# ON A TRIANGULATED CATEGORY WHICH BEHAVES LIKE A CLUSTER CATEGORY OF INFINITE DYNKIN TYPE, AND THE RELATION TO TRIANGULATIONS OF THE INFINITY-GON

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## 0. Introduction

This paper investigates a certain 2-Calabi-Yau triangulated category $\mathsf{D}$ whose Auslander-Reiten quiver is $\mathbb{Z}A_\infty$. We show that the cluster tilting subcategories of $\mathsf{D}$ form a so-called cluster structure, and we classify these subcategories in terms of what one may call 'triangulations of the $\infty$-gon'.

This is reminiscent of the cluster category $\mathsf{C}$ of type $A_n$ which is a 2-Calabi-Yau triangulated category whose Auslander-Reiten quiver is a quotient of $\mathbb{Z}A_n$; see [2] and [4]. The cluster tilting subcategories of $\mathsf{C}$ form a cluster structure and they are classified in terms of triangulations of the $(n+3)$-gon by [4].

The category $\mathsf{D}$ behaves like a 'cluster category of type $A_\infty$'.

Let us give some more details. The category $\mathsf{D}$ is the compact derived category $\mathsf{D}^c(A)$ of the Differential Graded cochain algebra $A = \mathrm{C}^*(S^2; k)$ where $S^2$ is the 2-sphere and $k$ is a field. In fact, $S^2$ is formal over any field $k$, so one can equally well use as $A$ the quasi-isomorphic DG algebra obtained by placing copies of $k$ in cohomological degrees 0 and 2.

The category $\mathsf{D}$ was studied in [9], and it follows from [9, prop. 4.4 and cor. 5.2] that it is a 2-Calabi-Yau category. This makes it natural to consider maximal 1-orthogonal subcategories $\mathscr{A}$ of $\mathsf{D}$. These are the subcategories which satisfy $\mathscr{A} = (\Sigma^{-1}\mathscr{A})^\perp = {}^\perp(\Sigma\mathscr{A})$ and they were introduced by Iyama; see [2], [6], [7], [8], and [11]. Our first main result is this.

**Theorem A.** *There is a bijection between maximal 1-orthogonal subcategories of $\mathsf{D}$ and triangulations of the $\infty$-gon.*

By a triangulation of the $\infty$-gon, we mean a maximal set of non-intersecting arcs connecting non-neighbouring integers: We adopt the philosophy that the integers can be viewed as the vertices of the $\infty$-gon, and that the arcs can be viewed as diagonals. There are two obvious ways to achieve such maximal sets; they are shown in the following two sketches where the arcs must be continued ad infinitum according to the indicated pattern. First a 'leapfrog' configuration which is locally

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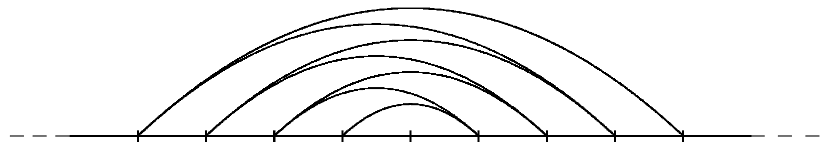
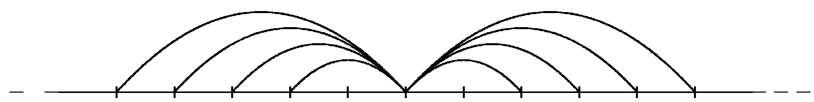
finite in the sense that only finitely many arcs end in each integer.



(1)

Then a 'fountain' where infinitely many arcs going to either side end in a single integer.

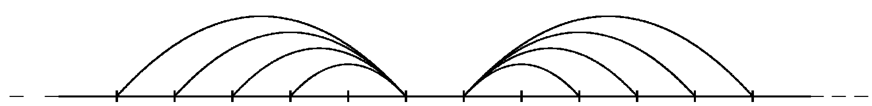


(2)

Maximal 1-orthogonal subcategories are particularly important if they are functorially finite. Then they are called cluster tilting subcategories and the corresponding quotient categories are abelian by [11, sec. 2] and [13, thm. 3.3]. Our second main result is the following.

**Theorem B.** *A maximal* 1*-orthogonal subcategory of* $\mathsf{D}$ *is functorially finite if and only if the corresponding triangulation of the* $\infty$*-gon is locally finite or has a fountain.*

The point is that there are triangulations of the $\infty$-gon like the following, which have a 'right-fountain' and a 'left-fountain' but do not satisfy the conditions of Theorem B.



(3)

This gives an example of a maximal 1-orthogonal subcategory of $\mathsf{D}$ which is not functorially finite; see Example 4.5.

If $\mathscr{A}$ is a cluster tilting subcategory of $\mathsf{D}$, then we will call the collection $A$ of indecomposable objects of $\mathscr{A}$ a cluster. Since $\mathscr{A} = \operatorname{add} A$, the subcategory and the corresponding cluster contain the same information. Our third main result is the following.

**Theorem C.** *The clusters form a cluster structure in* $\mathsf{D}$*.*

The notion of a cluster structure was introduced in [1] and we have reproduced it in Section 5. Some of the salient features are that if $A$ is a cluster and $a$ is an indecomposable object in $A$, then $a$ can be replaced with a unique other indecomposable object $a^*$ of $\mathsf{D}$ such that a new cluster $A^*$ results, and that passing from the Auslander-Reiten (AR) quiver of $\operatorname{add} A$ to the AR quiver of $\operatorname{add} A^*$ is given by Fomin-Zelevinsky mutation at $a$ in the sense of [5, sec. 8].

There are several viewpoints on the results of this paper.

(1) As mentioned, $\mathsf{D}$ behaves like a cluster category of type $A_\infty$.

(2) Theorems A and B show that $\mathsf{D}$ can be viewed as a categorification of triangulations of the $\infty$-gon. Such triangulations have not, to our knowledge, been studied elsewhere, but they seem to be interesting combinatorial objects.

(3) Theorem C shows that $\mathsf{D}$ provides a cluster tilting theory for the abelian categories of the form $\mathsf{D}/\mathscr{A}$ where $\mathscr{A}$ is a cluster tilting subcategory of $\mathsf{D}$. Namely, we have $\mathsf{D}/\mathscr{A} \simeq \mathsf{mod}\,\mathscr{A}$ by [11, sec. 2] and [13, cor. 4.4], and Theorem C says that the AR quivers of $\mathscr{A} = \operatorname{add} A$ and $\mathscr{A}^* = \operatorname{add} A^*$ are related by Fomin-Zelevinsky mutation at $a$, so passing from $\mathsf{D}/\mathscr{A} \simeq \mathsf{mod}\,\mathscr{A}$ to $\mathsf{D}/\mathscr{A}^* \simeq \mathsf{mod}\,\mathscr{A}^*$ can be viewed as 'cluster tilting at $a$'. Some of the categories $\mathsf{mod}\,\mathscr{A}$ are hereditary categories of the form $\operatorname{rep}\Gamma$ where $\Gamma$ is an infinite quiver; see Example 5.4. Such categories were investigated by Reiten and Van den Bergh in [15] and form an important branch in the taxonomy of hereditary categories.

Other aspects of the category $\mathsf{D}$ have been studied in the literature: It is equivalent to the category $\mathscr{C}_{\mathscr{H}}$ which appeared in [11, sec. 2.1] where a cluster tilting subcategory was also shown, its Hall algebra was computed in [12], and some relations with algebraic topology were investigated in [14].

Let us remark that we will actually obtain $\mathsf{D}$ by a different recipe from the one mentioned above. Namely, we will use $\mathsf{D} = \mathsf{D}^{\mathrm{f}}(k[T])$ where $k[T]$ is viewed as a DG algebra with $T$ placed in homological degree 1 and zero differential. This gives a category which is triangulated equivalent to the one above by [9, sec. 8], but some computations become easier.

The paper is organised as follows. Section 1 gives basic information on the category $\mathsf{D}$. Section 2 investigates the morphisms of $\mathsf{D}$. Section 3 gives the information we need on triangulations of the $\infty$-gon. Section 4 proves Theorems A and B, and Section 5 proves Theorem C.

Section 6 presents some questions; for instance, is it possible to define a 'cluster algebra of type $A_\infty$'?

**Notation 0.1.** The set of morphisms in $\mathsf{D}$ from $x$ to $y$ is denoted $\mathsf{D}(x, y)$.

We will join a common abuse of terminology by saying 'indecomposable object' when we mean 'isomorphism class of indecomposable objects', and by viewing two subcategories of $\mathsf{D}$ as equal if they have the same essential closure.

## 1. Basic properties of the category $\mathsf{D}$

This section defines the category $\mathsf{D}$ and recalls a few basic properties.

**Setup 1.1.** Throughout, $k$ is a field and $R = k[T]$ is the polynomial algebra. We view $R$ as a DG algebra with zero differential and $T$ placed in homological degree 1.

