
FINITE SYMMETRY GROUPS IN COMPLEX GEOMETRY

by

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Introduction. — On June 5, 2007 the second author delivered a talk at the Journées de l’Institut Élie Cartan entitled *Finite symmetry groups in complex geometry*. This paper begins with an expanded version of that talk which, in the spirit of the Journées, is intended for a wide audience. The later paragraphs are devoted both to the exposition of basic methods, in particular an equivariant minimal model program for surfaces, as well as an outline of recent work of the authors on the classification of K3-surfaces with special symmetry.

1. Riemann surfaces

Throughout this introductory section X denotes a Riemann surface, namely a connected, compact complex manifold of dimension one. If it is regarded as a real surface, then its *genus* $g = g(X)$, which can be defined by $2g$ being the rank of the first homology group $H_1(X, \mathbb{Z})$, is its only topological invariant. This means that two such surfaces are homeomorphic if and only if they are diffeomorphic and this holds if and only if they have the same genus. Starting with the case $g = 0$, we consider Riemann surfaces from the point of view of biholomorphic symmetry, i.e., we are interested in actions on X of subgroups of the group $\text{Aut}(X)$ of its holomorphic automorphisms.

1.1. The Riemann sphere. — If $g(X) = 0$, then it can be shown that X is biholomorphically equivalent to the Riemann sphere. As the name indicates, this is the 2-dimensional sphere which often is regarded as the compactification of the complex plane by adding the point at infinity. If z is the standard linear coordinate in a neighborhood of 0 in the complex plane, then $\zeta := \frac{1}{z}$ is a coordinate of the corresponding neighborhood of ∞ .

Here we regard the Riemann sphere as the space $\mathbb{P}_1(\mathbb{C})$ of 1-dimensional linear subspaces (lines) of \mathbb{C}^2 . Every point $(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ is contained in a unique such line which is denoted by $[z_0 : z_1]$. It follows that $[z_0 : z_1] = [w_0 : w_1]$ if and only if there exists $\lambda \in \mathbb{C}^*$ with $(w_0, w_1) = \lambda(z_0, z_1)$. Thus $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$ can be regarded as the quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by scalar multiplication equipped with the quotient topology. The sets $U_i := \{[z_0 : z_1] \in \mathbb{P}_1 \mid z_i \neq 0\}$ are open. On U_0, U_1 respectively, we define the coordinate z by $[z_0 : z_1] \mapsto \frac{z_1}{z_0} =: z$, respectively $[z_0 : z_1] \mapsto \frac{z_0}{z_1} =: \zeta$. Setting $0 = [1 : 0]$ and $\infty = [0 : 1]$, the description of the Riemann sphere as the set of lines in \mathbb{C}^2 is seen to coincide with the compactified complex plane.

A linear map $T \in \text{GL}_2(\mathbb{C})$ takes lines to lines and therefore induces a map $\mathbb{P}(T) : \mathbb{P}_1 \rightarrow \mathbb{P}_1$. One checks that $\mathbb{P}(T)$ is holomorphic and that the homomorphism $\mathbb{P} : \text{GL}_2(\mathbb{C}) \rightarrow \text{Aut}(X)$ is surjective.

First author’s research supported by grants from the Studienstiftung des deutschen Volkes and the Deutsche Forschungsgemeinschaft.

Second author’s research partially supported by grants from the Deutsche Forschungsgemeinschaft.

The kernel of \mathbb{P} is the group $\mathbb{C}^* = \mathbb{C}^* \cdot \text{Id}$. Consequently, if we restrict \mathbb{P} to $\text{SL}_2(\mathbb{C})$, it is still surjective and yields the exact sequence

$$0 \rightarrow \{\pm \text{Id}\} \rightarrow \text{SL}_2(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}_1) \rightarrow 0.$$

The subgroup $\{\pm \text{Id}\}$ is the center of $\text{SL}_2(\mathbb{C})$ and the quotient $\text{SL}_2(\mathbb{C})/\{\pm \text{Id}\} =: \text{PSL}_2(\mathbb{C}) \cong \text{Aut}(\mathbb{P}_1)$ is the associated *projective linear group*.

As a subgroup of $\text{SL}_2(\mathbb{C})$ the special unitary group SU_2 defined by the standard Hermitian structure on \mathbb{C}^2 acts on \mathbb{P}_1 . Since it contains the center $\{\pm \text{Id}\}$ which acts trivially, this defines an action of the quotient $\text{SU}_2/\{\pm \text{Id}\}$, which can be identified with the group $\text{SO}_3(\mathbb{R})$ of orientation preserving linear isometries of \mathbb{R}^3 .

Guided by our interest in finite symmetry groups we consider finite subgroups of $\text{Aut}(\mathbb{P}_1)$. For the moment we simplify the discussion and only consider finite subgroups of $\text{SL}_2(\mathbb{C})$. Note that if G is such a subgroup, then we may average the standard Hermitian structure to obtain a G -invariant Hermitian form and consequently G is conjugate in $\text{SL}_2(\mathbb{C})$ to a subgroup of SU_2 . If we perform this conjugation, which changes nothing essential, and project G to $\text{Aut}(\mathbb{P}_1)$, we may regard it as a group of Euclidean isometries of S^2 . Conversely, paying the price of the 2:1 central extension, we may consider the preimage of a group of Euclidean motions in SU_2 and regard it as acting by holomorphic transformations on \mathbb{P}_1 .

If X is a Riemann surface and $G \subset \text{Aut}(X)$ is a finite group, then the quotient X/G carries a unique structure of a Riemann surface with the property that the quotient map $X \rightarrow X/G$ is holomorphic. Following ideas of Felix Klein, we begin with a finite group G of rigid motions of the sphere, lift it to a group of holomorphic transformations of $X = \mathbb{P}_1$ in SU_2 and consider such a quotient. Since there is no nonconstant holomorphic map from \mathbb{P}_1 to some other Riemann surface, it follows that X/G is likewise \mathbb{P}_1 . Using this fact and looking closely at the ramified covering map $X \rightarrow X/G$, Klein listed *all* possible finite subgroups of $\text{Aut}(\mathbb{P}_1)$. Other than the cyclic and dihedral groups, these are the isometry groups of the tetrahedron, the octahedron and the icosahedron of order 12, 24 and 60.

The cyclic group C_n of order n can be realized as a group of diagonal matrices (rotations) in SU_2 . The dihedral group D_{2n} is a semidirect product $C_2 \rtimes C_n$. If C_n is a group of diagonal matrices in SU_2 , then conjugation with

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

acts on C_n by $x \rightarrow x^{-1}$ and $\langle w \rangle \rtimes C_n$ is a realization of D_{2n} in $\text{SU}_2/\{\pm \text{Id}\}$.

It is a rather simple matter to place the corner points of a regular tetrahedron and of a regular octahedron on \mathbb{P}_1 and then to write down the matrices in SU_2 which realize their isometry groups as subgroups of PSU_2 . The same can be done for the icosahedron, but this is a much more difficult task. The group of isometries of the icosahedron is isomorphic to the alternating group A_5 . It should be emphasized that the preimage in SU_2 of a group of rigid motions of a regular polyhedron is a *nontrivial* central extension. Particularly in the case of A_5 , it is an interesting exercise to find the 2-dimensional representation of this group!

1.2. Tori. — In order to identify a Riemann surface X of genus zero with \mathbb{P}_1 , one constructs a meromorphic function on X which has only one pole and that being of order one. Analogously, one would like to identify a Riemann surfaces of genus one with a complex torus. In order to do this, one must prove the existence of a nowhere vanishing holomorphic 1-form, i.e., a 1-form which in local coordinates is given by $f dz$ for a nowhere vanishing holomorphic function f . Integrating this form provides a biholomorphic map $\alpha : X \rightarrow \mathbb{C}/\Gamma$, where the lattice Γ is the additive subgroup of $(\mathbb{C}, +)$ defined by integrating the given 1-form over the closed curves in X .

Unlike the case of \mathbb{P}_1 , there is a 1-dimensional family of holomorphically inequivalent Riemann surfaces of genus one. One way of realizing this family is to choose a basis of the periods so that $\Gamma = \langle 1, \tau \rangle$, where τ is in the upper-halfplane H^+ . Then the 1-dimensional family is given by $H^+/\text{SL}_2(\mathbb{Z}) \cong \mathbb{C}$.

A torus $T = \mathbb{C}/\Gamma$ is a group and acts on itself by group multiplication. This defines an embedding $T \hookrightarrow \text{Aut}(T)$. Given any holomorphic automorphism $\varphi \in \text{Aut}(T)$ we lift it to a biholomorphic map $\hat{\varphi} : \mathbb{C} \rightarrow \mathbb{C}$ of the universal cover of T . Since $\hat{\varphi}$ is an affine map of the form $\hat{\varphi}(z) = az + b$ and we are free to conjugate it with a translation, we may assume that it is either a linear map, i.e., $z \mapsto az$, or a translation. The translations correspond to the action of T on itself mentioned above.

Every torus possesses the holomorphic automorphism σ defined by $t \mapsto -t$, which is also a group automorphism. Except for two special tori, the full automorphism of T is just $T \rtimes \langle \sigma \rangle$. These special tori are defined by the lattices $\langle 1, i \rangle$ and $\langle 1, e^{i\frac{\pi}{6}} \rangle$. In the former case $\text{Aut}(T) = T \rtimes C_4$, where the linear part C_4 is generated by a rotation by $\frac{\pi}{2}$, and in the latter case $\text{Aut}(T) = T \rtimes C_6$, where the C_6 is generated by rotation through 60 degrees.

In summary, in all cases $\text{Aut}(T) = T \rtimes L$ for a linear group L of rotations. Hence, given a finite subgroup $G \subset \text{Aut}(T)$, we can decompose it into its translation and linear parts.

1.3. Riemann surfaces of general type. — For the remaining Riemann surfaces, i.e., for most, we have the following observation.

Theorem 1.1. — *The automorphism group of a Riemann surface of genus at least two is finite.*

To prove this theorem one needs basic results on the existence of certain globally defined holomorphic tensors. For example, one knows that the space $\Omega(X)$ of holomorphic 1-forms is g -dimensional. Note that if X possesses a nowhere vanishing holomorphic 1-form ω_0 , then every other holomorphic 1-form ω is a multiple $\omega = f\omega_0$ where f is a globally defined holomorphic function and therefore constant. Thus $\Omega(X) = \mathbb{C}\omega_0$ and $g(X) = 1$. Conversely, if $g(X) > 1$ then every holomorphic 1-form vanishes at at least one point of X . It can be shown that, counting multiplicities, every $\omega \in \Omega(X)$ has exactly $2g(X) - 2$ zeros.

Another basic fact which is useful for the proof of the above theorem is that the group of holomorphic automorphisms of a compact complex manifold, in this case a Riemann surface, is a *complex Lie group* acting holomorphically on X . This means that $\text{Aut}(X)$ is itself a (paracompact) complex manifold having the property that the group operations and the action map $\text{Aut}(X) \times X \rightarrow X$, $(g, x) \mapsto gx$, are holomorphic. Note that if $\text{Aut}(X)$ is positive-dimensional and $\{g_t\}$ is a holomorphic 1-parameter subgroup, then differentiation with respect to this group defines a holomorphic vector field on X . Conversely, holomorphic vector fields on *compact* complex manifolds can be integrated to define 1-parameter groups.

If $\omega \in \Omega(X) \setminus \{0\}$ and ξ is a holomorphic vector field which is not identically zero, then $\omega(\xi)$ is a holomorphic function on X which is also not identically zero. Thus, if ω vanished at some point of X , we would have produced a nonconstant holomorphic function, contrary to X being compact. As a result we obtain the following weak version of the above theorem.

Proposition 1.2. — *The automorphism group of a Riemann surface of genus at least two is discrete.*

Proof of Theorem 1.1. — One can show that a Riemann surfaces X of genus at least two possess enough holomorphic forms, or holomorphic tensors of higher order, locally of the form $f(dz)^k$ with f holomorphic, to define a canonical embedding of X in a projective space: If V_k is the vector space of such k -tensors, then one considers the holomorphic map $\varphi_k : X \rightarrow \mathbb{P}(V_k^*)$ which is defined by sending a point $x \in X$ to the *hyperplane* H_x of tensors in V_k vanishing at x . For k large enough φ_k is a holomorphic embedding. In fact $k = 1$ is usually enough and at most $k = 3$ is required.

The image $Z_k := \varphi_k(X)$ is a complex submanifold of the projective space $\mathbb{P}(V_k^*)$. Applying Chow's theorem, it follows that Z_k is an algebraic submanifold, i.e., it is defined as the common zero-set of finitely many homogeneous polynomials.

Since $\text{Aut}(X)$ acts as a group of linear transformations on V_k , where the action is given by a representation $\rho : \text{Aut}(X) \rightarrow \text{GL}(V_k^*)$, it follows that φ_k is $\text{Aut}(X)$ -equivariant. In other words, for every $g \in \text{Aut}(X)$, it follows that $\varphi_k(gx) = \rho(g)(\varphi_k(x))$. Thus $\text{Aut}(X)$ can be regarded as the stabilizer of

Z_k in the projective linear group $\mathrm{PGL}(V_k^*)$. Since stabilizers of algebraic submanifolds are algebraic groups and algebraic groups have only finitely many components, it follows that $\mathrm{Aut}(X)$ is finite. \square

Remark 1.3. — In complex geometry the terminology “manifold of general type” refers to a compact complex manifold (usually algebraic) which has as many holomorphic tensors of a certain kind as possible. In the higher-dimensional case one considers holomorphic volume forms which are locally of the form $f dz_1 \wedge \dots \wedge dz_n$ and higher-order tensors which are described in local coordinates by $f(dz_1 \wedge \dots \wedge dz_n)^k$. One cannot quite require that the space V_k defines an embedding as above, but it does make sense to require that the analogous map φ_k is bimeromorphic onto its image. Such maps are embeddings outside small sets. This is the origin of our referring to Riemann surfaces of genus at least two as being of general type.

1.4. The Hurwitz estimate. — As in the previous section, we restrict our considerations to Riemann surfaces X of genus at least two. Having shown that $\mathrm{Aut}(X)$ is finite we would like to outline some ideas behind the proof of the following beautiful theorem.

Theorem 1.4. — *If X is a Riemann surface of general type, then*

$$|\mathrm{Aut}(X)| \leq 84(g - 1).$$

Before going into the ideas of the proof, we emphasize the qualitative meaning of this estimate: the topological Euler number of X is given by $e(X) = 2 - 2g(X)$ and consequently the estimate above is given by $-42e(X)$. In other words, the bound for $|\mathrm{Aut}(X)|$ is a linear function of the topological Euler number.

The key to the above estimate is the Riemann-Hurwitz formula which in our particular case of interest gives a precise relationship between the topological Euler numbers of X and X/G , where G is any finite group of automorphisms. In the figure below we have shown a possible example where the group G is the cyclic group C_6 of order six. The surface X is schematically represented by a collection of curves which come together at a number of ramification points. The map from upstairs to downstairs represents the quotient $\pi : X \rightarrow X/G$. The observation that with three exceptions the preimage of a point downstairs consists of six different points reflects the following general fact: If G is a finite group in $\mathrm{Aut}(X)$, then there is a finite subset R such that G acts freely on the complement $X \setminus R$.

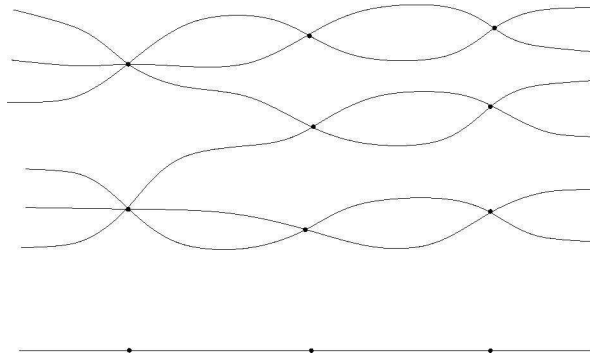


FIGURE 1. A ramified covering of Riemann surfaces

At a *ramification point* $x \in R$ the isotropy group $G_x = \{g \in G \mid g.x = x\}$ consists of more than just the identity. We note that the natural representation of G_x on the holomorphic tangent space $T_x X$ is faithful, and, since $\mathrm{GL}(T_x X) \cong \mathbb{C}^*$, it follows that G_x is cyclic. Let $\pi(R) =: B$ denote the *branch set* of the covering and note that for every point $b \in B$ we have the canonically defined numerical invariant $n(b) := |G_x|$, where x is any point in the preimage $\pi^{-1}(b)$. In the figure B consists of three points, two of which have $n(b) = 2$ and one of which has $n(b) = 3$.

Let us compare the topological Euler numbers of X and X/G by triangulating $Y := X/G$ so that set B of branch points is contained in the set of vertices of the triangles and let us lift this triangulation to X . We compute the Euler number as $v - e + f$, where v is the total number of vertices, e is the number of edges and f is the number of faces in the triangulation. In general,

$$e(X) = |G| \cdot e(Y) - \varepsilon_R,$$

where ε_R is a correction caused by ramification. In the case of the figure above, every face and every edge of the triangulation Y lifts to 6 faces and 6 edges in the triangulation of X . This is also true for the vertices which are not contained in B . In each of the two cases where $n(b) = 2$ we must subtract a correction term $3 = 3(2 - 1) = |G \cdot x| \cdot (n(b) - 1)$. In the case of $n(b) = 3$ this correction equals $4 = 2(3 - 1)$ and the precise formula for the figure is $e(X) = 6e(X/G) - 4 - 3 - 3$. In general, the Riemann-Hurwitz formula reads

$$\begin{aligned} 2 - 2g(X) &= |G| \cdot (2 - 2g(Y)) - \sum_{b \in B} \frac{|G|}{n(b)} (n(b) - 1) \\ &= |G| \cdot \left((2 - 2g(Y)) - \sum_{b \in B} \left(1 - \frac{1}{n(b)}\right) \right) \end{aligned}$$

The Hurwitz estimate for the maximal order of G , i.e., for the order of the full automorphism group $G = \text{Aut}(X)$ follows from experiments with the numbers $n(b)$. (see e.g. [Kob72], Theorem III.2.5)

1.5. Plane curves. — One important class of Riemann surfaces consists of those which can be realized as submanifolds of 2-dimensional projective space. These are often simply referred to as (complex) *curves*. Being 1-codimensional, a curve C can be described as the zero-set of a homogeneous polynomial which is unique up to a scalar factor and vanishes along C of order one. If $C = \{[z_0 : z_1 : z_2] \mid P_d(z_0, z_1, z_2) = 0\}$ where P_d is of degree d , then C is said to be of degree d . Remarkably, the genus of C can be directly computed from its degree:

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Example 1.5. — Curves of degree three are of genus one, i.e., they can be realized as tori $C = \mathbb{C}/\Gamma$ which have positive-dimensional automorphism groups. Very few of these automorphisms can be realized as restrictions of automorphisms of \mathbb{P}_2 which, in analogy to the case of \mathbb{P}_1 , are induced by linear transformations of \mathbb{C}^3 . To see this, let $S := \text{Stab}_{\text{Aut}(\mathbb{P}_2)}(C)$ be the subgroup of elements $T \in \text{Aut}(\mathbb{P}_2)$ with $T(C) = C$. Since C is defined by a complex polynomial equation, the group S is a complex subgroup of the complex Lie group $\text{Aut}(\mathbb{P}_2) = \text{SL}_3(\mathbb{C})/C_3$. Here C_3 is realized in SL_3 as its center, i.e, the group of diagonal matrices $\lambda \cdot \text{Id}_{\mathbb{C}^3}$ with $\lambda^3 = 1$.

