STRATIFYING TRIANGULATED CATEGORIES

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Abstract. A notion of stratification is introduced for any compactly generated triangulated category T endowed with an action of a graded commutative noetherian ring R. The utility of this notion is demonstrated by establishing diverse consequences which follow when T is stratified by R. Among them are a classification of the localizing subcategories of T in terms of subsets of the set of prime ideals in R; a classification of the thick subcategories of the subcategory of compact objects in T; and results concerning the support of the R-module of homomorphisms $\mathrm{Hom}^*_{\mathsf{T}}(C, D)$ leading to an analogue of the tensor product theorem for support varieties of modular representation of groups.

Contents

1. Introduction

Over the last few decades, the theory of support varieties has played an increasingly important role in various aspects of representation theory. The original context was Carlson's support varieties for modular representations of finite groups [12], but the method soon spread to restricted Lie algebras [14], complete intersections in commutative algebra [1, 2], Hochschild cohomological support for certain finite dimensional algebras [13], and finite group schemes [15, 16].

One of the themes in this development was the classification of thick or localizing subcategories of various triangulated categories of representations. This story started with Hopkins' classification [18] of thick subcategories of the perfect complexes over a commutative Noetherian ring R, followed by Neeman's classification [25] of localizing subcategories of the full derived category of R; both involved a notion of support for complexes living in the prime ideal spectrum of R. Somewhat

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later came the classification by Benson, Carlson and Rickard [6] of the thick subcategories of the stable module category of finite dimensional representation of a finite group G in terms of the spectrum of its cohomology ring.

In [8] we established an analogous classification theorem for the localizing subcategories of the stable module category of all representations of G. The strategy of proof is a series of reductions and involves a passage through various other triangulated categories admitting a tensor structure. To execute this strategy, it was important to isolate a property which would permit one to classify localizing subcategories in tensor triangulated categories, and could be tracked easily under changes of categories. This is the notion of *stratification* introduced in [8] for tensor triangulated categories, inspired by work of Hovey, Palmieri, and Strickland [19]. For the stable module category of G, this condition yields a parameterization of localizing subcategories reminiscent of, and enhancing, Quillen stratification [30] of the cohomology algebra of G, whence the name.

In this work we present a notion of stratification for any compactly generated triangulated category T, and establish a number of consequences which follow when this property holds for T. The context is that we are given an *action* of a graded commutative ring R on T, namely a map from R to the graded center of T. We write $\operatorname{Spec} R$ for the set of homogeneous prime ideals of R. In [7] we developed a theory of support for objects in T, based on a construction of exact functors $\varGamma_{\mathfrak{p}}\colon \mathsf{T} \to \mathsf{T}$ for each $\mathfrak{p} \in \operatorname{Spec} R$, which are analogous to local cohomology functors from commutative algebra. The support of any object X of T is the set

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} X \neq 0\}.$$

In this paper, we investigate in detail what is needed in order to classify localizing subcategories in this general context, in terms of the set $\operatorname{Spec} R$.

We separate out two essential ingredients of the process of classifying localizing subcategories. The first is the *local-global principle*: it states that for each object X of T, the localizing subcategory generated by X is the same as the localizing subcategory generated by the set of objects $\{\varGamma_{\mathfrak{p}} X \mid \mathfrak{p} \in \operatorname{Spec} R\}$. We prove that T has this property when, for example, the Krull dimension of R is finite.

When the local-global principle holds for T the problem of classifying localizing subcategories of T can be tackled one prime at a time. This is the content of the following result, which is part of Proposition 3.6.

Theorem 1.1. *When the local-global principle holds for T there is a one-to-one correspondence between localizing subcategories of T and functions assigning to each $\mathfrak{p} \in \operatorname{Spec} R$ a localizing subcategory of $\varGamma_{\mathfrak{p}}\mathsf{T}$. The function corresponding to a localizing subcategory S sends $\mathfrak{p}$ to $\mathsf{S} \cap \varGamma_{\mathfrak{p}}\mathsf{T}$.*

The second ingredient is that in good situations the subcategory $\varGamma_{\mathfrak{p}}\mathsf{T}$, which consists of objects supported at $\mathfrak{p}$, is either zero or contains no proper localizing subcategories. If this property holds for each $\mathfrak{p}$ and the local-global principle holds, then we say T is *stratified* by R. In this case, the map in Theorem 1.1 gives a one-to-one correspondence between localizing subcategories of T and subsets of $\operatorname{supp}_R \mathsf{T}$, which is the set of primes $\mathfrak{p}$ such that $\varGamma_{\mathfrak{p}}\mathsf{T} \neq 0$; see Theorem 4.2.

We draw a number of further consequences of stratification. The best statements are available when T, in addition to be being stratified by R, is *noetherian*, meaning that the R-module $\operatorname{End}^*_{\mathsf{T}}(C)$ is finitely generated for each compact object C in T.

Theorem 1.2. *If T is noetherian and stratified by R, then the map described in Theorem* 1.1 *gives a one-to-one correspondence between the thick subcategories of the compact objects in T and the specialization closed subsets of* $\operatorname{supp}_R \mathsf{T}$.

This result is a rewording of Theorem 6.1 and can be deduced from the classification of localizing subcategories of T, using an argument due to Neeman [25]. We give a different proof based on the following result, which is Theorem 5.1.

Theorem 1.3. *If T is noetherian and stratified by R, then for each pair of compact objects C, D in T there is an equality*

$$\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,D) = \operatorname{supp}_R C \cap \operatorname{supp}_R D\,.$$

When in addition $R^i = 0$ holds for $i < 0$, one has $\operatorname{Hom}^n_{\mathsf{T}}(C,D) = 0$ for $n \gg 0$ if and only if $\operatorname{Hom}^n_{\mathsf{T}}(D,C) = 0$ for $n \gg 0$.

The statement of this theorem is inspired by an analogous statement for modules over complete intersection local rings, due to Avramov and Buchweitz [2]. A stratification theorem is not yet available in this context; see however [21].

The stratification condition also implies that Ravenel's 'telescope conjecture' [31], sometimes called the 'smashing conjecture', holds for T.

Theorem 1.4. *If T is noetherian and stratified by R and $L\colon \mathsf{T} \to \mathsf{T}$ is a localization functor that preserves arbitrary coproducts, then the localizing subcategory $\operatorname{Ker} L$ is generated by objects that are compact in T.*

This result is contained in Theorem 6.3, which establishes also a classification of localizing subcategories of T that are also closed under products. Another application, Corollary 5.7, addresses a question of Rickard. If S is a localizing subcategory of T, write ${}^{\perp}\mathsf{S}$ for the full subcategory of objects X such that there are no nonzero morphisms from X to any object in S.

Theorem 1.5. *Suppose that T is noetherian and stratified by R, and that S is a localizing subcategory of T. Then ${}^{\perp}\mathsf{S}$ is the localizing subcategory corresponding to the set of primes $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathcal{V}(\mathfrak{p}) \cap \operatorname{supp}_R \mathsf{S} = \emptyset\}$.*

In Section 7 we consider the case when T has a structure of a tensor triangulated category compatible with the R-action, and discuss a notion of stratification suitable for this context. A noteworthy feature is that the analogue of the local-global principle always holds, so stratification concerns only whether each $\varGamma_{\mathfrak{p}}\mathsf{T}$ is minimal as tensor ideal localizing subcategory. When this property holds one has the following analogue of the tensor product theorem of modular representation theory as described in [5, Theorem 10.8]; cf. also Theorem 1.3.

Theorem 1.6. *Let T be a tensor triangulated category with a canonical R-action. If R stratifies T, then for any objects X, Y in T there is an equality*

$$\operatorname{supp}_R(X \otimes Y) = \operatorname{supp}_R X \cap \operatorname{supp}_R Y\,.$$

This result reappears as Theorem 7.3. One can establish analogues of other results discussed above for tensor triangulated categories, but we do not do so; the arguments required are the same, and in any case, many of these results appear already in [8], at least for triangulated categories associated to modular representations of finite groups.

Most examples of stratified triangulated categories that appear in this work are imported from elsewhere in the literature. The one exception is the derived category of differential graded modules over any graded-commutative noetherian ring A. In Section 8 we verify that this triangulated category is stratified by the canonical A-action, building on arguments from [8, §5] which dealt with the case A is a graded polynomial algebra over a field.

There are interesting classes of triangulated categories which cannot be stratified via a ring action, in the sense explained above; see Example 4.6. On the other hand, there are important contexts where it is reasonable to expect stratification, notably, modules over cocommutative Hopf algebras and modules over the Steenrod algebra, where analogues of Quillen stratification have been proved by Friedlander and Pevtsova [15] and Palmieri [29] respectively. One goal of [7] and the present work is to pave the way to such results.

2. Local cohomology and support

The foundation for this article is the work in [7] where we constructed analogues of local cohomology functors and support from commutative algebra for triangulated categories. In this section we further develop these ideas, as required, and along the way recall basic notions and constructions from *op. cit.*

Let T be a triangulated category admitting arbitrary coproducts.

Compact generation. A *localizing subcategory* of T is a full triangulated subcategory that is closed under taking coproducts. We write $\operatorname{Loc}_{\mathsf{T}}(\mathsf{C})$ for the smallest localizing subcategory containing a given class of objects C in T, and call it the localizing subcategory *generated* by C.

A object C in T is *compact* if the functor $\operatorname{Hom}_{\mathsf{T}}(C,-)$ commutes with all coproducts; we write T^{c} for the full subcategory of compact object in T. The category T is *compactly generated* if it is generated by a set of compact objects.

We recall some facts concerning localization functors; see, for example, [7, §3].

Localization. A *localization functor* $L\colon \mathsf{T}\to\mathsf{T}$ is an exact functor that admits a natural transformation $\eta\colon \operatorname{Id}_{\mathsf{T}}\to L$, called *adjunction*, such that $L(\eta X)$ is an isomorphism and $L(\eta X)=\eta(LX)$ for all objects $X\in\mathsf{T}$. A localization functor $L\colon \mathsf{T}\to\mathsf{T}$ is essentially uniquely determined by the corresponding full subcategory

$$\operatorname{Ker} L=\{X\in\mathsf{T}\mid LX=0\}\,.$$

Given such a localization functor L, the natural transformation $\operatorname{Id}_{\mathsf{T}}\to L$ induces for each object X in T a natural exact *localization triangle*

$$\varGamma X\longrightarrow X\longrightarrow LX\longrightarrow$$

This exact triangle gives rise to an exact functor $\varGamma\colon \mathsf{T}\to\mathsf{T}$ with

$$\operatorname{Ker} L=\operatorname{Im}\varGamma\quad\text{and}\quad \operatorname{Ker}\varGamma=\operatorname{Im} L\,.$$

Here $\operatorname{Im} F$, for any functor $F\colon \mathsf{T}\to\mathsf{T}$, denotes the *essential image*: the full subcategory of T formed by objects $\{X\in\mathsf{T}\mid X\cong FY \text{ for some } Y \text{ in } \mathsf{T}\}$.

The next lemma provides the existence of localization functors with respect to a fixed localizing subcategory; see [26, Theorem 2.1] for the special case that the localizing subcategory is generated by compact objects.

Lemma 2.1. *Let T be a compactly generated triangulated category. If a localizing subcategory S of T is generated by a set of objects, then there exists a localization functor $L\colon \mathsf{T}\to\mathsf{T}$ with $\operatorname{Ker} L=\mathsf{S}$.*

Proof. In [28, Corollary 4.4.3] it is shown that the collection of morphisms between each pair of objects in the Verdier quotient T/S form a set. The quotient functor $Q\colon \mathsf{T}\to\mathsf{T}/\mathsf{S}$ preserves coproducts, and a standard argument based on Brown's representability theorem [23, 27] yields an exact right adjoint Q_ρ. Note that Q_ρ is fully faithful; see [17, Proposition I.1.3]. It follows that the composite $L=Q_\rho Q$ is a localization functor satisfying $\operatorname{Ker} L=\mathsf{S}$; see [7, Lemma 3.1]. $\square$

Central ring actions. Let R be a graded-commutative ring; thus R is $\mathbb{Z}$-graded and satisfies $r \cdot s = (-1)^{|r||s|} s \cdot r$ for each pair of homogeneous elements r, s in R. We say that the triangulated category T is *R-linear*, or that R acts on T, if there is a homomorphism $R \to Z^*(\mathsf{T})$ of graded rings, where $Z^*(\mathsf{T})$ is the graded center of T. In this case, for all objects $X, Y \in \mathsf{T}$ the graded abelian group

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^i Y)$$

carries the structure of a graded R-module.

Henceforth R denotes a graded-commutative noetherian ring and T a compactly generated R-linear triangulated category with arbitrary coproducts.

