TILTING BUNDLES ON RATIONAL SURFACES AND QUASI-HEREDITARY ALGEBRAS

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Abstract. Let X be any rational surface. We construct a tilting bundle T on X. Moreover, we can choose T in such way that its endomorphism algebra is quasi-hereditary. In particular, the bounded derived category of coherent sheaves on X is equivalent to the bounded derived category of finitely generated modules over a finite dimensional quasi-hereditary algebra A. The construction starts with a full exceptional sequence of line bundles on X and uses universal extensions. If X is any smooth projective variety with a full exceptional sequence of coherent sheaves (or vector bundles, or even complexes of coherent sheaves) with all groups Ext^q for $q \geq 2$ vanishing, then X also admits a tilting sheaf (tilting bundle, or tilting complex, respectively) obtained as a universal extension of this exceptional sequence.

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1. Introduction

Tilting bundles were first constructed by Beilinson on the projective n–space $\mathbb{P}^n$ [Be]. Later Kapranov obtained tilting bundles on homogeneous spaces [Kap]. Moreover, many further examples are known for certain monoidal transformations and projective space bundles [Or]. More recently, tilting bundles consisting of line bundles were investigated by the authors [HP2] and exceptional sequences on stacky toric varieties were constructed by Kawamata [Kaw]. It is also known that varieties admitting a tilting bundle satisfy very strict conditions, its Grothendieck group of coherent sheaves is a finitely generated free abelian group and the Hodge diamond is concentrated on the diagonal (in characteristic zero) [BH]. However, we are still far from a classification of smooth (projective) algebraic varieties admitting a tilting bundle. The present note is a step forward in this direction for algebraic surfaces. The converse of our main result, if X is a surface admitting a tilting bundle then it is rational, is still an open problem. It can be shown for many surfaces that tilting bundles cannot exist using the classification (see e.g. [BPV]). However, there exist

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surfaces of general type that have all the strong properties we need: the canonical divisor has no global sections, the Grothendieck group of coherent sheaves is finitely generated and free, and the Hodge diamond is concentrated on the diagonal.

In this note X is a rational surface over an algebraically closed field k. We assume it is smooth and projective. Some results are valid also for any smooth projective algebraic variety, however our main interest concerns rational surfaces. The principal aim is to show, that any rational surface admits a tilting bundle. The proof is constructive and goes in two main steps. First, we construct on any rational surface a full exceptional sequence of line bundles (Section 2). This already follows from our previous work [HP2]. Moreover we show, that in such a sequences there are no Ext^2–groups between the line bundles. In a second step, we define a universal (co)extension for such sequences, and obtain a tilting bundle. The last step, if we use only universal extensions, coincide with a construction known in representation theory of so-called quasi-hereditary algebras, and is called standardization in [DR].

Since our methods work in a much broader context, we try to be as general as possible. In fact, the last step, the universal extension can be defined for any exceptional sequence of complexes (objects in the corresponding derived category). However, we obtain a tilting complex only if all higher (that is Ext^2 and higher groups) do vanish. Otherwise, we obtain at least a partial result (Theorem 1.4). We start with our main results and then explain the strategy of the proof together with the content of this work. A vector bundle T on an algebraic variety X is called *tilting bundle* if $\mathrm{Ext}^q(T, T) = 0$ for all $q > 0$ and T generates the derived category of coherent sheaves on X in the following sense: the smallest triangulated subcategory of the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X containing all direct summands of T is already $\mathcal{D}^b(X)$ itself. For further notions of generators we refer to [BV].

Theorem 1.1. *Any smooth, projective rational surface X admits a tilting bundle T on X.*

This, in particular, yields an equivalence $\mathbb{R}\operatorname{Hom}(T, -)$ between the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X and the bounded derived category $\mathcal{D}^b(A)$ of right modules over the finite dimensional endomorphism algebra A of T. In fact we will see in Section 6 that we have many choices to construct a tilting bundle T. First, we choose a sequence of blow ups and a standard augmentation (see Definition 2) to obtain a full exceptional sequence of line bundles on X. Then we can either use universal extensions or universal coextensions (we have again a choice for any Ext^1–block, see the final part in section 4) to obtain a tilting bundle. So it is desirable and possible to construct tilting bundles with further good properties. One possibility is to keep the ranks of the indecomposable direct summands of T small. This needs some detailed understanding of the non-vanishing extension groups and is based on our previous work [HP2]. The other way is to obtain an endomorphism algebra with good homological properties. One natural choice is to construct a tilting bundle T so that $A = \mathrm{End}(T)$ becomes a so–called quasi–hereditary algebra. Quasi–hereditary algebras have very nice and well–understood homological properties (see [DR] for a short introduction). In particular, there is the subcategory $\mathcal{F}(\Delta)$ of A–modules with a Δ–filtration, an additive subcategory closed under kernels and extensions. Such a choice is also of interest by a second reason. If we deal with exceptional sequences, where the higher Ext–groups do not vanish, we obtain a functor between the derived categories that is not an equivalence. Anyway, using quasi–hereditary algebras we can at least obtain an equivalence between certain subcategories (Theorem 1.3 and Theorem 1.4).

Theorem 1.2. *Let X be a rational surface then there exists a tilting bundle T on X with quasi-hereditary endomorphism algebra.*

Let ε be any set of objects in an abelian category. Then we define the subcategory $\mathcal{F}(\varepsilon)$ as the full subcategory of objects M admitting a filtration $F^0 = 0 \subseteq F^1 \subseteq F^2 \subseteq, \ldots \subseteq F^l = M$ for some integer l, so that for any $0 < i < l$ the quotient F^{i+1}/F^i is isomorphic to one object in ε. We consider this category in the particular case that ε is an exceptional sequence. If the abelian category is the category of finitely generated modules over a finite dimensional algebra A, an exceptional sequence is also called a set of standardizable modules (see [DR]). This particular case we consider in detail in Section 5. If we restrict, over a quasi-hereditary algebra, to the set of standard modules $\Delta(1), \ldots, \Delta(t)$ then these modules form an exceptional sequence and the category of modules with a Δ–filtration is also called the *category of good modules* over A. This category plays an important role under the equivalence above. In fact, we can obtain an equivalence between the subcategory of coherent sheaves $\mathcal{F}(\varepsilon)$ admitting a filtration by line bundles in the exceptional sequence with the well understood category of good modules over the quasi-hereditary endomorphism algebra. At this point it is desirable to have small ranks (thus line bundles) for the objects in ε to keep the category $\mathcal{F}(\varepsilon)$ large. In fact the next result is also constructive, however it needs more background that we develop only in the last section to formulate it in this way. Also note, that we can take any full exceptional sequence of line bundles obtained from the Hirzebruch surface by any sequence of standard augmentations (see Definition 2) in the following theorem. The tilting bundle T is then obtained as a universal extension of the line bundles in the exceptional sequence.