Note that no element of S fixes C pointwise, because such a linear transformation would necessarily pointwise fix the linear subspace of \mathbb{P}_2 spanned by C , i.e., \mathbb{P}_2 itself. Since $\text{Aut}(C)$ is compact, it follows that S is likewise compact. Thus the lift \hat{S} of S to $\text{SL}_3(\mathbb{C})$ can be regarded as a compact complex submanifold of the vector space $\text{Mat}(2 \times 2, \mathbb{C}) \cong \mathbb{C}^4$. Consequently, we obtain holomorphic functions on \hat{S} as restrictions of holomorphic functions on \mathbb{C}^4 . Since \hat{S} is compact, the maximum principle implies that these are constant on its components. As these restrictions clearly separate the points of \hat{S} we see that \hat{S} is finite. Hence we have proven the following proposition.

Proposition 1.6. — *If C is a cubic curve in \mathbb{P}_2 , then the subgroup S of $\text{Aut}(C)$ of automorphisms which extend to automorphisms of \mathbb{P}_2 is finite.*

The case of a curve C of degree four is completely different. In this case $g(C) = 3$ and the space V_1 of holomorphic 1-forms on C is 3-dimensional. The mapping $\varphi_1 : C \rightarrow \mathbb{P}(V_1^*)$ is an embedding and the original realization of C as a curve is, after a choice of coordinates, just $\text{Im}(\varphi_1)$. Consequently, curves of degree four are equivariantly embedded.

Proposition 1.7. — *If C is a quartic curve, then every automorphism of C extends to a unique automorphism of \mathbb{P}_2 .*

The study curves of genus three from the point of view of symmetry is therefore closely related to the classification and invariant theory of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ ([Bli17], see also [YY93])

Example 1.8. — We consider the quartic curve defined by $z_0z_1^3 + z_1z_2^3 + z_2z_0^3$ and refer to it as Klein's curve C_{Klein} . Although the general theory tells us that every automorphism of C_{Klein} is the restriction of an automorphism of \mathbb{P}_2 , not all of these automorphisms are immediately visible. In fact, $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{GL}_3(\mathbb{F}_2)$. This group, which is often denoted by $L_2(7)$, is the unique simple group of order 168. Note that $168 = 84(3 - 1)$ and therefore the automorphism group of Klein's curve attains the maximal order among Riemann surfaces of genus three allowed by the Hurwitz estimate. One can show that C_{Klein} is the unique genus three curve for which this upper bound is attained. The book [Lev99] is dedicated to various interesting aspects concerning the geometry of this curve and its automorphisms.

2. Manifolds of general type

The title of this work indicates our interest in the role of finite symmetry groups in arbitrary dimensions. Nevertheless, after the previous introductory section on Riemann surfaces, mostly all of our considerations are devoted to the case of compact complex surfaces, i.e., complex 2-dimensional, connected, compact complex manifolds. Before restricting to that case, we do comment on higher-dimensional manifolds of general type.

Recall that a Riemann surface X is of general type if it possesses sufficiently many globally defined holomorphic tensors, which are locally of the form $f(dz)^k$, so that the k -canonical map $\varphi_k : X \rightarrow \mathbb{P}(V_k^*)$ is a biholomorphic embedding for k sufficiently large. In the higher-dimensional case, $\dim_{\mathbb{C}} X = n$, the analogous objects in the case $k = 1$ are holomorphic n -forms which are locally of the form $f dz_1 \wedge \dots \wedge dz_n$. Here f is a holomorphic function on a coordinate chart with coordinates z_1, \dots, z_n . For arbitrary k , its k -tensors are locally of the form $f(dz_1 \wedge \dots \wedge dz_n)^k$. In a less archaic language, these are sections of the k -th power \mathcal{K}_X^k of the canonical line bundle and the space V_k of k -tensors is denoted by $\Gamma(X, \mathcal{K}_X^k)$.

It turns out to be appropriate to require that for some k the mapping $\varphi_k : X \rightarrow \mathbb{P}(V_k^*)$ is a meromorphic instead of holomorphic embedding. This condition is more conveniently described by an invariant of the associated function field: if s and t are two tensors of the same type, i.e., $s, t \in V_k$, where locally $s = f(dz_1 \wedge \dots \wedge dz_n)^k$ and $t = g(dz_1 \wedge \dots \wedge dz_n)^k$, then their ratio

$$m = \frac{s}{t} = \frac{f(dz_1 \wedge \dots \wedge dz_n)^k}{g(dz_1 \wedge \dots \wedge dz_n)^k} = \frac{f}{g}$$

has an interpretation as a globally defined meromorphic function on X .

We let $Q(V_k)$ be the quotient field generated by this procedure. From the theorem of Thimm-Siegel-Remmert we know that meromorphic functions on a compact complex manifold are analytically dependent if and only if they are algebraically dependent, and consequently the transcendence degree of $Q(V_k)$ over the field of constant functions, or equivalently, the maximal number of analytically independent meromorphic functions which can be constructed as quotients of tensors from V_k , is at most $\dim_{\mathbb{C}}(X)$. If for some k this number equals the dimension of X , then one says that X is of *general type*.

If X is of general type, then for some k , maybe not the one in the definition, the mapping φ_k is indeed a bimeromorphic embedding. Since φ_k is equivariant with respect to the full automorphism group, it follows that $\mathrm{Aut}(X)$ is represented as the subgroup of $\mathrm{Aut}(\mathbb{P}(V_k^*))$ which stabilizes the image $\mathrm{Im}(\varphi_k)$. Thus, by precisely the same type of argument as in the 1-dimensional case, we have the following fact.

Proposition 2.1. — *The automorphism group of a manifold of general type is finite.*

Even if the field $Q(V_k)$ does not have transcendence degree equal to the dimension of X , it certainly contains important information. As a very rough first invariant one defines the *Kodaira dimension* of X as the maximal transcendence degree attained by some V_k . In other words, the Kodaira dimension $\kappa(X)$ is the maximal number of analytically independent meromorphic functions which can be obtained as quotients of k -tensors. If for all k there are no such tensors, i.e., $V_k = 0$ for all k , then one lets $\kappa(X) := -\infty$. It follows that $\kappa(X) \in \{-\infty, 0, 1, \dots, \dim(X)\}$.

For $\dim(X) > \kappa(X) \geq 1$ the meromorphic maps $\varphi_k : X \rightarrow \mathbb{P}(V_k^*)$ can still be very interesting. However, it is quite possible that a nontrivial automorphism of X can act trivially on $\text{Im}(\varphi_k)$.

2.1. Surfaces of general type - Quotients by small subgroups. — Inspired by the 1-dimensional case one wishes to obtain bounds for the order of the automorphism groups of manifolds of general type. Below we give an outline of a simple method which has been used, for example, to obtain estimates of Hurwitz type (see [HS90]).

At the present time the following sharp estimate for surfaces of general type requires essentially more combinatorial work ([Xia94], [Xia95]).

Theorem 2.2. — *The order of the automorphism group of a surface of general type is bounded by $(42)^2 K^2$.*

Since the self-intersection number K^2 of a canonical divisor is a topological invariant, this is exactly the desired type of estimate.

Turning to the method mentioned above, given a finite group G we want analyze the possibilities of it acting on surfaces of general type with given topological invariants, in this case Chern numbers. We then look for a small subgroup S in G with an interesting normalizer N . The notion of interesting can vary. For example, this can mean that N is large with respect to G or that N has a rich group structure. The group S should be small in the sense of size and structure. Clearly, $S = C_2$ or some other small cyclic group would be a good choice.

The first step is to consider the quotient $X \rightarrow X/S =: Y$. Since the normalizer N acts on Y , we are presented with the new task of understanding Y as an N -variety and $X \rightarrow Y$ as an N -equivariant map, e.g. by studying the action of N on the ramification and branch loci. If this can be done, then we attempt to piece together the G -action on X from knowledge of the S -quotient and the N -action on Y .

Because we have been forced to transfer our consideration to the smaller group N , it might appear that we have even lost ground. However, there are at least two possible advantages of this approach. First, without being overly optimistic one can hope that the topological invariants have decreased in size so that if Y is still of general type, some inductive argument can be carried out. Alternatively, if Y is not of general type, then we come into a range of Kodaira dimension where new methods are available.

3. The Enriques Kodaira classification

From now on we restrict our considerations to compact complex surfaces X . Here $\kappa(X)$ can take on the values $-\infty, 0, 1$ and 2 . One would like to prove a classification theorem similar to that for Riemann surfaces with one big class consisting of the surfaces of general type and the remaining surfaces with $\kappa(X) \leq 1$ being precisely described.

This is almost possible with the final result being called the *Enriques Kodaira classification*. For a detailed exposition we refer the reader e.g. to [BHPV04]. In the case of algebraic surfaces, i.e., those compact complex surfaces which can be holomorphically embedded in some projective space, much of the essential work was carried out by members of the Italian school of algebraic geometry, in particular by Enriques. It should be noted that a surface is algebraic if and only if it possesses two analytically independent meromorphic functions.

Minimal models. — One complicating factor in the classification theory is that, given a surface X , one can *blow it up* to obtain a new surface \hat{X} and a holomorphic map $\hat{X} \rightarrow X$ which is almost biholomorphic. Conversely, given X , one may be able to *blow it down*, i.e., X is the blow up of some other surface. Let us briefly explain this process.

The simplest *blow up* is constructed as follows. Let $X = \mathbb{P}_2$ and choose $p := [1 : 0 : 0]$ as the point to be blown up. The projection $\pi : \mathbb{P}_2 \rightarrow \mathbb{P}_1, [z_0 : z_1 : z_2] \mapsto [z_1 : z_2]$ is well-defined and holomorphic outside of the base point p . So we remove p and consider the restricted map. Its fiber over a point $[a, b]$ is a copy of the complex plane parameterized by $t \mapsto [t : a : b]$. This continues to a map of the full projective line given by $[t_0 : t_1] \mapsto [t_0 : t_1 a : t_1 b]$. The map π realizes $\mathbb{P}_2 \setminus \{p\}$ as a bundle of lines: we say that the fibration $\mathbb{P}_2 \setminus \{p\} \rightarrow \mathbb{P}_1$ is a *line bundle*, a rank one holomorphic vector bundle. Each of its fibers is naturally isomorphic to \mathbb{C} and contains a natural choice of zero, namely $[0 : a : b]$, and a natural point at infinity $[1 : 0 : 0]$.

The point at infinity is the same for each of the lines. To resolve this problem, we formally add individual points at infinity to each of the lines, i.e., two different lines receive different points at infinity. One checks that this construction results in a complex manifold $\text{Bl}_p(\mathbb{P}_2)$ to which the line bundle fibration extends as a \mathbb{P}_1 -bundle $\text{Bl}_p(\mathbb{P}_2) \rightarrow \mathbb{P}_1$. The set E of points at infinity is a copy of \mathbb{P}_1 which is mapped to the point p by the natural projection $\text{Bl}_p(\mathbb{P}_2) \rightarrow \mathbb{P}_2$. Outside of E this projection is biholomorphic.

There are several first observations about this construction. For one, it should be noted that the construction is local. In other words, for any open neighborhood U of p we can define the blow up $\text{Bl}_p(U) \rightarrow U$. Thus, for any surface X and a point $p \in X$ we have $\text{Bl}_p(X) \rightarrow X$. One checks that up to biholomorphic transformations the construction is independent of the coordinate chart which is used. Secondly, regarding E as a homology class in $H_2(\text{Bl}_p(X), \mathbb{Z})$ which is equipped with its natural intersection pairing, one shows that $E \cdot E = -1$. Remarkably, the converse statement holds (see [Gra62]):

Theorem 3.1. — *Let X be a complex surface and E be a smooth curve in X which is holomorphically equivalent to \mathbb{P}_1 . Then there is a holomorphic map $X \rightarrow Y$ to a complex surface which realizes X as $\text{Bl}_p(Y)$ if and only if $E \cdot E = -1$.*

As a result of this theorem it is reasonable to classify only those surfaces which are *minimal* in the sense that they contain no (-1)-curves, i.e., curves E which are biholomorphic to \mathbb{P}_1 and satisfy $E \cdot E = -1$. We give a rough summary of this classification: Any minimal surface belongs to one of the following classes of surfaces, ordered according to Kodaira dimension.

$\kappa = -\infty$	Ruled surfaces, \mathbb{P}_2 , and exceptional nonalgebraic surfaces
$\kappa = 0$	Tori, K3-, Enriques-, Kodaira-, and bi-elliptic surfaces
$\kappa = 1$	Elliptic surfaces
$\kappa = 2$	Surfaces of general type

A ruled surface $X \neq \mathbb{P}_1 \times \mathbb{P}_1$ admits a canonical locally trivial holomorphic fibration onto a Riemann surface with generic fiber \mathbb{P}_1 . Elliptic surfaces possess canonically defined fibrations over \mathbb{P}_1 with the generic fiber being a 1-dimensional torus. In this situation different fibers can be biholomorphically different tori. Note that canonically defined fibrations $\pi : X \rightarrow Y$ are automatically equivariant, i.e., there is an action of $\text{Aut}(X)$ on Y so that for every $g \in \text{Aut}(X)$ it follows that $\pi \circ g = g \circ \pi$. Thus from the point of view of group actions it is particularly advantageous if the surface is either ruled or elliptic. To exemplify this, we present a detailed discussion of automorphisms of rational ruled surfaces, the Hirzebruch surfaces.

Hirzebruch surfaces. — The n -th Hirzebruch surface Σ_n is defined as the compactification of the total space of the n -th power H^n of the hyperplane bundle over \mathbb{P}_1 . The compactification is constructed by adding the point at infinity to each fiber. This makes sense because the structure group of a line bundle is $\text{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ whose action on the complex line canonically extends to an action on \mathbb{P}_1 . The surface Σ_0 is the compactification of the trivial bundle and is therefore $\mathbb{P}_1 \times \mathbb{P}_1$. We have seen above that Σ_1 is $\text{Bl}_p(\mathbb{P}_2)$.

By construction the \mathbb{P}_1 -bundle $\Sigma_n \rightarrow \mathbb{P}_1$ has a section E_n at infinity. Let us show that E_n can be blown down to a point which, except in the case of Σ_1 , is singular. For this it is convenient to recall that H^n is the quotient of H by the cyclic group C_n acting via the principal \mathbb{C}^* -action in the fibers of

H. This extends to Σ_1 to give us a diagram

$$\begin{array}{ccc} \Sigma_1 & \longrightarrow & \Sigma_n \\ \downarrow b_1 & & \downarrow b_n \\ \mathbb{P}_2 & \longrightarrow & \text{Cone}_n(\mathbb{P}_1). \end{array}$$

The map b_1 blows down the (-1) -curve E_1 . The horizontal maps are the C_n -quotients and b_n is induced by b_1 . Regarding \mathbb{P}_2 as a cone over the line $\{z_0 = 0\}$ at infinity, $\text{Cone}_n(\mathbb{P}_1)$ is defined as its quotient by C_n acting on its fibers.

Let us turn now to the automorphism groups of the Hirzebruch surfaces. For this we consider the standard $\text{GL}_2(\mathbb{C})$ -action on \mathbb{P}_2 fixing the point $p = [1 : 0 : 0]$ and stabilizing the hyperplane $\{z_0 = 0\}$. By construction, it lifts to the blow up $\Sigma_1 = \text{Bl}_p(\mathbb{P}_2)$. This action is centralized by the C_n -action discussed above and therefore there is an holomorphic action of $L_n := \text{GL}_2(\mathbb{C})/C_n$ on the quotient $\Sigma_n = \Sigma_1/C_n$.

The remaining automorphisms come from the sections $s \in \Gamma(\mathbb{P}_1, H^n)$ of H^n . In general if $s : X \rightarrow L$ is a section of a holomorphic line bundle $\pi : L \rightarrow X$, then $x \mapsto x + s(\pi(x))$ defines a holomorphic automorphism of the bundle space X which extends to the associated \mathbb{P}_1 -bundle. Thus, in the case at hand we may regard $\Gamma(\mathbb{P}_1, H^n)$ as a subgroup of $\text{Aut}(\Sigma_n)$. Conjugation by elements of L_n stabilizes $\Gamma(\mathbb{P}_1, H^n)$ and the semidirect product $L_n \rtimes \Gamma(\mathbb{P}_1, H^n)$ is a subgroup of $\text{Aut}(\Sigma_n)$. In fact there are no other automorphisms (see e.g. [HO84]).

Proposition 3.2. — $\text{Aut}(\Sigma_n) = L_n \rtimes \Gamma(\mathbb{P}_1, H^n)$.

Having identified $\text{Aut}(\Sigma_n)$ we wish to pin down its finite subgroups. First note that the maximal compact subgroups of a connected Lie group are unique up to conjugation. In the case of $\text{Aut}(\Sigma_n)$ we observe that the image of the unitary group U_2 is a maximal compact subgroup. If G is a finite subgroup of $\text{Aut}(\Sigma_n)$, then we may conjugate it to a subgroup of the image of the unitary group. Allowing the action to contain the kernel of $U_2(\mathbb{C}) \rightarrow U_2/C_n$, we characterize the finite group actions on Σ_n as being given by finite subgroups of U_2 .

Let us return the rough classification of minimal surfaces outlined above. Note that bi-elliptic surfaces admit a locally trivial elliptic fibration over an elliptic curve. The same holds for a Kodaira surface itself or an unramified covering. In analogy to our treatment of Hirzebruch surfaces, automorphisms of fibered surfaces can be investigated by using the structure given by the fibration.

Finite subgroups acting on \mathbb{P}_2 were classified at the turn of the 20th century by Blichfeld ([Bli17], see also [YY93]). Much later the finite groups acting on 2-dimensional tori were classified by Fujiki ([Fuj88]). Noting that the universal cover of an Enriques surface is a K3-surface, the interest of finite symmetry groups may be focussed on the remaining case of K3-surfaces.

From the point of view of symmetries the restriction of our considerations to minimal surfaces is not necessarily natural. This is due to the fact that the reduction from a surface to its minimal model may not be equivariant. The concept of minimal models therefore needs to be replaced by an equivariant analogue, namely an equivariant reduction procedure. The following section is dedicated to a detailed presentation of the equivariant minimal model program for surfaces formulated in the language of Mori theory.

4. Equivariant Mori reduction

This section is dedicated to a discussion of Example 2.18 in [KM98] (cf. also Section 2.3 in [Mor82]) which introduces a minimal model program for surfaces respecting finite groups of symmetries.