Support. We write $\operatorname{Spec} R$ for the set of homogeneous prime ideals of R. Given a homogeneous ideal $\mathfrak{a}$ in R, we set

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Let $\mathfrak{p}$ be a point in $\operatorname{Spec} R$ and M a graded R-module. We write $M_{\mathfrak{p}}$ for the homogeneous localization of M at $\mathfrak{p}$. When the natural map of R-modules $M \to M_{\mathfrak{p}}$ is bijective M is said to be $\mathfrak{p}$-local. This condition is equivalent to: $\operatorname{supp}_R M \subseteq \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$, where $\operatorname{supp}_R M$ is the support of M. The module M is $\mathfrak{p}$-torsion if each element of M is annihilated by a power of $\mathfrak{p}$; equivalently, if $\operatorname{supp}_R M \subseteq \mathcal{V}(\mathfrak{p})$; see [7, §2] for proofs of these assertions.

The *specialization closure* of a subset $\mathcal{U}$ of $\operatorname{Spec} R$ is the set

$$\operatorname{cl}\mathcal{U} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \text{there exists } \mathfrak{q} \in \mathcal{U} \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

The subset $\mathcal{U}$ is *specialization closed* if $\operatorname{cl}\mathcal{U} = \mathcal{U}$; equivalently, if $\mathcal{U}$ is a union of Zariski closed subsets of $\operatorname{Spec} R$. For each specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, we define the full subcategory of T of $\mathcal{V}$-torsion objects as follows:

$$\mathsf{T}_{\mathcal{V}} = \{X \in \mathsf{T} \mid \operatorname{Hom}^*_{\mathsf{T}}(C, X)_{\mathfrak{p}} = 0 \text{ for all } C \in \mathsf{T}^c,\ \mathfrak{p} \in \operatorname{Spec} R \setminus \mathcal{V}\}.$$

This is a localizing subcategory and there exists a localization functor $L_{\mathcal{V}} \colon \mathsf{T} \to \mathsf{T}$ such that $\operatorname{Ker} L_{\mathcal{V}} = \mathsf{T}_{\mathcal{V}}$; see [7, Lemma 4.3, Proposition 4.5]. For each object X in T the adjunction morphism $X \to L_{\mathcal{V}} X$ induces the exact localization triangle

$$\varGamma_{\mathcal{V}} X \longrightarrow X \longrightarrow L_{\mathcal{V}} X \longrightarrow . \tag{2.2}$$

This exact triangle gives rise to an exact *local cohomology functor* $\varGamma_{\mathcal{V}} \colon \mathsf{T} \to \mathsf{T}$. In [7] we established a number of properties of these functors, to facilitate working with them. We single out one that is used frequently in this work: They commute with all coproducts in T; see [7, Corollary 6.5].

For each $\mathfrak{p}$ in $\operatorname{Spec} R$ set $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$, so $\mathcal{V}(\mathfrak{p}) \setminus \mathcal{Z}(\mathfrak{p}) = \{\mathfrak{p}\}$, and

$$X_{\mathfrak{p}} = L_{\mathcal{Z}(\mathfrak{p})} X \quad \text{for each } X \in \mathsf{T}.$$

The notation is justified by the next result which enhances [7, Theorem 4.7].

Proposition 2.3. *Let $\mathfrak{p}$ be a point in $\operatorname{Spec} R$ and X, Y objects in T. The R-modules $\operatorname{Hom}^*_{\mathsf{T}}(X, Y_{\mathfrak{p}})$ and $\operatorname{Hom}^*_{\mathsf{T}}(X_{\mathfrak{p}}, Y)$ are $\mathfrak{p}$-local, so the adjunction morphism $Y \to Y_{\mathfrak{p}}$ induces a unique homomorphism of R-modules*

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y)_{\mathfrak{p}} \longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(X, Y_{\mathfrak{p}}).$$

This map is an isomorphism if X is compact.

Proof. The last assertion in the statement is [7, Theorem 4.7]. It implies that the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C, Y_{\mathfrak{p}})$ is $\mathfrak{p}$-local for each compact object C in T. It then follows that $\operatorname{Hom}^*_{\mathsf{T}}(X, Y_{\mathfrak{p}})$ is $\mathfrak{p}$-local for each object X, since X is in the localizing subcategory generated by the compact objects, and the subcategory of $\mathfrak{p}$-local modules is closed under taking products, kernels, cokernels and extensions; see [7, Lemma 2.5].

At this point we know that $\operatorname{End}^*_{\mathsf{T}}(X_{\mathfrak{p}})$ is $\mathfrak{p}$-local, and hence so is $\operatorname{Hom}^*_{\mathsf{T}}(X_{\mathfrak{p}}, Y)$, since the R-action on it factors through the homomorphism $R \to \operatorname{End}^*_{\mathsf{T}}(X_{\mathfrak{p}})$. □

Consider the exact functor $\varGamma_{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$ obtained by setting

$$\varGamma_{\mathfrak{p}} X = \varGamma_{\mathcal{V}(\mathfrak{p})}(X_{\mathfrak{p}})\,.$$

for each object X in T. The essential image of the functor $\varGamma_{\mathfrak{p}}$ is denoted by $\varGamma_{\mathfrak{p}}\mathsf{T}$, and an object X in T belongs to $\varGamma_{\mathfrak{p}}\mathsf{T}$ if and only if $\operatorname{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for every compact object C; see [7, Corollary 4.10]. From this it follows that $\varGamma_{\mathfrak{p}}\mathsf{T}$ is a localizing subcategory.

The *support* of an object X in T is a subset of $\operatorname{Spec} R$ defined as follows:

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} X \neq 0\}\,.$$

In addition to properties of the functors $\varGamma_{\mathcal{V}}$ and $L_{\mathcal{V}}$, and support, given in [7], we require also the following ones.

Lemma 2.4. *Let $\mathcal{V} \subseteq \operatorname{Spec} R$ be a specialization closed subset and $\mathfrak{p} \in \operatorname{Spec} R$. Then for each object X in T one has*

$$\varGamma_{\mathfrak{p}}(\varGamma_{\mathcal{V}} X) \cong \begin{cases} \varGamma_{\mathfrak{p}} X & \text{when } \mathfrak{p} \in \mathcal{V}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varGamma_{\mathfrak{p}}(L_{\mathcal{V}} X) \cong \begin{cases} \varGamma_{\mathfrak{p}} X & \text{when } \mathfrak{p} \notin \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply the exact functor $\varGamma_{\mathfrak{p}}$ to the exact triangle $\varGamma_{\mathcal{V}} X \to X \to L_{\mathcal{V}} X \to$. The assertion then follows from the fact that either $\varGamma_{\mathfrak{p}}(\varGamma_{\mathcal{V}} X) = 0$ or $\varGamma_{\mathfrak{p}}(L_{\mathcal{V}} X) = 0$; see [7, Theorem 5.6]. □

Further results involve a useful construction from [7, 5.10].

Koszul objects. Let $r \in R$ be a homogeneous element of degree d and X an object in T. We denote $X/\!\!/r$ any object that appears in an exact triangle

$$X \xrightarrow{\ r\ } \Sigma^d X \longrightarrow X/\!\!/r \longrightarrow \tag{2.5}$$

and call it a *Koszul object of r on X*; it is well defined up to (nonunique) isomorphism. Given a homogeneous ideal $\mathfrak{a}$ in R we write $X/\!\!/\mathfrak{a}$ for any Koszul object obtained by iterating the construction above with respect to some finite sequence of generators for $\mathfrak{a}$. This object may depend on the choice of the generating sequence for $\mathfrak{a}$. However one has the following uniqueness statements.

Lemma 2.6. *Let $\mathfrak{a}$ be a homogenous ideal in R and X an object in T.*

(1) $\operatorname{Thick}_{\mathsf{T}}(X/\!\!/\mathfrak{b}) \subseteq \operatorname{Thick}_{\mathsf{T}}(X/\!\!/\mathfrak{a})$ *for each ideal $\mathfrak{b}$ with $\mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a})$.*
(2) $\operatorname{supp}_R(X/\!\!/\mathfrak{a}) = \mathcal{V}(\mathfrak{a}) \cap \operatorname{supp}_R X$.

Proof. Note that $\mathcal{V}(\mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a})$ holds if and only if the radical of $\mathfrak{b}$ contains the radical of $\mathfrak{a}$. The proof of [19, Lemma 6.0.9] thus carries over to yield (1).

(2) We verify the claim for $\mathfrak{a} = (r)$; an obvious iteration gives the general result.

Fix a point $\mathfrak{p}$ in $\operatorname{Spec} R$ and a compact object C in T. Applying the exact functor $\varGamma_{\mathfrak{p}}$ to the exact triangle (2.5), and then the functor $\operatorname{Hom}^*_{\mathsf{T}}(C, -)$ results in an exact sequence of R-modules

$$\begin{aligned} &\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}} X) \xrightarrow{\mp r} \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}} X)[d] \longrightarrow \\ &\quad \longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}}(X/\!\!/r)) \longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}} X)[1] \xrightarrow{\pm r} \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}} X)[d+1]\,. \end{aligned}$$

Set $H = \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}} X)$. The R-module H is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion, see [7, Corollary 4.10], and this is used as follows. If $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}}(X/\!\!/r)) \neq 0$ holds, then $H \neq 0$ and $r \in \mathfrak{p}$ since H is $\mathfrak{p}$-local. On the other hand, $H \neq 0$ and $r \in \mathfrak{p}$ implies that $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{p}}(X/\!\!/r)) \neq 0$ since H is $\mathfrak{p}$-torsion. This implies the desired equality. □

The result below is [8, Proposition 3.5], except that there G is assumed to consist of a single object. The argument is however the same, so we omit the proof.

Proposition 2.7. *Let G be a set of compact objects which generate T, and let $\mathcal{V}$ be a specialization closed subset of $\operatorname{Spec} R$. For any decomposition $\mathcal{V} = \bigcup_{i\in I} \mathcal{V}(\mathfrak{a}_i)$ where each $\mathfrak{a}_i$ is an ideal in R, there are equalities*

$$\mathsf{T}_{\mathcal{V}} = \operatorname{Loc}_{\mathsf{T}}(\{C/\!\!/\mathfrak{a}_i \mid C \in \mathsf{G},\, i \in I\}) = \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathcal{V}(\mathfrak{a}_i)} C \mid C \in \mathsf{G},\, i \in I\})\,. \qquad \square$$

An element $r \in R^d$ is invertible on an R-module M if the map $M \xrightarrow{r} M[d]$ is an isomorphism. In the same vein, we say r is *invertible* on an object X in T if the natural morphism $X \xrightarrow{r} \Sigma^{|r|} X$ is an isomorphism; equivalently, if $X/\!\!/r$ is zero.

Lemma 2.8. *Let X be an object in T and $\mathcal{V} \subseteq \operatorname{Spec} R$ a specialization closed subset. Each element $r \in R$ with $\mathcal{V}(r) \subseteq \mathcal{V}$ is invertible on $L_{\mathcal{V}} X$, and hence on the R-modules $\operatorname{Hom}^*_{\mathsf{T}}(L_{\mathcal{V}} X, Y)$ and $\operatorname{Hom}^*_{\mathsf{T}}(Y, L_{\mathcal{V}} X)$, for any object Y in T.*

Proof. From [7, Theorem 5.6] and Lemma 2.6 one gets equalities

$$L_{\mathcal{V}}(X/\!\!/r) = \mathcal{V}(r) \cap \operatorname{supp}_R X \cap (\operatorname{Spec} R \setminus \mathcal{V}(r)) = \emptyset\,.$$

Therefore $L_{\mathcal{V}}(X/\!\!/r) = 0$, by [7, Theorem 5.2]. Applying $L_{\mathcal{V}}$ to the exact triangle (2.5) yields an isomorphism $L_{\mathcal{V}} X \xrightarrow{r} \Sigma^{|r|} L_{\mathcal{V}} X$, which is the first part of the statement. Applying $\operatorname{Hom}^*_{\mathsf{T}}(-, Y)$ and $\operatorname{Hom}^*_{\mathsf{T}}(Y, -)$ to it gives the second part. □

Homotopy colimits. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$ be a sequence of morphisms in T. Its *homotopy colimit*, denoted $\operatorname{hocolim} X_n$, is defined by an exact triangle

$$\bigoplus_{n\geqslant 1} X_n \xrightarrow{\theta} \bigoplus_{n\geqslant 1} X_n \longrightarrow \operatorname{hocolim} X_n \longrightarrow$$

where θ is the map $(\mathrm{id} - f_n)$; see [11].

Now fix a homogeneous element $r \in R$ of degree d. For each X in T and each integer n set $X_n = \Sigma^{nd} X$ and consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "r"] & X \arrow[r, equal] \arrow[d, "r^2"] & X \arrow[r, equal] \arrow[d, "r^3"] & \cdots \\ X_1 \arrow[r, "r"] \arrow[d] & X_2 \arrow[r, "r"] \arrow[d] & X_3 \arrow[r, "r"] \arrow[d] & \cdots \\ X/\!\!/r \arrow[r] & X/\!\!/r^2 \arrow[r] & X/\!\!/r^3 \arrow[r] & \cdots \end{tikzcd}$$

where each vertical sequence is given by the exact triangle defining $X/\!\!/r^n$, and the morphisms in the last row are the (non-canonical) ones induced by the commutativity of the upper squares. The gist of the next result is that the homotopy colimits of the horizontal sequences in the diagram compute $L_{\mathcal{V}(r)} X$ and $\varGamma_{\mathcal{V}(r)} X$.