Theorem 1.3. *On any rational surface X there exists a full exceptional sequence of line bundles $\varepsilon = (L_1, \ldots, L_t)$ and a tilting bundle T, so that under the equivalence $\mathbb{R}\operatorname{Hom}(T, -)$ between the derived categories above the category of coherent sheaves $\mathcal{F}(\varepsilon)$ with a filtration by the line bundles in ε is equivalent to the category $\mathcal{F}(\Delta)$ of good A-modules. Moreover, the functor $\mathbb{R}\operatorname{Hom}(T, -)$ maps L_i to $\Delta(i)$.*

Now it is desirable to have similar constructions for other varieties, in particular, in any dimension. We assume X is any smooth projective (or at least complete) algebraic variety and ε is an exceptional sequence. Then the theorem above generalizes to this situation. For we need to construct a quasi-hereditary algebra A. It appears as the endomorphism algebra of the universal extension $\overline{E}$ of ε. We discuss this construction in detail in Section 4. Here we only need to know, that there exists such a quasi-hereditary algebra A. The construction is completely parallel to the construction in [DR] for modules over finite dimensional algebras. Note that we do not need a full sequence anymore, however the category $\mathcal{F}(\varepsilon)$ can be rather small.

Theorem 1.4. *If ε is an exceptional sequence of sheaves on X, then there exists a quasi-hereditary algebra A so that the the category $\mathcal{F}(\varepsilon)$ of coherent sheaves with an ε–filtration is equivalent to the category $\mathcal{F}(\Delta)$ of Δ–good A–modules.*

Note that the equivalence in the theorem does, in general, not induce an equivalence between the corresponding derived categories. To obtain an equivalence of the derived categories it is necessary (and also sufficient, as the next theorem shows) that all higher extension groups vanish. Consequently, we eventually consider exceptional sequences ε with vanishing higher extension groups. For those sequences we can even construct a tilting object. In fact, the exceptional sequence we start with need not to be a sequence of sheaves, it can even consist of complexes of sheaves. However, for complexes we can not expect to get an equivalence for the filtered objects as above. For any exceptional sequence of complexes of coherent sheaves ε we define $\mathcal{D}(\varepsilon)$ to be the smallest full triangulated subcategory of $\mathcal{D}^b(X)$ containing all objects in ε. In case ε consists of coherent sheaves the category $\mathcal{F}(\epsilon)$ also generates $\mathcal{D}(\varepsilon)$.

Theorem 1.5. *Let $\varepsilon = (E_1, \ldots, E_t)$ be any exceptional sequence of objects in the bounded derived category of coherent sheaves on a smooth projective algebraic X with $\mathrm{Ext}^q(E_i, E_j) = 0$ for all $q < 0$ and all $q \geq 2$. Then the full triangulated subcategory $\mathcal{D}(\varepsilon)$ generated by ε admits a tilting object T, that is obtained as a universal extension by objects in ε. If the exceptional sequence ε consists of sheaves, then the tilting object is a sheaf as well, and if the exceptional sequence consists of vector bundles then T is also a vector bundle. Finally, if ε is full then T is a tilting object in $\mathcal{D}^b(X)$.*

Outline The reader just interested in a construction of a tilting bundle on a rational surface only needs to read Section 2 to Section 4. The construction gets more technical if one wants to construct tilting bundles with further properties or wants to obtain the more general results for any projective algebraic variety X. In fact the results in Section 3 and Section 4 have a nice interpretation in terms of differential graded algebras (DG-algebras). Given a DG-algebra as an endomorphism algebra of an exceptional sequence with only degree zero and degree one terms (that is $\mathrm{Ext}^q = 0$ for ≥ 2) then it is derived equivalent to an ordinary algebra (that is the endomorphism algebra of the universal extension).

In Section 2 we start with the construction of an exceptional sequence on any rational surface and prove some further vanishing results. In section 3 we consider universal (co)extensions of pairs of objects. Since we need to use the construction recursively, it is not sufficient to consider only exceptionl pairs. In Section 4 we define universal extensions of exceptional sequences and prove Theorem 1.1. In Section 5 we proceed with quasi-hereditary algebras. In fact we need this notion to define modules with good filtration. Moreover, using this notion we also get results for exceptional sequences on varieties of higher dimension. Finally, in Section 6 we construct one tilting bundle explicitly.

2. Exceptional sequences of line bundles on rational surfaces

In this section we construct on any rational surface a full exceptional sequence of line bundles $\varepsilon = (L_1, \ldots, L_t)$ that satisfies $\mathrm{Ext}^2(L_i, L_j) = 0$ for all $1 \leq i, j \leq t$. The construction is based on a form of a pull back of an exceptional sequence of line bundles for any blow up (Theorem 2.4), called standard augmentation in [HP2]. The main steps of the construction have already been proved in [HP2], Section 5. We only recall the main augmentation lemma and prove the vanishing of the higher Ext-groups. Since any rational surface, not isomorphic to $\mathbb{P}^2$, admits a blow down to a Hirzebruch surface in finitely many steps we can use induction on blow ups. For $\mathbb{P}^2$ such a full exceptional sequence is $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$. For the Hirzebruch surfaces we construct an infinite sequence of these sequences (where we can assume without loss of generality that L_1 is the trivial line bundles $\mathcal{O}$). Denote by $\mathbb{F}_m$ the mth Hirzebruch surface. It admits a natural projection to $\mathbb{P}^1$ with fiber $\mathbb{P}^1$ and a natural projection to $\mathbb{P}^2$ with exceptional fiber a prime divisor E. Then there is a prime divisor Q linear equivalent to $mP + E$. Computing the self-intersection numbers we obtain $P^2 = 0, E^2 = -m$, and $Q^2 = (mP + E)^2 = m$. Moreover, the Picard group of $\mathbb{F}_m$ is freely generated by $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ (or $\mathcal{O}(E)$, respectively), so any line bundle is isomorphic to $\mathcal{O}(\alpha P + \beta Q)$. The computation of the cohomology groups uses standard formulas in toric geometry (see [F], Section 3.5). The concrete results for the Hirzebruch surfaces can also be found in [H2].