Our main object of study is

$$\mathsf{D} = \mathsf{D}^{\mathrm{f}}(R),$$

the derived category of DG $R$-modules with finite dimensional homology over $k$. The suspension, Serre functor, and AR quiver of $\mathsf{D}$ will be denoted by $\Sigma$, $S$, and $Q$.

The next three remarks sum up some results on $\mathsf{D}$ from [9, section 8 in particular].

**Remark 1.2** (Basic properties)**.** The category $\mathsf{D}$ has finite dimensional Hom spaces over $k$ and split idempotents, so it is Krull-Schmidt. It is a 2-Calabi-Yau triangulated category, that is, its Serre functor is $S = \Sigma^2$ where $\Sigma$ denotes the suspension functor. Accordingly, the AR translation is $\tau = S\Sigma^{-1} = \Sigma$.

**Remark 1.3** (Indecomposable objects)**.** For each integer $r \geq 0$, we have a DG $R$-module

$$X_r = R/(T^{r+1})$$

which is concentrated in homological degrees from $0$ to $r$. The indecomposable objects of $\mathsf{D}$ are $\Sigma^j X_r$ for $j$, $r$ integers, $r \geq 0$.

There is an obvious short exact sequence of DG modules $0 \to \Sigma^{r+1}R \to R \to X_r \to 0$ which induces a distinguished triangle

$$\Sigma^{r+1}R \to R \to X_r \to \Sigma^{r+2}R \tag{4}$$

in $\mathsf{D}$. Hence the DG module $X_r$ is quasi-isomorphic to the mapping cone $C_r$ of $\Sigma^{r+1}R \to R$.

Denote by $(-)^\natural$ the operation of forgetting the differential. Then $R^\natural$ is a graded algebra, $C_r^\natural$ is a graded $R^\natural$-module, and the construction of the mapping cone gives

$$C_r^\natural = R^\natural \oplus \Sigma^{r+2}R^\natural.$$

Denoting the generators of the two copies of $R^\natural$ by $e_0$ and $e_{r+2}$, the differential of $C_r$ is given by

$$\partial(e_0) = 0, \quad \partial(e_{r+2}) = T^{r+1}e_0.$$

It is easy to see that $C_r$ is a minimal semi-free resolution of $X_r$.

**Remark 1.4** (Auslander-Reiten quiver)**.** The AR quiver $Q$ of $\mathsf{D}$ is $\mathbb{Z}A_\infty$ and the indecomposable objects are arranged in the quiver as follows.

$$
\begin{array}{ccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \\
\cdots & & \Sigma^0 X_3 & & \Sigma^{-1} X_3 & & \Sigma^{-2} X_3 & & \Sigma^{-3} X_3 & & \Sigma^{-4} X_3 & & \cdots \\
 & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & \\
 & \Sigma^1 X_2 & & \Sigma^0 X_2 & & \Sigma^{-1} X_2 & & \Sigma^{-2} X_2 & & \Sigma^{-3} X_2 & & \Sigma^{-4} X_2 & \\
\nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow \\
\cdots & & \Sigma^1 X_1 & & \Sigma^0 X_1 & & \Sigma^{-1} X_1 & & \Sigma^{-2} X_1 & & \Sigma^{-3} X_1 & & \cdots \\
 & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & \\
 & \Sigma^2 X_0 & & \Sigma^1 X_0 & & \Sigma^0 X_0 & & \Sigma^{-1} X_0 & & \Sigma^{-2} X_0 & & \Sigma^{-3} X_0 &
\end{array}
$$

We will use the following standard coordinate system on $Q$.

$$
\begin{array}{ccccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \\
\cdots & & (-5,0) & & (-4,1) & & (-3,2) & & (-2,3) & & (-1,4) & & \cdots \\
 & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & \\
 & (-5,-1) & & (-4,0) & & (-3,1) & & (-2,2) & & (-1,3) & & (0,4) & \\
\nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow \\
\cdots & & (-4,-1) & & (-3,0) & & (-2,1) & & (-1,2) & & (0,3) & & \cdots \\
 & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & \\
 & (-4,-2) & & (-3,-1) & & (-2,0) & & (-1,1) & & (0,2) & & (1,3) &
\end{array}
$$

Accordingly, coordinate pairs and indecomposable objects will be related by

$$(m,n) = \Sigma^{-n} X_{n-m-2}.$$

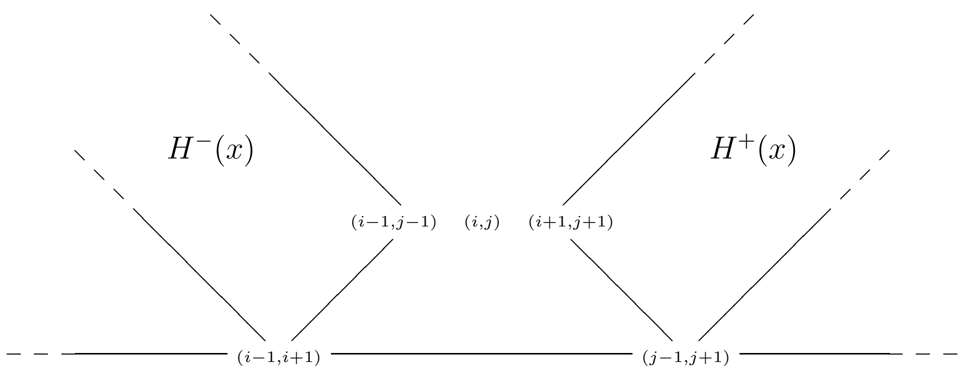
Note that in terms of coordinates, the actions of $\Sigma = \tau$ and $S = \Sigma^2$ on objects are

$$\Sigma(m,n) = (m-1,n-1), \quad S(m,n) = (m-2,n-2).$$

## 2. Morphisms in the category $\mathsf{D}$

This technical section provides detailed information on the morphisms of the category $\mathsf{D}$.

**Definition 2.1.** Let $x = (i,j)$ be a vertex of the AR quiver $Q$ of $\mathsf{D}$. We define (unbounded) subsets $H^-(x)$ and $H^+(x)$ of vertices of $Q$ which can be sketched as follows.




(5)

The subsets include the edges. In a more rigorous vein, we have

$$H^-(x) = \{\, (m,n) \mid m \le i-1, \;\; i+1 \le n \le j-1 \,\},$$
$$H^+(x) = \{\, (m,n) \mid i+1 \le m \le j-1, \;\; j+1 \le n \,\}.$$

We write $H(x) = H^-(x) \cup H^+(x)$.

The following Proposition says that an indecomposable object $x$ has non-zero morphisms to the indecomposable objects $y$ in two regions like the ones in figure (5). The object $x$ is at the leftmost vertex of the right hand region.

**Proposition 2.2.** *Let $x$ and $y$ be indecomposable objects of $\mathsf{D}$. Then*

$$\mathsf{D}(x,y) = \begin{cases} k & \text{for } y \in H(\Sigma x), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Using a power of $\Sigma$, we can shift $x$ and $y$ horizontally on the quiver without loss of generality, and so we can assume

$$x = (-r-2, 0) = X_r.$$

We will write

$$y = (m,n) = \Sigma^{-n} X_{n-m-2}.$$

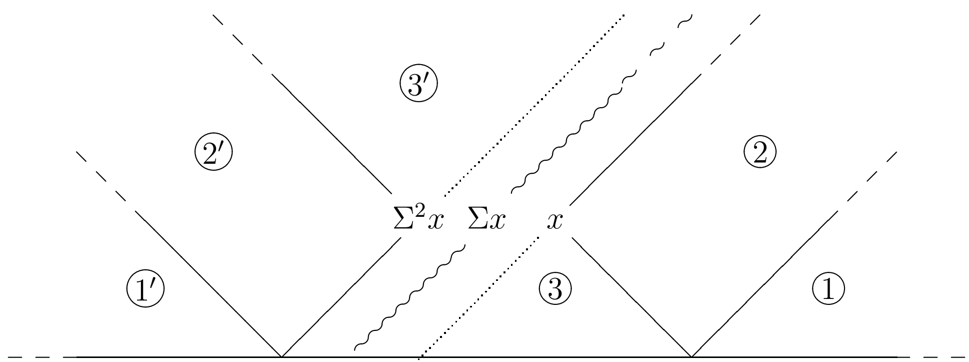
Taking Hom of the distinguished triangle (4) into the object $\Sigma^{-n}X_{n-m-2}$ gives a long exact sequence containing

$$\begin{aligned}\mathsf{D}(\Sigma^{r+2}R, \Sigma^{-n}X_{n-m-2}) &\to \mathsf{D}(X_r, \Sigma^{-n}X_{n-m-2}) \\ &\to \mathsf{D}(R, \Sigma^{-n}X_{n-m-2}) \to \mathsf{D}(\Sigma^{r+1}R, \Sigma^{-n}X_{n-m-2})\end{aligned}$$

which is

$$\begin{aligned}\mathrm{H}_{n+r+2}(X_{n-m-2}) &\to \mathsf{D}(x,y) \\ &\to \mathrm{H}_n(X_{n-m-2}) \to \mathrm{H}_{n+r+1}(X_{n-m-2}).\end{aligned} \tag{6}$$

Consider the sketch of the AR quiver below. It is cumbersome, but elementary, to verify from (6) that $\mathsf{D}(x,y)$ is $k$ when $y$ is in the region ② (which includes the edges and is equal to $H^+(\Sigma x)$). Also, $\mathsf{D}(x,y)$ is 0 when $y$ is in region ① (which does not include the diagonal edge) or region ③ (which includes the dotted edge, but not the other one).