Proposition 2.9. *Let $r \in R$ be a homogeneous element of degree d. For each X in T the adjunction morphisms $X \to L_{\mathcal{V}(r)}X$ and $\varGamma_{\mathcal{V}(r)}X \to X$ induce isomorphisms*

$$\operatorname{hocolim} X_n \xrightarrow{\sim} L_{\mathcal{V}(r)}X \quad \textit{and} \quad \operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n) \xrightarrow{\sim} \varGamma_{\mathcal{V}(r)}X\,.$$

Proof. Applying the functor $\varGamma_{\mathcal{V}(r)}$ to the middle row of the diagram above yields a sequence of morphisms $\varGamma_{\mathcal{V}(r)}X_1 \to \varGamma_{\mathcal{V}(r)}X_2 \to \cdots$. For each compact object C in T, this induces a sequence of morphisms of R-modules

$$\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_1) \xrightarrow{g_1} \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_2) \xrightarrow{g_2} \cdots$$

We claim that the colimit of this sequence, in the category of R-modules, satisfies:

$$\operatorname{colim} \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_n) = 0 \tag{2.10}$$

Indeed, each R-module $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_n)$ is (r)-torsion and, identifying this module with $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X)[nd]$, the map g_n is given by multiplication with r.

Applying the functor $L_{\mathcal{V}(r)}$ to the canonical morphism $\phi\colon X \to \operatorname{hocolim} X_n$ yields the following commutative square.

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\eta X"'] & \operatorname{hocolim} X_n \arrow[d, "\eta \operatorname{hocolim} X_n"] \\
L_{\mathcal{V}(r)}X \arrow[r, "L_{\mathcal{V}(r)}\phi"] & L_{\mathcal{V}(r)} \operatorname{hocolim} X_n
\end{tikzcd}$$

The morphism $\eta \operatorname{hocolim} X_n$ is an isomorphism since $\varGamma_{\mathcal{V}(r)} \operatorname{hocolim} X_n = 0$. The equality holds because, for each compact object C, there is a chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)} \operatorname{hocolim} X_n) &\cong \operatorname{Hom}^*_{\mathsf{T}}(C, \operatorname{hocolim} \varGamma_{\mathcal{V}(r)}X_n) \\
&\cong \operatorname{colim} \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_n) \\
&\cong 0
\end{aligned}$$

where the second one holds because C is compact and the last one is by (2.10).

On the other hand, $L_{\mathcal{V}(r)}\phi$ is an isomorphism, since

$$L_{\mathcal{V}(r)} \operatorname{hocolim} X_n \cong \operatorname{hocolim} L_{\mathcal{V}(r)}X_n$$

and r is invertible on $L_{\mathcal{V}(r)}X$, by Lemma 2.8. Thus $\operatorname{hocolim} X_n \cong L_{\mathcal{V}(r)}X$.

Now consider the canonical morphism $\psi\colon \operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n) \to X$. Applying the functor $\varGamma_{\mathcal{V}(r)}$ to it yields a commutative square:

$$\begin{tikzcd}
\varGamma_{\mathcal{V}(r)} \operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n) \arrow[r, "\varGamma_{\mathcal{V}(r)}\psi"] \arrow[d, "\theta \operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n)"'] & \varGamma_{\mathcal{V}(r)}X \arrow[d, "\theta X"] \\
\operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n) \arrow[r, "\psi"] & X
\end{tikzcd}$$

By [7, Lemma 5.11], each $X/\!\!/r^n$ is in $\mathsf{T}_{\mathcal{V}(r)}$ and hence so is $\operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n)$. Thus the morphism $\theta \operatorname{hocolim} \Sigma^{-1}(X/\!\!/r^n)$ is an isomorphism. It remains to show that $\varGamma_{\mathcal{V}(r)}\psi$ is an isomorphism; equivalently, that the map $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}\psi)$ is an isomorphism for each compact object C.

The exact triangle $X \to X_n \to X/\!\!/r^n \to$ induces an exact sequence of R-modules:

$$\begin{aligned}
\operatorname{Hom}^*_{\mathsf{T}}(C, \Sigma^{-1}\varGamma_{\mathcal{V}(r)}X_n) \longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(C, \Sigma^{-1}(\varGamma_{\mathcal{V}(r)}X/\!\!/r^n)) \longrightarrow \\
\longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X) \longrightarrow \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}X_n)
\end{aligned}$$

In view of (2.10), passing to their colimits yields that $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathcal{V}(r)}\psi)$ is an isomorphism, as desired. $\square$

Proposition 2.11. *Let $\mathfrak{a}$ be an ideal in R. For each object X in T the following statements hold:*

(1) *$X/\!\!/\mathfrak{a}$ is in $\operatorname{Thick}_{\mathsf{T}}(\varGamma_{\mathcal{V}(\mathfrak{a})}X)$ and $\varGamma_{\mathcal{V}(\mathfrak{a})}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X/\!\!/\mathfrak{a})$;*
(2) *$\operatorname{Loc}_{\mathsf{T}}(X/\!\!/\mathfrak{a}) = \operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathcal{V}(\mathfrak{a})}X)$;*
(3) *$\varGamma_{\mathcal{V}(\mathfrak{a})}X$ and $L_{\mathcal{V}(\mathfrak{a})}X$ are in $\operatorname{Loc}_{\mathsf{T}}(X)$.*

Proof. (1) By construction $X/\!\!/\mathfrak{a}$ is in $\operatorname{Thick}_{\mathsf{T}}(X)$. As $\varGamma_{\mathcal{V}(\mathfrak{a})}$ is an exact functor, one obtains that $\varGamma_{\mathcal{V}(\mathfrak{a})}(X/\!\!/\mathfrak{a})$ is in $\operatorname{Thick}_{\mathsf{T}}(\varGamma_{\mathcal{V}(\mathfrak{a})}X)$. This justifies the first claim in (1), since $X/\!\!/\mathfrak{a}$ is in $\mathsf{T}_{\mathcal{V}(\mathfrak{a})}$ by [7, Lemma 5.11].

Now we verify that $\varGamma_{\mathcal{V}(\mathfrak{a})}X$ is in the localizing subcategory generated by $X/\!\!/\mathfrak{a}$.

Consider the case where $\mathfrak{a}$ is generated by a single element, say a. It follows from Proposition 2.9 that $\varGamma_{\mathcal{V}(a)}X$ is the homotopy colimit of objects $\Sigma^{-1}X/\!\!/a^n$. Each $X/\!\!/a^n$ is in $\operatorname{Thick}_{\mathsf{T}}(X/\!\!/a)$ by Lemma 2.6, hence $\varGamma_{\mathcal{V}(a)}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X/\!\!/a)$.

Now suppose $\mathfrak{a} = (a_1, \dots, a_n)$, and set $\mathfrak{a}' = (a_1, \dots, a_{n-1})$. Then the equality $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(a_1) \cap \mathcal{V}(\mathfrak{a}')$ yields $\varGamma_{\mathcal{V}(\mathfrak{a})} = \varGamma_{\mathcal{V}(a_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}$ by[7, Proposition 6.1]. By induction on n one may assume that $\varGamma_{\mathcal{V}(\mathfrak{a}')}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X/\!\!/\mathfrak{a}')$. Therefore $\varGamma_{\mathcal{V}(\mathfrak{a})}X$ is in $\operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathcal{V}(a_n)}(X/\!\!/\mathfrak{a}'))$. The basis of the induction implies that $\varGamma_{\mathcal{V}(a_n)}(X/\!\!/\mathfrak{a}')$ is in the localizing subcategory generated by $(X/\!\!/\mathfrak{a}')/\!\!/a_n$, that is to say, by $X/\!\!/\mathfrak{a}$. Therefore, $\varGamma_{\mathcal{V}(\mathfrak{a})}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X/\!\!/\mathfrak{a})$, as claimed.

(2) is an immediate consequence of (1).

(3) Since $X/\!\!/\mathfrak{a}$ is in $\operatorname{Thick}_{\mathsf{T}}(X)$, it follows from (1) that $\varGamma_{\mathcal{V}(\mathfrak{a})}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$. The localization triangle (2.2) then yields that $L_{\mathcal{V}(\mathfrak{a})}X$ is also in $\operatorname{Loc}_{\mathsf{T}}(X)$. □

3. A local-global principle

We introduce a local-global principle for T and explain how, when it holds, the problem of classifying the localizing subcategories can be reduced to one of classifying localizing subcategories supported at a single point in $\operatorname{Spec} R$.

Recall that T is a compactly generated R-linear triangulated category. If for each object X in T there is an equality

$$\operatorname{Loc}_{\mathsf{T}}(X) = \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{Spec} R\})$$

we say that the *local-global principle holds* for T.

Theorem 3.1. *Let T be a compactly generated R-linear triangulated category. The local-global principle is equivalent to each of the following statements.*

(1) *For any $X \in \mathsf{T}$ and any localizing subcategory S of T, one has*
$$X \in \mathsf{S} \iff \varGamma_{\mathfrak{p}}X \in \mathsf{S} \text{ for each } \mathfrak{p} \in \operatorname{Spec} R\,.$$
(2) *For any $X \in \mathsf{T}$, one has $X \in \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{Spec} R\})$.*
(3) *For any $X \in \mathsf{T}$ and any specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, one has*
$$\varGamma_{\mathcal{V}}X \in \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \mathcal{V}\})\,.$$
(4) *For any $X \in \mathsf{T}$, one has $\operatorname{Loc}_{\mathsf{T}}(X) = \operatorname{Loc}_{\mathsf{T}}(\{X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\})$.*
(5) *For any $X \in \mathsf{T}$ and any localizing subcategory S of T, one has*
$$X \in \mathsf{S} \iff X_{\mathfrak{p}} \in \mathsf{S} \text{ for each } \mathfrak{p} \in \operatorname{Spec} R\,.$$
(6) *For any $X \in \mathsf{T}$, one has $X \in \operatorname{Loc}_{\mathsf{T}}(\{X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\})$.*

The proof uses some results, which may also be useful elsewhere.

Lemma 3.2. *Let X be an object in T. Suppose that for any specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, one has*

$$\varGamma_{\mathcal{V}}X \in \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \mathcal{V}\})\,.$$

Then $\varGamma_{\mathcal{V}}X$ and $L_{\mathcal{V}}X$ belong to $\operatorname{Loc}_{\mathsf{T}}(X)$ for every specialization closed $\mathcal{V} \subseteq \operatorname{Spec} R$.

Proof. It suffices to prove that $\varGamma_{\mathfrak{p}}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$ for each $\mathfrak{p}$, that is to say, that the set $\mathcal{U} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}}X \notin \operatorname{Loc}_{\mathsf{T}}(X)\}$ is empty. Assume $\mathcal{U}$ is not empty and choose a maximal element, say $\mathfrak{p}$, with respect to inclusion. This is possible since R is noetherian. Set $\mathcal{W} = \mathcal{V}(\mathfrak{p}) \setminus \{\mathfrak{p}\}$, and consider the localization triangle

$$\varGamma_{\mathcal{W}}X \longrightarrow \varGamma_{\mathcal{V}(\mathfrak{p})}X \longrightarrow \varGamma_{\mathfrak{p}}X \longrightarrow$$

of $\varGamma_{\mathcal{V}(\mathfrak{p})}X$ with respect to $\mathcal{W}$. The hypothesis implies the first inclusion below

$$\varGamma_{\mathcal{W}}X \in \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{q}}X \mid \mathfrak{q} \in \mathcal{W}\}) \subseteq \operatorname{Loc}_{\mathsf{T}}(X)\,,$$

and the second one follows from the choice of $\mathfrak{p}$. The object $\varGamma_{\mathcal{V}(\mathfrak{p})}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$, by Proposition 2.11, so the exact triangle above yields that $\varGamma_{\mathfrak{p}}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$. This contradicts the choice of $\mathfrak{p}$, and hence $\mathcal{U} = \emptyset$, as desired. □

Finite dimension. The *dimension* of a subset $\mathcal{U}$ of $\operatorname{Spec} R$, denoted $\dim \mathcal{U}$, is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in $\mathcal{U}$. The set $\mathcal{U}$ is called *discrete* if $\dim \mathcal{U} = 0$.