Proposition 2.1. *Any Hirzebruch surface $\mathbb{F}_m$ admits a full exceptional sequence of line bundles $\varepsilon = (\mathcal{O}, \mathcal{O}(P), \mathcal{O}(Q + aP), \mathcal{O}(Q + (a+1)P))$, where this sequence is strong precisely when $a \geq 0$.*

Proof. From the standard formula for cohomology of line bundles on toric varieties (see [F] or [O]) follows for $\beta = -1$, or $\beta \geq 0$ and $\alpha \geq -1$

$$H^1(\mathbb{F}_m, \mathcal{O}(\alpha P + \beta Q)) = 0$$

For line bundles in the the sequence above, the second cohomology group vanishes, since for any $\beta \geq -1$

$$H^2(\mathbb{F}_m, \mathcal{O}(\alpha P + \beta Q)) = 0.$$

Consequently, we have on any Hirzebruch surface an infinite family of exceptional sequences and an infinite family of strongly exceptional sequences. It remains to show that both are full. To show this claim it is sufficient that one exceptional sequence in the family is full: consider the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(P)^2 \longrightarrow \mathcal{O}(2P) \longrightarrow 0.$$

Then we can recursively show that $(\mathcal{O}, \mathcal{O}(P), \mathcal{O}(Q + aP), \mathcal{O}(Q + (a + 1)P))$ is full, precisely when $(\mathcal{O}, \mathcal{O}(P), \mathcal{O}(Q + (a-1)P), \mathcal{O}(Q + aP))$ is full, just tensor the sequence above with $\mathcal{O}(Q + (a-1)P)$.

It is well-known that the sequence above is full for $\mathbb{P}^1 \times \mathbb{P}^1$ and the first Hirzebruch surface $\mathbb{F}_1$. Moreover, using the projection $\mathbb{F}_m \longrightarrow \mathbb{P}^1$ and results in [Or] we see that ε is full on any Hirzebruch surface. □

Remark. On a Hirzebruch surface with $m \geq 3$ the set of sequences in Theorem 2.1 is already the complete classification of (strongly) exceptional sequences of line bundles (up to a twist with a line bundle). For $m = 0, 1$, and 2 there is a finite number of further sequences (see [H2]).

In the next step we consider blow ups $\widetilde{X} \longrightarrow X$ in one point $x \in X$ with exceptional divisor $E \subset X$. If L is a line bundle on X, then we denote the pull back of L under a blow up of X with the same letter. Since $H^q(X; \widetilde{L}) = H^q(X; L)$ for any line bundle L on X this notation is convenient and does not lead to any confusion if we compute extension groups. Let X be any surface with an exceptional sequence $\varepsilon = (L_1, \dots, L_t)$ of line bundles on X. Then we obtain a sequence $\widetilde{\varepsilon} := (L_1(E), \dots, L_{i-1}(E), L_i, L_i(E), L_{i+1}, \dots, L_t)$ on the blow up $\widetilde{X}$.

Definition. Given an exceptional sequence ε on X. We call the sequence $\widetilde{\varepsilon}^{(i)} := (L_1(E), \dots, L_{i-1}(E), L_i, L_i(E), L_{i+1}, \dots, L_t)$ on the blow up a *standard augmentation* of ε (at position i).

Note that we can choose any i to obtain a standard augmentation, so for any blow up we have t choices to produce a new sequence. We will show that $\widetilde{\varepsilon}$ is exceptional for each $i = 1, \dots, t$. If ε is strongly exceptional then in some cases the new sequence is even strongly exceptional. In this case, there must exist a section in $\mathrm{Hom}\,(L_j, L_k)$ for all $j \leq i$ and all $k \geq i$ that does not vanish in x (see [HP2], Theorem 5.8 and the proof). However, in general the new sequence is only exceptional. The more detailed analysis of when $\widetilde{\varepsilon}$ is strongly exceptional is needed only for the concrete construction of the tilting bundle T. So we leave this part to the end in Section 6.

Proposition 2.2. *Let X be a surface with an exceptional sequence ε. Then $\widetilde{\varepsilon}$ is an exceptional sequence on the blow up $\widetilde{X}$ in one point. If ε is full then $\widetilde{\varepsilon}$ is also full.*

Proof. We prove first the vanishing result. For we have to show that on $\widetilde{X}$

$$\mathrm{Ext}^q(L_j, L_k(E)) = 0 = \mathrm{Ext}^q(L_j, L_k) \text{ for all } q \text{ and } j > k.$$

The second vanishing is obvious, since it coincides with the same group on X. It remains to show the first vanishing. Using $\mathrm{Ext}^q(L_j, L_k(E)) = H^q(L_j^{-1} \otimes L_k(E))$ it is sufficient to show the following lemma. Moreover, also $\mathrm{Ext}^q(L_i(E), L_i) = H^q(\mathcal{O}(-E)) = 0$ for $q = 0, 1, 2$ follows from the exact sequence in the proof of the following lemma.

Finally, we need to show that $\widetilde{\varepsilon}$ is full, provided ε is. For we consider the semi–orthogonal decomposition of $\mathcal{D}^b(\widetilde{X})$ with respect to $\mathcal{D}^b(X)$ and $\mathcal{O}_E(-1)$ (see [Or]). Obviously, by assumption, the line bundles L_i generate $\mathcal{D}^b(X)$. Moreover, the bundles $L_i(E)$ and L_i generate $\mathcal{O}_E(-1)$. Then in the last step, the subcategory generated by $L_j(E)$ and $\mathcal{O}_E(-1)$ contains L_j. Consequently, $\widetilde{\varepsilon}$ generates $\mathcal{D}^b(X)$ and $\mathcal{O}_E(-1)$, thus also $\mathcal{D}^b(\widetilde{X})$. □

Lemma 2.3. *If L is a line bundle on a surface Y with $H^q(Y; L) = 0$ for all q and $E \simeq \mathbb{P}^1$ is a (-1)–curve on Y so that $L\mid_E$ is trivial then $H^q(Y; L(E)) = 0$ for all q.*

Note that this is exactly our situation. If we consider the pull back of a line bundle L to a blow up, then the restriction of L to the exceptional divisor is trivial and the exceptional divisor is a (-1)–curve.

Proof. We consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}}(E) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0$$

and tensor it with L. Then in the corresponding long exact sequence the following groups vanish: $H^q(Y; L)$ for all q and $H^q(\mathbb{P}^1, L\mid_E(-1)) = H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$ for all q. Consequently, the claim follows. □

Our main theorem in this section states that any rational surface X admits a full exceptional sequence of line bundles, so that all the groups $\mathrm{Ext}^2(L_i, L_j)$ vanish. Such a sequence can be obtained by recursive standard augmentation from an exceptional sequence on a Hirzebruch surface.

Theorem 2.4. *Let X be any rational surface. Then X admits full exceptional sequences of line bundles, obtained from a full exceptional sequence of line bundles on a Hirzbruch surface by applying any standard augmentation in each step of the blow up. For such a sequence the groups* Ext^2 *between any two members of the sequence vanish.*

We will see in the next two sections that any exceptional sequence with this property defines a tilting bundle. So, using the universal extension (to define in the next sections) we have proved Theorem 1.1.

Remark. Note that we defined standard augmentation in [HP2] in a more general sense and allowed to blow up several times in one step. This gave us even more flexibility in constructing exceptional sequences. However, if we perform a standard augmentation only for the blow up of one point (as we do in this note) then it is always admissible in the sense of [HP2], Section 5, and it is sufficient to prove our results.