Given a projective algebraic surface X with an action of a finite group G , in analogy to the usual minimal model program, one obtains from X a G -minimal model $X_{G\text{-min}}$ by a finite number of G -equivariant blow-downs, each contracting a finite number of disjoint (-1) -curves. The surface $X_{G\text{-min}}$ is either a conic bundle over a smooth curve, a Del Pezzo surface or has nef canonical bundle.

Equivariant Mori reduction and the theory of G -minimal models have applications in various different contexts and can also be generalized to higher dimensions. Initiated by Bayle and Beauville in [BaBe00], the methods have been employed in the classification of subgroups of the Cremona group $\text{Bir}(\mathbb{P}_2)$ of the plane for example by Beauville and Blanc [Bea07], [BeBl04], [Bla06], [Bla07], de Fernex [dF04], Dolgachev and Iskovskikh [DI06], [DI07], and Zhang [Zha01]. These references also provide certain relevant details regarding Example 2.18 in [KM98] and Section 2.3 in [Mor82]: e.g. the case $G \cong C_2$ is discussed in [BaBe00], the case $G \cong C_p$ for p prime in [dF04], the case of perfect fields is treated in [DI07].

For the convenience of the reader we here give a detailed exposition of the equivariant minimal model program for arbitrary finite groups acting on complex projective surfaces (see also Chapter 2 in [Fra08] for further details).

4.1. The cone of curves and the cone theorem. — Throughout this section we let X be a smooth projective algebraic surface and let $\text{Pic}(X)$ denote the group of isomorphism classes of line bundles on X . Here a curve is an irreducible 1-dimensional subvariety.

Definition 4.1. — A *divisor* on X is a formal linear combination of curves $C = \sum a_i C_i$ with $a_i \in \mathbb{Z}$. A *1-cycle* on X is a formal linear combination of curves $C = \sum b_i C_i$ with $b_i \in \mathbb{R}$. A 1-cycle is *effective* if $b_i \geq 0$ for all i . Extending the pairing $\text{Pic}(X) \times \{\text{divisors}\} \rightarrow \mathbb{Z}$, $(L, D) \mapsto L \cdot D = \deg(L|_D)$ by linearity, we obtain a pairing $\text{Pic}(X) \times \{\text{1-cycles}\} \rightarrow \mathbb{R}$. Two 1-cycles C, C' are called *numerically equivalent* if $L \cdot C = L \cdot C'$ for all $L \in \text{Pic}(X)$. We write $C \equiv C'$. The numerical equivalence class of a 1-cycle C is denoted by $[C]$. The space of 1-cycles modulo numerical equivalence is a real vector space denoted by $N_1(X)$. Note that $N_1(X)$ is finite-dimensional. A line bundle L is called *nef* if $L \cdot C \geq 0$ for all curves C . We set

$$NE(X) = \left\{ \sum a_i [C_i] \mid C_i \subset X \text{ irreducible curve, } 0 \leq a_i \in \mathbb{R} \right\} \subset N_1(X).$$

The closure $\overline{NE}(X)$ of $NE(X)$ in $N_1(X)$ is called *Kleiman-Mori cone* or *cone of curves*. For a line bundle L , we write $\overline{NE}(X)_{L \geq 0} = \{[C] \in N_1(X) \mid L \cdot C \geq 0\} \cap \overline{NE}(X)$. Analogously, we define $\overline{NE}(X)_{L \leq 0}$, $\overline{NE}(X)_{L > 0}$, and $\overline{NE}(X)_{L < 0}$.

Using this notation we phrase Kleiman's ampleness criterion (cf. Theorem 1.18 in [KM98]) as follows: A line bundle L on X is ample if and only if $\overline{NE}(X)_{L > 0} = \overline{NE}(X) \setminus \{0\}$.

Definition 4.2. — A subset $N \subset V$ of a finite-dimensional real vector space V is called *cone* if $0 \in N$ and N is closed under multiplication by positive real numbers. A subcone $M \subset N$ is called *extremal* if $u, v \in N$ satisfy $u, v \in M$ whenever $u + v \in M$. An extremal subcone is also referred to as an *extremal face*. A 1-dimensional extremal face is called *extremal ray*. For subsets $A, B \subset V$ we define $A + B := \{a + b \mid a \in A, b \in B\}$.

The cone of curves $\overline{NE}(X)$ is a convex cone in $N_1(X)$ and the following cone theorem, stated here only for surfaces, describes its geometry (cf. Theorem 1.24 in [KM98]).

Theorem 4.3. — Let X be a smooth projective surface and let \mathcal{K}_X denote the canonical line bundle on X . There are countably many rational curves $C_i \in X$ such that $0 < -\mathcal{K}_X \cdot C_i \leq \dim(X) + 1$ and

$$\overline{NE}(X) = \overline{NE}(X)_{\mathcal{K}_X \geq 0} + \sum_i \mathbb{R}_{\geq 0} [C_i].$$

For any $\varepsilon > 0$ and any ample line bundle L

$$\overline{NE}(X) = \overline{NE}(X)_{(\mathcal{K}_X + \varepsilon L) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0} [C_i].$$

4.2. Surfaces with group action and the cone of invariant curves. — Let X be a smooth projective surface and let $G \subset \text{Aut}(X)$ be a group of holomorphic transformations of X . For $g \in G$ and an irreducible curve C_i we denote by gC_i the image of C_i under g . For a 1-cycle $C = \sum a_i C_i$ we define $gC = \sum a_i (gC_i)$. This defines a G -action on the space of 1-cycles. Since two 1-cycles C_1, C_2 are numerically equivalent if and only if $gC_1 \equiv gC_2$ for any $g \in G$, we can define a G -action on $N_1(X)$ by setting $g[C] := [gC]$ and extending by linearity. We write $N_1(X)^G = \{[C] \in N_1(X) \mid [C] = [gC] \text{ for all } g \in G\}$, the set of invariant 1-cycles modulo numerical equivalence. This space is a linear subspace of $N_1(X)$. The cone $NE(X)$ is a G -invariant set and so is its closure $\overline{NE}(X)$. The subset of invariant elements in $\overline{NE}(X)$ is denoted by $\overline{NE}(X)^G$.

Remark 4.4. — $\overline{NE}(X)^G = \overline{NE(X) \cap N_1(X)^G} = \overline{NE}(X) \cap N_1(X)^G$.

The subcone $\overline{NE}(X)^G$ of $\overline{NE}(X)$ inherits the geometric properties of $\overline{NE}(X)$ established by the cone theorem. Note however that the extremal rays of $\overline{NE}(X)^G$, which we refer to as G -extremal rays, are in general neither extremal in $\overline{NE}(X)$ (cf. Figure 2) nor generated by classes of curves but by classes of 1-cycles.

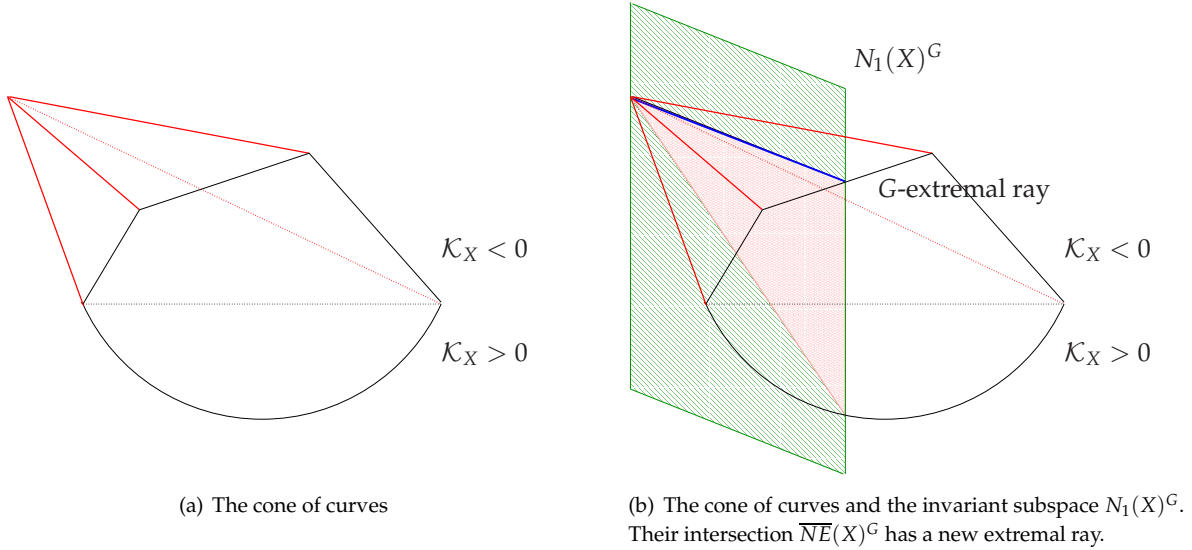


FIGURE 2. The extremal rays of $\overline{NE}(X)^G$ are not extremal in $\overline{NE}(X)$

Lemma 4.5. — Let G be a finite group and let R be a G -extremal ray with $\mathcal{K}_X \cdot R < 0$. Then there exists a rational curve C_0 such that R is generated by the class of the 1-cycle $C = \sum_{g \in G} gC_0$.

Proof. — Consider a G -extremal ray $R = \mathbb{R}_{\geq 0}[E]$. By the cone theorem (Theorem 4.3) $[E] \in \overline{NE}(X)^G \subset \overline{NE}(X)$ can be written as $[E] = [\sum_i a_i C_i] + [F]$, where $\mathcal{K}_X \cdot F \geq 0$, $a_i \geq 0$ and C_i are rational curves. Let $|G|$ denote the order of G and let $[GF] = G[F] = \sum_{g \in G} g[F]$. Since $g[E] = [E]$ for all $g \in G$ we can write

$$|G|[E] = \sum_{g \in G} g[E] = \sum_{g \in G} ([\sum_i a_i gC_i] + g[F]) = \sum_i a_i G[C_i] + G[F].$$

The element $[\sum_i a_i (GC_i)] + [GF]$ of the G -extremal ray $\mathbb{R}_{\geq 0}[E]$ is decomposed as the sum of two elements in $\overline{NE}(X)^G$. Since R is extremal in $\overline{NE}(X)^G$ both must lie in $R = \mathbb{R}_{\geq 0}[E]$. Consider $[GF] \in R$. Since $g^* \mathcal{K}_X \equiv \mathcal{K}_X$ for all $g \in G$, we obtain

$$\mathcal{K}_X \cdot (GF) = \sum_{g \in G} \mathcal{K}_X \cdot (gF) = \sum_{g \in G} (g^* \mathcal{K}_X) \cdot F = |G| \mathcal{K}_X \cdot F \geq 0.$$

As $\mathcal{K}_X \cdot R < 0$ by assumption, this implies $[F] = 0$ and $\mathbb{R}_{\geq 0}[E] = \mathbb{R}_{\geq 0}[\sum a_i(GC_i)]$. Again using the fact that R is extremal in $\overline{NE}(X)^G$, we conclude that each summand of $[\sum a_i(GC_i)]$ must be contained in $R = \mathbb{R}_{\geq 0}[E]$ and the extremal ray $\mathbb{R}_{\geq 0}[E]$ is therefore generated by $[GC_i]$ for some C_i chosen such that $[GC_i] \neq 0$. This completes the proof of the lemma. \square

4.3. The contraction theorem and minimal models of surfaces. — In this subsection, we state the contraction theorem for smooth projective surfaces. The proof of this theorem can be found e.g. in [KM98] and needs to be modified slightly in order to give an equivariant contraction theorem in the next subsection.

Definition 4.6. — Let X be a smooth projective surface and let $F \subset \overline{NE}(X)$ be an extremal face. A morphism $\text{cont}_F : X \rightarrow Z$ is called the *contraction of F* if

- $(\text{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$ and
- $\text{cont}_F(C) = \{\text{point}\}$ for an irreducible curve $C \subset X$ if and only if $[C] \in F$.

The following result is known as the contraction theorem (cf. Theorem 1.28 in [KM98]).

Theorem 4.7. — Let X be a smooth projective surface and $R \subset \overline{NE}(X)$ an extremal ray such that $\mathcal{K}_X \cdot R < 0$. Then the contraction morphism $\text{cont}_R : X \rightarrow Z$ exists and is one of the following types:

1. Z is a smooth surface and X is obtained from Z by blowing up a point.
2. Z is a smooth curve and $\text{cont}_R : X \rightarrow Z$ is a minimal ruled surface over Z .
3. Z is a point and \mathcal{K}_X^{-1} is ample.

The contraction theorem leads to a minimal model program for surfaces: Starting from X , if \mathcal{K}_X is not nef, i.e, there exists an irreducible curve C such that $\mathcal{K}_X \cdot C < 0$, then $\overline{NE}(X)_{\mathcal{K}_X < 0}$ is nonempty and there exists an extremal ray R which can be contracted. The contraction morphisms either gives a new surface Z (in case 1) or provides a structure theorem for X which is then either a minimal ruled surface over a smooth curve (case 2) or isomorphic to \mathbb{P}^2 (case 3). Note that the contraction theorem as stated above only implies \mathcal{K}_X^{-1} ample in case 3. It can be shown that X is in fact \mathbb{P}^2 . This is omitted here since this statement does not transfer to the equivariant setup. In case 1, we can repeat the procedure if \mathcal{K}_Z is not nef. Since the Picard number drops with each blow down, this process terminates after a finite number of steps. The surface obtained from X at the end of this program is called a *minimal model* of X .

Remark 4.8. — Let E be a (-1)-curve on X and C be any irreducible curve on X . Then $E \cdot C < 0$ if and only if $C = E$. It follows that $\overline{NE}(X) = \text{span}(\mathbb{R}_{\geq 0}[E], \overline{NE}(X)_{E \geq 0})$. Now $E^2 = -1$ implies $E \notin \overline{NE}(X)_{E \geq 0}$ and E is seen to generate an extremal ray in $\overline{NE}(X)$. By adjunction, $\mathcal{K}_X \cdot E < 0$. The contraction of the extremal ray $R = \mathbb{R}_{\geq 0}[E]$ is precisely the contraction of the (-1)-curve E . Conversely, each extremal contraction of type 1 above is the contraction of a (-1)-curve generating the extremal ray R .

4.4. Equivariant contraction theorem and G -minimal models. — In this subsection we prove an equivariant contraction theorem for smooth projective surfaces with finite groups of symmetries. Most steps in the proof are carried out in analogy to the proof of the standard contraction theorem.

Definition 4.9. — Let G be a finite group, let X be a smooth projective surface with G -action and let $R \subset \overline{NE}(X)^G$ be G -extremal ray. A morphism $\text{cont}_R^G : X \rightarrow Z$ is called the *G -equivariant contraction of R* if

- cont_R^G is equivariant with respect to G
- $(\text{cont}_R^G)_* \mathcal{O}_X = \mathcal{O}_Z$ and
- $\text{cont}_R^G(C) = \{\text{point}\}$ for an irreducible curve $C \subset X$ if and only if $[GC] \in R$.

Theorem 4.10. — Let G be a finite group, let X be a smooth projective surface with G -action and let R be a G -extremal ray with $\mathcal{K}_X \cdot R < 0$. Then R can be spanned by the class of $C = \sum_{g \in G} gC_0$ for a rational curve C_0 , the equivariant contraction morphism $\text{cont}_R^G : X \rightarrow Z$ exists and is one of the following three types:

1. $C^2 < 0$ and gC_0 are smooth disjoint (-1) -curves. The map $\text{cont}_R^G : X \rightarrow Z$ is the equivariant blow down of the disjoint union $\bigcup_{g \in G} gC_0$.
2. $C^2 = 0$ and any connected component of C is either irreducible or the union of two (-1) -curves intersecting transversally at a single point. The map $\text{cont}_R^G : X \rightarrow Z$ defines an equivariant conic bundle over a smooth curve.
3. $C^2 > 0$, $N_1(X)^G = \mathbb{R}$ and \mathcal{K}_X^{-1} is ample, i.e., X is a Del Pezzo surface. The map $\text{cont}_R^G : X \rightarrow Z$ is constant, Z is a point.

Proof. — Let R be a G -extremal ray with $\mathcal{K}_X \cdot R < 0$. By Lemma 4.5 the ray R can be spanned by a 1-cycle of the form $C = GC_0$ for a rational curve C_0 . Let $n = |GC_0|$ and write $C = \sum_{i=1}^n C_i$ where the C_i correspond to gC_0 for some $g \in G$. We distinguish three cases according to the sign of the self-intersection of C .

The case $C^2 < 0$. — We write $0 > C^2 = \sum_i C_i^2 + \sum_{i \neq j} C_i \cdot C_j$. Since C_i are effective curves we know $C_i \cdot C_j \geq 0$ for all $i \neq j$. Since all curves C_i have the same negative self-intersection and by assumption, $\mathcal{K}_X \cdot C = \sum_i \mathcal{K}_X \cdot C_i = n(\mathcal{K}_X \cdot C_i) < 0$ the adjunction formula reads $2g(C_i) - 2 = -2 = \mathcal{K}_X \cdot C_i + C_i^2$. Consequently, $\mathcal{K}_X \cdot C_i = -1$ and $C_i^2 = -1$. It remains to show that all C_i are disjoint. We assume the contrary and without loss of generality $C_1 \cap C_2 \neq \emptyset$. Now $gC_1 \cap gC_2 \neq \emptyset$ for all $g \in G$ and $\sum_{i \neq j} C_i \cdot C_j \geq n$. This is however contrary to $0 > C^2 = \sum_i C_i^2 + \sum_{i \neq j} C_i \cdot C_j = -n + \sum_{i \neq j} C_i \cdot C_j$.

We let $\text{cont}_R^G : X \rightarrow Z$ be the blow-down of $\bigcup_{i=1}^n C_i$ which is equivariant with respect to the induced action on Z and fulfills $(\text{cont}_R^G)_* \mathcal{O}_X = \mathcal{O}_Z$. If D is an irreducible curve such that $\text{cont}_R^G(D) = \{\text{point}\}$, then $D = gC_0$ for some $g \in G$. In particular, $GD = GC_0 = \lambda C$ and $[GD] \in R$. Conversely, if $[GD] \in R$ for some irreducible curve D , then $[GD] = \lambda[C]$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Now $(GD) \cdot C = \lambda C^2 < 0$. It follows that D is an irreducible component of C .

The case $C^2 > 0$. — This case is treated in precisely the same way as the corresponding case in the standard contraction theorem. Our aim is to show that $[C]$ is in the interior of $\overline{NE}(X)^G$. This is a consequence of the following lemma (Corollary 1.21 in [KM98]).

Lemma 4.11. — Let X be a projective surface and let L be an ample line bundle on X . Then the set $Q = \{[E] \in N_1(X) \mid E^2 > 0\}$ has two connected components $Q^+ = \{[E] \in Q \mid L \cdot E > 0\}$ and $Q^- = \{[E] \in Q \mid L \cdot E < 0\}$. Moreover, $Q^+ \subset \overline{NE}(X)$.

This lemma follows from the Hodge Index Theorem and the fact that $E^2 > 0$ implies that either E or $-E$ is effective.