Proposition 3.3. *Let X be an object of T and set $\mathcal{U} = \operatorname{supp}_R X$. If $\mathcal{U}$ is discrete, then there are natural isomorphisms*

$$X \xleftarrow{\sim} \coprod_{\mathfrak{p} \in \mathcal{U}} \varGamma_{\mathcal{V}(\mathfrak{p})}X \xrightarrow{\sim} \coprod_{\mathfrak{p} \in \mathcal{U}} \varGamma_{\mathfrak{p}}X\,.$$

Proof. Arguing as in the proof of [7, Theorem 7.1] one gets that the morphisms $\varGamma_{\mathcal{V}(\mathfrak{p})}X \to X$ induce the isomorphism on the left, in the statement above. The isomorphism on the right holds since for each $\mathfrak{p} \in \mathcal{U}$ the morphism $\varGamma_{\mathcal{V}(\mathfrak{p})}X \to \varGamma_{\mathfrak{p}}X$ is an isomorphism by Lemma 2.4. □

Theorem 3.4. *Let T be a compactly generated R-linear triangulated category and X an object of T. If $\dim \operatorname{supp}_R X < \infty$, then X is in $\operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{supp}_R X\})$.*

Proof. Set $\mathcal{U} = \operatorname{supp}_R X$ and $\mathsf{S} = \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \mathcal{U}\})$. The proof is an induction on $n = \dim \mathcal{U}$. The case $n = 0$ is covered by Proposition 3.3. For $n > 0$ set $\mathcal{U}' = \mathcal{U} \setminus \min \mathcal{U}$, where $\min \mathcal{U}$ is the set of minimal elements with respect to inclusion in $\mathcal{U}$, and set $\mathcal{V} = \operatorname{cl} \mathcal{U}'$. It follows from Lemma 2.4 that $\operatorname{supp}_R \varGamma_{\mathcal{V}}X = \mathcal{U}'$. Since $\dim \mathcal{U}' = \dim \mathcal{U} - 1$, the induction hypothesis yields that $\varGamma_{\mathcal{V}}X$ is in S. On the other hand, $\operatorname{supp}_R L_{\mathcal{V}}X = \min \mathcal{U}$ is discrete and therefore $L_{\mathcal{V}}X$ belongs to S by Proposition 3.3 and Lemma 2.4. Thus X is in S, in view of the localization triangle (2.2). □

Proof of Theorem 3.1. It is easy to check that the local-global principle is equivalent to (1). Also, the implications (1) $\Longrightarrow$ (2) and (4) $\Longleftrightarrow$ (5) $\Longrightarrow$ (6) are obvious.

(2) $\Longrightarrow$ (3): Fix $X \in \mathsf{T}$ and a specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$. Then

$$\varGamma_{\mathcal{V}}X \in \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}\varGamma_{\mathcal{V}}X \mid \mathfrak{p} \in \operatorname{Spec} R\}) = \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \mathcal{V}\})$$

hold, where the last equality follows from Lemma 2.4.

(3) $\Longrightarrow$ (1): Since $\varGamma_{\mathfrak{p}} = \varGamma_{\mathcal{V}(\mathfrak{p})}L_{\mathcal{Z}(\mathfrak{p})}$, it follows from condition (3) and Lemma 3.2 that $\varGamma_{\mathfrak{p}}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$. This implies $\operatorname{Loc}_{\mathsf{T}}(X) \supseteq \operatorname{Loc}_{\mathsf{T}}(\{\varGamma_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{Spec} R\})$ and the reverse inclusion holds by condition (3) for $\mathcal{V} = \operatorname{Spec} R$. Thus the local-global principle, which is equivalent to condition (1), holds.

(3) $\Longrightarrow$ (4): We have $\varGamma_{\mathfrak{p}}X = \varGamma_{\mathcal{V}(\mathfrak{p})}X_{\mathfrak{p}} \in \operatorname{Loc}_{\mathsf{T}}(X_{\mathfrak{p}})$ for each prime ideal $\mathfrak{p}$ by Proposition 2.11 and hence the hypothesis implies $X \in \operatorname{Loc}_{\mathsf{T}}(\{X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\})$. On the other hand, $X_{\mathfrak{p}} \in \operatorname{Loc}_{\mathsf{T}}(X)$ for each prime ideal $\mathfrak{p}$ by Lemma 3.2.

(6) $\Longrightarrow$ (2): Fix $X \in \mathsf{T}$. For every prime ideal $\mathfrak{p}$, one has, for example from Lemma 2.4, that $\operatorname{supp}_R X_{\mathfrak{p}}$ is a subset of $\{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$. In particular, it

is finite dimensional, since R is noetherian, so $X_{\mathfrak p} \in \operatorname{Loc}_{\mathsf T}(\{\varGamma_{\mathfrak q} X_{\mathfrak p} \mid \mathfrak q \in \operatorname{Spec} R\})$ holds, by Theorem 3.4. Thus

$$\begin{aligned} X \in \operatorname{Loc}_{\mathsf T}(\{X_{\mathfrak p} \mid \mathfrak p \in \operatorname{Spec} R\}) &\subseteq \operatorname{Loc}_{\mathsf T}(\{\varGamma_{\mathfrak q} X_{\mathfrak p} \mid \mathfrak p, \mathfrak q \in \operatorname{Spec} R\}) \\ &= \operatorname{Loc}_{\mathsf T}(\{\varGamma_{\mathfrak q} X \mid \mathfrak q \in \operatorname{Spec} R\}), \end{aligned}$$

where the last equality follows from Lemma 2.4. □

The result below is an immediate consequence of Theorems 3.4 and 3.1.

Corollary 3.5. *When* $\dim \operatorname{Spec} R$ *is finite the local-global principle holds for* $\mathsf T$. □

Classifying localizing subcategories. Localizing subcategories of $\mathsf T$ are related to subsets of $\mathcal V = \operatorname{supp}_R \mathsf T$ via the following maps

$$\left\{ \begin{matrix} \text{Localizing} \\ \text{subcategories of } \mathsf T \end{matrix} \right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{ \begin{matrix} \text{Families } (\mathsf S(\mathfrak p))_{\mathfrak p \in \mathcal V} \text{ with } \mathsf S(\mathfrak p) \text{ a} \\ \text{localizing subcategory of } \varGamma_{\mathfrak p}\mathsf T \end{matrix} \right\}$$

which are defined by $\sigma(\mathsf S) = (\mathsf S \cap \varGamma_{\mathfrak p}\mathsf T)$ and $\tau(\mathsf S(\mathfrak p)) = \operatorname{Loc}_{\mathsf T}\big(\mathsf S(\mathfrak p) \mid \mathfrak p \in \mathcal V\big)$.

The next result expresses the local-global principle in terms of these maps.

Proposition 3.6. *The following conditions are equivalent.*

(1) *The local-global principle holds for* $\mathsf T$.
(2) *The map* σ *is bijective, with inverse* τ.
(3) *The map* σ *is one-to-one.*

Proof. We repeatedly use the fact that $\varGamma_{\mathfrak p}$ is an exact functor preserving coproducts. For each localizing subcategory $\mathsf S$ of $\mathsf T$ and each $\mathfrak p$ in $\operatorname{Spec} R$ there is an inclusion

$$\mathsf S \cap \varGamma_{\mathfrak p}\mathsf T \subseteq \varGamma_{\mathfrak p}\mathsf S\,. \tag{3.7}$$

We claim that $\sigma\tau$ is the identity, that is to say, that for any family $(\mathsf S(\mathfrak p))_{\mathfrak p \in \mathcal V}$ of localizing subcategories with $S(\mathfrak p) \subseteq \varGamma_{\mathfrak p}\mathsf T$ the localizing subcategory generated by all the $S(\mathfrak p)$, call it $\mathsf S$, satisfies

$$\mathsf S \cap \varGamma_{\mathfrak p}\mathsf T = \mathsf S(\mathfrak p)\,, \quad \text{for each } \mathfrak p \in \mathcal V.$$

To see this, note that $\varGamma_{\mathfrak p}\mathsf S = \mathsf S(\mathfrak p)$ holds, since $\varGamma_{\mathfrak p}\varGamma_{\mathfrak q} = 0$ when $\mathfrak p \neq \mathfrak q$. Hence (3.7) yields an inclusion $\mathsf S \cap \varGamma_{\mathfrak p}\mathsf T \subseteq \mathsf S(\mathfrak p)$. The reverse inclusion is obvious.

(1) $\Longrightarrow$ (2): It suffices to show that $\tau\sigma$ equals the identity, since $\sigma\tau = \mathrm{id}$ holds. Fix a localizing subcategory $\mathsf S$ of $\mathsf T$. It is clear that $\tau\sigma(\mathsf S) \subseteq \mathsf S$. As to the reverse inclusion: Fixing X in $\mathsf S$, it follows from Theorem 3.1(1) that $\varGamma_{\mathfrak p} X$ is in $\mathsf S \cap \varGamma_{\mathfrak p}\mathsf T$ and hence in $\tau\sigma(\mathsf S)$, for each $\mathfrak p \in \operatorname{Spec} R$. Thus, X is in $\tau\sigma(\mathsf S)$, again by Theorem 3.1(1).

(2) $\Longrightarrow$ (3): Clear.

(3) $\Longrightarrow$ (1): Since $\sigma\tau = \mathrm{id}$ and σ is one-to-one, one gets $\tau\sigma = \mathrm{id}$. For each object X in $\mathsf T$ there is thus an equality:

$$\begin{aligned} \operatorname{Loc}_{\mathsf T}(X) &= \operatorname{Loc}_{\mathsf T}(\{\operatorname{Loc}_{\mathsf T}(X) \cap \varGamma_{\mathfrak p}\mathsf T \mid \mathfrak p \in \operatorname{Spec} R\}) \\ &\subseteq \operatorname{Loc}_{\mathsf T}(\{\varGamma_{\mathfrak p} X \mid \mathfrak p \in \operatorname{Spec} R\}) \end{aligned}$$

The inclusion follows from (3.7). Now apply Theorem 3.1. □

The local-global principle focuses attention on the subcategory $\varGamma_{\mathfrak p}\mathsf T$. Next we describe some of its properties, even though these are not needed in the sequel.

Local structure. Let $\mathfrak{p}$ be a point in $\operatorname{Spec} R$. In analogy with the case of R-modules, we say that an object X in T is $\mathfrak{p}$-*local* if an inclusion

$$\operatorname{supp}_R X \subseteq \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$$

holds, and that X is $\mathfrak{p}$-*torsion* if there is an inclusion:

$$\operatorname{supp}_R X \subseteq \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supseteq \mathfrak{p}\}.$$

The objects of $\varGamma_{\mathfrak{p}}\mathsf{T}$ are precisely those that are both $\mathfrak{p}$-local and $\mathfrak{p}$-torsion; see [7, Corollary 5.9] for alternative descriptions. Set

$$X(\mathfrak{p}) = (X/\!\!/\mathfrak{p})_{\mathfrak{p}}\,.$$

The subcategory $\operatorname{Thick}_{\mathsf{T}}(X(\mathfrak{p}))$ is independent of the choice of a generating set for the ideal $\mathfrak{p}$ used to construct $X/\!\!/\mathfrak{p}$; this follows from Lemma 2.6.

Lemma 3.8. *The following statements hold for each $X \in \mathsf{T}$ and $\mathfrak{p} \in \operatorname{Spec} R$.*

1. *$X(\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion.*
2. *$\operatorname{Loc}_{\mathsf{T}}(X(\mathfrak{p})) = \operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}X)$.*
3. *$\operatorname{Hom}_{\mathsf{T}}(W, X(\mathfrak{p})) = 0$ for any object W that is $\mathfrak{q}$-local and $\mathfrak{q}$-torsion with $\mathfrak{q} \neq \mathfrak{p}$.*

Proof. The argument is based on the fact that the localization functor that takes an object X to $X_{\mathfrak{p}}$ is exact and preserves coproducts.

(1) Exactness of localization implies $(X/\!\!/\mathfrak{p})_{\mathfrak{p}}$ can be be realized as $X_{\mathfrak{p}}/\!\!/\mathfrak{p}$. Hence $X(\mathfrak{p})$ belongs to $\operatorname{Thick}_{\mathsf{T}}(X_{\mathfrak{p}})$, so that it is $\mathfrak{p}$-local; it is $\mathfrak{p}$-torsion by [7, Lemma 5.11].

(2) Applying the localization functor to the equality $\operatorname{Loc}_{\mathsf{T}}(X/\!\!/\mathfrak{p}) = \operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathcal{V}(\mathfrak{p})}X)$ in Proposition 2.11 yields (2).

(3) If $\mathfrak{q} \not\subseteq \mathfrak{p}$ holds, then $\varGamma_{\mathcal{V}(\mathfrak{q})}(X(\mathfrak{p})) = 0$ and hence the desired claim follows from the adjunction isomorphism $\operatorname{Hom}_{\mathsf{T}}(W, \varGamma_{\mathcal{V}(\mathfrak{q})}X(\mathfrak{p})) \cong \operatorname{Hom}_{\mathsf{T}}(W, X(\mathfrak{p}))$. If $\mathfrak{q} \subseteq \mathfrak{p}$, then the R-module $\operatorname{Hom}^*_{\mathsf{T}}(W, X(\mathfrak{p}))$ is $\mathfrak{q}$-local, by Proposition 2.3, and $\mathfrak{p}$-torsion, by [7, Lemma 5.11], and hence zero since $\mathfrak{q} \neq \mathfrak{p}$. □

Proposition 3.9. *For each $\mathfrak{p}$ in $\operatorname{Spec} R$ and each compact object C in T, the object $C(\mathfrak{p})$ is compact in $\varGamma_{\mathfrak{p}}\mathsf{T}$, and both $\{C(\mathfrak{p}) \mid C \in \mathsf{T}^{\mathsf{c}}\}$ and $\{\varGamma_{\mathfrak{p}}C \mid C \in \mathsf{T}^{\mathsf{c}}\}$ generate the triangulated category $\varGamma_{\mathfrak{p}}\mathsf{T}$. Furthermore, the R-linear structure on T induces a natural structure of an $R_{\mathfrak{p}}$-linear triangulated category on $\varGamma_{\mathfrak{p}}\mathsf{T}$.*

Proof. Recall that $\varGamma_{\mathfrak{p}}T$ is a localizing subcategory of T, so the coproduct in it is the same as the one in T. Each object X in $\varGamma_{\mathfrak{p}}\mathsf{T}$ is $\mathfrak{p}$-local, so there is an isomorphism

$$\operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), X) \cong \operatorname{Hom}_{\mathsf{T}}(C/\!\!/\mathfrak{p}, X)\,.$$

When C is compact in T, so is $C/\!\!/\mathfrak{p}$. Thus the isomorphism above implies that $C(\mathfrak{p})$ is compact in $\varGamma_{\mathfrak{p}}\mathsf{T}$. Furthermore, the collection of objects $C/\!\!/\mathfrak{p}$ with C compact in T generates $\mathsf{T}_{\mathcal{V}(\mathfrak{p})}$ by Proposition 2.7, and hence the $C(\mathfrak{p})$ generate $\varGamma_{\mathfrak{p}}\mathsf{T}$.