Serre duality says $\mathsf{D}(a,b) \cong \mathsf{D}(b, \Sigma^2 a)^\vee$. By applying this to the previous results, we get that $\mathsf{D}(x,y)$ is $k$ when $y$ is in the region ②′ (which includes the edges and is equal to $H^-(\Sigma x)$). Also, $\mathsf{D}(x,y)$ is 0 when $y$ is in region ①′ (which does not include the diagonal edge) or region ③′ (which includes the dotted edge but not the other one).

To complete the proof, we must show that $\mathsf{D}(x,y)$ is 0 when $y$ is on the wavy line through $\Sigma x$. The vertices on this line have the form $(-r-3, -r-1+t)$ for $t \geq 0$, that is, they are the objects $\Sigma^{r+1-t}X_t$ for $t \geq 0$, and we must show that a morphism $X_r \to \Sigma^{r+1-t}X_t$ in $\mathsf{D}$ is 0. Such a morphism is a homotopy class of morphisms of DG modules

$$\gamma : C_r \to \Sigma^{r+1-t}X_t$$

where $C_r$ is the minimal semi-free resolution of $X_r$ from Remark 1.3.

Recall that $C_r$ has generators $e_0$ and $e_{r+2}$ in homological degrees 0 and $r+2$. The DG module $\Sigma^{r+1-t}X_t$ is concentrated in homological degrees from $r+1-t$ to $r+1$.

If $r+1-t > 0$, then $\Sigma^{r+1-t}X_t$ is 0 in each degree where $C_r$ has a generator, so $\gamma = 0$ is clear.

If $r+1-t \leq 0$, then $\Sigma^{r+1-t}X_t$ is 0 in degree $r+2$, but it is $k$ in degree 0. Potentially, $\gamma(e_0)$ could be non-zero. However,

$$T^{r+1}\gamma(e_0) = \gamma(T^{r+1}e_0) = \gamma\partial(e_{r+2}) = \partial\gamma(e_{r+2}) \overset{\text{(a)}}{=} \partial(0) = 0$$

where (a) is because $\Sigma^{r+1-t}X_t$ is 0 in degree $r+2$. Since $\Sigma^{r+1-t}X_t$ is equal to $k$ in degree $r+1$, it follows that $\gamma(e_0) = 0$ so $\gamma = 0$. $\square$

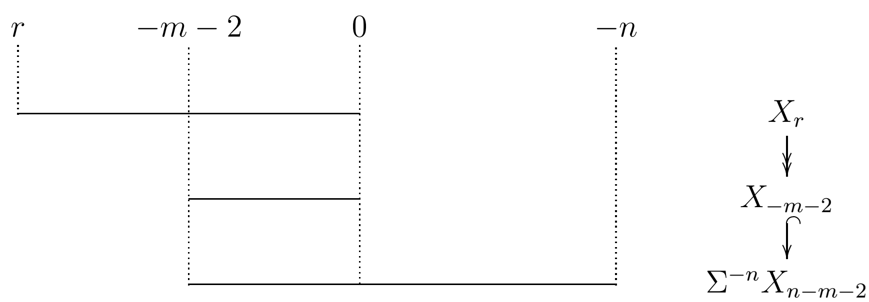
**Corollary 2.3.** *Let $x$ and $y$ be indecomposable objects of $\mathsf{D}$. The following are equivalent.*

(i) $\mathsf{D}(x, y) \neq 0$.
(ii) $\mathsf{D}(x, y) = k$.
(iii) $y \in H(\Sigma x)$.
(iv) $x \in H(\Sigma^{-1} y)$.

*Proof.* (i), (ii), and (iii) are equivalent by Proposition 2.2. Using Serre duality, (i) is equivalent to $\mathsf{D}(y, \Sigma^2 x) \neq 0$. Using (iii), this is equivalent to $\Sigma^2 x \in H(\Sigma y)$, that is $x \in H(\Sigma^{-1} y)$, and this is (iv). $\square$

**Remark 2.4** (Forward morphisms)**.** Proposition 2.2 and Corollary 2.3 show that there are two distinct types of non-zero morphisms going from $x$ to indecomposable objects of $\mathsf{D}$: Those going to objects in $H^+(\Sigma x)$ will be called *forward morphisms*, and those going to objects in $H^-(\Sigma x)$ will be called *backward morphisms*. The backward morphisms cannot be seen in the AR quiver; they are in the infinite radical of $\mathsf{D}$.

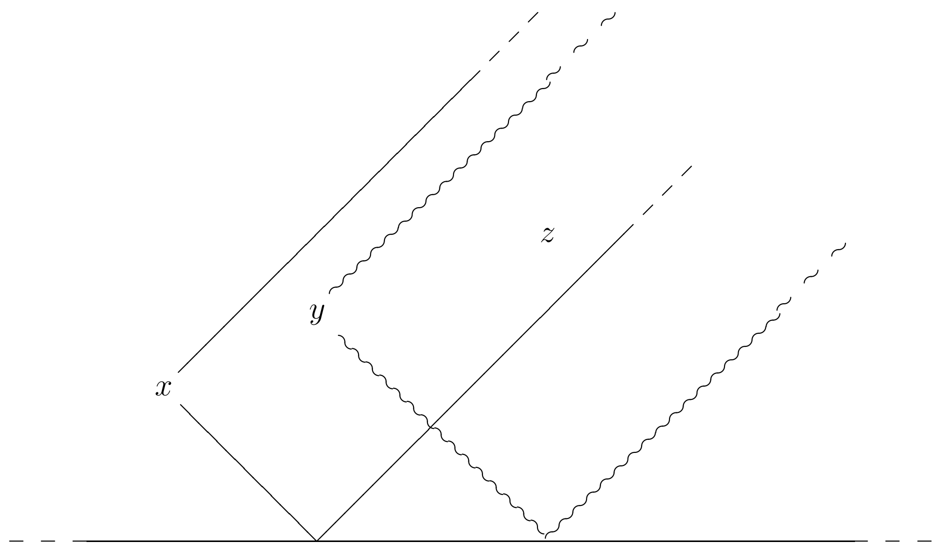
The forward morphisms have an easy model: Up to multiplication by a non-zero scalar, they are induced by certain canonical morphisms of DG modules. Namely, consider again the case $x = (-r-2, 0) = X_r$. Then $x$ is a DG module which is concentrated in homological degrees from 0 to $r$. Let $y = (m, n) = \Sigma^{-n} X_{n-m-2}$ be in the region $H^+(\Sigma x)$ whence $-r-2 \leq m \leq -2$ and $n \geq 0$. Then $y$ is a DG module which is concentrated in homological degrees from $-n$ to $-m-2$, and we have $-n \leq 0$ and $0 \leq -m-2 \leq r$. We can sketch the non-zero parts of the DG modules $x = X_r$ and $y = \Sigma^{-n} X_{n-m-2}$ as follows, where the numbers at the top are homological degrees and where each horizontal line indicates the degrees where a module has non-zero components.




We have included the DG module $X_{-m-2}$ in the sketch, and it is clear that there is a surjective and an injective morphism of DG modules as indicated. Their composition is a canonical morphism of DG modules which induces a forward morphism $x \to y$ in $\mathsf{D}$.

Observe that the canonical morphism of DG modules induces a (non-zero) forward morphism $x \to y$ in $\mathsf{D}$ if and only if there is a homological degree where both $x$ and $y$ have a non-zero component. Indeed, it is easy to check that this happens if and only if $y$ is in $H^+(\Sigma x)$.