We consider an effective cycle $C = \sum C_i$ with $C^2 > 0$. By the above lemma, $[C]$ is contained in Q^+ which is an open subset of $N_1(X)$ contained in $\overline{NE}(X)$. In particular, $[C]$ lies in the interior of $\overline{NE}(X)$. The G -extremal ray $R = \mathbb{R}_{\geq 0}[C]$ can only lie in the interior if $\overline{NE}(X)^G = R$. By assumption $\mathcal{K}_X \cdot R < 0$, hence \mathcal{K}_X is negative on $\overline{NE}(X)^G \setminus \{0\}$ and therefore on $\overline{NE}(X) \setminus \{0\}$. The anticanonical bundle \mathcal{K}_X^{-1} is ample by Kleiman ampleness criterion and X is a Del Pezzo surface.

We can define a constant map cont_R^G mapping X to a point Z which is the equivariant contraction of $R = \overline{NE}(X)^G$ in the sense of Definition 4.9.

The case $C^2 = 0$. — Our aim is to show that for some $m > 0$ the linear system $|mC|$ defines a conic bundle structure on X . The argument is separated into a number of lemmata. For the convenience of the reader, we include also the proofs of well-known preparatory lemmata which do not involve group actions.

Lemma 4.12. — $H^2(X, \mathcal{O}(mC)) = 0$ for $m \gg 0$.

Proof. — By Serre's duality, $h^2(X, \mathcal{O}(mC)) = h^0(\mathcal{O}(-mC) \otimes \mathcal{K}_X)$. Since C is an effective divisor on X , it follows that $h^0(\mathcal{O}(-mC) \otimes \mathcal{K}_X) = 0$ for $m \gg 0$. \square

Lemma 4.13. — For $m \gg 0$ the dimension $h^0(X, \mathcal{O}(mC))$ of $H^0(X, \mathcal{O}(mC))$ is at least two.

Proof. — Let m be such that $h^2(X, \mathcal{O}(mC)) = 0$. For a line bundle L on X we denote by $\chi(L) = \sum_i (-1)^i h^i(X, L)$ the Euler characteristic of L . Using the theorem of Riemann-Roch we obtain

$$\begin{aligned} h^0(X, \mathcal{O}(mC)) &\geq h^0(X, \mathcal{O}(mC)) - h^1(X, \mathcal{O}(mC)) \\ &= h^0(X, \mathcal{O}(mC)) - h^1(X, \mathcal{O}(mC)) + h^2(X, \mathcal{O}(mC)) \\ &= \chi(\mathcal{O}(mC)) \\ &= \chi(\mathcal{O}) + \frac{1}{2}(\mathcal{O}(mC) \otimes \mathcal{K}_X^{-1}) \cdot (mC) \\ &\stackrel{C^2=0}{=} \chi(\mathcal{O}) - \frac{m}{2} \mathcal{K}_X \cdot C. \end{aligned}$$

Now $\mathcal{K}_X C < 0$ implies the desired behaviour of $h^0(X, \mathcal{O}(mC))$. \square

For a divisor D on X we denote by $|D|$ the complete *linear system* of D , i.e., the space of all effective divisors linearly equivalent to D . A point $p \in X$ is called a *base point* of $|D|$ if $p \in \text{support}(C)$ for all $C \in |D|$.

Lemma 4.14. — There exists $m' > 0$ such that the linear system $|m'C|$ is base point free.

Proof. — Let m be chosen such that $h^0(X, \mathcal{O}(mC)) \geq 2$. We denote by B the *fixed part* of the linear system $|mC|$, i.e., the biggest divisor B such that each $D \in |mC|$ can be decomposed as $D = B + E_D$ for some effective divisor E_D . The support of B is the union of all positive dimensional components of the set of base points of $|mC|$. We assume that B is nonempty. The choice of m guarantees that $|mC|$ is not fixed, i.e., there exists $D \in |mC|$ with $D \neq B$. Since $\text{supp}(B) \subset \{s = 0\}$ for all $s \in \Gamma(X, \mathcal{O}(mC))$, each irreducible component of $\text{supp}(B)$ is an irreducible component of C and G -invariance of C implies G -invariance of the fixed part of $|mC|$. It follows that $B = m_0 C$ for some $m_0 < m$. Decomposing $|mC|$ into the fixed part $B = m_0 C$ and the remaining *free part* $|(m - m_0)C|$ shows that some multiple $|m'C|$ for $m' > 0$ has no fixed components. The linear system $|m'C|$ has no isolated base points since these would correspond to isolated points of intersection of divisors linearly equivalent to $m'C$. Such intersections are excluded by $C^2 = 0$. \square

We consider the base point free linear system $|m'C|$ and the associated morphism $\varphi = \varphi_{|m'C|} : X \rightarrow \varphi(X) \subset \mathbb{P}(\Gamma(X, \mathcal{O}(m'C))^*)$. Since C is G -invariant, it follows that φ is an equivariant map with respect to action of G on $\mathbb{P}(\Gamma(X, \mathcal{O}(m'C))^*)$ induced by pullback of sections.

Let z be a linear hyperplane in $\Gamma(X, \mathcal{O}(m'C))$. By definition, $\varphi^{-1}(z) = \bigcap_{s \in z} (s)_0$ where $(s)_0$ denotes the zero set of the section s . Since $(s)_0$ is linearly equivalent to $m'C$ and $C^2 = 0$, the intersection $\bigcap_{s \in z} (s)_0$ does not consist of isolated points but all $(s)_0$ with $s \in z$ have a common component. In particular, each fiber is one-dimensional. Let $f : X \rightarrow Z$ be the Stein factorization of $\varphi : X \rightarrow \varphi(X)$. The space Z is normal and 1-dimensional, i.e., Z is a Riemann surface. Note that there is a G -action on Z such that f is equivariant.

Lemma 4.15. — The map $f : X \rightarrow Z$ defines an equivariant conic bundle, i.e., an equivariant fibration with general fiber isomorphic to \mathbb{P}_1 .

Proof. — Let F be a smooth fiber of f . By construction, F is a component of $(s)_0$ for some element $s \in \Gamma(X, \mathcal{O}(m'C))$. We can find an effective 1-cycle D such that $(s)_0 = F + D$. Averaging over the

group G we obtain $\sum_{g \in G} gF + \sum_{g \in G} gD = \sum_{g \in G} g(s)_0$. Recalling $(s)_0 \sim m'C$ and $[C] \in \overline{NE}(X)^G$ we deduce

$$\left[\sum_{g \in G} gF + \sum_{g \in G} gD \right] = \left[\sum_{g \in G} g(s)_0 \right] = m' \left[\sum_{g \in G} gC \right] = m|G|[C].$$

This shows that $[\sum_{g \in G} gF + \sum_{g \in G} gD]$ is contained in the G -extremal ray generated by $[C]$. By the definition of extremality $[\sum_{g \in G} gF] = \lambda[C] \in \mathbb{R}^{>0}[C]$ and therefore $\mathcal{K}_X \cdot (\sum_{g \in G} gF) < 0$. This implies $\mathcal{K}_X \cdot F < 0$.

In order to determine the self-intersection of F , we first observe $(\sum_{g \in G} gF)^2 = \lambda^2 C^2 = 0$. Since F is a fiber of a G -equivariant fibration, we know that $\sum_{g \in G} gF = kF + kF_1 + \dots + kF_l$ where F, F_1, \dots, F_l are distinct fibers of f and $k \in \mathbb{N}^{>0}$. Now $0 = (\sum_{g \in G} gF)^2 = (l+1)k^2 F^2$ shows $F^2 = 0$. The adjunction formula implies $g(F) = 0$ and F is isomorphic to \mathbb{P}_1 . \square

The map $\text{cont}_R^G := f$ is equivariant and fulfills $f_*\mathcal{O}_X = \mathcal{O}_Z$ by Stein's factorization theorem. Let D be an irreducible curve in X such that f maps D to a point, i.e., D is contained in a fiber of f . Going through the same arguments as above one checks that $[GD] \in R$. Conversely, if D is an irreducible curve in X such that $[GD] \in R$ it follows that $(GD) \cdot C = 0$. If D is not contracted by f , then $f(D) = Z$ and D meets every fiber of f . In particular, $D \cdot C > 0$, a contradiction. It follows that D must be contracted by f .

This completes the proof of the equivariant contraction theorem. \square

A *conic* is a divisor defined by a homogeneous polynomial of degree two in \mathbb{P}_2 . It is therefore either a smooth curve of degree two and multiplicity one, two projective lines of multiplicity one which intersect transversally in one point, or a single line of multiplicity two. A smooth conic is isomorphic to \mathbb{P}_1 . A conic bundle $X \rightarrow Z$ is, as the name suggests, a "bundle" of conics. Its possible degenerations correspond precisely to the degenerations of conics.

The singular fibers of the conic bundle in case (2) of the theorem above are characterized by the following lemma stating that only conic degenerations of the first kind may occur.

Lemma 4.16. — *Let $R = \mathbb{R}^{>0}[C]$ be a \mathcal{K}_X -negative G -extremal ray with $C^2 = 0$. Let $\text{cont}_R^G := f : X \rightarrow Z$ be the equivariant contraction of R defining a conic bundle structure on X . Then every singular fiber of f is a union of two (-1) -curves intersecting transversally.*

Proof. — Let F be a singular fiber of f . The same argument as in the previous lemma yields that $\mathcal{K}_X \cdot F < 0$ and $F^2 = 0$. Since F is connected, the arithmetic genus of F is zero and $\mathcal{K}_X \cdot F = -2$. The assumption on F being singular implies that F must be reducible. Let $F = \sum F_i$ be the decomposition into irreducible components and note that $g(F_i) = 0$ for all i .

We apply the same argument as above to the component F_i of F : after averaging over G we deduce that GF_i is in the G -extremal ray R and $\mathcal{K}_X \cdot F_i < 0$. Since $-2 = \mathcal{K}_X \cdot F = \sum \mathcal{K}_X \cdot F_i$, we may conclude that $F = F_1 + F_2$ and $F_i^2 = -1$. The desired result follows. \square

It should be remarked that G -equivariant conic bundles with or without singular fibers can be studied by considering the G -action on the base and the actions of the isotropy groups of points of the base on the corresponding fibers.

4.5. G -minimal models of surfaces. — Let X be a surface with an action of a finite group G such that \mathcal{K}_X is not nef, i.e., $\overline{NE}(X)_{\mathcal{K}_X < 0}$ is nonempty.

Lemma 4.17. — *There exists a G -extremal ray R such that $\mathcal{K}_X \cdot R < 0$.*

Proof. — Let $[C] \in \overline{NE}(X)_{\mathcal{K}_X < 0} \neq \emptyset$ and consider $[GC] \in \overline{NE}(X)^G$. The G -orbit or G -average of a \mathcal{K}_X -negative effective curve is again \mathcal{K}_X -negative. It follows that $\overline{NE}(X)_{\mathcal{K}_X < 0}^G$ is nonempty. Let L be a

G -invariant ample line bundle on X . By the cone theorem, for any $\varepsilon > 0$

$$(1) \quad \overline{NE}(X)^G = \overline{NE}(X)_{(\mathcal{K}_X + \varepsilon L) \geq 0}^G + \sum_{\text{finite}} \mathbb{R}_{\geq 0} G[C_i].$$

with $\mathcal{K}_X \cdot C_i < 0$ for all i . Since $\overline{NE}(X)_{\mathcal{K}_X < 0}^G$ is nonempty, we may choose $\varepsilon > 0$ such that $\overline{NE}(X)^G \neq \overline{NE}(X)_{(\mathcal{K}_X + \varepsilon L) \geq 0}^G$. If the ray $R_1 = \mathbb{R}_{\geq 0} G[C_1]$ is not extremal in $\overline{NE}(X)^G$, then its generator $G[C_1]$ can be decomposed as a sum of elements of $\overline{NE}(X)^G$ not contained in R_1 . It follows that the ray R_1 is superfluous in the formula (1). Since $\overline{NE}(X)^G \neq \overline{NE}(X)_{(\mathcal{K}_X + \varepsilon L) \geq 0}^G$ by assumption, we may not remove all rays R_i from the formula and at least one ray $R_i = \mathbb{R}_{\geq 0} G[C_i]$ is G -extremal. \square

We apply the equivariant contraction theorem to X : In case (1) we obtain a new surface Z from X by blowing down a G -orbit of disjoint (-1) -curves. There is a canonically defined holomorphic G -action on Z such that the blow-down is equivariant. If K_Z is not nef, we repeat the procedure which will stop after a finite number of steps. In case (2) we obtain an equivariant conic bundle structure on X . In case (3) we conclude that X is a Del Pezzo surface with G -action. We call the G -surface obtained from X at the end of this procedure a G -minimal model of X .

As a special case, we consider a rational surface X with G -action. Since the canonical bundle \mathcal{K}_X of a rational surface X is never nef, a G -minimal model of X is an equivariant conic bundle over a smooth rational curve Z or a Del Pezzo surface with G -action. This proves the well-known classification of G -minimal models of rational surfaces (cf. [Man67], [Isk80]).

Although this classification does classically not rely on Mori theory, the proof given above is based on Mori's approach. We therefore refer to an equivariant reduction $Y \rightarrow Y_{\min}$ as an *equivariant Mori reduction*.

5. Del Pezzo surfaces

The equivariant minimal model program for surfaces presents us with the task of studying automorphism groups of Del Pezzo surfaces.

A Del Pezzo surface is defined as a connected compact complex surface whose anticanonical line bundle is ample. Using Kleiman's ampleness criterion we have identified the class of Del Pezzo surfaces as a class of G -minimal surfaces. Here we wish to replace this very abstract notion of ampleness by a definition involving the notions of bundles and sections.

We let X be a connected compact complex surface, T_X the holomorphic tangent bundle, T_X^* the cotangent bundle, and $\mathcal{K}_X := \Lambda^2 T_X^*$ its top exterior power, the canonical line bundle. The *anticanonical bundle* is given by $\mathcal{K}_X^{-1} = \Lambda^2 T_X$.

In this terminology the space V_k , which we have discussed in a naive fashion up to this point, is the space of sections $\Gamma(X, \mathcal{K}_X^k)$. The requirement that \mathcal{K}_X^{-1} is *ample* means that some power $(\mathcal{K}_X^{-1})^k =: -k\mathcal{K}_X$ has many sections. More precisely, one requires that for some k the map $X \rightarrow \mathbb{P}(\Gamma(X, -k\mathcal{K}_X)^*)$ is a holomorphic embedding.

Using wedge-products of holomorphic vector fields one easily shows that the anticanonical bundles of \mathbb{P}_2 and $\mathbb{P}_1 \times \mathbb{P}_1$ are ample. In order to characterize the remaining Del Pezzo surfaces it is convenient to introduce the *degree* of a Del Pezzo surface as the self-intersection number d of an anticanonical divisor. For \mathbb{P}_2 it is simple to compute this degree: $\mathcal{K}_{\mathbb{P}_2}^{-1}$ is the 3rd power H^3 of the hyperplane bundle and its sections are homogeneous polynomials of degree three. By Bezout's theorem, the intersection of two such cubics consists of nine points counted with multiplicity and the degree d of the Del Pezzo surface \mathbb{P}_2 equals $d = 9$. The possible degrees of Del Pezzo surfaces range from one to nine.

The following theorem (cf. Theorem 24.4 in [Man74]) gives a classification of Del Pezzo surfaces according to their degree.

Theorem 5.1. — *Let Z be a Del Pezzo surface of degree d .*

- *If $d = 9$, then Z is isomorphic to \mathbb{P}_2 .*
- *If $d = 8$, then Z is isomorphic to either $\mathbb{P}_1 \times \mathbb{P}_1$ or the blow-up of \mathbb{P}_2 in one point.*
- *If $1 \leq d \leq 7$, then Z is isomorphic to the blow-up of \mathbb{P}_2 in $9 - d$ points in general position, i.e., no three points lie on one line and no six points lie on one conic.*

Using the theorem above we may identify a Del Pezzo surface $X \neq \mathbb{P}_1 \times \mathbb{P}_1$ with the blow up $X_{\{p_1, \dots, p_m\}}$ of \mathbb{P}_2 at each point of $\{p_1, \dots, p_m\}$ for $m \in \{0, \dots, 8\}$. We carry the points p_1, \dots, p_m in the notation, because for $m \geq 5$ the complex structure really does depend on the points which are blown up. For example, if $m = 4$, using automorphisms of \mathbb{P}_2 we can move the points to a standard location, e.g., $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$, $[1 : 1 : 1]$. Of course such a normal form is even easier to achieve if $m < 4$. On the other hand, if $m > 4$, then we put p_1, \dots, p_4 in this normal form, but p_5, \dots, p_m are allowed to vary. As these points vary the complex structure of the Del Pezzo surface varies. So for $m \geq 5$ Del Pezzo surfaces come in families.

5.1. Automorphism groups of Del Pezzo surfaces. — Let us now turn to a study of automorphism groups of Del Pezzo surfaces. In analogy to the case of surfaces of general type we begin with the following first remark.

Proposition 5.2. — *The automorphism group $\text{Aut}(X)$ of a Del Pezzo surface is an algebraic group acting algebraically on X .*

Proof. — For k sufficiently large the map $X \rightarrow \mathbb{P}(\Gamma(X, -k\mathcal{K}_X)^*)$ is an $\text{Aut}(X)$ -equivariant, holomorphic embedding. We denote its image by Z . The group $\text{Aut}(X)$ can be realized as the stabilizer of Z in the automorphism group of the ambient projective space, which is itself an algebraic group. By the theorem of Chow the space Z is an algebraic subvariety. Since the stabilizer of an algebraic subvariety is an algebraic group acting algebraically, the result follows. \square

There are Del Pezzo surfaces with positive-dimensional automorphism groups, e.g., \mathbb{P}_2 and $\mathbb{P}_1 \times \mathbb{P}_1$ are even homogeneous. Since the stabilizer in $\text{Aut}(\mathbb{P}_2)$ of $\{p_1, \dots, p_m\}$ has an open orbit for $m \leq 3$, the following shows in particular that the Del Pezzo surfaces of degree at least six are almost homogeneous, i.e., their automorphism groups have an open orbit.

Proposition 5.3. — *If $\pi : X_{\{p_1, \dots, p_m\}} \rightarrow \mathbb{P}_2$ is the defining blow up of the Del Pezzo surface $X_{\{p_1, \dots, p_m\}}$ and $g \in \text{Aut}(\mathbb{P}_2)$ stabilizes the set $\{p_1, \dots, p_m\}$, then there exists a uniquely defined automorphism $\hat{g} \in \text{Aut}(X_{\{p_1, \dots, p_m\}})$ so that $g \circ \pi = \pi \circ \hat{g}$.*

Proof. — If E_j is the π -preimage of p_j , $j = 1, \dots, m$, then there exists a unique automorphism \hat{g} of $X_{\{p_1, \dots, p_m\}} \setminus \bigcup_j E_j$ with the desired property. Thus it is a matter of extending \hat{g} to the full Del Pezzo surface. It is enough to show that it extends across $E = E_1$. For notational simplicity we may assume that $gp_1 = p_1 =: p$. Since points of E correspond to tangent lines through $\pi(E) = p$ and $gp = p$, it follows that \hat{g} extends continuously across E . The fact that this extension is holomorphic is guaranteed by Riemann's Hebbarkeitssatz. It is an automorphism since g^{-1} can likewise be lifted to $X_{\{p_1, \dots, p_m\}}$. \square

Furthermore, the considerations in this section will benefit from the following converse of the above statement.