The class of compact objects C generate T hence the objects $\varGamma_{\mathfrak{p}}C$ generate $\varGamma_{\mathfrak{p}}\mathsf{T}$.

Proposition 2.3 implies that for each pair of objects X, Y in $\varGamma_{\mathfrak{p}}\mathsf{T}$ the R-module $\operatorname{Hom}^*_{\mathsf{T}}(X, Y)$ is $\mathfrak{p}$-local, so that they admit a natural $R_{\mathfrak{p}}$-module structure. This translates to an action of $R_{\mathfrak{p}}$ on $\varGamma_{\mathfrak{p}}\mathsf{T}$. □

4. Stratification

In this section we introduce a notion of stratification for triangulated categories with ring actions. It is based on the concept of a minimal subcategory introduced by Hovey, Palmieri, and Strickland [19, §6].

As before T is a compactly generated R-linear triangulated category.

Minimal subcategories. A localizing subcategory of T is said to be *minimal* if it is nonzero and has no proper nonzero localizing subcategories.

Lemma 4.1. *A nonzero localizing subcategory* S *of* T *is minimal if and only if for any nonzero objects* X, Y *in* S *one has* $\operatorname{Hom}^*_{\mathsf{T}}(X, Y) \neq 0$.

Proof. When S is minimal and X a nonzero object in it $\operatorname{Loc}_{\mathsf{T}}(X) = \mathsf{S}$, by minimality, so if $\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0$ for some Y in S, then $\operatorname{Hom}^*_{\mathsf{T}}(Y, Y) = 0$, that is to say, $Y = 0$.

Suppose S contains a nonzero proper localizing subcategory S'; we may assume $\mathsf{S}' = \operatorname{Loc}_{\mathsf{T}}(X)$ for some nonzero object X. For each object W in T there is then an exact triangle $W' \xrightarrow{\theta} W \xrightarrow{\eta} W'' \to$ with $W' \in \mathsf{S}'$, $\operatorname{Hom}^*_{\mathsf{T}}(X, W'') = 0$, and θ invertible if and only if and W is in S'; see Lemma 2.1. It remains to pick an object W in $\mathsf{S} \setminus \mathsf{S}'$, set $Y = W''$, and note that Y is in S and nonzero. □

Stratification. We say that T is *stratified by* R if the following conditions hold:

(S1) The local-global principle, discussed in Section 3, holds for T.
(S2) For each $\mathfrak{p} \in \operatorname{Spec} R$ the localizing subcategory $\varGamma_{\mathfrak{p}}\mathsf{T}$ is either zero or minimal.

The crucial condition here is (S2); for example, (S1) holds when the Krull dimension of R is finite, by Corollary 3.5. Since the objects in $\varGamma_{\mathfrak{p}}\mathsf{T}$ are precisely the $\mathfrak{p}$-local and $\mathfrak{p}$-torsion ones in T, condition (S2) is that each nonzero $\mathfrak{p}$-local $\mathfrak{p}$-torsion object builds every other such object.

Given a localizing subcategory S of T and a subset $\mathcal{U}$ of $\operatorname{Spec} R$ set

$$\operatorname{supp}_R \mathsf{S} = \bigcup_{X \in \mathsf{S}} \operatorname{supp}_R X \quad \text{and} \quad \operatorname{supp}_R^{-1} \mathcal{U} = \{X \in \mathsf{T} \mid \operatorname{supp}_R X \subseteq \mathcal{U}\}.$$

Observe that supp_R and $\operatorname{supp}_R^{-1}$ both preserve inclusions.

Theorem 4.2. *Let* T *be a compactly generated* R*-linear triangulated category. If* T *is stratified by* R*, then there are inclusion preserving inverse bijections:*

$$\left\{ \begin{gathered} \textit{Localizing} \\ \textit{subcategories of } \mathsf{T} \end{gathered} \right\} \underset{\operatorname{supp}_R^{-1}}{\overset{\operatorname{supp}_R}{\rightleftarrows}} \left\{ \textit{Subsets of } \operatorname{supp}_R \mathsf{T} \right\}$$

Conversely, if the map supp_R *is injective, then* T *must be stratified by* R.

Proof. For each $\mathfrak{p} \in \operatorname{Spec} R$ the subcategory $\operatorname{Ker} \varGamma_{\mathfrak{p}}$ is localizing. This implies that for any subset $\mathcal{U}$ of $\operatorname{Spec} R$ the subcategory $\operatorname{supp}_R^{-1} \mathcal{U}$ is localizing, for

$$\operatorname{supp}_R^{-1} \mathcal{U} = \bigcap_{\mathfrak{p} \notin \mathcal{U}} \operatorname{Ker} \varGamma_{\mathfrak{p}}\,.$$

Moreover, it is clear that $\operatorname{supp}_R(\operatorname{supp}_R^{-1} \mathcal{U}) = \mathcal{U}$ for each subset $\mathcal{U}$ of $\operatorname{supp}_R \mathsf{T}$, and that $\mathsf{S} \subseteq \operatorname{supp}_R^{-1}(\operatorname{supp}_R \mathsf{S})$ holds for any localizing subcategory S. The moot point is whether S contains $\operatorname{supp}_R^{-1}(\operatorname{supp}_R \mathsf{S})$; equivalently, whether supp_R is one-to-one.

The map supp_R factors as $\sigma'\sigma$ with σ as in Proposition 3.6 and σ' the map

$$\left\{ \begin{gathered} \text{Families } (\mathsf{S}(\mathfrak{p}))_{\mathfrak{p} \in \operatorname{supp}_R \mathsf{T}} \text{ with } \mathsf{S}(\mathfrak{p}) \\ \text{a localizing subcategory of } \varGamma_{\mathfrak{p}}\mathsf{T} \end{gathered} \right\} \longrightarrow \left\{ \text{Subsets of } \operatorname{supp}_R \mathsf{T} \right\}$$

where $\sigma'(\mathsf{S}(\mathfrak{p})) = \{\mathfrak{p} \in \operatorname{Spec} R \mid S(\mathfrak{p}) \neq \{0\}\}$. Evidently σ' is one-to-one if and only if it is bijective, if and only if the minimality condition (S2) holds. The map σ is also one-to-one if and only if it is bijective; moreover this holds precisely when the local-global principle holds for T, by Proposition 3.6. The desired result follows. □

Corollary 4.3. *If R stratifies T and G is a set of generators for T, then each localizing subcategory S of T is generated by the set $\mathsf{S}\cap\{\Gamma_{\mathfrak{p}}X \mid X \in \mathsf{G},\ \mathfrak{p}\in \operatorname{Spec} R\}$. In particular, there exists a localization functor $L\colon \mathsf{T}\to\mathsf{T}$ such that $\mathsf{S}=\operatorname{Ker} L$.*

Proof. The first assertion is an immediate consequence of Theorem 4.2, since S and the localizing subcategory generated by the given set of objects have the same support. Given this, the second one follows from Lemma 2.1. □

Other consequences of stratification are given in Sections 5 and 6. Now we provide examples of triangulated categories that are stratified; see also Example 7.4.

Example 4.4. Let A be a commutative noetherian ring and $\mathsf{D}(A)$ the derived category of the category of A-modules. The category $\mathsf{D}(A)$ is compactly generated, A-linear, and triangulated. This example is discussed in [7, §8], where it is proved that the notion of support introduced in [7] coincides with the usual one, due to Foxby and Neeman; see [7, Theorem 9.1]. In view of Theorem 4.2, one can reformulate [25, Theorem 2.8] as: The A-linear triangulated category $\mathsf{D}(A)$ is stratified by A. This example will be subsumed in Theorem 8.1.

Example 4.5. Let k be a field and Λ an exterior algebra over k in finitely many indeterminates of negative odd degree; the grading is upper. We view Λ as a dg algebra, with differential zero. In [8, §6] we introduced the homotopy category of graded-injective dg Λ-modules and proved that it is stratified by a natural action of its cohomology algebra, $\operatorname{Ext}^*_\Lambda(k,k)$.

The next example shows that there are triangulated categories which cannot be stratified by any ring action.

Example 4.6. Let k be a field and Q a quiver of Dynkin type; see, for example, [4, Chapter 4]. The path algebra kQ is a finite dimensional hereditary algebra of finite representation type. It is easily checked that the graded center of the derived category $\mathsf{D}(kQ)$ is isomorphic to k. In fact, each object in $\mathsf{D}(kQ)$ is a direct sum of indecomposable objects, and $\operatorname{End}^*_{\mathsf{D}(kQ)}(X)\cong k$ for each indecomposable object X. The localizing subcategories of $\mathsf{D}(kQ)$ are parameterized by the noncrossing partitions associated to Q; this can be deduced from work of Ingalls and Thomas [20]. Thus the triangulated category $\mathsf{D}(kQ)$ is stratified by some ring acting on it if and only if the quiver consists of one vertex and has no arrows.

5. Orthogonality

Let X and Y be objects in T. The discussion below is motivated by the question: when is $\operatorname{Hom}^*_{\mathsf{T}}(X,Y)=0$? The orthogonality property [7, Corollary 5.8] says that if $\operatorname{cl}(\operatorname{supp}_R X)$ and $\operatorname{supp}_R Y$ are disjoint, then one has the vanishing. What we seek are converses to this statement, ideally in terms of the supports of X and Y. Lemma 4.1 suggests that one can expect satisfactory answers only when T is stratified. In this section we establish some results addressing this question and give examples which indicate that these may be the best possible.

For any graded R-module M set $\operatorname{Supp}_R M=\{\mathfrak{p}\in\operatorname{Spec} R \mid M_{\mathfrak{p}}\neq 0\}$. This subset is sometimes referred to as the 'big support' of M to distinguish it from its 'homological' support, $\operatorname{supp}_R M$. Analogously, for any object X in T, we set

$$\operatorname{Supp}_R X=\bigcup_{C\in\mathsf{T}^c}\operatorname{Supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,X)\,.$$

It follows from [7, Theorem 5.15(1) and Lemma 2.2(1)] that there is an equality:

$$\operatorname{Supp}_R X=\operatorname{cl}(\operatorname{supp}_R X)\,.$$

We use this equality without further comment.

Theorem 5.1. *Let T be a compactly generated R-linear triangulated category. If R stratifies T, then for each compact object C and each object Y, there is an equality*

$$\operatorname{Supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,Y) = \operatorname{Supp}_R C \cap \operatorname{Supp}_R Y\,.$$

The proof requires only stratification condition (S2), never (S1).