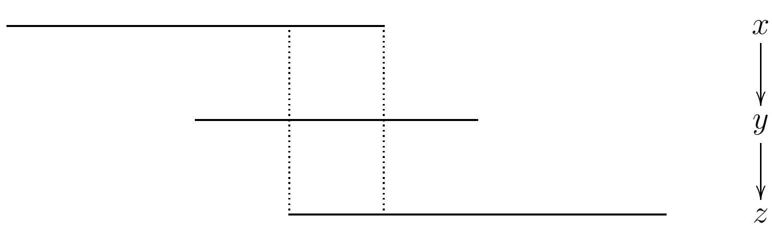
**Lemma 2.5.** *Let $x$, $y$, and $z$ be indecomposable objects of $\mathsf{D}$ such that $y, z \in H^+(\Sigma x)$ and $z \in H^+(\Sigma y)$, for instance as in the following sketch.*




(i) *The composition of non-zero morphisms $x \to y$ and $y \to z$ is non-zero.*

(ii) *Let $y \xrightarrow{f} z$ be a non-zero morphism. Then each morphism $x \to z$ factors as $x \to y \xrightarrow{f} z$.*

*Proof.* (i). Since we have $y \in H^+(\Sigma x)$ and $z \in H^+(\Sigma y)$, the non-zero morphisms $x \to y$ and $y \to z$ in $\mathsf{D}$ are forward morphisms. By Remark 2.4, up to multiplication by non-zero scalars which can be ignored, they are induced by canonical morphisms of DG modules which we can indicate as follows.




It is clear that these compose to a canonical morphism $x \to z$.

Since we have $z \in H^+(\Sigma x)$, Remark 2.4 gives that there is a homological degree where both the DG modules $x$ and $z$ have a non-zero component, and hence the canonical morphism $x \to z$ induces a (non-zero) forward morphism $x \to z$ in $\mathsf{D}$ as desired.

(ii). We must show that $\mathsf{D}(x, f) : \mathsf{D}(x, y) \to \mathsf{D}(x, z)$ is surjective. Since each non-zero Hom set is isomorphic to $k$, it is enough to see that $\mathsf{D}(x, f)$ is non-zero, and this follows from part (i) because it sends $x \to y$ to the composition $x \to y \xrightarrow{f} z$. $\square$

**Lemma 2.6.** *Let $x$, $y$, and $z$ be indecomposable objects of $\mathsf{D}$.*

(i) *$y \in H^+(\Sigma x) \Leftrightarrow Sx \in H^-(\Sigma y)$.*
(ii) *$z \in H^-(\Sigma x) \Leftrightarrow Sx \in H^+(\Sigma z)$.*

*Proof.* (i) Suppose $y \in H^+(\Sigma x)$; then there is a non-zero morphism $x \to y$ in $\mathsf{D}$ by Corollary 2.3. The Serre duality isomorphism $\mathsf{D}(x, y) \cong \mathsf{D}(y, Sx)^\vee$ implies that there is a non-zero morphism $y \to Sx$ so we have $Sx \in H(\Sigma y)$. To establish the implication $\Rightarrow$, it remains to see that $Sx$ is in $H^-(\Sigma y)$, not $H^+(\Sigma y)$.

However, if $x = (i, j)$ then the shape of the region $H^+(\Sigma x)$ implies $y = (i+p, j+q)$ for some $p, q \geq 0$. This again means that the points of $H^+(\Sigma y)$ have the form $(i+p+p', j+q+q')$ for some $p', q' \geq 0$, but $Sx = (i-2, j-2)$ is not of this form so we must have $Sx$ in $H^-(\Sigma y)$.

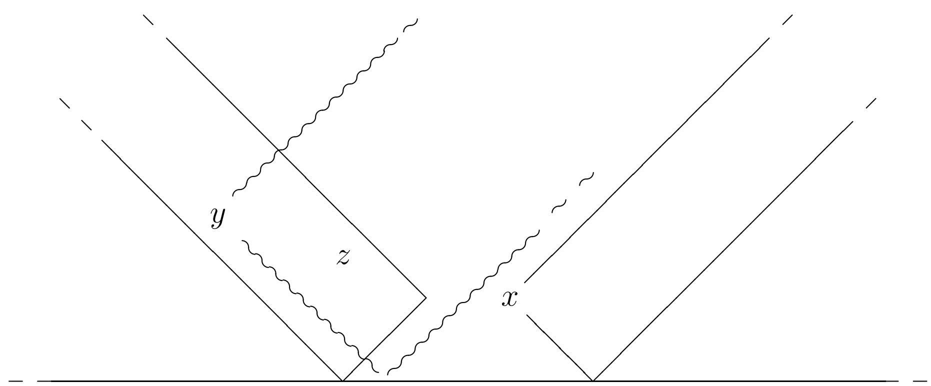
The implication $\Leftarrow$ is proved by a similar argument.

(ii) We have

$$z \in H^-(\Sigma x) \Leftrightarrow SS^{-1}z \in H^-(\Sigma x) \Leftrightarrow x \in H^+(\Sigma S^{-1}z) \Leftrightarrow Sx \in H^+(\Sigma z)$$

where the second biimplication is by part (i). □

**Lemma 2.7.** *Let $x$, $y$, and $z$ be indecomposable objects of $\mathsf{D}$ such that $y, z \in H^-(\Sigma x)$ and $z \in H^+(\Sigma y)$, for instance as in the following sketch.*




*Let $y \xrightarrow{f} z$ be a non-zero morphism. Then each morphism $x \to z$ factors as $x \to y \xrightarrow{f} z$.*

*Proof.* We must show that $\mathsf{D}(x, f) : \mathsf{D}(x, y) \to \mathsf{D}(x, z)$ is surjective. Using Serre duality, it is the same to show that $\mathsf{D}(f, Sx) : \mathsf{D}(z, Sx) \to \mathsf{D}(y, Sx)$ is injective. This map sends $z \xrightarrow{\zeta} Sx$ to the composition $y \xrightarrow{f} z \xrightarrow{\zeta} Sx$. However, we have $z \in H^-(\Sigma x)$ so Lemma 2.6(ii) says $Sx \in H^+(\Sigma z)$. Hence, if $\zeta$ is non-zero then it is a forward morphism. So is $f$ since $z \in H^+(\Sigma y)$, and then $y \xrightarrow{f} z \xrightarrow{\zeta} Sx$ is non-zero by Lemma 2.5(i) since we have $Sx \in \mathrm{H}^+(\Sigma y)$; this holds by Lemma 2.6(ii) again since $y \in H^-(\Sigma x)$. □

## 3. Triangulations of the $\infty$-gon

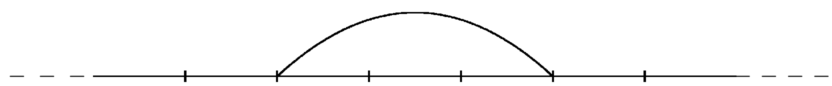
This section studies triangulations of the $\infty$-gon, that is, maximal sets of non-intersecting arcs, and their relation with the category $\mathsf{D}$.

**Definition 3.1.** An *arc* is a pair $(m, n)$ of integers with $m \leq n - 2$.

The arc $(m, n)$ is said to *end* in each of the integers $m$ and $n$.

Two arcs $(m, n)$ and $(p, q)$ are said to *intersect* if we have either $m < p < n < q$ or $p < m < q < n$.

The definition is intended to capture our geometric intuition in which an arc is drawn as a curve between two integers on the number line as follows.



Two arcs can be drawn as non-intersecting curves precisely if they do not intersect in the sense of the definition, with the proviso that curves which only meet at their end points are not viewed as intersecting.

In an informal sense, it is reasonable to view the integers as being the vertices of an $\infty$-gon and to view arcs as being diagonals between vertices. Hence a maximal set of non-intersecting arcs can be viewed as a triangulation of the $\infty$-gon. Some typical ways of achieving such maximal sets are shown in sketches (1), (2), and (3) in the Introduction. The sketches inspire the following definition.

**Definition 3.2.** Let $\mathfrak{A}$ be a set of arcs. If for each integer $n$ there are only finitely many arcs in $\mathfrak{A}$ which end in $n$, then $\mathfrak{A}$ is called *locally finite*.

If $n$ is an integer such that $\mathfrak{A}$ contains infinitely many arcs of the form $(m, n)$, then $n$ is called a *left-fountain* of $\mathfrak{A}$.

If $n$ is an integer such that $\mathfrak{A}$ contains infinitely many arcs of the form $(n, p)$, then $n$ is called a *right-fountain* of $\mathfrak{A}$.

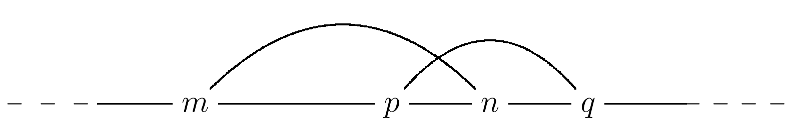
If $n$ is both a left- and a right-fountain of $\mathfrak{A}$, then it is called a *fountain*.

It turns out that if a maximal set of non-intersecting arcs has a right-fountain then it also has a left-fountain and vice versa; we owe this observation to Collin Bleak. However, all we need here is the following more modest result.