Proof. Since we have already shown the existence of a full exceptional sequence for any recursive blow up of a Hirzebruch surface (Proposition 2.2) we get a full exceptional sequence on any rational surface X. Then, using only recursive standard augmentations, we obtain a full exceptional sequence $\varepsilon = (L_1, \dots, L_t)$ with

$\mathrm{Ext}^2(L_i, L_j) = 0$ for all $1 \leq i, j \leq t$: just apply the proof of Lemma 2.3 also to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(E) \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

Then $\widetilde{\varepsilon}$ has no Ext^2 between any members of the sequence since $H^2(\widetilde{X}; L(-E)) = 0$ and thus also $\mathrm{Ext}^2(L_i(E), L_j) = 0 = \mathrm{Ext}^2(L_i, L_j)$. □

For later use in Section 6 we also need to prove some further vanishing results. In particular, we need to compute the cohomology of $\mathcal{O}(R_i - R_j)$, where R_i and R_j are both divisors (not prime in general) of self-intersection -1. So let X be any rational surface, not $\mathbb{P}^2$ and fix a sequence of blow downs of a (-1)–curve in X_i step by step to a Hirzebruch surface

$$X = X_t \longrightarrow X_{t-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 = \mathbb{F}_m.$$

Note that the rank of the Grothendieck group of $\mathbb{F}_m$ is 4 so for X it is just $t + 4$, where $t+4$ is also the length of the full exceptional sequence. In each X_i (for $i > 0$) we a distinguished (-1)-curve E_i blown down under $X_i \longrightarrow X_{i-1}$. We define R_i to be the divisor on X obtained as the pull back of E_i to X. Then we need to compute $H^q(X, \mathcal{O}(R_i - R_j))$.

Definition. We say R_i is above R_j if E_i is blown down to a point $P_i \in X_{i-1}$ that is on the inverse image of E_j in X_{i-1}. Then we also write $i \succ j$.

If R_i is above R_j then i must be larger than j,

Lemma 2.5. *The cohomology groups $H^1(X, \mathcal{O}(R_i - R_j))$ and $H^0(X, \mathcal{O}(R_i - R_j))$ both equal k precisely when $i \succ j$, otherwise both vanish. The second cohomology group $H^2(X, \mathcal{O}(R_i - R_j))$ is always zero.*

Proof. First note that $H^2(X, \mathcal{O}(R_i - R_j)) = H^2(X_i, \mathcal{O}(E_i - R_j))$ vanishes, since $H^2(X_j, \mathcal{O}(-R_j)) = H^2(X_i, \mathcal{O}(-R_j))$ is zero and we have Proposition 2.2. Further note that by Riemann-Roch the Euler characteristic of $\mathcal{O}(E_i - R_j)$ is zero, thus $H^0(X, \mathcal{O}(R_i - R_j))$ and $H^1(X, \mathcal{O}(R_i - R_j))$ have the same dimension.
We start to compute $H^0(X, \mathcal{O}(R_i - R_j))$. If R_i is above R_j then $R_i - R_j$ is effective, thus $H^0(X, \mathcal{O}(R_i - R_j)) \neq 0$. On the other hand $H^0(X, \mathcal{O}(R_i)) = H^0(X_i, \mathcal{O}(E_i))$ is one-dimensional and the dimension of $H^0(X, \mathcal{O}(R_i - R_j))$ can not exceed the dimension of $H^0(X, \mathcal{O}(R_i)$. A similar argument applies to $H^0(X, \mathcal{O}(R_i - R_j))$ for R_i not above R_j, then we obtain $H^0(X, \mathcal{O}(R_i - R_j)) = 0$. □

3. Universal extension of pairs

In this section we study universal extensions of objects in an abelian category. We work with sheaves, however all the techniques developed here work whenever the groups Ext^1 are finite dimensional. The principal aim of this part is to show, that any pair of objects with certain Ext–groups vanishing can be transformed, using universal (co)extensions, in a pair with vanishing Ext^1–group. Roughly spoken we apply a certain partial mutation (in the sense of [Bo]) to such a pair and the first extension group vanishes. However, the price we have to pay is, that we create new homomorphisms between the new objects. In particular, whenever we have a non-vanishing extension group we create, using universal extensions, additional homomorphism in the opposite direction. Thus the result is no longer an exceptional sequence. Even worse, the new object has nontrivial endomorphisms.

Definition. Let (E, F) be a pair of coherent sheaves. Then we define the *universal extension* $\overline{E}$ of E by F (respectively, the *universal coextension* $\overline{F}$ of F by E) by the following extension

$$0 \longrightarrow F \otimes \mathrm{Ext}^1(E, F)^* \longrightarrow \overline{E} \longrightarrow E \longrightarrow 0,$$

respectively

$$0 \longrightarrow F \longrightarrow \overline{F} \longrightarrow E \otimes \mathrm{Ext}^1(E,F)) \longrightarrow 0.$$

If we consider the second exact sequence as a triangle in the derived category, the boundary map is the canonical evaluation map $\mathrm{Ext}^1(E,F) \otimes E \longrightarrow F[1]$. This map induces, just by taking the adjoint, a canonical map $E \longrightarrow \mathrm{Ext}^1(E,F)^* \otimes F[1]$. The mapping cone over this map defines the first exact sequence. Thus, both exact sequences are unique up to isomorphism. The first exact sequence can be characterized by the following property: if we apply $\mathrm{Hom}\,(-,F)$ in the long exact sequence we get an induced map $\mathrm{Hom}\,(F,F) \otimes \mathrm{Ext}^1(E,F) \longrightarrow \mathrm{Ext}^1(E,F)$ that is the Yoneda product. This map is surjective, since $\mathrm{id} \otimes \xi$ maps to ξ. In a similar way one can characterize the second exact sequence. If we apply $\mathrm{Hom}\,(E,-)$ then we get a surjective map $\mathrm{Hom}\,(E,E) \otimes \mathrm{Ext}^1(E,F) \longrightarrow \mathrm{Ext}^1(E,F)$ that is just the ordinary Yoneda product. The following lemma is formulated in more generality than actually needed.

Lemma 3.1. *a) Let (E,F) be a pair of objects with $\mathrm{Ext}^q(F,E) = \mathrm{Ext}^q(E,E) = \mathrm{Ext}^q(F,F) = 0$ for all $q > 0$. Then $(E,\overline{F})$ and $(\overline{E},F)$ satisfy $\mathrm{Ext}^1(E,\overline{F}) = 0 = \mathrm{Ext}^q(\overline{F},E)$ and $\mathrm{Ext}^1(\overline{E},F) = 0 = \mathrm{Ext}^q(F,\overline{E})$ for all $q > 0$. Moreover, $\mathrm{Ext}^1(\overline{E},\overline{E}) = 0 = \mathrm{Ext}^1(\overline{F},\overline{F})$.*
b) If in addition we have $\mathrm{Ext}^q(E,F) = 0$ for some $q \geq 2$ then $\mathrm{Ext}^q(E,\overline{F}) = 0 = \mathrm{Ext}^q(\overline{F},E)$ and $\mathrm{Ext}^q(F,\overline{E}) = 0 = \mathrm{Ext}^q(\overline{E},F)$. Moreover, $\mathrm{Ext}^q(\overline{E},\overline{E}) = 0 = \mathrm{Ext}^q(\overline{F},\overline{F})$.