Proof. Fix a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$. Suppose $\operatorname{Hom}^*_{\mathsf{T}}(C,Y)_{\mathfrak{p}} \neq 0$; by definition, one then has $\mathfrak{p} \in \operatorname{Supp}_R Y$. Moreover $\operatorname{End}^*_{\mathsf{T}}(C)_{\mathfrak{p}} \neq 0$ since the R-action on $\operatorname{Hom}^*_{\mathsf{T}}(C,Y)_{\mathfrak{p}}$ factors through it, hence $\mathfrak{p}$ is also in $\operatorname{Supp}_R C$. Thus there is an inclusion

$$\operatorname{Supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,Y) \subseteq \operatorname{Supp}_R C \cap \operatorname{Supp}_R Y\,.$$

Now suppose $\operatorname{Hom}^*_{\mathsf{T}}(C,Y)_{\mathfrak{p}} = 0$. One has to verify that that for any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ either $\varGamma_{\mathfrak{q}} C = 0$ or $\varGamma_{\mathfrak{q}} Y = 0$. By [7, Theorem 4.7], see also Proposition 2.3, since C is compact the adjunction morphism $Y \to Y_{\mathfrak{q}}$ induces an isomorphism

$$0 = \operatorname{Hom}^*_{\mathsf{T}}(C,Y)_{\mathfrak{q}} \cong \operatorname{Hom}^*_{\mathsf{T}}(C,Y_{\mathfrak{q}})\,.$$

As $\varGamma_{\mathcal{V}(\mathfrak{q})} Y$ is in $\operatorname{Loc}_{\mathsf{T}}(Y)$, by Proposition 2.11, one obtains that $\varGamma_{\mathfrak{q}} Y$ is in $\operatorname{Loc}_{\mathsf{T}}(Y_{\mathfrak{q}})$, hence the calculation above yields $\operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{q}} Y) = 0$. As $\varGamma_{\mathfrak{q}} Y$ is $\mathfrak{q}$-local the adjunction morphism $C \to C_{\mathfrak{q}}$ induces the isomorphism below

$$\operatorname{Hom}^*_{\mathsf{T}}(C_{\mathfrak{q}}, \varGamma_{\mathfrak{q}} Y) \cong \operatorname{Hom}^*_{\mathsf{T}}(C, \varGamma_{\mathfrak{q}} Y) = 0\,.$$

Using now the fact that $\varGamma_{\mathfrak{q}} C$ is in $\operatorname{Loc}_{\mathsf{T}}(C_{\mathfrak{q}})$ one gets $\operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{q}} C, \varGamma_{\mathfrak{q}} Y) = 0$. Our hypothesis was that R stratifies T. Thus one of $\varGamma_{\mathfrak{q}} C$ or $\varGamma_{\mathfrak{q}} Y$ is zero. □

The example below shows that the conclusion of the preceding theorem need not hold when C is not compact. See also Example 5.9

Example 5.2. Let A be a commutative noetherian ring with Krull dimension at least one and $\mathfrak{m}$ a maximal ideal of A that is not also a minimal prime. For example, take $A = \mathbb{Z}$ and $\mathfrak{m} = (p)$, where p is a prime number.

Let T be the derived category of A-modules, viewed as an A-linear category; see Example 4.4. Let E be the injective hull of $A/\mathfrak{m}$. The A-module $\operatorname{Hom}^*_{\mathsf{T}}(E,E)$ is then the $\mathfrak{m}$-adic completion of A, so it follows that

$$\operatorname{Supp}_A \operatorname{Hom}^*_{\mathsf{T}}(E,E) = \{\mathfrak{p} \subseteq \mathfrak{m} \mid \mathfrak{p} \in \operatorname{Spec} R\} \supsetneq \{\mathfrak{m}\} = \operatorname{Supp}_A E\,.$$

Observe that $\operatorname{supp}_A \operatorname{Hom}^*_{\mathsf{T}}(E,E) = \operatorname{Supp}_A \operatorname{Hom}^*_{\mathsf{T}}(E,E)$ and $\operatorname{supp}_A E = \operatorname{Supp}_A E$.

One drawback of Theorem 5.1 is that it involves the big support Supp_R, while one is mainly interested in supp_R. Next we identify a rather natural condition on T under which one can obtain results in the desired form.

Noetherian categories. We call a compactly generated R-linear triangulated category *noetherian* if for any compact object C in T the R-module $\operatorname{End}^*_{\mathsf{T}}(C)$ is finitely generated. This is equivalent to the condition that for all compact objects C, D the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C,D)$ is finitely generated: consider $\operatorname{End}^*_{\mathsf{T}}(C \oplus D)$. If C generates T, then T is noetherian if and only if the R-module $\operatorname{End}^*_{\mathsf{T}}(C)$ is noetherian.

As a consequence of Theorem 5.1 one gets:

Corollary 5.3. *If T is noetherian and stratified by R, then for each pair of compact objects C, D in T there is an equality*

$$\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,D) = \operatorname{supp}_R C \cap \operatorname{supp}_R D\,.$$

When in addition $R^i = 0$ holds for $i < 0$, one has $\operatorname{Hom}^n_{\mathsf{T}}(C,D) = 0$ for $n \gg 0$ if and only if $\operatorname{Hom}^n_{\mathsf{T}}(D,C) = 0$ for $n \gg 0$.

Proof. In view of the noetherian hypothesis and [7, Lemma 2.2(1), Theorem 5.5(2)], the desired equality follows from Theorem 5.1. It implies in particular that

$$\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,D) = \operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(D,C)\,.$$

When $R^i = 0$ holds for $i < 0$ and M is a noetherian R-module one has $M^n = 0$ for $n \gg 0$ if and only if $\operatorname{supp}_R M \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq R^{\geqslant 1}\}$; see [10, Proposition 2.4]. The last part of the corollary now follows from the equality above. □

There is a version of the preceding result where the objects C and D need not be compact. This is the topic of the next theorem. As preparation for its proof, and for later applications, we further develop the material in [7, Definition 4.8]. Let C be a compact object in T. For each injective R-module I, the Brown representability theorem [23, 27] yields an object $T_C(I)$ in T such that there is a natural isomorphism:

$$\operatorname{Hom}_{\mathsf{T}}(-, T_C(I)) \cong \operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{T}}(C,-), I)\,.$$

Moreover, the assignment $I \mapsto T_C(I)$ defines a functor $T_C\colon \mathsf{Inj}\,R \to \mathsf{T}$ from the category of injective R-modules to T.

Proposition 5.4. *Let C be a compact object in T. The functor $T_C\colon \mathsf{Inj}\,R \to \mathsf{T}$ preserves products. If the R-linear category T is noetherian, each $I \in \mathsf{Inj}\,R$ satisfies:*

$$\operatorname{supp}_R T_C(I) = \operatorname{supp}_R C \cap \operatorname{supp}_R I = \operatorname{supp}_R \operatorname{End}^*_{\mathsf{T}}(C) \cap \operatorname{supp}_R I\,.$$

In particular, for each $\mathfrak{p} \in \operatorname{Spec} R$ the object $T_C(E(R/\mathfrak{p}))$ is in $\varGamma_{\mathfrak{p}}\mathsf{T}$.

Proof. It follows by construction that T_C preserves products. For each compact object D in T, there is an isomorphism of R-modules

$$\operatorname{Hom}^*_{\mathsf{T}}(D, T_C(I)) \cong \operatorname{Hom}^*_R(\operatorname{Hom}^*_{\mathsf{T}}(C,D), I)\,.$$

When T is noetherian, so that the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C,D)$ is finitely generated, the isomorphism above gives the first equality below:

$$\begin{aligned}\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(D, T_C(I)) &= \operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,D) \cap \operatorname{supp}_R I \\ &= \operatorname{supp}_R C \cap \operatorname{supp}_R D \cap \operatorname{supp}_R I\,.\end{aligned}$$

The second equality holds by Corollary 5.3. Lemma 5.5 below then yields the first of the desired equalities; the second one holds by [7, Theorem 5.5(2)]. □

The following lemma provides an alternative description of the support of an object in T. Note that T need not to be noetherian.

Lemma 5.5. *Let X an object in T. If a subset $\mathcal{U}$ of $\operatorname{supp}_R \mathsf{T}$ satisfies an equality*

$$\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(C,X) = \mathcal{U} \cap \operatorname{supp}_R C$$

for each compact object C, then $\operatorname{supp}_R X = \mathcal{U}$.

Proof. It follows from [7, Theorem 5.2] that $\operatorname{supp}_R X \subseteq \mathcal{U}$.

Fix $\mathfrak{p}$ in $\mathcal{U}$ and choose a compact object D with $\mathfrak{p}$ in $\operatorname{supp}_R D$. Then $\mathfrak{p}$ is in $\operatorname{supp}_R(D/\!\!/\mathfrak{p})$, so the hypothesis yields that $\mathfrak{p}$ is in $\operatorname{supp}_R \operatorname{Hom}^*_{\mathsf{T}}(D/\!\!/\mathfrak{p}, X)$. Hence $\mathfrak{p}$ belongs to $\operatorname{supp}_R X$, by [7, Proposition 5.12]. □

For a compact object C, the functor $\operatorname{Hom}^*_{\mathsf{T}}(C,-)$ vanishes on $\operatorname{Loc}_{\mathsf{T}}(Y)$ if and only if $\operatorname{Hom}^*(C,Y) = 0$. Using this observation, it is easy to verify that the theorem below is an extension of Corollary 5.3. Compare it also with [7, Corollary 5.8].

Theorem 5.6. *Let T be an R-linear triangulated category that is noetherian and stratified by R. For any X and Y in T the conditions below are equivalent:*

(1) $\operatorname{Hom}^*_{\mathsf{T}}(X, Y') = 0$ *for any* Y' *in* $\operatorname{Loc}_{\mathsf{T}}(Y)$*;*
(2) $\operatorname{cl}(\operatorname{supp}_R X) \cap \operatorname{supp}_R Y = \emptyset$.

Proof. (1) $\Longrightarrow$ (2): Let $\mathfrak{p}$ be a point in $\operatorname{supp}_R Y$ and C a compact object in T. Proposition 5.4 yields that $T_C(E(R/\mathfrak{p}))$ is in $\varGamma_{\mathfrak{p}}\mathsf{T}$, and hence also in $\operatorname{Loc}_{\mathsf{T}}(Y)$; the last assertion holds by Theorem 4.2. This explains the equality below:

$$\operatorname{Hom}^*_R(\operatorname{Hom}^*_{\mathsf{T}}(C, X), E(R/\mathfrak{p})) \cong \operatorname{Hom}^*_{\mathsf{T}}(X, T_C(E(R/\mathfrak{p}))) = 0\,,$$

while the isomorphism follows from the definition of T_C. Thus $\operatorname{Hom}^*_{\mathsf{T}}(C, X)_{\mathfrak{p}} = 0$. Since C was arbitrary, this means that $\mathfrak{p}$ is not in $\operatorname{cl}(\operatorname{supp}_R X)$.

(2) $\Longrightarrow$ (1): One has $\operatorname{supp}_R Y' \subseteq \operatorname{supp}_R Y$ for Y' in $\operatorname{Loc}_{\mathsf{T}}(Y)$, since the functor $\varGamma_{\mathfrak{p}}$ is exact and preserves coproducts. The orthogonality property of supports, [7, Corollary 5.8] thus implies that if condition (2) holds, then $\operatorname{Hom}^*_{\mathsf{T}}(X, Y') = 0$. □

Recall that the left orthogonal subcategory of S, denoted ${}^{\perp}\mathsf{S}$, is the localizing subcategory $\{X \in \mathsf{T} \mid \operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \text{ for all } Y \in \mathsf{S}\}$. As a straightforward consequence of Theorem 5.6 one obtains a description of the support of the left orthogonal of a localizing category, answering a question raised by Rickard.[1]

Corollary 5.7. *For each localizing subcategory* S *of* T *the following equality holds:* $\operatorname{supp}_R({}^{\perp}\mathsf{S}) = \{\mathfrak{p} \in \operatorname{supp}_R \mathsf{T} \mid \mathcal{V}(\mathfrak{p}) \cap \operatorname{supp}_R \mathsf{S} = \emptyset\}$. □

Remark 5.8. In the context of Theorem 5.6, for any compact object C one has

$$\operatorname{Hom}^*_{\mathsf{T}}(C, Y) = 0 \text{ if and only if } \operatorname{supp}_R C \cap \operatorname{supp}_R Y = \emptyset\,.$$

The next example shows that one cannot do away entirely with the hypothesis that C is compact; the point being that $\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0$ does not imply that $\operatorname{Hom}^*(X, -)$ is zero on $\operatorname{Loc}_{\mathsf{T}}(Y)$, unless X is compact.

Example 5.9. Let A be a complete local domain and Q its field of fractions. For example, take A to be the completion of $\mathbb{Z}$ at a prime p. It follows from a result of Jensen [22, Theorem 1] that $\operatorname{Ext}^*_A(Q, A) = 0$. Thus, with T the derived category of A, one gets $\operatorname{supp}_A \operatorname{Hom}^*_{\mathsf{T}}(Q, A) = \emptyset$ while $\operatorname{supp}_A Q \cap \operatorname{supp}_A A$ consists of the zero ideal. Note that Q is in $\operatorname{Loc}_{\mathsf{T}}(A)$, so there is no contradiction with Theorem 5.6.

6. Classifying thick subcategories

In this section we prove that when T is noetherian and stratified by R its thick subcategories of compact objects are parameterized by specialization closed subsets of $\operatorname{supp}_R \mathsf{T}$. As before, R is a graded-commutative noetherian ring and T is a compactly generated R-linear triangulated category.

Thick subcategories. One can deduce the next result from the classification of localizing subcategories, Theorem 4.2, as in [25, §3]. We give a different proof.

Theorem 6.1. *Let* T *be a compactly generated* R*-linear triangulated category that is noetherian and stratified by* R*. The map*

$$\left\{\begin{matrix} \textit{Thick subcategories} \\ \textit{of } \mathsf{T}^c \end{matrix}\right\} \xrightarrow{\ \operatorname{supp}_R\ } \left\{\begin{matrix} \textit{Specialization closed} \\ \textit{subsets of } \operatorname{supp}_R \mathsf{T} \end{matrix}\right\}$$

is bijective. The inverse map sends a specialization closed subset $\mathcal{V}$ *of* $\operatorname{Spec} R$ *to the subcategory* $\{C \in \mathsf{T}^c \mid \operatorname{supp}_R C \subseteq \mathcal{V}\}$*.*

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[1]After a talk by Iyengar at the workshop 'Homological methods in group theory', MSRI 2008.