**Lemma 3.3.** *Let $\mathfrak{A}$ be a maximal set of non-intersecting arcs. Then $\mathfrak{A}$ has at most one right-fountain and at most one left-fountain.*

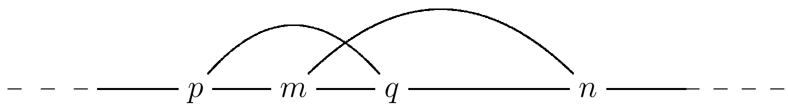
*Proof.* Suppose that $\mathfrak{A}$ is not locally finite and let $m$ be an integer where infinitely many arcs of $\mathfrak{A}$ end. So $m$ is either a right- or a left-fountain of $\mathfrak{A}$ and we can suppose the former without loss of generality.

We must show that $m$ is the only right-fountain of $\mathfrak{A}$, so let $p \neq m$ be an integer. If $p > m$ then we can pick $n > p$ such that $(m, n)$ is in $\mathfrak{A}$. An arc $(p, q)$ will intersect $(m, n)$ as soon as $q > n$.




So $\mathfrak{A}$ contains only finitely many arcs of the form $(p, q)$ and $p$ is not a right-fountain.

If $p < m$ then an arc $(p, q)$ can only be in $\mathfrak{A}$ if $q \leq m$, for if $q > m$ then it is possible to pick an arc $(m, n)$ in $\mathfrak{A}$ with $n > q$, and then $(m, n)$ and $(p, q)$ intersect.




Again $\mathfrak{A}$ contains only finitely many arcs of the form $(p, q)$ and $p$ is not a right-fountain. $\square$

**Remark 3.4.** An ordered pair of integers $(m, n)$ with $m \leq n - 2$ can be viewed as an arc. Using the coordinate system of Remark 1.4, it can also be viewed as a vertex of the AR quiver $Q$ of $\mathsf{D}$, that is, an indecomposable object of $\mathsf{D}$.

So there is a bijection between arcs and indecomposable objects of $\mathsf{D}$.

This induces a bijection between sets of arcs and sets of indecomposable objects of $\mathsf{D}$. But such sets correspond bijectively to subcategories of $\mathsf{D}$ which are closed under direct sums and direct summands, the bijection being given by $A \mapsto \operatorname{add} A$.

So there is a bijection between sets of arcs and subcategories of $\mathsf{D}$ which are closed under direct sums and direct summands.

It is easy to check that the bijection plays together with the regions $H(x)$ as follows.

**Lemma 3.5.** *Let $x$ and $y$ be indecomposable objects of $\mathsf{D}$. The following conditions are equivalent.*

(i) *$x \in H(y)$.*
(ii) *$y \in H(x)$.*
(iii) *The arcs corresponding to $x$ and $y$ intersect.*

The following is an immediate consequence.

**Lemma 3.6.** *Let $x$ and $y$ be indecomposable objects of $\mathsf{D}$. Then*

$$\mathsf{D}(x, y) \neq 0 \;\Leftrightarrow\; \text{the arcs corresponding to } x \text{ and } \Sigma^{-1}y \text{ intersect.}$$

*Proof.* By Corollary 2.3, we have $\mathsf{D}(x, y) \neq 0$ if and only if $x \in H(\Sigma^{-1}y)$, and by Lemma 3.5, this is the same as for the arcs corresponding to $x$ and $\Sigma^{-1}y$ to intersect. $\square$

## 4. Cluster tilting subcategories of $\mathsf{D}$

This section proves Theorems A and B from the Introduction; see Theorems 4.3 and 4.4.

The distinction between maximal 1-orthogonal and cluster tilting subcategories in the following definition is not standard in the literature, but Theorem B means that it is useful for this paper.

**Definition 4.1.** Let $\mathscr{A}$ be a subcategory of $\mathsf{D}$. We write

$$\mathscr{A}^{\perp} = \{\, d \in \mathsf{D} \,|\, \mathsf{D}(a,d) = 0 \text{ for each } a \in \mathscr{A} \,\},$$
$$^{\perp}\mathscr{A} = \{\, d \in \mathsf{D} \,|\, \mathsf{D}(d,a) = 0 \text{ for each } a \in \mathscr{A} \,\}.$$

A subcategory $\mathscr{A}$ is called *maximal* 1*-orthogonal* if it satisfies $\mathscr{A} = (\Sigma^{-1}\mathscr{A})^{\perp}$ and $\mathscr{A} = {}^{\perp}(\Sigma\mathscr{A})$. (In fact, either equality implies the other because $\mathsf{D}$ is a 2-Calabi-Yau category.)

A subcategory $\mathscr{A}$ is called *cluster tilting* if it is maximal 1-orthogonal and functorially finite.

**Remark 4.2.** Let $\mathscr{A}$ be a subcategory of $\mathsf{D}$ which is closed under direct sums and direct summands. The inclusion

$$\mathscr{A} \subseteq (\Sigma^{-1}\mathscr{A})^{\perp} \tag{7}$$

holds precisely if the presence of an indecomposable object $a$ in $\mathscr{A}$ forbids an indecomposable object $x$ from being in $\mathscr{A}$ when there is a non-zero morphism $\Sigma^{-1}a \to x$.

It hence follows from Corollary 2.3 that the inclusion (7) is equivalent to the following condition: If $a$ is in $\mathscr{A}$ then the indecomposable objects in $H(\Sigma\Sigma^{-1}a) = H(a)$ are forbidden from $\mathscr{A}$.

We therefore sometimes refer to the $H(a)$ as *forbidden regions*. Note that, in particular, a maximal 1-orthogonal subcategory of $\mathsf{D}$ satisfies (7).

**Theorem 4.3.** *Let $\mathscr{A}$ be a subcategory of $\mathsf{D}$ which is closed under direct sums and direct summands. Let $\mathfrak{A}$ be the corresponding set of arcs under the bijection of Remark 3.4.*

*Then $\mathscr{A}$ is a maximal* 1*-orthogonal subcategory of $\mathsf{D}$ if and only if $\mathfrak{A}$ is a maximal set of non-intersecting arcs.*

*Proof.* By Remark 4.2, the inclusion (7) is equivalent to the condition that if $a$ is in $\mathscr{A}$ then the objects in $H(a)$ are forbidden from $\mathscr{A}$.

An indecomposable object $a$ corresponds to an arc $\mathfrak{a}$, and by Lemma 3.5 the indecomposable objects in $H(a)$ correspond precisely to arcs intersecting $\mathfrak{a}$. So the subcategory $\mathscr{A}$ satisfies (7) if and only if it corresponds to a set of non-intersecting arcs.

It follows that subcategories $\mathscr{A}$ maximal among the ones satisfying (7) correspond to maximal sets of non-intersecting arcs. But it is not hard to check that such maximal subcategories are precisely the ones with $\mathscr{A} = (\Sigma^{-1}\mathscr{A})^{\perp}$, and these are the maximal 1-orthogonal subcategories of $\mathsf{D}$. □

**Theorem 4.4.** *Let $\mathscr{A}$ be a maximal* 1*-orthogonal subcategory of $\mathsf{D}$. Let $\mathfrak{A}$ be the corresponding maximal set of non-intersecting arcs under the bijection of Remark 3.4.*

*Then $\mathscr{A}$ is functorially finite (that is, $\mathscr{A}$ is a cluster tilting subcategory of $\mathsf{D}$) if and only if $\mathfrak{A}$ is* (i) *locally finite, or* (ii) *has a fountain.*

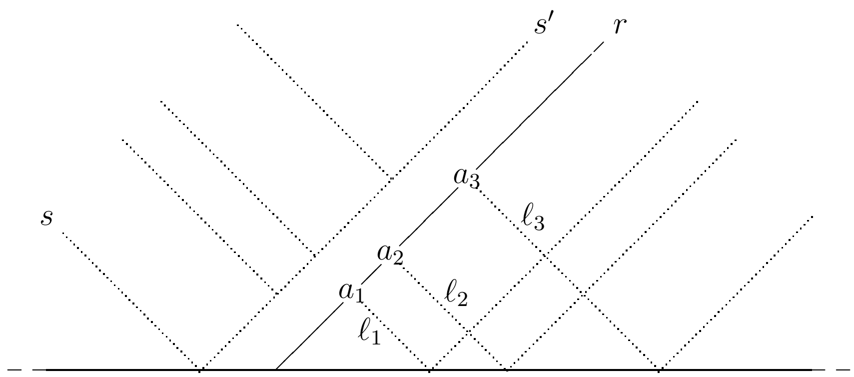
*Proof.* We remind the reader of Corollary 2.3 on the relation between existence of non-zero morphisms in $\mathsf{D}$ and membership of the regions $H^+$ and $H^-$. This will be used repeatedly in the proof.

We must show that $\mathscr{A}$ is functorially finite if and only if $\mathfrak{A}$ satisfies condition (i) or (ii) in the theorem. By [13, lem. 3.2.3] and its dual, it is enough to show that $\mathscr{A}$ is precovering or preenveloping if and only if $\mathfrak{A}$ satisfies (i) or (ii).

Suppose that (i) holds. Then it follows easily from Lemma 3.6 that for each indecomposable object $x$ of $\mathsf{D}$, only finitely many indecomposable objects of $\mathscr{A}$ have non-zero morphisms to $x$, and this implies that $\mathscr{A}$ is precovering.

