a group

$$P(d) := \{f \in \mathrm{Aut}\,(\oplus_{i=1}^t V_i) \mid f(V_j) \subseteq \oplus_{i=1}^j V_i \text{ for all } j = 1, \dots, t\}$$

and a vector space

$$\mathfrak{p}_u(d) := \{f \in \mathrm{End}(\oplus_{i=1}^t V_i) \mid f(V_j) \subseteq \oplus_{i=1}^{j-1} V_i \text{ for all } j = 1, \dots, t\}.$$

The group $P(d)$ is a standard parabolic subgroup in the General Linear Group and $\mathfrak{p}_u(d)$ is the Lie algebra of the unipotent radical of $P(d)$. The group $P(d)$ acts on $\mathfrak{p}_u(d)$, its derived Lie algebras

$$\mathfrak{p}_u(d)^{(l)} := \{f \in \mathrm{End}(\oplus_{i=1}^t V_i) \mid f(V_j) \subseteq \oplus_{i=1}^{j-l-1} V_i \text{ for all } j = 1, \dots, t\},$$

and also the quotients $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$ (and $\mathfrak{p}_u(d)^{(k)}/\mathfrak{p}_u(d)^{(l)}$ for $k < l$) via conjugation. By a classical result of Richardson ([Rc]) the group $P(d)$ acts with a dense orbit on $\mathfrak{p}_u(d)$ and consequently also with a dense orbit on $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$ for all $l > 1$. In [BH] we describe the complement of the dense orbit explicitely using certain rank conditions on the matrix $A \in \mathfrak{p}_u(d)$. Thus it is desirable to obtain a similar elementary description of the complement of the dense orbit in the case $k = l - 1$. In fact this case corresponds to a disjoint union of equioriented Dynkin quivers of type $\mathbb{A}$.

The case $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$ can be handled using a variation of the line diagrams introduced in [BH].

In contrast, the case $\mathfrak{p}_u(d)^{(l)}$ is still open in general. It is even not known for general d whether there exists a dense orbit. A first idea to attack the problem can be found in [H1].

6.2. **Irreducible components for the quotients $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$.** The combinatorics with line diagrams allows to describe also the components of the complement of the dense orbit in all quotients $\mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$. For this, we define subvarieties $Z_{i,j} \subset \mathfrak{p}_u(d)/\mathfrak{p}_u(d)^{(l)}$ by certain rank conditions as in [BH]. We claim that there are sets $I^{(l)}(d) \subseteq J^{(l)}(d)$ so that

i) $Z_{i,j}$ is irreducible precisely when $(i, j) \in J^{(l)}(d)$ and

ii) $Z_{i,j}$ is an irreducible component in the complement of the Richardson orbit, if and only if $(i, j) \in I^{(l)}(d)$.

The definition of $Z_{i,j}$ and the construction of the sets $I^{(l)}(d) \subseteq J^{(l)}(d)$ can be read off from line diagrams with connections of length at most l. Note that this specializes to the situation in this note for $l = 1$ and to the construction in [BH] for l sufficiently large.

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