Proposition 5.4. — *The defining projection $\pi : X_{\{p_1, \dots, p_m\}} \rightarrow \mathbb{P}_2$ is equivariant with respect to the connected component $\text{Aut}(X_{\{p_1, \dots, p_m\}})^\circ$ containing the identity.*

Proof. — Let g_t be a 1-parameter subgroup in $\text{Aut}(X_{\{p_1, \dots, p_m\}})$ and let $E := E_j$ be the π -preimage of $p := p_j$. Given a small neighborhood U of p , if t is sufficiently small, then $\pi(g_t E)$ is contained in U . Since π is a proper holomorphic map, it follows from Remmert's mapping theorem that $\pi(g_t E)$ is a compact analytic subset of U . If U is sufficiently small, its only analytic subsets are discrete. Hence,

$\pi(g_t E)$ is a single point and $g_t E = E$. As a result every 1-parameter subgroup of $\text{Aut}(X_{\{p_1, \dots, p_m\}})$ stabilizes every E_j and consequently the same is true for the full connected component $\text{Aut}(X_{\{p_1, \dots, p_m\}})^\circ$. Thus every element of $\text{Aut}(X_{\{p_1, \dots, p_m\}})^\circ$ defines a map of \mathbb{P}_2 fixing $\{p_1, \dots, p_m\}$ which is holomorphic outside $\{p_1, \dots, p_m\}$ and the desired result follows from Riemann's Hebbarkeitssatz. \square

Corollary 5.5. — *If $m \geq 4$, then the automorphism group of the Del Pezzo surface $X_{\{p_1, \dots, p_m\}}$ is finite.*

Detailed lists of the automorphism groups Del Pezzo surfaces can be found in Dolgachev's book ([Dol08]). For the convenience of the reader we present a brief road map here.

The projective plane. — The action of $\text{SL}_3(\mathbb{C})$ on \mathbb{C}^3 defines a surjective homomorphism $\text{SL}_3(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}_2)$ and $\text{Aut}(\mathbb{P}_2)$ can be identified with the quotient $\text{SL}_3(\mathbb{C})/C_3$. Here C_3 is embedded as the center of $\text{SL}_3(\mathbb{C})$ which consists of matrices of the form λId with $\lambda^3 = 1$. In general, the action of a finite group G on \mathbb{P}_2 is given by a 3-dimensional linear representation of a nontrivial central extension of G . An interesting example is provided by the Valentiner group defining an action of the alternating group A_6 on \mathbb{P}_2 .

The space $\mathbb{P}_1 \times \mathbb{P}_1$. — The group $S := \text{Aut}(\mathbb{P}_1) \times \text{Aut}(\mathbb{P}_1)$ acts on the Del Pezzo surface $X = \mathbb{P}_1 \times \mathbb{P}_1$ in an obvious fashion. In addition, $\text{Aut}(X)$ contains the holomorphic involution σ which exchanges the factors. The group S coincides with the subgroup of all $h \in \text{Aut}(X)$ such that both projections are h -equivariant. At the level of homology classes F_i of fibers, $hF_i = F_i$ for each $h \in S$. In fact S can be identified with the subgroup of elements of $\text{Aut}(X)$ having this property. For an arbitrary $g \in \text{Aut}(X)$, since $(gF_i)^2 = 0$, either $gF_i = F_i$ for $i = 1, 2$ or $gF_1 = F_2$. So either $g \in S$ or $\sigma \circ g \in S$ and $\text{Aut}(\mathbb{P}_1 \times \mathbb{P}_1) = S \rtimes \langle \sigma \rangle$.

In the following we denote by X_d a Del Pezzo surface of degree d which arises as the blow up of \mathbb{P}_2 in $9 - d$ points. We have already pointed out that any Del Pezzo surface except $\mathbb{P}_1 \times \mathbb{P}_1$ is of this form.

The simple blow up of \mathbb{P}_2 . — The surface X_8 is the simple blow up $\text{Bl}_p(\mathbb{P}_2)$. Since the exceptional curve E of this blow up is the unique (-1)-curve in X_8 , it follows that every automorphism of X_8 stabilizes E and $\text{Aut}(X_8)$ can be identified with the isotropy group at p in $\text{Aut}(\mathbb{P}_2)$

The blow up of \mathbb{P}_2 in two points. — Let $p_1, p_2 \in \mathbb{P}_2$ be the two points which are blown up to obtain X_7 . Note that there exists an automorphism σ of \mathbb{P}_2 , more precisely a holomorphic involution, which exchanges p_1 and p_2 , and note that by the arguments given above it can be lifted to an automorphism of X_2 . Denoting this lift by $\hat{\sigma}$, it follows that $\text{Aut}(X_2)$ can be identified with the semidirect product $T \rtimes \langle \hat{\sigma} \rangle$ where T is the subgroup of $\text{Aut}(\mathbb{P}_2)$ consisting of transformations fixing both p_1 and p_2 .

The three-point blow up of \mathbb{P}_2 . — Let X_6 be the Del Pezzo surface defined by blowing up the three points $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, and $p_3 = [0 : 0 : 1]$. As we know, the connected component of the identity in $\text{Aut}(X_6)$ is the connected component of the stabilizer of $\{p_1, p_2, p_3\}$. This is the group of diagonal matrices in $\text{SL}_3(\mathbb{C})$ modulo the center of $\text{SL}_3(\mathbb{C})$ and is therefore isomorphic to $(\mathbb{C}^*)^2$. The full permutation group of $\{p_1, p_2, p_3\}$ can also be realized in $\text{Aut}(\mathbb{P}_2)$. We see that the subgroup of automorphisms of X_6 which are equivariant with respect to the defining map $X_6 \rightarrow \mathbb{P}_2$ is isomorphic to $(\mathbb{C}^*)^2 \rtimes S_3$.

The proper transform \hat{L}_{ij} in X_6 of the projective line L_{ij} joining p_i and p_j is a (-1)-curve. This is due to the fact that $L_{ij}^2 = 1$ and L_{ij} contains exactly two points which are blown up. One can show that the only (-1)-curves in X_6 are the exceptional curves E_i obtained from blowing up the points p_i and the "lines" \hat{L}_{ij} . The graph of this configuration of curves is a hexagon H . Restriction yields a homomorphism $R : \text{Aut}(X_6) \rightarrow I(H)$, where $I(H) \cong D_{12}$ is the group of rigid motions of the hexagon and the kernel of R is the connected component $\text{Aut}(X_6)^\circ \cong (\mathbb{C}^*)^2$ discussed above.

We have already seen that the permutation group S_3 is contained in the image of R and will now show that there is an additional involution in this image so that in fact R is surjective. In order to determine this involution it is useful to regard X_6 as the blow up of $\mathbb{P}_1 \times \mathbb{P}_1$ in the *antidiagonal corner*

points $c_1 = ([1 : 0], [0 : 1])$ and $c_2 = ([0 : 1], [1 : 0])$. If one draws $\mathbb{P}_1 \times \mathbb{P}_1$ as a square, then the hexagon H consists of the (-1) -curves arising from blowing up c_1 and c_2 together with the proper transforms of the four edges of the square. Using elementary intersection arguments one can show that by blowing down either of the two configurations of three disjoint curves in this hexagon one obtains \mathbb{P}_2 with three points in general position. Hence, this blow up is indeed X_6 and the additional involution discussed above is defined by lifting the involution of $\mathbb{P}_1 \times \mathbb{P}_1$ which exchanges the factors. Thus we have shown that $\text{Aut}(X_6)$ is naturally isomorphic to $(\mathbb{C}^*)^2 \times I(H) = (\mathbb{C}^*)^2 \times D_{12}$.

The Del Pezzo surface of degree five. — We may define X_5 by blowing up $p_4 := [1 : 1 : 1]$ and p_1, p_2, p_3 as above. Defining \hat{L}_{ij} as before we obtain a configuration of ten (-1) -curves which in fact is the entire collection of (-1) -curves in X_5 . The dual graph of this configuration is known as the Petersen graph P .

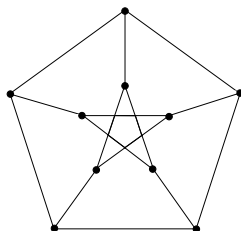


FIGURE 3. The Petersen graph

A point in the graph represents a (-1) -curve and two such curves intersect (transversally) if and only if the corresponding points in the graph are connected by a line segment. The connected component of $\text{Aut}(X_5)$ is trivial and therefore the restriction map $R : \text{Aut}(X_5) \rightarrow I(P)$ realizes the automorphism group of X_5 as a subgroup of the graph automorphism group $I(P) \cong S_5$.

There are various collections of four disjoint curves in P which can be blown down to obtain a copy of \mathbb{P}_2 with four distinguished points. By Proposition 5.3 their permutation group S_4 can be identified with a subgroup of $\text{Aut}(X_5)$. For two configurations of disjoint curves in P which have one curve in common we observe that the two corresponding copies of S_4 do not define the same subgroup of $\text{Aut}(X_5)$. Thus together they generate a subgroup of S_5 properly containing S_4 . Since the index of S_4 in S_5 is prime, it follows that they generate the full group S_5 . We have shown $\text{Aut}(X_5) \cong I(P) \cong S_5$.

In the remaining cases we use a number of basic general facts about Del Pezzo surfaces. For their proofs, we refer the reader, e.g., to [Man74] and [Dol08]. Here we present outlines of the arguments required to identify $\text{Aut}(X_d)$ for $d \leq 4$.

Five-point blow ups of \mathbb{P}_2 . — We define a surface X_4 by blowing up the four points p_1, \dots, p_4 as above and in addition a fifth point p_5 . As p_5 moves so does the complex structure of X_4 . Thus, it is to be expected that the automorphisms group of X_4 depend on the position of p_5 .

In the following, we strongly use the fact that X_4 is embedded by \mathcal{K}_X^{-1} in \mathbb{P}_4 as a surface which is the transversal intersection of two nondegenerate quadric 3-folds. Choosing coordinates appropriately we may assume that these quadrics are defined by $Q_1 := \sum z_j^2$ and $Q_2 := \sum a_j z_j^2$ with $a_i \neq a_j$ for $i \neq j$. Since the embedding in \mathbb{P}_4 is $\text{Aut}(X_4)$ -equivariant, the group $\text{Aut}(X_4)$ can be identified with the stabilizer S in $\text{Aut}(\mathbb{P}_4)$ of the subspace $V := \text{Span}(Q_1, Q_2)$ in the space of all quadratic forms. Computing in $\text{SL}_5(\mathbb{C})$ one sees that S is the normalizer of the group of diagonal matrices T modulo the center C_5 of $\text{SL}_5(\mathbb{C})$. This is the group $T \rtimes S_5$ where S_5 is acting by permuting the coordinate functions z_0, \dots, z_4 .

The meromorphic map $\mathbb{P}_4 \rightarrow \mathbb{P}(V)$ defined by Q_1, Q_2 is S -equivariant and therefore defines a homomorphism $S \rightarrow S/I \hookrightarrow \text{Aut}(\mathbb{P}_1)$. The kernel I consists of those transformations which act on Q_1 and Q_2 by the same character. Since $a_i \neq a_j$ for $i \neq j$, it follows that $I \cong C_2^4$ is generated by the elements of T of the form $\text{Diag}(\pm 1, \dots, \pm 1)$. Since S_5 normalizes this group, we see that $S = C_2^4 \rtimes S/I$, where S/I is on the one hand a subgroup of S_5 and on the other a subgroup of $\text{Aut}(\mathbb{P}_1)$. Using this

information one can directly compute all possibilities for $\text{Aut}(X_4)$, namely $S/I \in \{C_2, C_4, S_3, D_{10}\}$. (see § 10.2.2 in [Dol08]).

Cubic surfaces. — The case of Del Pezzo surfaces of degree three is conceptually simple, but computationally complicated. In this case \mathcal{K}_X^{-1} is still very ample and embeds X_3 as a cubic surface in \mathbb{P}_3 , i.e., as the zero-set of a cubic polynomial P_3 . Since this embedding is $\text{Aut}(X_3)$ -equivariant, $\text{Aut}(X_3)$ can be identified with the stabilizer in $\text{Aut}(\mathbb{P}_3)$ of the line $\mathbb{C} \cdot P_3$ in the space of all cubics. Consequently, the classification of automorphism groups of Del Pezzo surfaces of degree three amounts to the determination of the invariants of actions of the finite subgroups of $\text{SL}_4(\mathbb{C})$ on the space of cubic homogeneous polynomials. This is carried out in [Dol08] where the results are presented in Table 10.3.

Double covers ramified over a quartic. — A Del Pezzo surface X_2 of degree two, can be realized as a 2:1 cover ramified over a smooth curve C of degree four by the anticanonical map $\varphi_{\mathcal{K}_X^{-1}} : X_2 \rightarrow \mathbb{P}_2$. Conversely, if $X \rightarrow \mathbb{P}_2$ is a 2:1 cover ramified over a smooth quartic curve, then X is a Del Pezzo surface of degree two.

A smooth quartic curve C is abstractly a Riemann surface of genus three which is embedded as a quartic curve in \mathbb{P}_2 by its canonical bundle. This embedding is equivariant and consequently $\text{Aut}(C)$ is the stabilizer of C in $\text{Aut}(\mathbb{P}_2)$. Furthermore, $\text{Aut}(C)$ is acting canonically on the bundle space H of the hyperplane section bundle because its restriction to C is the canonical line bundle.

The 2:1 cover X_2 of \mathbb{P}_2 ramified along C is constructed in the bundle space H^2 where $\text{Aut}(C)$ also acts. So on the one hand, the equivariant covering map $X_2 \rightarrow \mathbb{P}_2$ defines a homomorphism of $\text{Aut}(X_2)$ onto a subgroup of $\text{Aut}(C)$, and on the other hand, $\text{Aut}(C)$ lifts to a subgroup of $\text{Aut}(X_2)$. Since the kernel of the surjective homomorphism $\text{Aut}(X_2) \rightarrow \text{Aut}(C)$ is generated by the covering transformation, it follows that we have a canonical splitting $\text{Aut}(X_2) = \text{Aut}(C) \times C_2$. Since C is equivariantly embedded in \mathbb{P}_2 and $\text{Aut}(C)$ is acting as a subgroup of $\text{SL}_3(\mathbb{C})$, the classification of the automorphism groups of Del Pezzo surfaces of degree two results from the classification of the finite subgroups of $\text{SL}_3(\mathbb{C})$ ([Bli17, YY93]) and the invariant theory of their representations on the space of homogeneous polynomials of degree four (see Table 10.4 in [Dol08]).

Del Pezzo surfaces of degree one. — In this case the anticanonical map is a meromorphic map $\varphi_{\mathcal{K}_X^{-1}} : X_1 \rightarrow \mathbb{P}_1$ with exactly one point p of indeterminacy, a so-called base point. Thus p is fixed by $\text{Aut}(X_1)$ and, since it is a finite group and in particular compact, the linearization of $\text{Aut}(X_1)$ on $T_p X_1$ is a faithful representation. This already places a strong limitation on the group $\text{Aut}(X_1)$. Furthermore, the map $X_1 \rightarrow \mathbb{P}_3$ defined by $-2\mathcal{K}_X$ has no points of indeterminacy and realizes X_1 as a 2:1 ramified cover over a quadric cone. A study of these two maps, both of which are equivariant, leads to a precise description of all possible automorphism groups of Del Pezzo surface of degree one (Table 10.5 in [Dol08])

6. K3-surfaces with special symmetry

In this section a setting is considered where the equivariant minimal model program has been implemented to prove classification theorems for K3-surfaces with finite symmetry groups. Additional techniques which aid in simplifying the combinatorial geometry involved in the Mori reduction are outlined and recent results which appear in [Fra08] are sketched. Details of a concrete situation involving the group A_6 are given in next section.

6.1. Maximal groups. — A K3-surface X is a simply-connected compact complex surface admitting a globally defined nowhere-vanishing holomorphic 2-form ω . A transformation $g \in \text{Aut}(X)$ is said to be *symplectic* if $g^*\omega = \omega$ and the group of symplectic automorphisms is denoted by $\text{Aut}_{\text{sym}}(X)$. If $\chi : \text{Aut}(X) \rightarrow \mathbb{C}^*$ denotes the character defined by $g^*\omega = \chi(g)\omega$, then $\text{Aut}_{\text{sym}}(X) = \text{Ker}(\chi)$. For a finite subgroup $G \subset \text{Aut}(X)$ we have the exact sequence

$$1 \rightarrow G_{\text{sym}} \rightarrow G \rightarrow C_n \rightarrow 1,$$

where the homomorphism $G \rightarrow C_n$ is the restriction $\chi|_G$. The group G can be regarded as a coextension of G_{sym} by C_n . Although we restrict here to the case where G is finite, it should be underlined that the full group $\text{Aut}(X)$ may not be finite.

Example 6.1. — Let T_1 be the 1-dimensional torus defined by the lattice $\langle 1, i \rangle_{\mathbb{Z}}$ and let $T = T_1 \times T_1 = \mathbb{C}^2/\Lambda$. The group $\Gamma := \text{SL}_2(\mathbb{Z})$ is contained in $\text{Aut}(T)$ and centralizes the involution $\sigma := -\text{Id}$. Thus Γ acts as a group of holomorphic transformations on the quotient $Y := T/\sigma$. The set of singular points in Y consists of 16 ordinary double points. The desingularization $\text{Kum}(T) \rightarrow Y$ blows up each of these points, replacing them by copies E of \mathbb{P}_1 with $E \cdot E = -2$. The group Γ lifts to act as a group of holomorphic transformations on the *Kummer surface* $\text{Kum}(T)$. The holomorphic 2-form $dz \wedge dw$ on T is σ -invariant and defines a nowhere vanishing holomorphic 2-form on $\text{Kum}(T)$ which is Γ -invariant. Since $X = \text{Kum}(T)$ is simply-connected, it follows that it is a K3-surface with $\Gamma \subset \text{Aut}_{\text{sym}}(X)$.

We are interested in finite subgroups G of $\text{Aut}(X)$ where G_{sym} is either large or possesses interesting group structure and the following theorem of Mukai ([Muk88]) is of particular relevance (See [Kon98] for an alternative proof.).

Theorem 6.2. — *If G_{sym} is a finite group of symplectic transformations of a K3-surface X , then it is contained in one of the groups M listed in the following table.*

	M	$ M $	Structure
1	$L_2(7)$	168	$\text{PSL}_2(\mathbb{F}_7) = \text{GL}_3(\mathbb{F}_2)$
2	A_6	360	even permutations
3	S_5	120	$A_6 \rtimes C_2$
4	M_{20}	960	$C_2^4 \rtimes A_5$
5	F_{384}	384	$C_2^4 \rtimes S_4$
6	$A_{4,4}$	288	$C_2^4 \rtimes A_{3,3}$
7	T_{192}	192	$(Q_8 * Q_8) \rtimes S_3$
8	H_{192}	192	$C_2^4 \rtimes D_{12}$
9	N_{72}	72	$C_3^2 \rtimes D_8$
10	M_9	72	$C_3^2 \rtimes Q_8$
11	T_{48}	48	$Q_8 \rtimes S_3$

For brevity we refer to the groups which are listed in this table as *Mukai groups*. It should be mentioned that this result is sharp in two senses. First, given two *maximal* groups M_1 and M_2 which are listed in this table, M_1 can not be realized as a subgroup of M_2 and vice versa. Secondly, given a group M listed in the table, there exists a K3-surface X with $M \subset \text{Aut}_{\text{sym}}(X)$.