Suppose that (i) does not hold; that is, $\mathfrak{A}$ has a right- or a left-fountain. Without loss of generality we can suppose that $\mathfrak{A}$ has a right-fountain which by Lemma 3.3 is the only right-fountain of $\mathfrak{A}$. We must show that $\mathscr{A}$ is precovering if and only if the right-fountain is also a left-fountain.

Suppose first that $\mathscr{A}$ is precovering. The right-fountain of $\mathfrak{A}$ is an integer $n$ for which there are infinitely many arcs of the form $(n, p)$ in $\mathfrak{A}$. These arcs give a collection $P$ of indecomposable objects in $\mathscr{A}$ which sit on a diagonal half line $r$ in the AR quiver $Q$ of $\mathsf{D}$. The following sketch of $Q$ shows $r$ along with some of the indecomposable objects $a$ of $P$ and, in dotted lines, their regions $H(\Sigma a)$.




(8)

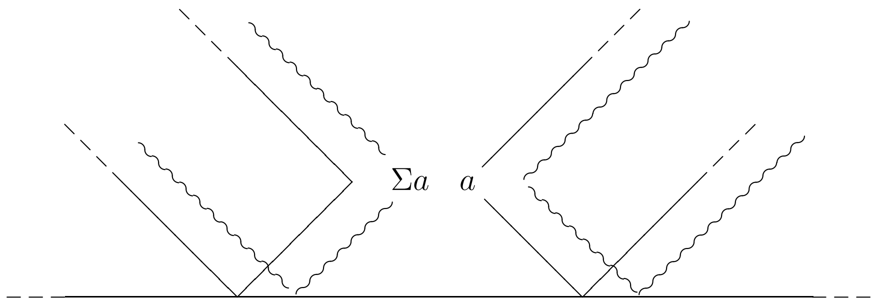
Note that the regions $H^-(\Sigma a)$ for $a$ in $P$ share the half line $s$ as a common edge, while each of the line segments $\ell_i$ is an edge of a region $H^+(\Sigma a)$ with $a$ in $P$.

We are aiming to show that $n$ is also a left-fountain, that is, there are infinitely many arcs in $\mathfrak{A}$ of the form $(m, n)$. This is the same as showing that there are infinitely many indecomposable objects of $\mathscr{A}$ which are on the half line $s$.

Let $x$ be an indecomposable object in the region bounded by the diagonal half lines $s$ and $s'$ and let $b \to x$ be an $\mathscr{A}$-precover. We can assume that the morphism $b \to x$ is non-zero on each direct summand of $b$; in particular, each direct summand of $b$ belongs to $H(\Sigma^{-1}x)$.

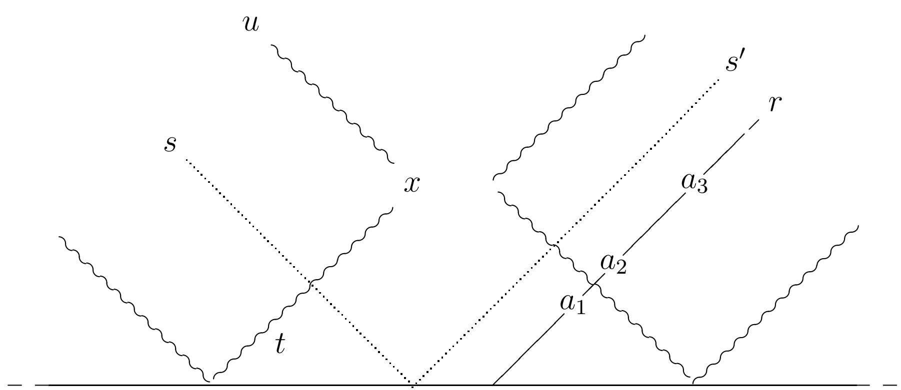
It is easy to see that $x$ is in $H^-(\Sigma a)$ for infinitely many $a$ in $P$, so there are infinitely many $a$ in $P$ with a non-zero morphism $a \to x$. Each of these morphisms factors through $b \to x$, so there is an indecomposable direct summand $c$ of $b$ such that infinitely many $a$ in $P$ have non-zero morphisms to $c$. Hence $c$ is certainly in $H(\Sigma a)$ for some $a$ in $P$. Moreover, since $c$ is in $\mathscr{A}$, Remark 4.2 says that $c$ must be outside the forbidden region $H(a)$ for each $a$ in $P$. The following sketch shows the regions

$H(\Sigma a)$ (ordinary lines) and $H(a)$ (wavy lines) for an indecomposable object $a$.




Combining this with the previous sketch shows that there are only three possible places for $c$: It is either on one of the line segments $\ell_i$, or on the half line $r$, or on the half line $s$.

Now, there are infinitely many $a$ in $P$ with non-zero morphisms to $c$; that is, infinitely many $a$ in $P$ which are in $H(\Sigma^{-1}c)$. And it is easy to see from the sketch (8) that this does not happen if $c$ is on an $\ell_i$ or on $r$, so $c$ must be on $s$. Moreover, $c$ is a direct summand of $b$, so $c$ is in $H(\Sigma^{-1}x)$. Combining the sketch (8) with $H(\Sigma^{-1}x)$, indicated in wavy lines, gives the following.




This shows that the indecomposable object $c$ must be on $s$ and above the line segment $t$.

For each $x$ we get a $c$ in $\mathscr{A}$ which is on $s$ and above the line segment $t$ corresponding to $x$. By moving $x$ out along the half line $u$, we can clearly force infinitely many distinct $c$'s. It follows that, as desired, there are infinitely many indecomposable objects of $\mathscr{A}$ which are on $s$.

Suppose next that the right-fountain of $\mathfrak{A}$ is also a left-fountain. Let $x$ be an indecomposable object of $\mathsf{D}$; we will construct a $\mathscr{A}$-precover of $x$. Consider again the sketch (8). The arcs going right, respectively left from the fountain of $\mathfrak{A}$ correspond to indecomposable objects of $\mathscr{A}$ on the half lines $r$, respectively $s$, so each of $r$ and $s$ contains infinitely many indecomposable objects of $\mathscr{A}$. The other arcs in $\mathfrak{A}$ correspond to indecomposable objects of $\mathscr{A}$ away from $r$ and $s$.

Consider the set of indecomposable objects $a$ of $\mathscr{A}$ which have non-zero morphisms to $x$, and divide it into disjoint subsets $R$, $S$, and $T$ according to whether $a$ is on $r$, $s$, or neither. We will construct a morphism $a_r \to x$ with $a_r \in \mathscr{A}$ such that each $a \to x$ with $a \in R$ factors as $a \to a_r \to x$. We will also construct morphisms

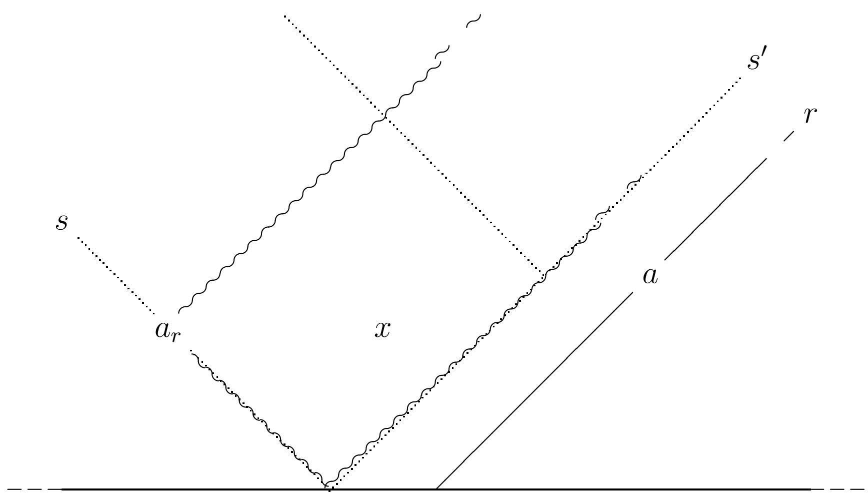
$a_s \to x$ and $a_t \to x$ with the analogous properties with respect to $S$ and $T$; then an $\mathscr{A}$-precover can be obtained as $a_r \oplus a_s \oplus a_t \to x$.

If a set $R$, $S$, or $T$ is finite, then the construction of the corresponding morphism $a_r \to x$, $a_s \to x$, or $a_t \to x$ is trivial.