Observe that in the proof the injectivity of the map supp_R requires only that T satisfies the stratification condition (S2), while the surjectivity uses only the hypothesis that T is noetherian.

Proof. First we verify that $\operatorname{supp}_R \mathsf{C}$ is specialization closed for any thick subcategory C of T^c. For any compact object C the R-module $\operatorname{End}^*_{\mathsf{T}}(C)$ is finitely generated, and this implies $\operatorname{supp}_R C = \operatorname{supp}_R \operatorname{End}^*_{\mathsf{T}}(C)$, by [7, Theorem 5.5]. Thus $\operatorname{supp}_R C$ is a closed subset of $\operatorname{Spec} R$, and therefore $\operatorname{supp}_R \mathsf{C}$ is specialization closed.

To verify that the map supp_R is surjective, let $\mathcal{V}$ be a specialization closed subset of $\operatorname{supp}_R \mathsf{T}$ and set $\mathsf{C} = \{C/\!\!/\mathfrak{p} \mid C \in \mathsf{T}^c,\ \mathfrak{p} \in \mathcal{V}\}$. One then has that $\operatorname{Loc}_{\mathsf{T}}(\mathsf{C}) = \mathsf{T}_{\mathcal{V}}$ by [7, Theorem 6.4], and therefore the following equalities hold

$$\operatorname{supp}_R \mathsf{C} = \operatorname{supp}_R \mathsf{T}_{\mathcal{V}} = \mathcal{V} \cap \operatorname{supp}_R \mathsf{T} = \mathcal{V}\,.$$

It remains to prove that supp_R is injective. Let C be a thick subcategory of T and set $\mathsf{D} = \{D \in \mathsf{T}^c \mid \operatorname{supp}_R D \subseteq \operatorname{supp}_R \mathsf{C}\}$. We need to show that $\mathsf{C} = \mathsf{D}$. Evidently, an inclusion $\mathsf{C} \subseteq \mathsf{D}$ holds. To establish the other inclusion, let $L\colon \mathsf{T} \to \mathsf{T}$ be the localization functor with $\operatorname{Ker} L = \operatorname{Loc}_{\mathsf{T}}(\mathsf{C})$; see Lemma 2.1 for its existence. Let D be an object in D. Each object C in C satisfies $\operatorname{Hom}^*_{\mathsf{T}}(C, LD) = 0$, so Theorem 5.1 implies $\operatorname{Supp}_R C \cap \operatorname{Supp}_R LD = \emptyset$. Hence $LD = 0$, that is to say, D belongs to $\operatorname{Loc}_{\mathsf{T}}(\mathsf{C})$. It then follows from [25, Lemma 2.2] that D is in C. □

Smashing subcategories. Next we prove that when T is stratified and noetherian, the telescope conjecture [31] holds for T. In preparation for its proof, we record an elementary observation.

Lemma 6.2. *Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in $\operatorname{Spec} R$. The injective hull $E(R/\mathfrak{p})$ of $R/\mathfrak{p}$ is a direct summand of a product of shifted copies of $E(R/\mathfrak{q})$.*

Proof. The shifted copies of $E(R/\mathfrak{q})$ form a set of injective cogenerators for the category of $\mathfrak{q}$-local modules. This implies the desired result. □

A subset $\mathcal{U}$ of $\operatorname{Spec} R$ is said to be *closed under generalization* if $\operatorname{Spec} R \setminus \mathcal{U}$ is specialization closed. More explicitly: $\mathfrak{q} \in \mathcal{U}$ and $\mathfrak{p} \subseteq \mathfrak{q}$ imply $\mathfrak{p} \in \mathcal{U}$.

Theorem 6.3. *Let T be an R-linear triangulated category that is noetherian and stratified by R. There is then a bijection*

$$\left\{\begin{gathered}\textit{Localizing subcategories of } \mathsf{T}\\ \textit{closed under all products}\end{gathered}\right\} \xrightarrow{\ \operatorname{supp}_R\ } \left\{\begin{gathered}\textit{Subsets of } \operatorname{supp}_R \mathsf{T}\\ \textit{closed under generalization}\end{gathered}\right\}$$

Moreover, if $L\colon \mathsf{T} \to \mathsf{T}$ is a localization functor that preserves arbitrary coproducts, then the localizing subcategory $\operatorname{Ker} L$ is generated by objects that are compact in T.

Remark 6.4. The inverse map of supp_R takes a generalization closed subset $\mathcal{U}$ of $\operatorname{Spec} R$ to the category of objects X of T with $\operatorname{supp}_R X \subseteq \mathcal{U}$; in other words, the category of $L_{\mathcal{V}}$-local objects, where $\mathcal{V} = \operatorname{Spec} R \setminus \mathcal{U}$.

Proof. Let S be a localizing subcategory of T that is closed under arbitrary products. We know from Theorem 4.2 that S is determined by its support $\operatorname{supp}_R \mathsf{S}$. Thus we need to show that it is closed under generalization.

Fix prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{supp}_R \mathsf{T}$ and suppose that $\mathfrak{q}$ is in $\operatorname{supp}_R \mathsf{S}$. It follows from Theorem 4.2 that $\Gamma_{\mathfrak{p}}\mathsf{T} \subseteq \mathsf{S}$ holds. Pick a compact object C such that $\operatorname{supp}_R C$ contains $\mathfrak{p}$; this is possible since $\operatorname{supp}_R \mathsf{T}^c = \operatorname{supp}_R \mathsf{T}$. Since T is noetherian, $\operatorname{supp}_R C$ is a closed subset of $\operatorname{Spec} R$, by [7, Theorem 5.5], and hence contains also $\mathfrak{q}$. Let $E(R/\mathfrak{q})$ be the injective hull of the R-module $R/\mathfrak{q}$. Since T is noetherian, Proposition 5.4 yields that $T_C(E(R/\mathfrak{q}))$ is in $\Gamma_{\mathfrak{q}}\mathsf{T}$ and hence in S. The functor T_C preserves products, so Lemma 6.2 implies that $T_C(E(R/\mathfrak{p}))$ is a direct summand of $T_C(E(R/\mathfrak{q}))$ and hence it is also in S, because the latter is a localizing subcategory closed under products. Another application of Proposition 5.4 shows that $\operatorname{supp}_R T_C(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$, so that $\mathfrak{p} \in \operatorname{supp}_R \mathsf{S}$ holds, as desired

Next let $\mathcal{U}$ be a generalization closed subset of $\operatorname{Spec} R$ and set $\mathcal{V} = \operatorname{Spec} R \setminus \mathcal{U}$. Let S be the category of $L_{\mathcal{V}}$-local objects, so that $\operatorname{supp}_R \mathsf{S} = \mathcal{U}$ holds, by [7, Corollary 5.7]. By construction, the category S is triangulated and closed under arbitrary products; it is localizing because the localization functor $L_{\mathcal{V}}$ preserves arbitrary coproducts, by [7, Corollary 6.5].

This completes the proof that supp_R induces the stated bijection.

Finally, let $L\colon \mathsf{T} \to \mathsf{T}$ be a localization functor that preserves arbitrary coproducts. The category of L-local objects, which always is closed under products, is then also a localizing subcategory of T. The first part of this proof shows that $L \cong L_{\mathcal{V}}$ for some specialization closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, because the localization functor L is determined by the category of L-local objects. It remains to note that $\operatorname{Ker} L$, which is the category $\mathsf{T}_{\mathcal{V}}$, is generated by compact objects, by [7, Theorem 6.4]. $\square$

7. Tensor triangulated categories

In this section we discuss special properties of triangulated categories which hold when they have a tensor structure. The main result here is Theorem 7.2, which says that the local-global principle holds for such categories, when the action of the tensor product is also taken into account.

Let $\mathsf{T} = (\mathsf{T}, \otimes, \mathbb{1})$ be a tensor triangulated category as defined in [7, §8]. In particular, T is a compactly generated triangulated category endowed with a symmetric monoidal structure; $\otimes$ is its tensor product and $\mathbb{1}$ the unit of the tensor product. It is assumed that $\otimes$ is exact in each variable, preserves coproducts, and that $\mathbb{1}$ is compact.

The symmetric monoidal structure ensures that the endomorphism ring $\operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$ is graded commutative. This ring acts on T via homomorphisms

$$\operatorname{End}^*_{\mathsf{T}}(\mathbb{1}) \xrightarrow{X\otimes -} \operatorname{End}^*_{\mathsf{T}}(X)\,,$$

In particular, any homomorphism $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$ of rings with R graded commutative induces an action of R on T. We say that an R action on T is *canonical* if it arises from such a homomorphism. In that case there are for each specialization closed subset $\mathcal{V}$ and point $\mathfrak{p}$ of $\operatorname{Spec} R$ natural isomorphisms

$$\varGamma_{\mathcal{V}} X \cong X \otimes \varGamma_{\mathcal{V}}\mathbb{1}\,, \quad L_{\mathcal{V}} X \cong X \otimes L_{\mathcal{V}}\mathbb{1}\,, \quad \text{and} \quad \varGamma_{\mathfrak{p}} X \cong X \otimes \varGamma_{\mathfrak{p}}\mathbb{1}\,. \tag{7.1}$$

These isomorphisms are from [7, Theorem 8.2, Corollary 8.3].[2]

Tensor ideal localizing subcategories. A localizing subcategory S of T said to be *tensor ideal* if for each $X \in \mathsf{T}$ and $Y \in \mathsf{S}$, the object $X \otimes Y$, hence also $Y \otimes X$, is in S. The smallest tensor ideal localizing subcategory containing a subcategory S is denoted $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$. Evidently there is always an inclusion $\operatorname{Loc}_{\mathsf{T}}(\mathsf{S}) \subseteq \operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$; equality holds when the unit $\mathbb{1}$ generates T.

The following result is proved in [8, Theorem 3.6] under the additional assumption that T has a single compact generator. The same argument carries over; except that, instead of [8, Proposition 3.5] use Proposition 2.7 above. We omit details.

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[2]For these results to hold, the R action should be canonical, for the R-linearity of the adjunction isomorphism $\operatorname{Hom}_{\mathsf{T}}(X \otimes Y, Z) \cong \operatorname{Hom}_{\mathsf{T}}(X, \mathcal{H}om(Y, Z))$ is used in the arguments.

Theorem 7.2. *Let T be a tensor triangulated category with a canonical R-action. For each object X in T there is an equality*

$$\operatorname{Loc}^{\otimes}_{\mathsf{T}}(X) = \operatorname{Loc}^{\otimes}_{\mathsf{T}}\left(\mathit{\Gamma}_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{Spec} R\right).$$

In particular, when $\mathbb{1}$ generates T, the local global principle holds for T. □

Stratification. For each $\mathfrak{p}$ in $\operatorname{Spec} R$, the localizing subcategory $\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T}$, consisting of $\mathfrak{p}$-local and $\mathfrak{p}$-torsion objects, is tensor ideal; this is immediate from (7.1). We say that T is *stratified* by R when for each $\mathfrak{p}$, the category $\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T}$ is either zero or has no proper *tensor ideal* localizing subcategories. Note the analogy with condition (S2) in Section 4; the analogue of (S1) need not be imposed thanks to Theorem 7.2.

There are analogues, for tensor triangulated categories, of results in Sections 5 and 6; the proofs are similar, see also [7, §11]. One has in addition also the following 'tensor product theorem'.

Theorem 7.3. *Let T be a tensor triangulated category with a canonical R-action. If R stratifies T, then for any objects X, Y in T there is an equality*

$$\operatorname{supp}_R(X \otimes Y) = \operatorname{supp}_R X \cap \operatorname{supp}_R Y\,.$$

Proof. Fix a point $\mathfrak{p}$ in $\operatorname{Spec} R$. From 7.1 it is easy to verify that there are isomorphisms $\mathit{\Gamma}_{\mathfrak{p}}(X \otimes Y) \cong \mathit{\Gamma}_{\mathfrak{p}}X \otimes \mathit{\Gamma}_{\mathfrak{p}}Y \cong \mathit{\Gamma}_{\mathfrak{p}}X \otimes Y$. These will be used without further ado. They yield an inclusion:

$$\operatorname{supp}_R(X \otimes Y) \subseteq \operatorname{supp}_R X \cap \operatorname{supp}_R Y\,.$$

When $\mathit{\Gamma}_{\mathfrak{p}}X \neq 0$ the stratification condition yields $\mathit{\Gamma}_{\mathfrak{p}}\mathbb{1} \in \operatorname{Loc}^{\otimes}(\mathit{\Gamma}_{\mathfrak{p}}X)$, and hence also $\mathit{\Gamma}_{\mathfrak{p}}Y \in \operatorname{Loc}^{\otimes}(\mathit{\Gamma}_{\mathfrak{p}}X \otimes Y)$. Thus when $\mathit{\Gamma}_{\mathfrak{p}}Y \neq 0$ also holds, $\mathit{\Gamma}_{\mathfrak{p}}(X \otimes Y) \neq 0$ holds, which justifies the reverse inclusion. □

Example 7.4. Let G be a finite group, k a field of characteristic p, where p divides the order of G, and kG the group algebra. The homotopy category of complexes of injective kG-modules, $\mathsf{K}(\mathsf{Inj}\, kG)$, is a compactly generated tensor triangulated category with a canonical action of the cohomology ring $H^*(G, k)$. One of the main results of [8], Theorem 9.7, is that $\mathsf{K}(\mathsf{Inj}\, kG)$ is stratified by this action. The same is true also of the stable module category $\mathsf{StMod}\, kG$; see [8, Theorem 10.3].