6.2. K3-surfaces with antisymplectic involutions. — An element $\sigma \in \text{Aut}(X)$ of order two with $\sigma^*\omega = -\omega$ is called an *antisymplectic involution*. For various reasons K3-surfaces equipped with such involutions are of particular interest, e.g., from the point of view of moduli spaces and associated automorphic forms (see [Yos04]).

Our study of these surfaces was motivated by an attempt to understand the K3-surfaces which possess finite groups G of automorphism where G_{sym} is large, e.g., where G_{sym} is maximal in the sense of Mukai's Theorem. In that setting there are strong restrictions on the group structure of the coextension $1 \rightarrow G_{\text{sym}} \rightarrow G \rightarrow C_n \rightarrow 1$ and the size of n which show that understanding the case $n = 2$ is of particular importance. Thus, as a starting point, we undertook the classification project in the case where $G = G_{\text{sym}} \times C_2$ and where the antisymplectic involution σ which generates C_2 has a nonempty set of fixed points.

Before turning to an outline of the main results of [Fra08] we would like to emphasize that our work was motivated by a number very interesting works of Keum, Oguiso, Zhang (see [OZ02],[KOZ05, KOZ07]) and of course depends on the foundational results of Nikulin ([Nik80] and Mukai ([Muk88]).

Simplifying the notation, we are interested in classifying triples (X, G, σ) where X is a K3-surface on which G is acting as a group of symplectic automorphisms centralized by an antisymplectic involution σ . We assume that G is acting effectively, i.e., that the only element of G which fixes every point of X is the identity, and we wish to classify these triples up to equivariant isomorphism. The fixed point set $\text{Fix}(\sigma)$ is either empty or 1-dimensional and G acts naturally on the quotient $Y := Y/\sigma$. If $\text{Fix}(X) = \emptyset$, then Y is an Enriques surface.

If $\text{Fix}(\sigma) \neq \emptyset$, the quotient Y is a smooth rational surface. Thinking in terms of the method of quotients by small subgroups which was discussed in § 2.1, we have moved to a G -manifold Y of lower Kodaira-dimension. In this case we apply the equivariant minimal model program to obtain an equivariant Mori reduction $Y \rightarrow Y_1 \rightarrow \dots \rightarrow Y_N = Y_{\min}$. It follows that Y_{\min} is either a Del Pezzo surface or an equivariant conic bundle over \mathbb{P}^1 . Thus, one can understand Y_{\min} as a G -manifold, describe the combinatorial geometry of the steps in the Mori reduction and then reconstruct the 2:1 cover $X \rightarrow Y$. If G is either large or has sufficiently complicated group structure, then the combinatorial geometry simplifies and fine classification results can be proved. In this regard we now mention two results from [Fra08].

Theorem 6.3. — *Let X be a K3-surface and G be a finite group of symplectic automorphisms of X which is centralized by an antisymplectic involution σ with $R = \text{Fix}(\sigma) \neq \emptyset$. Then, if $|G| > 96$, it follows that the quotient $Y = X/\sigma$ is a G -minimal Del Pezzo surface and R is a Riemann surface of general type.*

Given our detailed knowledge of all of the surfaces Y_{\min} and their automorphism groups, it would in principle be possible to explicitly determine the K3-surfaces which arise in this theorem.

Although not all Mukai groups are large in the sense of this theorem, those which are not have a structure which is sufficiently complicated to allow for a precise classification. This result can be formulated as follows.

Theorem 6.4. — *The K3-surfaces which are equipped with an effective and symplectic action of a Mukai group G centralized by an antisymplectic involution σ with $\text{Fix}(\sigma) \neq \emptyset$ are classified up to equivariant equivalence in the table below.*

	G	$ G $	K3-surface X
1a	$L_2(7)$	168	$\{x_1^3x_2 + x_2^3x_3 + x_3^3x_1 + x_4^4 = 0\} \subset \mathbb{P}_3$
1b	$L_2(7)$	168	Double cover of \mathbb{P}_2 branched along $\{x_1^5x_2 + x_3^5x_1 + x_2^5x_3 - 5x_1^2x_2^2x_3^2 = 0\}$
2	A_6	360	Double cover of \mathbb{P}_2 branched along $\{10x_1^3x_2^3 + 9x_1^5x_3 + 9x_2^3x_3^3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6 = 0\}$
3a	S_5	120	$\{\sum_{i=1}^5 x_i = \sum_{i=1}^6 x_1^2 = \sum_{i=1}^5 x_i^3 = 0\} \subset \mathbb{P}_5$
3b	S_5	120	Double cover of \mathbb{P}_2 branched along $\{F_{S_5} = 0\}$
9	N_{72}	72	$\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0\} \subset \mathbb{P}_4$
10	M_9	72	Double cover of \mathbb{P}_2 branched along $\{x_1^6 + x_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3) = 0\}$
11a	T_{48}	48	Double cover of \mathbb{P}_2 branched along $\{x_1x_2(x_1^4 - x_2^4) + x_3^6 = 0\}$
11b	T_{48}	48	Double cover of $\{x_0x_1(x_0^4 - x_1^4) + x_2^3 + x_3^2 = 0\} \subset \mathbb{P}(1, 1, 2, 3)$ branched along $\{x_2 = 0\}$

Examples 1a, 3a, 9, 10, and 11a appear in [Muk88] whereas the remaining provide additional examples of K3-surfaces with maximal symplectic symmetry.

There are several points concerning the above table which need to be clarified. First, the polynomial F_{S_5} in Example 3b can be written as

$$F_{S_5} = 2(x^4yz + xy^4z + xyz^4) - 2(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^4z^2 + y^2z^4) \\ + 2(x^3y^3 + x^3z^3 + y^3z^3) + x^3y^2z + x^3yz^2 + x^2yz^3 + xy^2z^3 - 6x^2y^2z^2.$$

The curve $C := \{F_{S_5} = 0\} \subset \mathbb{P}_2$ is singular at the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and $[1 : 1 : 1]$. The proper transform \hat{C} of C inside the Del Pezzo surface X_5 obtained by blowing up these four points is the normalization of C and defined by a section of $-2\mathcal{K}_{X_5}$. The double cover X of X_5 branched along \hat{C} is the minimal desingularization of the double cover of \mathbb{P}_2 branched along C and X is a K3-surface with an action of $S_5 \times C_2$. As is implied by Theorem 6.3 the Mori reduction with respect to the full group $G = S_5$ is such that $X_5 = Y = Y_{\min}$. The map $Y \rightarrow \mathbb{P}_2$ is the equivariant Mori reduction of the Del Pezzo surface Y with respect to the subgroup S_4 which acts as the permutation group of the four points which are blown up.

From the defining equation of Example 1a one can see that this K3-surface is a C_4 -cover of \mathbb{P}_2 which is branched over Klein's curve C . The preimage \hat{C} of C in the K3-surface is the fixed point set of C_4 . In this case σ generates the unique copy of C_2 in C_4 and the quotient $X/\sigma = Y = X_2$ is a Del Pezzo surface and minimal with respect to the action of $L_2(7)$. The group C_4/C_2 acts on X_2 and realizes X_2 as 2:1 cover of \mathbb{P}_2 branched over C . Here X_2 can also be realized as the blow up $b : X_2 \rightarrow \mathbb{P}_2$ at the seven singular points of the sextic $\{3x^2y^2z^2 - (x^5y + y^5z + z^5x) = 0\}$. Its proper transform in X_2 coincides with the branch locus of the map $X \rightarrow Y = X_2$. The map b is the Mori reduction of X_2 with respect to a maximal subgroup $C_3 \times C_7$ of $L_2(7)$.

The K3-surface in Example 3a is equivariantly embedded in \mathbb{P}_5 where S_5 acts by permuting the first five variables of \mathbb{C}^6 and by the character sgn on the sixth. The antisymplectic involution acts by $\sigma[x_1 : \dots : x_6] = [x_1 : \dots : x_5 : -x_6]$. The quotient $X \rightarrow X/\sigma = Y = Y_{\min}$ is defined by the projection $[x_1 : \dots : x_6] \mapsto [x_1 : \dots : x_5]$. Thus Y is the Del Pezzo surface of degree three defined by the equations $\sum_1^5 y_i = \sum_1^5 y_i^3 = 0$ in \mathbb{P}_4 known as *Clebsch cubic*. By a similar construction, Example 9 is seen to be a double cover of the *Fermat cubic*.

Finally, Example 2, the A_6 -covering of \mathbb{P}_2 , deserves special mention. In this case the action of A_6 on \mathbb{P}_2 is given by its unique central extension by C_3 , which is its preimage in $SL_3(\mathbb{C})$. This was constructed by Valentiner in the 19th century and remains of interest today (see e.g. [Cra99]).

6.3. Combinatorial geometry. — The simple nature of the classification results outlined above is at least in part due to the fact that the possibilities for the combinatorial geometry of a Mori reduction $Y \rightarrow Y_1 \rightarrow \dots \rightarrow Y_{\min}$ can be described in explicit ways. An indication of this can be found in the example in the next section. Here we close this section by listing the key facts which play a role in handling this combinatorial geometry.

- A basic result of Nikulin ([Nik83]) shows that the branch set B of the covering $X \rightarrow X/\sigma = Y$, which is the image in Y of the fixed point set of an antisymplectic involution σ , is either empty, consists of two disjoint linearly equivalent elliptic curves, is a union of rational curves or is the union of rational curves and a Riemann surface of genus at least one. In the last case it is possible, and quite often happens, that B consists of only a Riemann surface of genus at least one and no rational curves.
- By a result due to Zhang ([Zha98]) the number of connected components of B is at most ten.
- We refer to a rational curve E in Y as being a *Mori fiber* if it is blown down to a point at some stage $Y_k \rightarrow Y_{k+1}$ of the reduction to a minimal model. It can be shown that every Mori fiber intersects the branch set in at most two points.
- If a Mori-fiber E intersects B in two points, then both points of intersection are transversal, i.e., $E \cdot B = 2$.
- If $(E \cdot B)_p = 2$, then $E \cap B = \{p\}$.

7. The alternating group of degree six

In the previous section we considered K3-surfaces with a symplectic action of a finite group G_{sym} centralized by an antisymplectic involution, i.e., all groups under consideration were of the form $G = G_{\text{sym}} \times C_2$.

In this section we wish to discuss more general finite automorphisms groups \tilde{G} : if \tilde{G} contains an antisymplectic involution σ with fixed points, then as before, we consider the quotient by σ . However, if σ does not centralize the group \tilde{G}_{sym} inside \tilde{G} , the action of \tilde{G}_{sym} does *not* descend to the quotient surface. We therefore restrict our consideration to the centralizer $Z_{\tilde{G}}(\sigma)$ of σ inside \tilde{G} and study its action on the quotient surface. If we are able to describe the family of K3-surfaces with $Z_{\tilde{G}}(\sigma)$ -symmetry, it remains to identify the surfaces with \tilde{G} -symmetry inside this family.

We consider a situation where the group \tilde{G} contains the alternating group of degree six. Although, a precise classification cannot be obtained at present, we achieve an improved understanding of the equivariant geometry of K3-surfaces with \tilde{G} -symmetry and classify families of K3-surfaces with $Z_{\tilde{G}}(\sigma)$ -symmetry (cf. Theorem 7.24). In this sense, this section, which is an abbreviated version of Chapter 7 of [Fra08], serves as an outlook on how the method of equivariant Mori reduction allows generalization to more advanced classification problems.

7.1. The group \tilde{A}_6 . — We let \tilde{G} be any finite group containing the alternating group of degree six and in the following consider a K3-surface X with an effective action of \tilde{G} . This particular situation is considered by Keum, Oguiso, and Zhang in [KOZ05] and [KOZ07] with special emphasis on the maximal possible choice of \tilde{G} : they consider a group $\tilde{G} = \tilde{A}_6$ characterized by the exact sequence

$$(2) \quad \{\text{id}\} \rightarrow A_6 \rightarrow \tilde{A}_6 \xrightarrow{\alpha} C_4 \rightarrow \{\text{id}\}.$$

It follows from the fact that A_6 is simple and a Mukai group that the group of symplectic automorphisms \tilde{G}_{sym} in \tilde{G} coincides with A_6 . Let $N := \text{Inn}(\tilde{A}_6) \subset \text{Aut}(A_6)$ denote the group of inner automorphisms of \tilde{A}_6 and let $\text{int} : \tilde{A}_6 \rightarrow N$ be the homomorphism mapping an element $g \in \tilde{A}_6$ to conjugation with g . It can be shown that the group \tilde{A}_6 is a semidirect product $A_6 \rtimes C_4$ embedded in $N \times C_4$ by the map (int, α) (Theorem 2.3 in [KOZ07]). By Theorem 4.1 in [KOZ07] the group N is isomorphic to M_{10} and the isomorphism class of \tilde{A}_6 is uniquely determined by (2) and the condition that it acts on a K3-surface.

In [KOZ05] a lattice-theoretic proof of the following classification result (Theorem 5.1, Theorem 3.1, Proposition 3.5) is given.

Theorem 7.1 ([KOZ05]). — *A K3 surface X with an effective action of \tilde{A}_6 is isomorphic to the minimal desingularization of the surface in $\mathbb{P}_1 \times \mathbb{P}_2$ given by*

$$S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.$$

The existence of an isomorphism from a K3-surface with \tilde{A}_6 -symmetry to the surface defined by the equation above follows abstractly since both surfaces are shown to have the same transcendental lattice and the action of \tilde{A}_6 on the later is hidden. It is therefore desirable to obtain an explicit realization of X where the action of \tilde{A}_6 is visible.

We let the generator of the factor C_4 in $\tilde{A}_6 = A_6 \rtimes C_4$ be denoted by τ . It is nonsymplectic and has fixed points, the antisymplectic involution $\sigma := \tau^2$ fulfils $\text{Fix}_X(\sigma) \neq \emptyset$. Since σ is mapped to the trivial automorphism in $\text{Out}(A_6) = \text{Aut}(A_6)/\text{int}(A_6) \cong C_2 \times C_2$ there exists $h \in A_6$ such that $\text{int}(h) = \text{int}(\sigma) \in \text{Aut}(A_6)$. The antisymplectic involution $h\sigma$ centralizes A_6 in \tilde{A}_6 .

Remark 7.2. — If $\text{Fix}_X(h\sigma) \neq \emptyset$, the K3-surface X is an A_6 -equivariant double cover of \mathbb{P}_2 where A_6 acts as Valentiner's group and the branch locus is given by $F_{A_6}(x_1, x_2, x_3) = 10x_1^3x_2^3 + 9x_1^5x_3 + 9x_2^3x_3^3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6$ (cf. Theorem 6.4). By construction, there is an evident action of $A_6 \times C_2$ on this *Valentiner surface*, it is however not clear whether this surface admits the larger symmetry group \tilde{A}_6 .

In the following we assume that $h\sigma$ acts without fixed points on X as otherwise the remark above yields an A_6 -equivariant classification of X .

The centralizer G of σ in \tilde{A}_6 . — We study the quotient $\pi : X \rightarrow X/\sigma = Y$. As mentioned above, the action of the centralizer of σ descends to an action on Y . We therefore start by identifying the centralizer $G := Z_{\tilde{A}_6}(\sigma)$ of σ in \tilde{A}_6 . It follows from direct computation⁽¹⁾ and from the equality $\text{int}(\sigma) = \text{int}(h)$ that the group G equals $Z_{A_6}(\sigma) \rtimes C_4$ and $Z_{A_6}(\sigma) = Z_{A_6}(h)$. In the following, we wish to identify the group $Z_{A_6}(h)$. Since $\text{int}(\sigma) = \text{int}(h)$ and $\sigma^2 = \text{id}$, it follows that h^2 commutes with any element in A_6 . As $Z(A_6) = \{\text{id}\}$, we see that h is of order two. There is only one conjugacy class of elements of order two in A_6 . We calculate $Z_{A_6}(h)$ for one particular choice of $h = (13)(24) \in A_6$. Let $c = (1234)(56)$ and $g = (24)(56)$. Then $c^2 = h$ and both c and g centralize h . The group generated by c and g is seen to be a dihedral group of order eight; $\langle g \rangle \rtimes \langle c \rangle = D_8 < Z_{A_6}(h)$. Now direct computations in $S_4 < A_6$ yield $\langle g \rangle \rtimes \langle c \rangle = D_8 = Z_{A_6}(h)$.

Using the assumption that σh acts freely on X and by choosing the appropriate generator of $\langle c \rangle$ we find that the action of τ on $Z_{A_6}(h) = D_8$ given by $\tau g \tau^{-1} = c^3 g$ and $\tau c \tau^{-1} = c^3$. Furthermore, note that the commutator subgroup G' of G equals $G' = \langle c \rangle$.

The group $H = G/\langle \sigma \rangle$. — We consider the quotient $Y = X/\sigma$ equipped with the action of $G/\sigma =: H = Z_{\tilde{A}_6}(\sigma)/\langle \sigma \rangle = D_8 \rtimes C_2$. The group C_2 is generated by $[\tau]_\sigma$. For simplicity, we transfer the above notation from G to H by writing e.g. τ for $[\tau]_\sigma$. Since $\tau g \tau^{-1} = c^3 g = gc$, it follows that $H' = \langle c \rangle$.

Let $K < G$ be the cyclic group of order eight generated by $g\tau$. We denote the image of K in G/σ by the same symbol. Since $[\sigma c]_\sigma = [c]_\sigma \in K$ it contains $H' = \langle c \rangle$ and we can write $H = \langle \tau \rangle \rtimes K = D_{16}$.

Lemma 7.3. — *There is no nontrivial normal subgroup N in H with $N \cap H' = \{\text{id}\}$.*

Proof. — If such a group exists, first consider the case $N \cap K = \{\text{id}\}$. Then $N \cong C_2$ and $H = K \times N$ would be Abelian, a contradiction. If $N \cap K \neq \{\text{id}\}$ then $N \cap K = \langle (g\tau)^k \rangle$ for some $k \in \{1, 2, 4\}$. This implies $(g\tau)^4 = c^2 \in N$ and contradicts $N \cap H' = N \cap \langle c \rangle = \emptyset$. \square

The following observations strongly rely the assumption that σh acts freely on X .