The set $T$ is always finite: Suppose that $a$ is in $T$ and let $\mathfrak{a}$ be the arc corresponding to $a$. There is a non-zero morphism $a \to x$ so Lemma 3.6 gives that $\mathfrak{a}$ intersects the arc $\mathfrak{x} = (i, j)$ corresponding to $\Sigma^{-1}x$. Hence $\mathfrak{a}$ ends in an integer $m$ with $i < m < j$. Since $a$ is in $T$, it is in $\mathscr{A}$ but not on one of the half lines $r$ and $s$; this means that $\mathfrak{a}$ is an arc which is in $\mathfrak{A}$ but does not end in the fountain of $\mathfrak{A}$. In particular, $m$ is not the fountain of $\mathfrak{A}$. We conclude that each of the finitely many possible values of $m$ is an integer where only finitely many arcs of $\mathfrak{A}$ end, and it follows that there are only finitely many arcs $\mathfrak{a}$ as described. That is, $T$ has finitely many elements.

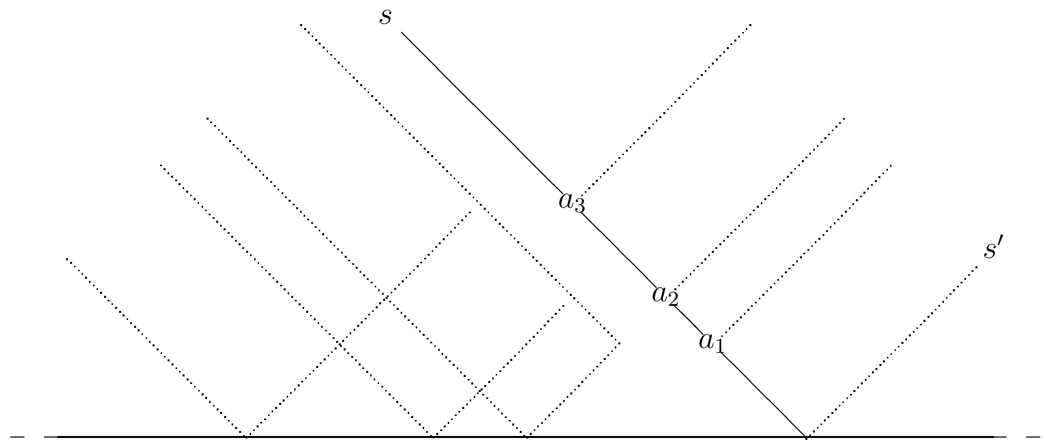
We are left to deal with the cases of $R$ and $S$ being infinite.

Suppose that $R$ is infinite. So there are infinitely many indecomposable objects $a$ of $\mathscr{A}$ on the half line $r$ with non-zero morphisms to $x$, that is, with $x$ in $H(\Sigma a)$. By inspecting the sketch (8) it can be seen that $x$ is in the region bounded by the half lines $s$ and $s'$. However, there are infinitely many indecomposable objects of $\mathscr{A}$ on the half line $s$, and so we can chose one, $a_r$, which has $x \in H^+(\Sigma a_r)$ as indicated here.

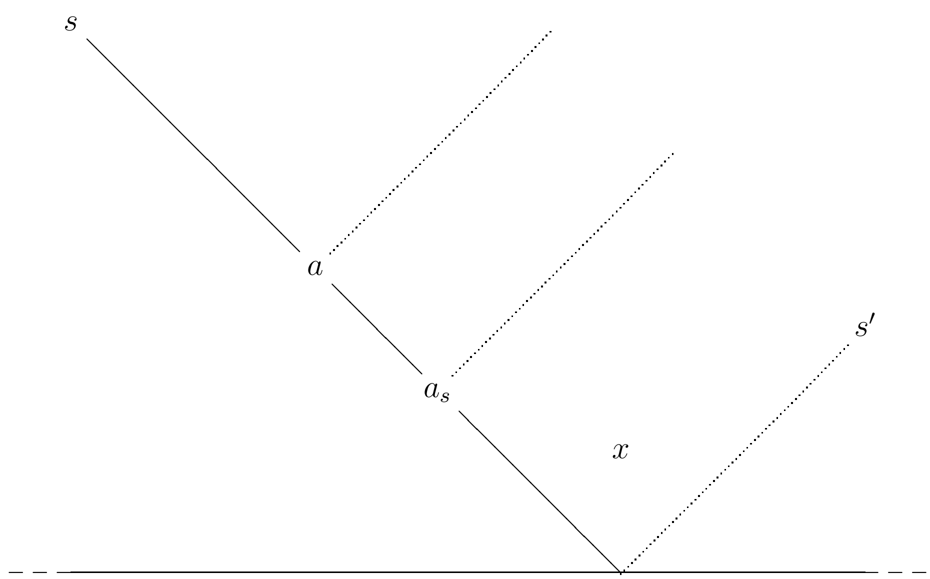



Pick a non-zero morphism $a_r \to x$. If $a$ is in $R$ then it has a non-zero morphism $a \to x$, and then we have $x \in H^-(\Sigma a)$ as in the sketch. But it is clear that $a_r \in H^-(\Sigma a)$ and so Lemma 2.7 says that a morphism $a \to x$ factors like $a \to a_r \to x$ as desired.

Suppose that $S$ is infinite. So there are infinitely many indecomposable objects $a$ of $\mathscr{A}$ on the half line $s$ with non-zero morphisms to $x$, that is, with $x$ in $H(\Sigma a)$. The following sketch shows some of the indecomposable objects $a$ on $s$ and, in dotted lines, their regions $H(\Sigma a)$. Since $x$ is in infinitely many of these regions, it can be seen that it is again in the region bounded by the half lines $s$ and $s'$.



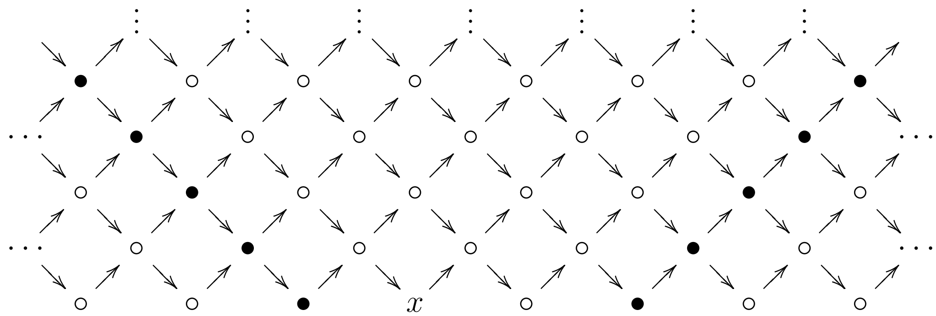

Let $a_s$ be the indecomposable object in $S$ which is closest to the end of $s$ and pick a non-zero morphism $a_s \to x$. It is clear that we have $x \in H^+(\Sigma a_s)$. If $a$ is in $S$ then it has a non-zero morphism $a \to x$, and again $x \in H^+(\Sigma a)$. The following sketch shows the whole situation.




But it is clear that $a_s \in H^+(\Sigma a)$ and so Lemma 2.5(ii) says that each morphism $a \to x$ factors like $a \to a_s \to x$ as desired. □

**Example 4.5.** Theorem 4.4 shows that there are maximal 1-orthogonal subcategories of $\mathsf{D}$ which are not cluster tilting; that is, they are not functorially finite.

A concrete example comes from the maximal set of non-intersecting arcs in the sketch (3) in the Introduction, which corresponds to the maximal 1-orthogonal subcategory $\mathscr{A}$ with the indecomposable objects marked by bullets.




In fact, it is not hard to adapt the arguments in the proof of Theorem 4.4 to show that there is no $\mathscr{A}$-precover of the indecomposable object $x$.

## 5. The cluster structure of $\mathsf{D}$

This section proves Theorem C from the Introduction; see Theorem 5.2.

**Definition 5.1.** For each cluster tilting subcategory $\mathscr{A}$ of $\mathsf{D}$ we can consider the set $A$ of indecomposable objects of $\mathscr{A}$ whence $\mathscr{A} = \operatorname{add} A$. We will refer to the sets $A$ as *clusters.*

The clusters are said to form a *cluster structure* if the following conditions are satisfied; cf. [1].

(i) If $A$ is a cluster, then each of its indecomposable objects $a$ can be replaced with a unique other indecomposable object $a^*$ of $\mathsf{D}$ such that a new cluster $A^*$ results.

(ii) There are distinguished triangles $a^* \to b \to a$ and $a \to b' \to a^*$ in $\mathsf{D}$ where the left-hand morphisms are $\operatorname{add}(A \setminus \{a\})$-envelopes and the right-hand morphisms are $\operatorname{add}(A \setminus \{a\})$-covers.

(iii) If $A$ is a cluster, then the AR quiver of $\operatorname{add} A$ has no loops or 2-cycles.

(iv) Passing from the AR quiver of $\operatorname{add} A$ to the AR quiver of $\operatorname{add} A^*$ is given by Fomin-Zelevinsky mutation at $a$ in the sense of [5, sec. 8].