Roughly spoken we can replace any exceptional pair with only non-vanishing Hom and Ext^1 by a pair with only non-vanishing Hom. If we perform this universal extension recursively we can replace any full exceptional sequence with vanishing Ext^q for $q > 1$ by a tilting object (for the details see Section 4).

PROOF. The proof is just a standard diagram chasing, we only prove the crucial step for a universal extension. The proof is analogous for coextensions. Note that the vanishing of $\mathrm{Ext}^q(\overline{E},F)$ follows from the long exact sequence for $\mathrm{Hom}\,(-,F)$ applied to the universal extension of F by E. We show that $\mathrm{Ext}^1(F,\overline{E})$ vanishes. We apply $\mathrm{Hom}\,(-,F)$ to the universal extension. Since the boundary map $\mathrm{Hom}\,(F,F) \otimes \mathrm{Ext}^1(E,F) \longrightarrow \mathrm{Ext}^1(E,F)$ is surjective and $\mathrm{Ext}^1(F,F) = 0$ we obtain $\mathrm{Ext}^1(\overline{E},F) = 0$. To obtain $\mathrm{Ext}^1(\overline{E},\overline{E}) = 0$ we apply $\mathrm{Ext}^1(\overline{E},-)$ to the universal extension. Since $\mathrm{Ext}^1(\overline{E},E) = 0$ by assumption and $\mathrm{Ext}^1(\overline{E},F) = 0$ by the previous argument we obtain the desired vanishing. This finishes the proof of a) for the universal extension.
To show b) we only need to apply $\mathrm{Hom}\,(-,F)$ to the universal extension and get the vanishing form the long exact sequence. For the second vanishing we apply $\mathrm{Ext}^q(\overline{E},-)$ to the universal extension, as we did for $q = 1$ in a). □

REMARK. Note that $\overline{E}$, respectively $\overline{F}$ need not to be indecomposable. However, the above vanishing result holds for any direct summand as well. If we deal with an exceptional pair (E,F), then there exists a unique indecomposable direct summand $\overline{E}_1$ of $\overline{E}$ with $\dim \mathrm{Hom}\,(\overline{E}_1,E) = \dim \mathrm{Hom}\,(\overline{E},E) = \dim \mathrm{Hom}\,(E,E) = 1$. This allows later to distinguish certain indecomposable direct summands of our universal (co)extension.

The following lemma is useful for a construction of a tilting bundle with small rank. In fact we will show in Section 6 that we can on any rational surface construct full exceptional sequences of line bundles with $\dim \mathrm{Ext}^1(L_i,L_j) \leq 1$ for all i,j (there are at most one-dimensional extension groups).

Lemma 3.2. *Assume (E, F) is a pair of coherent sheaves on X with $\dim \operatorname{Ext}^1(E, F) = 1$, then the universal extension $\overline{E}$ is isomorphic to the universal coextension $\overline{F}$.*

Proof. In this case the end terms of the two defining short exact sequences coincide. Any non-trivial element $\xi \in \operatorname{Ext}^1(E, F) = k$ defines the same middle term. Since $\overline{E}$ and $\overline{F}$ are both unique up to isomorphism, they must be isomorphic. □

4. Universal extensions of exceptional sequences

In this section we start with an exceptional sequence and perform universal extension or coextensions recursively. We explain the construction for universal extensions, for universal coextensions the construction is dual.

Definition. Let $\varepsilon = (E_1, \ldots, E_t)$ be any exceptional sequence, then we define $E_i(1) := E_i(2) := \ldots := E_i(i) := E_i$ and $E_i(j)$ to be the universal extension of $E_i(j-1)$ by E_j for $j > i$. Thus we have exact sequences

$$0 \longrightarrow E_j \otimes \operatorname{Ext}^1(E_i(j-1), E_j)^* \longrightarrow E_i(j) \longrightarrow E_i(j-1) \longrightarrow 0$$

for all $t \geq j > i$ (for $i \leq j$ the sequences are always trivial, since the first term vanishes). Thus we have defined new objects $E_i(t)$ for $1 \leq i \leq t$ that are not necessarily indecomposable. We define $\overline{E}_i$ to be the unique indecomposable direct summand of $E_i(t)$ with the property that $k = \operatorname{Hom}(E_i(t), E_i) = \operatorname{Hom}(\overline{E}_i, E_i)$ and denote by $\overline{E}$ the direct sum $\oplus_{i=1}^t \overline{E}_i$. We call $\overline{E}$ the *universal extension* of the exceptional sequence ε and $\overline{E}_i$ the universal extension of E_i by $E_{i+1}, \ldots, E_t$.

In a dual way we define universal coextensions $\underline{E}$ and $\underline{E}_i$ (here we need to perform recursive universal coextensions with $E_{i-1}, \ldots, E_1$ instead).

Remark. Note that we can also split the exceptional sequence into two subsets and perform universal extensions in the first and universal coextensions in the second subset.

Theorem 4.1. *Let $\varepsilon = (E_1, \ldots, E_t)$ be an exceptional sequence, $E := \oplus E_i$ the direct sum of the elements in ε and $\overline{E}$ the direct sum of the objects $\overline{E}_i$ constructed above. Then $\operatorname{Ext}^1(\overline{E}, \overline{E}) = 0$. If, moreover, $\operatorname{Ext}^q(E, E) = 0$ for some $q > 1$ then also $\operatorname{Ext}^q(\overline{E}, \overline{E}) = 0$.*

Proof. We first show $\operatorname{Ext}^1(E_i(j), E_l) = 0$ for all $j \geq l$. Assume $j = l$ then this follows from Lemma 3.1, since

$$0 \to E_j \otimes \operatorname{Ext}^1(E_i(j-1), E_j)^* \to E_i(j) \to E_i(j-1) \longrightarrow 0$$

is a universal extension by E_j. Then use induction over j, the induction step just follows from applying $\operatorname{Ext}^1(-, E_l)$ to the defining exact sequence above.