8. Formal differential graded algebras

The goal of this section is to prove that the derived category of differential graded (henceforth abbreviated to 'dg') modules over a formal commutative dg algebra is stratified by its cohomology algebra, when that algebra is noetherian. This result specializes to one of Neeman's [25] concerning rings, which may be viewed as dg algebras concentrated in degree 0.

For basic notions concerning dg algebras and dg modules over them we refer the reader to Mac Lane [24, §6.7]. A *quasi-isomorphism* between dg algebras A and B is a morphism $\varphi\colon A \to B$ of dg algebras such that $H^*(\varphi)$ is bijective; A and B are *quasi-isomorphic* if there is a chain of quasi-isomorphisms linking them. The multiplication on A induces one on its cohomology, $H^*(A)$. We say that A is *formal* if it is quasi-isomorphic to $H^*(A)$, viewed as a dg algebra with zero differential.

We write $\mathsf{D}(A)$ for the derived category of dg modules over a dg algebra A; it is a triangulated category, generated by the compact object A; see, for instance, [23].

A dg algebra A is said to be *commutative* if its underlying ring is graded commutative. In this case the derived tensor product of dg modules, denoted $\otimes^{\mathbf{L}}$, endows $\mathsf{D}(A)$ with a structure of a tensor triangulated category, with unit A. One is thus in the framework of Section 7.

The next theorem generalizes [8, Theorem 5.2], which deals with the case of graded algebras of the form $k[x_1, \dots, x_n]$, where k is a field and $x_1, \dots, x_n$ are indeterminates, of even degree if the characteristic of k is not 2.

Theorem 8.1. *Let A be a commutative dg algebra such that the ring $H^*(A)$ is noetherian. If A is formal, then $\mathsf{D}(A)$ is stratified by the canonical $H^*(A)$-action.*

In the proof we use a totalization functor from complexes over a graded ring to dg modules over the ring viewed as a dg algebra with differential zero; see [24, §10.9], where this functor is called condensation, and [23, §3.3].

Totalization. Let A be a graded algebra. For each graded A-module N and integer d we write $N[d]$ for the graded A-module with $N[d]^i = N^{d+i}$, and multiplication the same as the one on N.

Let F be a complex of graded A-modules with differential δ; so each F^i is a graded A-module, $\delta^i \colon F^i \to F^{i+1}$ are morphisms of graded A-modules, and $\delta^{i+1}\delta^i = 0$. We write $F^{i,j}$ for the component of degree j in the graded module F^i. The *totalization* of F, denoted $\operatorname{tot} F$, is the dg abelian group with

$$
\begin{aligned}
(\operatorname{tot} F)^n &= \bigoplus_{i+j=n} F^{i,j} \quad \text{for each } n \in \mathbb{Z} \\
\partial(f) &= \delta^i(f) \quad \text{for each } f \in F^{i,j}
\end{aligned}
$$

We consider $\operatorname{tot} F$ as a graded A-module with multiplication defined by

$$a \cdot f = (-1)^{di} af \quad \text{for each } a \in A^d \text{ and } f \in F^{i,j}.$$

A routine calculation shows that $\operatorname{tot} F$ is then a dg A-module, where A is viewed as dg algebra with zero differential, and that each morphism $\alpha \colon F \to G$ of complexes of graded A-modules induces a morphism $\operatorname{tot}\alpha \colon \operatorname{tot} F \to \operatorname{tot} G$ of dg A-modules. Moreover, there are equalities of dg A-modules:

- $\operatorname{tot} A = A$;
- $\operatorname{tot} N[-d] = \Sigma^d \operatorname{tot} N$ for each graded A-module N and integer d;
- $\operatorname{tot} \Sigma^n F = \Sigma^n \operatorname{tot} F$.

One thus get an additive functor from the category of complexes of graded A-modules to the category of dg A-modules. It follows from [23, Lemma 3.3] that if the complex F is acyclic so is $\operatorname{tot} F$, hence tot induces an exact functor

$$\operatorname{tot} \colon \mathsf{D}(\mathsf{GrMod}\, A) \longrightarrow \mathsf{D}(A)\,.$$

of triangulated categories; here $\mathsf{D}(\mathsf{GrMod}\, A)$ is the derived category of graded A-modules, while $\mathsf{D}(A)$ is the derived category of dg A-modules.

Lemma 8.2. *Let E be the Koszul complex on a sequence $\boldsymbol{a} = a_1, \dots, a_c$ of homogenous central elements in A. Then $\operatorname{tot} E \cong \Sigma^d A/\!\!/\boldsymbol{a}$ in $\mathsf{D}(A)$, where $d = \sum_n |a_n|$.*

Proof. Indeed, since tot preserves exact triangles, and both E and $A/\!\!/\boldsymbol{a}$ can be obtained as iterated mapping cones, it suffices to verify the statement for the Koszul complex on a single element, say a. The desired result is then immediate from the properties of tot listed above. $\square$

We require also some elementary results concerning transfer of stratification along exact functors; a detailed study is taken up in [9].

Change of categories. As before R is a graded commutative noetherian ring and T is a compactly generated R-linear triangulated category. Let $F\colon \mathsf{U} \to \mathsf{T}$ be an equivalence of triangulated categories. Observe that U is then compactly generated; it is also R-linear with action given by the isomorphism of graded abelian groups

$$\operatorname{Hom}^*_{\mathsf{U}}(X,Y) \cong \operatorname{Hom}^*_{\mathsf{T}}(FX,FY)$$

induced by F, for all X,Y in U.

Proposition 8.3. *The ring R stratifies U if and only if it stratifies T.*

Proof. Using [7, Corollary 5.9], it is easy to verify that for each $\mathfrak{p}$ in $\operatorname{Spec} R$ and X in U, there is an isomorphism $F(\varGamma_{\mathfrak{p}}X) \cong \varGamma_{\mathfrak{p}}(FX)$, and that the induced functor $\varGamma_{\mathfrak{p}}\mathsf{U} \to \varGamma_{\mathfrak{p}}\mathsf{T}$ is an equivalence of triangulated categories. Given this, it is immediate from definitions that R stratifies U if and only if it stratifies T. $\square$

When $A \to B$ is a quasi-isomorphism of dg algebras, $B \otimes^{\mathbf{L}}_A -\colon \mathsf{D}(A) \to \mathsf{D}(B)$ is an equivalence of categories, with quasi-inverse the restriction of scalars; see, for example, [3, 3.6], or [23, 6.1]. The preceding result thus yields:

Corollary 8.4. *Let A and B be quasi-isomorphic dg algebras. If $\mathsf{D}(A)$ is stratified by an action of R, then $\mathsf{D}(B)$ is stratified by the induced R-action.* $\square$

Proof of Theorem 8.1. Let $R = H^*(A)$. The category $\mathsf{D}(A)$ is tensor triangulated so it admits an R-action induced by the isomorphism $R \cong \operatorname{Hom}^*_{\mathsf{D}(A)}(A,A)$. The dg algebras A and $H^*(A)$ are quasi-isomorphic, as A is formal, so it suffices to prove that $\mathsf{D}(H^*(A))$ is stratified by the induced R-action; see Corollary 8.4. It is easy to verify that the homomorphism $R \to \operatorname{Hom}^*_{\mathsf{D}(H^*(A))}(H^*(A),H^*(A)) = H^*(A)$ induced by this R-action is bijective, and hence that $\mathsf{D}(H^*(A))$ is stratified by R if and only if it is stratified by the canonical $H^*(A)$-action.

In summary, replacing A by $H^*(A)$ we may thus assume the differential of A is zero. Set $\mathsf{D} = \mathsf{D}(A)$. Since A is a unit and a generator of this tensor triangulated category, its localizing subcategories are tensor closed. The local-global principle then holds for D, by Theorem 7.2. It remains to verify stratification condition (S2).

Fix a $\mathfrak{p}$ in $\operatorname{Spec} A$. Since A is a compact generator for D, a dg A-module M is in $\varGamma_{\mathfrak{p}}\mathsf{D}$ if and only if the A-module $H^*(M) = \operatorname{Hom}^*_{\mathsf{D}}(A,M)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. Hence for such an M the localization map $M \to M_{\mathfrak{p}}$ is an isomorphism; here $M_{\mathfrak{p}}$ denotes the usual (homogenous) localization of M at $\mathfrak{p}$. Localizing A at $\mathfrak{p}$ we may thus assume that it is local with maximal ideal $\mathfrak{p}$; set $k = A/\mathfrak{p}$, which is a graded field. Setting $\mathcal{V} = \mathcal{V}(\mathfrak{p})$, one has an isomorphism of functors $\varGamma_{\mathfrak{p}} \cong \varGamma_{\mathcal{V}}$.

Evidently, k is in $\varGamma_{\mathcal{V}}\mathsf{D}$, so to verify condition (S2) it suffices to verify that

$$\operatorname{Loc}_{\mathsf{D}}(M) = \operatorname{Loc}_{\mathsf{D}}(k) \tag{8.5}$$

holds for each M in $\varGamma_{\mathcal{V}}\mathsf{D}$ with $H^*(M) \neq 0$.

It is enough to prove that (8.5) holds for $M = \varGamma_{\mathcal{V}}A$. Indeed, applying the functor $- \otimes^{\mathbf{L}}_A M$ would then yield the second equality below:

$$\operatorname{Loc}_{\mathsf{D}}(M) = \operatorname{Loc}_{\mathsf{D}}(\varGamma_{\mathcal{V}}A \otimes^{\mathbf{L}}_A M) = \operatorname{Loc}_{\mathsf{D}}(k \otimes^{\mathbf{L}}_A M)\,,$$

while the first one holds, by (7.1), since $M \cong \varGamma_{\mathfrak{p}}M$; in particular, $H^*(k \otimes^{\mathbf{L}}_A M) \neq 0$. Since k is a graded field and the action of A on $k \otimes^{\mathbf{L}}_A M$ factors through k, this implies $\operatorname{Loc}_{\mathsf{D}}(k \otimes^{\mathbf{L}}_A M) = \operatorname{Loc}_{\mathsf{D}}(k)$. Combining with the equality above gives (8.5).

Now we verify (8.5) for $M = \varGamma_{\mathcal{V}}A$. The dg module k is isomorphic to $\varGamma_{\mathcal{V}}A \otimes^{\mathbf{L}}_A k$ and hence in $\operatorname{Loc}_{\mathsf{D}}(\varGamma_{\mathcal{V}}A)$. It remains to prove that $\varGamma_{\mathcal{V}}A$ is in $\operatorname{Loc}_{\mathsf{D}}(k)$, or, equivalently, that $A/\!\!/\mathfrak{p}$ is in $\operatorname{Loc}_{\mathsf{D}}(k)$; see Proposition 2.11. Let $\boldsymbol{a} = a_1,\dots,a_c$ be a homogeneous set of generators for the ideal $\mathfrak{p}$, and let $\boldsymbol{a}^2$ denote the sequence $a_1^2,\dots,a_c^2$.

Note that the elements a_i are central in A, since they are of even degree, and that the radical of the ideal $(\boldsymbol{a}^2)$ equals $\mathfrak{p}$; thus, by Lemma 2.6, it suffices to prove that

$$A /\!\!/ \boldsymbol{a}^2 \in \operatorname{Thick}_{\mathsf{D}}(k)\,. \tag{8.6}$$

Let $\operatorname{tot}\colon \mathsf{D}(\mathsf{GrMod}\,A) \to \mathsf{D}$ be the totalization functor described above and E in $\mathsf{D}(\mathsf{GrMod}\,A)$ the Koszul complex on the sequence $\boldsymbol{a}^2$. The complex E is bounded, consists of finitely generated graded A-modules, and satisfies $(\boldsymbol{a}^2)\cdot H^*(E) = 0$. Since k is a graded field, the subquotients of the filtration $\{0\} \subseteq (\boldsymbol{a})H^*(E) \subseteq H^*(E)$ are thus finite direct sums of shifts of k. Hence there are inclusions

$$E \in \operatorname{Thick}(H^*(E)) \subseteq \operatorname{Thick}(k)$$

in $\mathsf{D}(\mathsf{GrMod}\,A)$; see, for example, [3, Theorem 6.2(3)]. Since tot is an exact functor, it follows that $\operatorname{tot} E$ is in $\operatorname{Thick}(\operatorname{tot} k)$ in D. It remains to note that $\operatorname{tot} k = k$ and that $\operatorname{tot} E$ is isomorphic to a suspension of $A /\!\!/ \boldsymbol{a}^2$, by Lemma 8.2.

This justifies (8.6) and hence completes the proof of the theorem. □

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