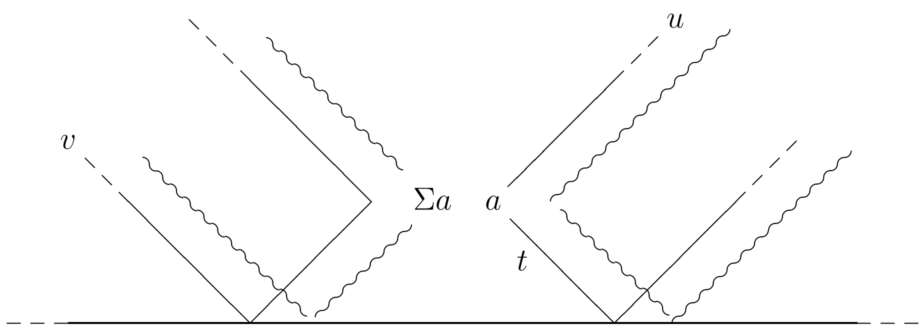
**Theorem 5.2.** *The clusters form a cluster structure on* $\mathsf{D}$*.*

*Proof.* Remark 1.2 says that $\mathsf{D}$ is a 2-Calabi-Yau category and it follows from Theorem 4.4 that there exist cluster tilting subcategories of $\mathsf{D}$. Hence by [1, thm. I.1.6] it is enough to show that for each cluster $A$, there are no loops or 2-cycles in the AR quiver of the cluster tilting subcategory $\mathscr{A} = \operatorname{add} A$.

It is clear that there are no loops: If $a$ is in $A$, then $\mathscr{A}(a,a) = \mathsf{D}(a,a) = k$ by Corollary 2.3, so each non-zero morphism $a \to a$ is an isomorphism and so not irreducible.

To show that there are no 2-cycles in the AR quiver of $\mathscr{A} = \operatorname{add} A$, we will show the stronger claim that given $a$ and $b$ in $A$ with $\mathscr{A}(a,b) \neq 0$, it follows that $\mathscr{A}(b,a) = 0$.

The following sketch shows the regions $H(\Sigma a)$ (straight lines) and $H(a)$ (wavy lines).

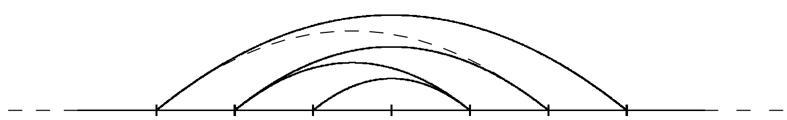



Since $\mathsf{D}(a, b) = \mathscr{A}(a, b) \neq 0$, the indecomposable object $b$ is in the region $H(\Sigma a)$ by Corollary 2.3. Since $a$ and $b$ are both in the cluster tilting subcategory $\mathscr{A}$, the object $b$ is outside the region $H(a)$ by Remark 4.2.

It follows that in the sketch, $b$ must be either on the line segment $t$ or on one of the half lines $u$ and $v$, and in any of these cases it is easy to verify that $a$ is outside $H(\Sigma b)$, that is, $\mathscr{A}(b, a) = \mathsf{D}(b, a) = 0$. $\square$

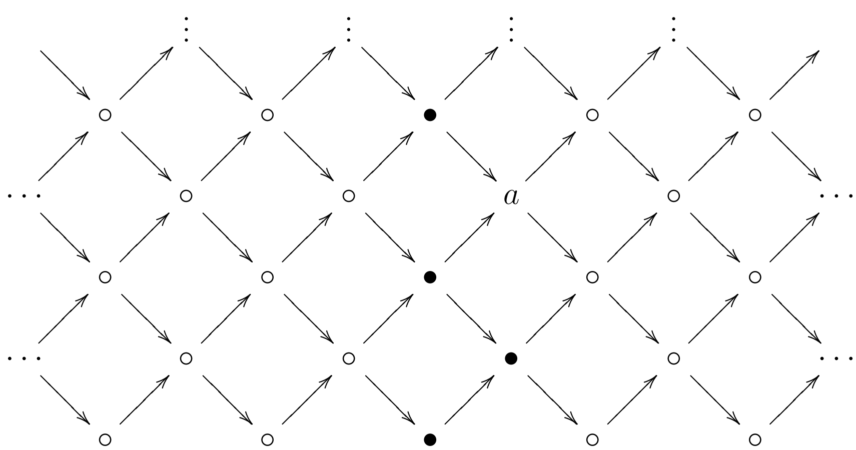
**Example 5.3.** Passing from the cluster $A$ to $A^*$ is referred to as cluster mutation at $a$. It corresponds to an obvious combinatorial mutation of maximal sets of arcs.

For instance, recall the leapfrog configuration.



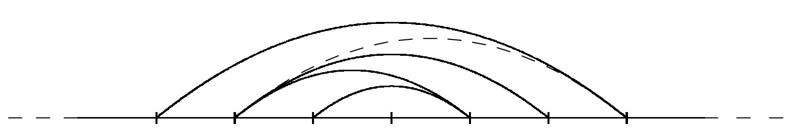
(9)

Under the bijection of Remark 3.4, this corresponds to the following cluster $A$ in $\mathsf{D}$; the broken arc corresponds to the object $a$.

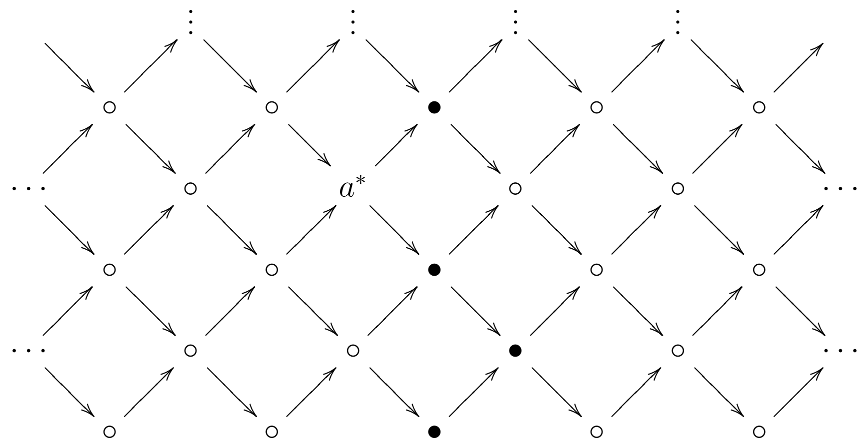



(10)

Removing the broken arc from (9) creates a 'quadrangle', and there is clearly a unique other arc which bisects it to form a new maximal set of non-intersecting arcs.



Under the bijection of Remark 3.4, this corresponds to the cluster $A^*$; the broken arc corresponds to the object $a^*$.




**Example 5.4.** If $\mathscr{A}$ is a cluster tilting subcateory of $\mathsf{D}$, then $\mathsf{D}/\mathscr{A}$ is an abelian category by [11, sec. 2] and [13, thm. 3.3], and we have $\mathsf{D}/\mathscr{A} \simeq \mathsf{mod}\,\mathscr{A}$ by [11, sec. 2] and [13, cor. 4.4].

In the case of the $\mathscr{A}$ given by the sketch (10) above, it is easy to check that $\mathscr{A}$ is the path category of its AR quiver $\Gamma$.

$$\Gamma = \ \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow a \longleftarrow \bullet \longrightarrow \cdots$$

So $\mathsf{mod}\,\mathscr{A}$ is equivalent to $\operatorname{rep}\Gamma$, the category of finitely presented representations of $\Gamma$. Such hereditary categories were studied in [15].

Likewise, $\mathscr{A}^*$ is the path category of its AR quiver $\Gamma^*$.

$$\Gamma^* = \ \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow a^* \longrightarrow \bullet \longrightarrow \cdots$$

So $\mathsf{mod}\,\mathscr{A}^*$ is equivalent to $\operatorname{rep}\Gamma^*$, and cluster mutation at $a$ has changed $\operatorname{rep}\Gamma$ to $\operatorname{rep}\Gamma^*$.

## 6. Questions

Let us end the paper by posing some questions which seem natural in the light of the results presented here.

(1) The category $\mathsf{D}$ behaves like a cluster category of type $A_\infty$. Is it possible to define a cluster algebra of type $A_\infty$?

(2) Section 5 gives the means to do cluster tilting of abelian categories of the form $\mathsf{D}/\mathscr{A}$ where $\mathscr{A}$ is a cluster tilting subcategory. Which abelian categories have this form? In particular, which hereditary abelian categories do?

(3) Can $\mathsf{D}$ be viewed as a covering category for the tubular 2-Calabi-Yau categories studied in [3, sec. 2]?

(4) The AR quiver of $\mathsf{D}$ is $\mathbb{Z}A_\infty$. Is there a similar category with AR quiver $\mathbb{Z}\Delta$ when $\Delta$ is another infinite Dynkin quiver than $A_\infty$?

(5) Is it possible to define 'higher cluster categories of type $A_\infty$'? See [10], [16], and [17] for the type $A_n$ case.

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