As a consequence we obtain $\operatorname{Ext}^1(E_i(t), E_j) = 0$ for all $1 \leq j \leq t$. Now we apply $\operatorname{Ext}^1(E_i(t), -)$ to the defining sequence for $E_i(j)$ and obtain an exact sequence

$$\operatorname{Ext}^1(E_i(t), E_j) \otimes \operatorname{Ext}^1(E_i(j-1), E_j)^* \longrightarrow \operatorname{Ext}^1(E_i(t), E_i(j)) \longrightarrow \operatorname{Ext}^1(E_i(t), E_i(j-1)).$$

The first and the last term are zero by induction, thus the middle term vanishes for all $j \geq i$, in particular, it vanishes for $j = t$. In a similar way we can, using $\operatorname{Ext}^q(\overline{E}, -)$, show the last claim. □

Theorem 4.2. *Assume $\varepsilon = (E_1, \ldots, E_t)$ is a full exceptional sequence of sheaves (or complexes of sheaves), $E = \oplus_{i=1}^t E_i$, with $\operatorname{Ext}^q(E, E) = 0$ for all $q \neq 0, 1$. Then the universal extension $\overline{E}$ is a tilting sheaf (or tilting object).*

PROOF. Using the defining exact sequences we see that E and $\overline{E}$ generate the same subcategory in the derived category $\mathcal{D}^b(X)$ (or the corresponding triangulated category of the abelian category we work with). Moreover, $\mathrm{Ext}^q(\overline{E}, \overline{E}) = 0$ for all $q \neq 0$ by the previous result. □

sProof. (of Theorem 1.5) The tilting object is just the universal extension of the exceptional sequence ε. □

Now we are interested in sequences where we can also perform both, universal extensions and universal coextensions. To explain this we collect our objects into Ext^1–blocks. Let $\varepsilon = (E_1, \dots, E_t)$ be an exceptional sequence.

DEFINITION. We define a graph $\Gamma(\varepsilon)$, called *Ext-graph* with vertices i for $1 \leq i \leq t$ (the indices of the objects in ε) and an edge between i and j, whenever $\mathrm{Ext}^1(E_i, E_j)$ or $\mathrm{Ext}^1(E_j, E_i)$ does not vanish. Then an Ext–*block* consists of a connected component in $\Gamma(\varepsilon)$.

In this way we define certain subsets, for each subset we can choose either a universal extension or a universal coextension. Note that we get different results only if at least one subset consists of at least three elements or there are two-dimensional extension groups.

In the last section we discuss in more detail how to choose an exceptional sequence of line bundles on a rational surface X depending on the sequence of blow ups from a Hirzebruch surface (or the projective plane). Then we also know precisely the Ext^1–blocks. We can use this to minimize the non-vanishing extension groups so that $\overline{E}$ has a small rank.

5. Quasi-hereditary algebras

Let X be a rational surface, or even any algebraic variety, with a tilting sheaf that is the universal extension of a full exceptional sequence (with all higher Ext–groups vanishing) on X. Then we claim that the endomorphism algebra of this sequence satisfies a well-understood and extensively studied property: it is quasi-hereditary. Note that a quasi-hereditary algebra is an algebra with an order on its primitive orthogonal idempotents (or equivalently on its isomorphism classes of indecomposable projective modules). This order is just induced from the natural order in the exceptional sequence we started with.

If we used universal coextensions then we get the dual notion of so-called ∇–modules. If we use both, universal extensions in some Ext^1–blocks and universal coextensions in the remaining Ext^1–blocks, then the endomorphism algebra is not quasi-hereditary (except the blocks only consist of two members).

In this section we review some of the main properties on quasi-hereditary algebras. In particular, we use the so-called standardization introduced by Dlab and Ringel in [DR] to prove Theorem 1.2 and, more important, Theorem 1.4.

Let A be the endomorphism algebra of a sheaf T that is obtained as a universal extension of an exceptional sequence $\varepsilon = (E_1, \dots, E_t)$ of sheaves on X. We decompose T into indecomposable direct summands $T = \oplus_{i=1}^t T(i)$. Then the natural order in ε defines an order on the indecomposable projective A-modules $P(i) := \mathrm{Hom}\,(T, T(i))$. Moreover, we define $\Delta(i)$ to be the quotient of $P(i)$ by the maximal submodule generated by any direct sum $\oplus_{j<i} P(j)^{a(j)}$. This submodule is a proper submodule.

DEFINITION. The algebra A is called *quasi-hereditary* (with this order) if each $P(i)$ is in $\mathcal{F}(\Delta)$ (see [DR]).

Remark. Note that our definition is slightly different to the one in [DR], since an exceptional sequence of sheaves on a complete variety X has all the properties of a standardizable set.

Theorem 5.1. *Let $\varepsilon = (E_1, \ldots, E_t)$ be any exceptional sequence of sheaves and T the sheaf obtained from ε by its universal extension. Then the endomorphism algebra of T is quasi-hereditary with Δ–modules $\Delta(i) = \mathrm{Hom}(T, E_i)$.*

Proof. We need to show that any finitely generated projective A-module has a filtration by the modules $\Delta(i)$ defined as a quotient of $P(i)$. Note that the recursive universal extensions provides us with such a filtration for the objects $\overline{E}_i$ by induction: we consider the defining exact sequence (from the previous section)

$$0 \longrightarrow E_j \otimes \mathrm{Ext}^1(E_i(j-1), E_j)^* \longrightarrow E_i(j) \longrightarrow E_i(j-1) \longrightarrow 0.$$

If we apply $\mathrm{Hom}\,(T, -)$ we obtain an exact sequence of A-modules

$$0 \to \mathrm{Hom}\,(T, E_j) \otimes \mathrm{Ext}^1(E_i(j-1), E_j)^* \to \mathrm{Hom}\,(T, E_i(j)) \to \mathrm{Hom}\,(T, E_i(j-1)) \to 0.$$

(it is exact, since $\mathrm{Ext}^1(T, E_j) = 0$). Thus, by induction over j the right A–module $\mathrm{Hom}\,(T, E_i(t))$ admits a filtration by the modules $\Delta(i)$. Now we use that $\mathcal{F}(\Delta)$ is closed under direct summands (see e.g. the characterization in [DR], Theorem 1). □

Theorem 5.2. *Let $\varepsilon = (E_1, \ldots, E_t)$ be a full exceptional sequence of sheaves and T its universal extension. Then the functor $\mathrm{Hom}\,(T, -)$ induces an equivalence between $\mathcal{F}(\varepsilon)$ and $\mathcal{F}(\Delta)$ mapping E_i to $\Delta(i)$.*