Lemma 7.4. — *The subgroup H' acts freely on the branch set $B = \pi(\text{Fix}_X(\sigma))$ in Y .*

Proof. — If for some $b \in B$ the isotropy group H'_b is nontrivial, then $c^2(b) = h(b) = b$ and σh fixes the corresponding point $\tilde{b} \in X$. \square

Corollary 7.5. — *The subgroup H' acts freely on the set \mathcal{R} of rational branch curves of the covering $\pi : X \rightarrow Y$. In particular, the number of rational branch curves n is a multiple of four.*

Corollary 7.6. — *The subgroup H' acts freely on the set of τ -fixed points in Y .*

Proof. — We show $\text{Fix}_Y(\tau) \subset B$. Since $\sigma = \tau^2$ on X , a $\langle \tau \rangle$ -orbit $\{x, \tau x, \sigma x, \tau^3 x\}$ in X gives rise to a τ -fixed point y in the quotient $Y = X/\sigma$ if and only if $\sigma x = \tau x$. Therefore, τ -fixed points in Y correspond to τ -fixed points in X . By definition $\text{Fix}_X(\tau) \subset \text{Fix}_X(\sigma)$ and the claim follows. \square

7.2. H -minimal models of Y . — Since $\text{Fix}_X(\sigma) \neq \emptyset$, the quotient surface Y is a smooth rational H -surface to which we apply the equivariant minimal model program. We denote by Y_{\min} an H -minimal model of Y . It is known that Y_{\min} is either a Del Pezzo surface or an H -equivariant conic bundle over \mathbb{P}_1 .

Theorem 7.7. — *An H -minimal model Y_{\min} does not admit an H -equivariant \mathbb{P}_1 -fibration. In particular, Y_{\min} is a Del Pezzo surface.*

⁽¹⁾For this and details of other technical arguments which have been omitted here see Chapter 7 of [Fra08]

In order to prove this statement we begin with the following general fact which follows from the observation that the action of a cyclic group on a Mori fiber has two fixed points contracting to a single fixed point.

Lemma 7.8. — *If $Y \rightarrow Y_{\min}$ is an H -equivariant Mori reduction and A a cyclic subgroup of H , then $|\text{Fix}_Y(A)| \geq |\text{Fix}_{Y_{\min}}(A)|$.*

Suppose that some Y_{\min} is an H -equivariant conic bundle, i.e., there is an H -equivariant fibration $p : Y_{\min} \rightarrow \mathbb{P}_1$ with generic fiber \mathbb{P}_1 and let $p_* : H \rightarrow \text{Aut}(\mathbb{P}_1)$ be the associated homomorphism.

Lemma 7.9. — $\text{Ker}(p_*) \cap H' = \{\text{id}\}$.

Proof. — The elements of $\text{Ker}(p_*)$ fix two points in every generic p -fiber. If $h = c^2 \in H' = \langle c \rangle$ fixes points in every generic p -fiber, then h acts trivially on a one-dimensional subset $C \subset Y$. Since $h = c^2$ acts symplectically on X it has only isolated fixed points in X . Therefore, on the preimage $\tilde{C} = \pi^{-1}(C) \subset X$, the action of h coincides with the action of σ . But then $\sigma h|_{\tilde{C}} = \text{id}|_{\tilde{C}}$ contradicts the assumption that σh acts freely on X . \square

Proof of Theorem 7.7. — Since there are no nontrivial normal subgroups in H which have trivial intersection with H' (Lemma 7.3), it follows from Lemma 7.9 that $\text{Ker}(p_*) = \{\text{id}\}$, i.e., the group H acts effectively on the base. We regard H as the semidirect product $H = \langle \tau \rangle \rtimes K$, where $K = C_8$ is described above. The automorphism τ exchanges the K -fixed points. We will obtain a contraction by analyzing the K -actions on the fibers F and τF over its two fixed points. By Lemma 4.16 there are two situations which we must consider:

1. F is a regular fiber of $Y_{\min} \rightarrow \mathbb{P}_1$.
2. $F = F_1 \cup F_2$ is the union of two (-1)-curves intersecting transversally in one point.

We study the fixed points of c , $h = c^2$ and $g\tau$ in Y_{\min} . Note that in X the symplectic transformation c has precisely four fixed points and h has precisely eight fixed points. This set of eight points is stabilized by the full centralizer of h , in particular by K . Since $h\sigma$ acts by assumption freely on X , it follows that σ acts freely on the set of h -fixed points in X . If $hy = y$ for some $y \in Y$, then the preimage of y in X consists of two elements $x_1, \sigma x_1 = x_2$. If these form an $\langle h \rangle$ -orbit, then both are σh -fixed, a contradiction. It follows that $\{x_1, x_2\} \subset \text{Fix}_X(h)$ and the number of h -fixed points in Y is precisely four. In particular, h acts effectively on any curve in Y .

Let us first consider case (2) where $F = F_1 \cup F_2$ is reducible. Since $\langle c \rangle$ is the only subgroup of index two in K , it follows that $\langle c \rangle$ stabilizes F_i and both c and h have three fixed points in F (two on each irreducible component, one is the point of intersection $F_1 \cap F_2$), i.e., six fixed points on $F \cup \tau F \subset Y_{\min}$. This is contrary to Lemma 7.8 because h has at most four fixed points in Y_{\min} .

If F is regular (case (1)), then the cyclic group K has two fixed points on the rational curve F . Since $h \in K$, the four K -fixed points on $F \cup \tau F$ are contained in the set of h -fixed points on Y_{\min} . As $|\text{Fix}_{Y_{\min}}(h)| \leq 4$, the K -fixed points coincide with the four h -fixed points in Y_{\min} , i.e., $\text{Fix}_{Y_{\min}}(h) = \text{Fix}_{Y_{\min}}(K)$. In particular, the Mori reduction does not affect the four h -fixed points $\{y_1, \dots, y_4\}$ in Y . By equivariance of the reduction, the group K acts trivially on this set of four points. Passing to the double cover X , we conclude that the action of $g\tau \in K$ on a preimage $\{x_i, \sigma x_i\}$ of y_i is either trivial or coincides with the action of σ . In both cases it follows that $(g\tau)^2 = c\sigma$ acts trivially on the set of h -fixed points in X . As $\text{Fix}_X(c) \subset \text{Fix}_X(h)$, this is contrary to the fact that σ acts freely on $\text{Fix}_X(h)$. \square

In the following we wish to identify the Del Pezzo surface Y_{\min} . For this, we use the Euler characteristic formulas,

$$24 = e(X) = 2e(Y) - 2n + \underbrace{2g - 2}_{\text{if } D_g \text{ is present}},$$

where $D_g \subset B$ is a branch curve of general type, $g = g(D_g) \geq 2$, and $e(Y) = e(Y_{\min}) + m$. Here $m = |\mathcal{E}|$ denotes the total number of Mori fibers of the reduction $Y \rightarrow Y_{\min}$ and n denotes the total number of rational curves in $\text{Fix}(\sigma)$. For convenience we introduce the difference $\delta = m - n$. If a branch curve D_g of general type is present, then $13 - g - \delta = e(Y_{\min})$ and if it is not present $12 - \delta = e(Y_{\min})$. The inequality $e(Y_{\min}) \leq 3$ implies $m \leq n + 9$.

Proposition 7.10. — *For every Mori fiber E the orbit $H.E$ consists of at least four Mori fibers.*

Proof. — A Mori fiber E can either be disjoint from B , contained in B , or intersect B in one or two points. We denote by $|HE|$ the number of disjoint curves in the orbit HE .

First assume $E \cap B \neq \emptyset$ and $E \not\subset B$. Since H' acts freely on the branch curves and E meets B in at most two points (cf. Section 6.3), we know $|H'E| \geq 2$. If $|HE| = 2$, then the isotropy group H_E is a normal subgroup of index two which necessarily contains the commutator group H' , a contradiction.

If $E \subset B$, we show that the H' -orbit of E consists of four Mori fibers. If it consisted of less than four Mori fibers, the stabilizer $H'_E \neq \{\text{id}\}$ of E in H' would fix two points in $E \subset B$. This contradicts Lemma 7.4.

All Mori fibers disjoint from B have self-intersection (-2) and meet exactly one Mori fiber of the previous steps of the reduction in exactly one point. If $E \cap B = \emptyset$ there is a chain of Mori fibers $E_1, \dots, E_k = E$ connecting E and B . The Mori fiber E_1 is the only one to have nonempty intersection with B and is the first curve of this configuration to be blown down in the reduction process. The H -orbit of this union of Mori fibers consists of at least four copies of this chain. This is due to that fact that the H -orbit of E_1 consists of at least four Mori fibers by Case 1. In particular, the H -orbit of E consists of at least four copies of E . \square

Corollary 7.11. — *The difference δ is a non-negative multiple $4k$ of four.*

Proof. — Above we have shown that m and n are multiples of four. Therefore $\delta = 4k$. If δ was negative, i.e., $m < n$, there is no configuration of Mori fibers meeting the rational branch curves, which have self-intersection -4 , such that the corresponding contractions transform them to curves on a Del Pezzo surface Y_{\min} where the self-intersection of any curve is at least -1 . It follows that δ is non-negative. \square

Theorem 7.12. — *Any H -minimal model Y_{\min} of Y is $\mathbb{P}_1 \times \mathbb{P}_1$.*

Proof. — If $\delta = 0$, then $n = m = 0$ and $Y = Y_{\min}$. The commutator subgroup $H' \cong C_4$ acts freely on the branch locus B implying $e(B) \in \{0, -8, -16, \dots\}$. Since the Euler characteristic of the Del Pezzo surface Y is at least 3 and at most 11, we only need to consider the cases $e(Y) \in \{4, 8\}$.

If $\delta \neq 0$, then since $\delta \geq 4$, it follows that $e(Y_{\min}) = 13 - g - \delta \leq 7$ if a branch curve D_g of general type is present, and $e(Y_{\min}) = 12 - \delta \leq 8$ if not.

We go through the list of Del Pezzo surfaces with $e(Y_{\min}) \leq 8$.

- If $e(Y_{\min}) = 8$, then $\text{deg}(Y_{\min}) = 4$ and $\text{Aut}(Y_{\min}) = C_2^4 \rtimes \Gamma$ for $\Gamma \in \{C_2, C_4, S_3, D_{10}\}$. If $D_{16} < C_2^4 \rtimes \Gamma$ then $A := D_{16} \cap C_2^4 \triangleleft D_{16}$ and A is either trivial or isomorphic to C_2 . In both cases D_{16}/A is not a subgroup of Γ in any of the possibilities listed above. Therefore, $e(Y) \neq 8$.
- If $e(Y_{\min}) = 7$, then $\text{Aut}(Y_{\min}) = S_5$. Since 120 is not divisible by 16, we see that a Del Pezzo surface of degree five does not admit an effective action of the group H .
- If $e(Y_{\min}) = 6$, then $A := \text{Aut}(Y_{\min}) = (C^*)^2 \rtimes D_{12}$. We denote by $A^\circ \cong (C^*)^2$ the connected component of A and consider $q: A \rightarrow A/A^\circ$. Now $q(H') < q(A)' \cong C_3$ and $H' = C_4 < A^\circ$. We may realize Y_{\min} as \mathbb{P}_2 blown up at the three corner points and A° as the set of diagonal matrices in $\text{SL}_3(\mathbb{C})$. Every possible representation of C_4 in this group has ineffectivity along one of the lines joining corner points. But, as we have seen before, the elements of H' , in particular $c^2 = h$, have only isolated fixed points in Y_{\min} .

- A Del Pezzo surface obtained by blowing up one or two points in \mathbb{P}_2 is never H -minimal and therefore does not occur
- Finally, $Y_{\min} \neq \mathbb{P}_2$: If $e(Y_{\min}) = 3$ then either $\delta = 9$ (if D_g is not present), a contradiction to $\delta = 4k$, or $g + \delta = 10$. In the later case, $\delta = 4, 8$ forces $g = 6, 2$. In both cases, the Euler characteristic $2 - 2g$ of D_g is not divisible by 4. This contradicts the fact that H' acts freely on D_g .

We have hereby excluded all possible Del Pezzo surfaces except $\mathbb{P}_1 \times \mathbb{P}_1$ and the proposition follows. \square

7.3. Branch curves and Mori fibers. — We let $M : Y \rightarrow Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$ denote an H -equivariant Mori reduction of Y .

Lemma 7.13. — *The number of Mori fibers in an H -orbit is at least eight.*

Proof. — Consider the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$. Both canonical projections are equivariant with respect to the commutator subgroup $H' = \langle c \rangle \cong C_4$. Since $c^2 \in H'$ does not act trivially on any curve in Y or Y_{\min} , it follows that H' has precisely four fixed points in $Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$. Since $h = c^2$ has precisely four fixed points in Y and $\text{Fix}_Y(H') = \text{Fix}_Y(c) \subset \text{Fix}_Y(c^2)$, we conclude that H' has precisely four fixed points in Y and it follows that the Mori fibers do not pass through H' -fixed points. Note that the H' -fixed points in Y coincide with the h -fixed points.

Suppose there is an H -orbit HE of Mori fibers of length strictly less than eight and let $p = M(E)$. We obtain an H -orbit Hp in $\mathbb{P}_1 \times \mathbb{P}_1$ with $|Hp| \leq 4$. Now $|Kp| \leq 4$ implies that $Kp \neq \{\text{id}\}$, in particular, $h = c^2 \in Kp$. It follows that p is an h -fixed point. This contradicts the fact that the Mori fibers do not pass through fixed points of h . \square

A total number of 24 or more Mori fibers would require 16 rational curves in B . This contradicts the fact that the number of connected components of the fixed point set of an antisymplectic involution on a K3-surface is at most ten (cf. Section 6.3). It therefore follows from the above lemma that the total number m of Mori fibers equals 0, 8, or 16.

Recalling that the number of rational branch curves is a multiple of four, i.e., $n \in \{0, 4, 8\}$ and using the fact $m \in \{0, 8, 16\}$ along with $m \leq n + 9$, we conclude that the surface Y is of one of the following types.

1. $m = 0$

The quotient surface Y is H -minimal. The map $X \rightarrow Y \cong \mathbb{P}_1 \times \mathbb{P}_1$ is branched along a single curve B . This curve B is a smooth H -invariant curve of bidegree $(4, 4)$.

2. $m = 8$ and $e(Y) = 12$

The surface Y is the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in an H -orbit consisting of eight points.

- (a) If the branch locus B of $X \rightarrow Y$ contains no rational curves, then $e(B) = 0$ and B is either an elliptic curve or the union of two elliptic curves defining an elliptic fibration on X .
- (b) If the branch locus B of $X \rightarrow Y$ contains rational curves, their number is exactly four (Eight or more rational branch curves of self-intersection -4 cannot be modified sufficiently and mapped to curves on a Del Pezzo surface by contracting eight Mori fibers). It follows that the branch locus is the disjoint union of an invariant curve of higher genus and four rational curves.

3. $m = 16$ and $e(Y) = 20$

The map $X \rightarrow Y$ is branched along eight disjoint rational curves.

We may simplify the above situation by studying rational curves in B , their intersection with Mori fibers and their images in $\mathbb{P}_1 \times \mathbb{P}_1$.

Proposition 7.14. — *If $e(Y) = 12$, then $n = 0$.*

Proof. — Suppose $n \neq 0$ and let $C_i \subset Y$ be a rational branch curve. Since $C_i^2 = -4$ and $M(C_i) \subset \mathbb{P}_1 \times \mathbb{P}_1$ has self-intersection ≥ 0 it must meet the union of Mori fibers $\bigcup E_j$. All possible configurations

of Mori fibers yield image curves $M(C_i)$ of self-intersection ≤ 4 . Adjunction on $\mathbb{P}_1 \times \mathbb{P}_1$ implies that $g(M(C_i)) = 0$ and $M(C_i)$ must be nonsingular. Hence each Mori fiber meets C_i in at most one point. It follows that C_i meets four Mori fibers, each in one point, and $(M(C_i))^2 = 0$. In particular, $M(C_i)$ are fibers of the canonical projections $\mathbb{P}_1 \times \mathbb{P}_1 \rightarrow \mathbb{P}_1$. The curve C_1 meets four Mori fibers E_1, \dots, E_4 and each of these Mori fibers meets some C_i for $i \neq 1$. After renumbering, we may assume that E_1 and E_2 meet C_2 and therefore $M(C_1)$ and $M(C_2)$ meet in more than one point, a contradiction. It follows that $e(Y) = 12$ implies $n = 0$ \square

Proposition 7.15. — *If $e(Y) = 20$, then Y is the blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in sixteen points $\{p_1, \dots, p_{16}\} = (\bigcup_{i=1}^4 F_i) \cap (\bigcup_{i=5}^8 F_i)$, where F_1, \dots, F_4 are fibers of the canonical projection π_1 and F_5, \dots, F_8 are fibers of π_2 . The branch locus is given by the proper transform of $\bigcup F_i$ in Y .*

Proof. — We denote the eight rational branch curves by C_1, \dots, C_8 . The Mori reduction can have two steps. A slightly more involved study of possible configurations of Mori fibers shows that $0 \leq (M(C_i))^2 \leq 4$. As above $M(C_i)$ is seen to be nonsingular and each Mori fiber can meet C_i in at most one point. Any configuration of curves with this property yields $(M(C_i))^2 = 0$ and $F_i = M(C_i)$ is a fiber of a canonical projection $\mathbb{P}_1 \times \mathbb{P}_1 \rightarrow \mathbb{P}_1$.

If there are Mori fibers disjoint from B these are blown down in the second step of the Mori reduction. Let E_1, \dots, E_8 denote the Mori fibers of the first step and $\tilde{E}_1, \dots, \tilde{E}_8$ those of the second step. We label them such that \tilde{E}_i meets E_i . Each curve E_i meets two rational branch curves C_i and C_{i+4} and their images $F_i = M(C_i)$ and $F_{i+4} = M(C_{i+4})$ meet with multiplicity ≥ 2 . This is contrary to the fact that they are fibers of the canonical projections. It follows that there are no Mori fibers disjoint from B and all 16 Mori fibers are contracted simultaneously. There is precisely one possible configuration of Mori fibers on Y such that all rational branch curves are mapped to fibers of the canonical projections of $\mathbb{P}_1 \times \mathbb{P}_1$: The curves C_1, \dots, C_4 are mapped to fibers of π_1 and C_5, \dots, C_8 are mapped to fibers of π_2 . The Mori reduction contracts 16 curves to the 16 points of intersection $\{p_1, \dots, p_{16}\} = (\bigcup_{i=1}^4 F_i) \cap (\bigcup_{i=5}^8 F_i) \subset \mathbb{P}_1 \times \mathbb{P}_1$. \square

Let us now restrict our attention to the case where the branch locus B is the union of two linearly equivalent elliptic curves and exclude this case.

Two elliptic branch curves. — In this paragraph we prove:

Theorem 7.16. — *$\text{Fix}_X(\sigma)$ is not the union of two elliptic curves.*

We assume the contrary, let $\text{Fix}_X(\sigma) = D_1 \cup D_2$ with D_i elliptic and let $f : X \rightarrow \mathbb{P}_1$ denote the elliptic fibration defined by the curves D_1 and D_2 . Note that σ acts effectively on the base \mathbb{P}_1 as otherwise σ would act trivially in a neighbourhood of D_i by a linearization argument. It follows that the group of order four generated by τ acts effectively on \mathbb{P}_1 .