Proof. Let F be any sheaf in $\mathcal{F}(\varepsilon)$. Then $\mathrm{Ext}^1(T, F) = 0$ since $\mathrm{Ext}^1(T, E_i) = 0$ for all i. Using the exact sequences defining the filtration of F recursively, we get a filtration of $\mathrm{Hom}\,(T, F)$ by $\Delta(i) = \mathrm{Hom}\,(T, E_i)$. Thus $\mathrm{Hom}\,(T, F)$ is in $\mathcal{F}(\Delta)$. Conversely, let M be an A-module in $\mathcal{F}(\Delta)$, then M has a projective presentation $P^1 \xrightarrow{f} P^0 \longrightarrow M \longrightarrow 0$. This defines an induced map $T^1 \xrightarrow{f^+} T^0$, where T^1 and T^0 are direct sums of direct summands of T. The map f^+ is just defined using the equality $A = \mathrm{Hom}_A(A, A) = \mathrm{Hom}_X(T, T)$ and the fact that P^1 and P^0 are direct sums of direct summands of A. Define $F(M)$ to be the cokernel of f^+. Note that f is injective precisely when f^+ is injective. Thus the Δ-filtration of M induces an filtration of $F(M)$ showing $F(M)$ is in $\mathcal{F}(\varepsilon)$. Note that under the functor $M \mapsto F(M)$ the module $\Delta(i)$ is mapped to E_i. Thus this functor is inverse to $\mathrm{Hom}\,(T, -)$ finishing the proof. □

Proof. (of Theorem 1.4)
The proof of Theorem 5.2 provides us with an explicit construction for the algebra A as the endomorphism algebra of a universal extension T of an exceptional sequence. □

Proof. (of Theorem 1.2)
If $\varepsilon = (E_1, \ldots, E_t)$ is any exceptional sequence with $\mathrm{Ext}^2(E_i, E_j) = 0$ for $1 \leq i, j \leq t$ on a surface X, then its universal extension has a quasi-hereditary endomorphism algebra by the theorem above. Such a sequence, consisting even of line bundles, exists by Theorem 2.4. □

Proof. (of Theorem 1.3)
This result was proved above, where we replace any sheaf E_i just by a line bundle L_i. □

Remark. The principal idea of the theorem above can be found in [DR], 3. standardization. Therein is a similar construction for any abelian category. In fact, such a construction, even in greater generality, can be performed in any abelian k-category with finite dimensional extension groups.

6. Construction of tilting bundle on rational surfaces

In this section we use the previous constructions to obtain a particular tilting bundle on any rational surfaces. We like to obtain a tilting bundle of small rank and a tilting bundle with a quasi-hereditary endomorphism algebra. Note that we are not optimal with our construction (compare for example with [HP2], Theorem 5.8), however, to avoid to many technical details and case by case considerations we present one construction that works for any rational surface X.

It is convenient to start with a strongly exceptional sequence on a Hirzebruch surface $\varepsilon = (\mathcal{O}, \mathcal{O}(P), \mathcal{O}(Q + aP), \mathcal{O}(Q + (a+1)P))$, where we can assume a is sufficiently large.

Then we chose a sequence of blow ups, where we allow to blow up finitely many different points in each step (so we use a slightly different notation than in Section 2)

$$X = X^m \xrightarrow{\pi_m} X^{m-1} \xrightarrow{\pi_{m-1}} \dots X^2 \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 = X_0 = \mathbb{F}_a$$

of X to a Hirzebruch surface. Note that we still have choices with this notation. To make the morphisms unique (for a chosen surface X with a fixed morphism π to $\mathbb{F}_a$) we blow up in the first step as many points as possible and proceed in this way. Thus if $x_v \in X^i$ is a point not blown up under $\pi_{i+1} : X^{i+1} \longrightarrow X^i$ then also its preimage in any X^j for $j > i$ is not blown up. Moreover, if $x_v \in X^i$ is blown up, we call $l(v) = i + 1$ its level and denote by E_v the corresponding exceptional divisor in X^{i+1}.

Next we define the blow up graph G as follows. Its vertices G_0 are the points x_v in X_i that are blown up under $\pi_i : X_{i+1} \longrightarrow X_i$. There is an edge between x_v and x_w, whenever $x_w \in E_v$ (or vice versa). This way, we get a levelled graph, that is for each edge v — w we have $|l(v) - l(w)| = 1$. The blow up graph is precisely the Hasse diagram (Hasse graph) for the partial order $\succ$ defined in Section 2. To construct an exceptional sequence of line bundles on X we also need the divsors R_v defined as the pull back of E_v in X. Note that the strict transform $\overline{E}_v$ of E_v is an irreducible component of the divisor R_v. For the self-intersection numbers on $X^{l(v)}$ we get $R_v^2 = -1 = E_v^2$ and on X we obtain $R_v^2 = -1$ and $\overline{E}_v^2 = -1 - a_v$ where a_v is the number of points in E_v blown up under π_{i+2}.

To start with the construction and to avoid to many notation we assume first X is the recursiv blow up of one point, so $\pi_i : X^i \longrightarrow X^{i-1}$ is the blow up of one point x_{i-1} on the exceptional divisor E_{i-1} for $i = 1, \dots, t = m$. With R_i we denote the pull back of E_i to X. Then we consider the full exceptional sequence

$$\varepsilon = (\mathcal{O}, \mathcal{O}(R_t), \mathcal{O}(R_{t-1}), \dots, \mathcal{O}(R_2), \mathcal{O}(R_1), \mathcal{O}(P), \mathcal{O}(aP + Q), \mathcal{O}((a+1)P + Q))$$

that is obtained by recursive standard augmentations in the first place. Using Lemma 2.5 we obtain non-vanishing extension groups $\operatorname{Ext}^1(\mathcal{O}(R_i), \mathcal{O}(R_j)) = k$ for all $i > j$. All other extension groups vanish. Then we can define recursively vector bundles V_i via $V_1 = \mathcal{O}(R_1)$ and

$$0 \longrightarrow \mathcal{O}(R_i) \longrightarrow V_i \longrightarrow V_{i-1} \longrightarrow 0$$

with $\operatorname{Ext}^1(V_{i-1}, \mathcal{O}(R_i)) = k$. In this way we define vector bundles V_i with non-trivial endomorphism ring. In fact the direct sum of all V_i has an endomorphism ring isomorphic to the Auslander algebra of $k[\alpha]/\alpha^t$, a quasi-hereditary algebra considered in [DR], Section 7.

Lemma 6.1. *The coherent sheaf V_i is a vector bundle of rank i and it is indecomposable with endomorphism ring isomorphic to $k[\alpha]/\alpha^i$. The direct sum $\oplus\mathcal{O} \oplus \bigoplus_i V_i \oplus \mathcal{O}(P) \oplus \mathcal{O}(aP+Q) \oplus \mathcal{O}((a+1)P+Q)$ is isomorphic to the universal extension $\overline{E}$ of the exceptional sequence ε.*

Proof. Clearly V_i is a vector bundle of rank i. To identify it with the universal extension of $\mathcal{O}(R_i)$ in the sequence ε we need to compute $\mathrm{Ext}^1_X(\mathcal{O}(R_i), V_{i-1}) = \mathrm{Ext}^1_{X^i}(\mathcal{O}(E_i), V_{i-1}) = k$. This can be shown recursively, just apply $\mathrm{Hom}\,(\mathcal{O}(E_i), -)$ to the defining sequence and we obtain $\mathrm{Hom}\,(\mathcal{O}(E_i), \mathcal{O}(R_j)) = k = \mathrm{Hom}\,(\mathcal{O}(E_i), V_j)$ and $\mathrm{Ext}^1(\mathcal{O}(E_i), \mathcal{O}(R_j)) = k = \mathrm{Ext}^1(\mathcal{O}(E_i), V_j)$ for all $j < i$. This follows directly from the equivalence in Theorem 1.4 for the exceptional sequence

$$(\mathcal{O}(R_t), \mathcal{O}(R_{t-1}), \ldots, \mathcal{O}(R_2), \mathcal{O}(R_1)).$$

Note that the quasi-hereditary algebra A for this sequence is the Auslander algebra of $k[\alpha]/\alpha^i$ considered in [DR], Section 7. However, one might check the claim also directly using the defining exact sequences and the vanishing of Ext^2. □

Now we consider the general case, let X be any rational surface together with a sequence of blow ups from a Hirzebruch surface. Then the Ext–blocks correspond to the points in $X^0 = \mathbb{F}_a$ that are blown up under π_1. Consequently, the Ext–blocks correspond to the connected components in the blow up graph, together with the four bundles $\mathcal{O}, \mathcal{O}(P), \mathcal{O}(aP+Q)$ and $\mathcal{O}((a+1)P+Q)$ we started with. To be precise, for our exceptional sequence the Ext–graph of $(\mathcal{O}(R_t), \mathcal{O}(R_{t-1}), \ldots, \mathcal{O}(R_2), \mathcal{O}(R_1))$ contains the blow up graph, by Lemma 2.5. Moreover, they have the same connected components. Note that we get non-trivial universal extensions only between two objects in the same connected component. For any point $x_j \in X^{l(j)-1}$ blown up under some morphism $\pi_{l(j)}$ we define the universal extension V_j of $\mathcal{O}(R_j)$ by all bundles $\mathcal{O}(R_i)$ with x_j is blown down to x_i under some composition of the maps π. This bundle V_i, according to Lemma 6.1, is a direct summand of the universal extension $\overline{E}$ for the exceptional sequence ε. The arguments above for the particular case also apply here, so we get a tilting bundle on X satisfying the following properties.

Theorem 6.2. *Let $\overline{E}$ be the universal extension of the full exceptional sequence ε above, then the direct summmands of $\overline{E}$ are isomorphic to the vector bundles V_i. In particular, $\mathcal{O} \oplus \bigoplus_i V_i \oplus \mathcal{O}(P) \oplus \mathcal{O}(aP+Q) \oplus \mathcal{O}((a+1)P+Q)$ is a tilting bundle on X. If $x_i \in X^j$ then $\mathrm{rk}\, V_i = j$. Thus $\overline{E}$ consists of vector bundles of rank at most t. Moreover, $\mathrm{Hom}\,(V_i, V_j) \neq 0$ precisely when x_i blows down to $x_j \in X^l$. In this case $\mathrm{Hom}\,(V_i, V_j)$ is l–dimensional.*

Proof. First note that the universal extension of $\mathcal{O}(R_i)$ with respect to ε coincides with the universal extension for the exceptional sequence consisting of all $\mathcal{O}(R_j)$ with x_i is mapped to xj under the composition $X^{l(i)} \longrightarrow X^{l(j)}$, since all other extension groups vanish. Thus we can apply the lemma above to show that V_i is indecomposable of rank $l(i)$ and $\overline{E}$ consists of the four line bundles on $\mathbb{F}_a$ and the bundles V_i. Consequently, $\overline{E}$ are isomorphic to the vectorbundles V_i. In particular, $\mathcal{O} \oplus \bigoplus_i V_i \oplus \mathcal{O}(P) \oplus \mathcal{O}(aP+Q) \oplus \mathcal{O}((a+1)P+Q)$ is a tilting bundle on X with $\mathrm{rk}\, V_i$ equals the level of x_i. Finally, the claim for the Hom–groups can be shown by induction using the defining exact sequences. □

Remark. We have chosen a simple example to construct at least one particular tilting bundle. In fact we have many other choices. First, we can use different projections to different Hirzebruch surfaces. Then we can choose the position of any standard augmentation and, eventually, we can chose to perform either extensions or coextensions. Moreover, we do not need to start with line bundles, in fact also the structure sheaf on any (-1)-curve can be used, since it is exceptional. However, apart from this a construction of other exceptional sheaves becomes more technical and the computation of the extensions groups might be more difficult as well. Thus line bundles are a nice, but not the only choice.

References

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4. Springer-Verlag, Berlin, 1984.

[Be] A. A. Beilinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra.* (Russian) Funktsional. Anal. i Prilozhen. **12**, no. 3 (1978), 68 – 69.

[Bo] A. I. Bondal, *Representations of associative algebras and coherent sheaves.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **53** no. 1 (1989), 25 – 44; translation in Math. USSR-Izv. **34**, no. 1 (1990), 23 – 42

[BV] A. Bondal,M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry.* Mosc. Math. J. **3**, no. 1 (2003), 1 – 36.

[BH] R. O. Buchweitz, L. Hille, *Hochschild (Co-)Homology of Schemes with Tilting Object* preprint, arXiv:1003.4201.

[DR] V. Dlab, C. M. Ringel, *The module theoretical approach to quasi-hereditary algebras.* Representations of algebras and related topics (Kyoto, 1990), 200 – 224, London Math. Soc. Lecture Note Ser. 168, Cambridge Univ. Press Cambridge, 1992.

[F] W. Fulton, *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[H1] L. Hille, *Consistent algebras and special tilting sequences.* Math. Z. 220 (1995), no. 2, 189 – 205.

[H2] L. Hille, *Exceptional sequences of line bundles on toric varieties.* Mathematisches Institut, Georg-August-Universität Göttingen: Seminars 2003/2004, 175 – 190, Universitätsdrucke Göttingen, Göttingen, 2004.

[HP1] L. Hille, M. Perling, *A counterexample to King's conjecture.* Compos. Math. 142 (2006), no. 6, 1507 – 1521.

[HP2] L. Hille, M. Perling *Exceptional Sequences of Invertible Sheaves on Rational Surfaces.* preprint, arXiv:0810.1936.

[Hu] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[Kap] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces.* Invent. Math. **92**, no. 3 (1988), 479 – 508.

[Kaw] Y. Kawamata, *Derived categories of toric varieties.* Michigan Math. J. **54**, no. 3 (2006), 517 – 535.

[O] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 15. Springer-Verlag, Berlin, 1988.

[Or] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 4, 852–862; translation in Russian Acad. Sci. Izv. Math. 41 (1993), no. 1, 131 – 141

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