Since the group G does not contain a cyclic group of order 16, it is neither cyclic nor dihedral and therefore cannot act effectively on \mathbb{P}_1 . It follows that the ineffectivity I of the induced G -action on the base \mathbb{P}_1 is nontrivial. We regard $G = C_4 \times D_8$ where $C_4 = \langle \tau \rangle$ and D_8 is the centralizer of σ in A_6 (cf. Section 7.1) and define $J := I \cap D_8$.

Using explicitly the groups structure of G along with the assumption that σh acts freely on X one finds that J is nontrivial. In the following, we consider the different possibilities for the order of J and show that in fact none of these occur.

If $|J| = 8$ then $D_8 \subset I$. Recall that any automorphism group of an elliptic curve splits into an Abelian part acting freely and a cyclic part fixing a point. Since D_8 is not Abelian, any D_8 -action on the fibers of f must have points with nontrivial isotropy. This gives rise to a positive-dimensional fixed point set of some subgroup of D_8 on X , contradicting the fact that D_8 acts symplectically on X . It follows that the maximal possible order of J is four.

Lemma 7.17. — *The ineffectivity I does not contain $\langle c \rangle$.*

Proof. — Assume the contrary and consider the fixed points of c^2 . If a c^2 -fixed point lies at a smooth point of a fiber of f , then the linearization of the c^2 -action at this fixed point gives rise to a positive-dimensional fixed point set in X and yields a contradiction. It follows that the fixed points of c^2 are contained in the singular f -fibers. Since $\langle \tau \rangle$ normalizes $\langle c \rangle$ and the $\langle \tau \rangle$ -orbit of a singular fiber consists of four such fibers, we must only consider two cases:

1. The eight c^2 -fixed points are contained in four singular fibers (one $\langle \tau \rangle$ -orbit of fibers), each of these fibers contains two c^2 -fixed points.
2. The eight c^2 -fixed points are contained in eight singular fibers (two $\langle \tau \rangle$ -orbits).

Note that $\langle c^2 \rangle$ is normal in I and therefore I acts on the set of $\langle c^2 \rangle$ -fixed points. In the second case, all eight c^2 -fixed points are also c -fixed. This is contrary to c having only four fixed points and therefore the second case does not occur.

The first case does not occur for similar reasons: If c^2 has exactly two fixed points x_1 and x_2 in some fiber F , then $\langle c \rangle$ either acts transitively on $\{x_1, x_2\}$ or fixes both points. Since $\text{Fix}_X(c) \subset \text{Fix}_X(c^2)$ and $\langle c \rangle$ must have exactly one fixed point on F , this is impossible. \square

Corollary 7.18. — $|J| \neq 4$.

Proof. — Assume $|J| = 4$. Using τ we check that no subgroup of D_8 isomorphic to $C_2 \times C_2$ is normal in G . It follows that the group $\langle c \rangle$ is the only order four subgroup of D_8 which is normal in G and therefore $J = \langle c \rangle$. By the lemma above this is however impossible. \square

It remains to consider the case where $|J| = 2$. The only normal subgroup of order two in D_8 is $J = \langle h \rangle$.

Lemma 7.19. — *If $|J| = 2$, then $I = \langle \sigma c \rangle$.*

Proof. — We first show that $|J| = 2$ implies $|I| = 4$: If $|I| = 2$, then $I = \langle h \rangle$ and $G/I = C_4 \times (C_2 \times C_2)$. Since this group does not act effectively on \mathbb{P}_1 , this is a contradiction. If $|I| \geq 8$, then G/I is Abelian and therefore I contains the commutator subgroup $G' = \langle c \rangle$. This contradicts Lemma 7.17. It follows that $|I| = 4$ and either $I \cong C_4$ or $I \cong C_2 \times C_2$. In the later case, the only possible choice is $I = \langle \sigma \rangle \times \langle h \rangle$ which contradicts the fact that σ acts effectively on the base. It follows that $I = \langle \sigma \zeta \rangle$, where $\zeta^2 = h$ and therefore $\zeta = c$. \square

Let us now consider the action of G on X with $I = \langle \sigma c \rangle$. Recall that the cyclic group $\langle \tau \rangle$ acts effectively on the base and has two fixed points there. Since $\sigma = \tau^2$, these are precisely the two σ -fixed points. In particular, $\langle \tau \rangle$ stabilizes both σ -fixed point curves D_1 and D_2 in X . Furthermore, the transformations σc and c stabilize D_i for $i = 1, 2$. Since the only fixed points of c in \mathbb{P}_1 are the images of D_1 and D_2 ,

$$\text{Fix}_X(c) \subset D_1 \cup D_2 = \text{Fix}_X(\sigma).$$

On the other hand, we know that $\text{Fix}_X(c) \cap \text{Fix}_X(\sigma) = \emptyset$. Thus $I = \langle \sigma c \rangle$ is not possible and the case $|J| = 2$ does not occur.

We have hereby eliminated all possibilities for $|J|$ and completed the proof of Theorem 7.16.

7.4. Rough classification of X . — We summarize the observations of the previous section in the following classification result.

Theorem 7.20. — *Let X be a K3-surface with an effective action of the group G such that $\text{Fix}_X(h\sigma) = \emptyset$. Then X is one of the following types:*

1. a double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along a smooth H -invariant curve of bidegree $(4,4)$.
2. a double cover of a blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ in eight points and branched along a smooth elliptic curve B . The image of B in $\mathbb{P}_1 \times \mathbb{P}_1$ has bidegree $(4,4)$ and eight singular points.
3. a double cover of a blow-up Y of $\mathbb{P}_1 \times \mathbb{P}_1$ in sixteen points $\{p_1, \dots, p_{16}\} = (\bigcup_{i=1}^4 F_i) \cap (\bigcup_{i=5}^8 F_i)$, where F_1, \dots, F_4 are fibers of the canonical projection π_1 and F_5, \dots, F_8 are fibers of π_2 . The branch locus is given by the proper transform of $\bigcup F_i$ in Y . The set $\bigcup F_i$ is an H -invariant reducible subvariety of bidegree $(4,4)$.

Proof. — It remains to consider case (2) and show that the image of B in $\mathbb{P}_1 \times \mathbb{P}_1$ has bidegree (4,4) and eight singular points. We prove that each Mori fiber E meets the branch locus B either in two points or once with multiplicity two, i.e., we need to check that E may not meet B transversally in exactly one point. If this was the case, the image $M(B)$ of the branch curve is a smooth H -invariant curve of bidegree (2,2). The double cover X' of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along the smooth curve $M(B) = C_{(2,2)}$ is a smooth surface. Since X is K3 and therefore minimal the induced birational map $X \rightarrow X'$ is an isomorphism. This is a contradiction since X' is not a K3-surface.

As each Mori fiber meets B with multiplicity two, the self-intersection number of $M(B)$ is 32 and $M(B)$ is a curve of bidegree (4,4) with eight singular points. These singularities are either nodes or cusps depending on the kind of intersection of E and B . \square

In order to obtain a description of possible branch curves, we study the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ and its invariants.

7.5. The action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ and invariant curves of bidegree (4,4).— Recall that we consider the dihedral group $H \cong D_{16}$ generated by τg of order eight and τ . The following proposition can be obtained from direct computations:

Proposition 7.21. — *In appropriately chosen coordinates the action of H on $\mathbb{P}_1 \times \mathbb{P}_1$ is given by*

$$\begin{aligned} c([z_0 : z_1], [w_0 : w_1]) &= ([iz_0 : z_1], [-iw_0 : w_1]) \\ \tau([z_0 : z_1], [w_0 : w_1]) &= ([z_1 : z_0], [iw_1 : w_0]) \\ g([z_0 : z_1], [w_0 : w_1]) &= ([w_0 : w_1], [z_0 : z_1]). \end{aligned}$$

Given this action of H on $\mathbb{P}_1 \times \mathbb{P}_1$, we wish to study the invariants and semi-invariants of bidegree (4,4). The space of (a, b) -bihomogeneous polynomials in $[z_0 : z_1][w_0 : w_1]$ is denoted by $\mathbb{C}_{(a,b)}([z_0 : z_1][w_0 : w_1])$.

An invariant curve C is given by a D_{16} -eigenvector $f \in \mathbb{C}_{(4,4)}([z_0 : z_1][w_0 : w_1])$. The kernel of the D_{16} -representation on the line $\mathbb{C}f$ spanned by f contains the commutator subgroup $H' = \langle c \rangle$ and f is an appropriate linear combination of c -invariant monomials of bidegree (4,4). It follows from the explicit form of the H -action that an H -invariant curve of bidegree (4,4) in $\mathbb{P}_1 \times \mathbb{P}_1$ is one of the following three types

$$\begin{aligned} C_a &= \{a_1 f_1 + a_2 f_2 + a_3 f_3 = 0\}, \\ C_b &= \{b_1 g_1 + b_3 g_3 + b_4 g_4 = 0\}, \\ C_0 &= \{g_2 = 0\}. \end{aligned}$$

7.6. Refining the classification of X . — Using the above description of invariant curves of bidegree (4,4) we may refine Theorem 7.20.

Theorem 7.22. — *Let X be a K3-surface with an effective action of the group G such that $\text{Fix}_X(h\sigma) = \emptyset$. If $e(X/\sigma) = 20$, then X/σ is equivariantly isomorphic to the blow up of $\mathbb{P}_1 \times \mathbb{P}_1$ in the singular points of the curve $C = \{f_1 - f_2 = 0\}$ and $X \rightarrow Y$ is branched along the proper transform of C in Y .*

Proof. — It follows from Theorem 7.20 that X is the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ blown up in sixteen points. These sixteen points are the points of intersection of eight fibers of $\mathbb{P}_1 \times \mathbb{P}_1$, four for each of fibration. By invariance these fibers lie over the base points $[1 : 1], [1 : -1], [1 : i], [1 : -i]$ and the configurations of eight fibers is defined by the invariant polynomial $f_1 - f_2$. The double cover $X \rightarrow Y$ is branched along the proper transform of this configuration of eight rational curves. This proper transform is a disjoint union of eight rational curves in Y , each with self-intersection (-4). \square

Theorem 7.23. — *Let X be a K3-surface with an effective action of the group G such that $\text{Fix}_X(h\sigma) = \emptyset$. If $X/\sigma \cong \mathbb{P}_1 \times \mathbb{P}_1$, then after a change of coordinates the branch locus is C_a for some $a_1, a_2, a_3 \in \mathbb{C}$.*

Proof. — The surface X is a double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along a smooth H -invariant curve of bidegree $(4,4)$. The invariant $(4,4)$ -curves C_b and C_0 discussed above are seen to be singular at $([1 : 0], [1 : 0])$ or $([1 : 0], [0 : 1])$. \square

Note that the general curve C_a is smooth. We obtain a 2-dimensional family $\{C_a\}$ of smooth branch curves and a corresponding family of K3-surfaces $\{X_{C_a}\}$. It remains to consider the case (2) of the classification. Our aim is to find an example of a K3-surface X such that $X/\sigma = Y$ has a nontrivial Mori reduction $M : Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_1 = Z$ contracting a single H -orbit of Mori fibers consisting of precisely 8 curves. In this case the branch locus $B \subset Y$ is mapped to a singular $(4,4)$ -curve $C = M(B)$ in Z . The curve C is irreducible and has precisely 8 singular points along a single H -orbit in Z .

As we have noted above, many of the curves C_a, C_b, C_0 are seen to be singular at $([1 : 0], [1 : 0])$ or $([1 : 0], [0 : 1])$. Since both points lie in H -orbits of length two, these curves are not candidates for our construction. This argument excludes the curves C_b, C_0 and C_a if $a_1 = 0$ or $a_2 = 0$.

For C_a with $a_3 = 0$ one checks that C_a has singular points if and only if $a_1 = -a_2$, i.e., if C_a is reducible. It therefore remains to consider curves C_a where all coefficients $a_i \neq 0$. By considering the H -action on the irreducible component of C_a one verifies that in this case C_a must be irreducible. We choose $a_3 = 1$.

One possible choice of an orbit of length eight is given by the orbit through a τ -fixed point $p_\tau = ([1 : 1], [\pm\sqrt{i} : 1])$. One checks that $p_\tau \in C_a$ for any choice of a_i . However, if we want C_a to be singular in p_τ , then $a_2 = 0$ and therefore C_a is singular at points outside Hp_τ . It has more than eight singular points and is therefore reducible.

All other orbits of length eight are given by orbits through g -fixed points $p_x = ([1 : x], [1 : x])$ for $x \neq 0$. One can choose coefficients $a_i(x)$ such that $C_{a(x)}$ is singular at p_x if and only if $x^8 \neq 1$. If the curve $C_{a(x)}$ is irreducible, then it has precisely eight singular points Hp_x of multiplicity 2 (cusps or nodes) and the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along $C_{a(x)}$ is a singular surface X_{sing} with precisely eight singular points. Its minimal desingularization X is a K3-surface. We obtain a diagram

$$\begin{array}{ccc}
 X_{\text{sing}} & \xleftarrow{\text{desing.}} & X \\
 \downarrow 2:1 & & \downarrow 2:1 \\
 C_{(4,4)} \subset \mathbb{P}_1 \times \mathbb{P}_1 & \xleftarrow{M} & Y \supset B
 \end{array}$$

If p_x is a node in $C_{a(x)}$, then the corresponding singularity of X_{sing} is resolved by a single blow-up. The (-2) -curve in X obtained from this desingularization is a double cover of a (-1) -curve in Y meeting B in two points. If p_x is a cusp in $C_{a(x)}$, then the corresponding singularity of X_{sing} is resolved by two blow-ups. The union of the two intersecting (-2) -curves in X obtained from this desingularization is a double cover of a (-1) -curve in Y tangent to B in one point. The information determining whether p_x is a cusp or a node is encoded in the rank of the Hessian of the equation of $C_{a(x)}$ at p_x . The condition that this rank equals one gives a nontrivial polynomial condition. For a general irreducible member of the family $\{C_{a(x)} \mid x \neq 0, x^8 \neq 1\}$ the singularities of $C_{a(x)}$ are nodes.

We let q be the polynomial in x that vanishes if and only if the rank of the Hessian of $C_{a(x)}$ at p_x is one. It has degree 24, but 16 of its solutions give rise to reducible curves $C_{a(x)}$. The remaining eight solutions give rise to four different irreducible curves. These are identified by the action of the normalizer of H in $\text{Aut}(\mathbb{P}_1 \times \mathbb{P}_1)$ and therefore define equivalent K3-surfaces.

We summarize the discussion in the following main classification theorem.

Theorem 7.24. — *Let X be a K3-surface with an effective action of the group G such that $\text{Fix}_X(h\sigma) = \emptyset$. Then X is an element of one the following families of K3-surfaces:*

1. *the two-dimensional family $\{X_{C_a}\}$ for C_a smooth,*
2. *the one-dimensional family of minimal desingularizations of double covers of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along curves in $\{C_{a(x)} \mid x \neq 0, x^8 \neq 1\}$. The general curve $C_{a(x)}$ has precisely eight nodes along an H -orbit. Up to natural equivalence there is a unique curve $C_{a(x)}$ with eight cusps along an H -orbit.*
3. *the trivial family consisting only of the minimal desingularization of the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ branched along the curve $C_a = \{f_1 - f_2 = 0\}$ where $a_1 = 1, a_2 = -1, a_3 = 0$.*

Corollary 7.25. — *Let X be a K3-surface with an effective action of the group \tilde{A}_6 . If $\text{Fix}_X(h\sigma) = \emptyset$, then X is an element of one the families (1)-(3) above. If $\text{Fix}_X(h\sigma) \neq \emptyset$, then X is A_6 -equivariantly isomorphic to the Valentiner surface.*

7.7. Summary and outlook. — Our initial goal in this section was the description of K3-surfaces with \tilde{A}_6 -symmetry. Using the group structure of \tilde{A}_6 this problem is now divided into two possible cases corresponding to the question whether $\text{Fix}_X(h\sigma)$ is empty or not. If it is nonempty, the K3-surface with \tilde{A}_6 -symmetry is the Valentiner surface (Remark 7.2). If it is empty, our discussion in the previous sections has reduced the problem to finding the \tilde{A}_6 -surface in the families of surfaces X_{C_a} with D_{16} -symmetry. It is known that a K3-surface with \tilde{A}_6 -symmetry has maximal Picard rank 20. This follows from a criterion due to Mukai ([Muk88]) and is explicitly shown in [KOZ05]. All surfaces X_{C_a} for $C_a \subset \mathbb{P}_1 \times \mathbb{P}_1$ a (4,4)-curve are elliptic since the natural fibration of $\mathbb{P}_1 \times \mathbb{P}_1$ induces an elliptic fibration on the double cover (or is desingularization).

A possible approach for finding the \tilde{A}_6 -example inside our families is to find those surfaces with maximal Picard number by studying the elliptic fibration. It would be desirable to apply the following formula for the Picard rank of an elliptic surface $f : X \rightarrow \mathbb{P}_1$ with a section (cf. [SI77]):

$$\rho(X) = 2 + \text{rank}(MW_f) + \sum_i (m_i - 1),$$

where the sum is taken over all singular fibers, m_i denotes the number of irreducible components of the singular fiber and $\text{rank}(MW_f)$ is the rank of the Mordell-Weil group of sections of f .

First, one has to ensure that the fibration under consideration has a section. One approach to find sections is to consider the quotient $q : \mathbb{P}_1 \times \mathbb{P}_1 \rightarrow \mathbb{P}_2$ and the image of the curve C_a inside \mathbb{P}_2 . For an appropriate bitangent to $q(C_a)$ its preimage in the double cover of $\mathbb{P}_1 \times \mathbb{P}_1$ is reducible and both its components define sections of the elliptic fibration. For the special curve C_a with eight nodes the existence of a section (two sections) follows from an application of the Plücker formula to the curve $q(C_a)$ with 3 cusps and its dual curve.

As a next step, one wishes to understand the singular fibers of the elliptic fibrations. Singular fibers occur whenever the branch curve C_a intersects a fiber F of the $\mathbb{P}_1 \times \mathbb{P}_1$ in less than four points. Depending on the nature of intersection $F \cap C_a$ one can describe the corresponding singular fiber of the elliptic fibration. For C_a the curve with eight cusps one finds precisely eight singular fibers of type I_3 , i.e., three rational curves forming a closed cycle. In particular, the contribution of all singular fibers $\sum_i (m_i - 1)$ in the formula above is 16. In the case where C_a is smooth or has eight nodes, this contribution is less.

In order to determine the number $\rho(X_{C_a})$ it is necessary to either understand the Mordell-Weil group or to find curves giving additional contribution to $\text{Pic}(X_{C_a})$ not included in $2 + \sum_i (m_i - 1)$.

In conclusion, the method of equivariant Mori reduction applied to quotients X/σ yields an explicit description of families of K3-surfaces with $D_{16} \times \langle \sigma \rangle$ -symmetry and by construction, the K3-surface with \tilde{A}_6 -symmetry is contained in one of these families. It remains to find criteria to characterize this particular surface inside these families. The possible approach by understanding the function $a \mapsto \rho(X_{C_a})$ using the elliptic structure of X_{C_a} requires a detailed analysis of the Mordell-Weil group.

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