

A NOTION OF INDUCTION-RESTRICTION DEPTH OF MULTIMATRIX ALGEBRA INCLUSIONS APPLIED TO SUBGROUPS

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ABSTRACT. We define a notion of depth for an inclusion of multimatrix algebras $B \subseteq A$ based on a comparison of powers of the induction-restriction table M (and its transpose matrix). The depth of the semisimple subalgebra B in the semisimple algebra A is the least positive integer $n \geq 2$ for which $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$. We prove that a depth two subalgebra is a normal subalgebra, and conversely. As a corollary, a depth n subalgebra is a normal subalgebra of its $(n-2)$ 'nd iterated endomorphism algebra in a tower above $B \subseteq A$. Odd and even depth may be expressed in terms of graphical distance between B -simples in the inclusion diagram of $B \subseteq A$. Applied to subgroups via complex group algebras, the depth of a subgroup H in a finite group G is bounded above by $2n$ if the core is an intersection of n conjugates of H in G ; with upper bound $2n-1$ if the core is trivial. We prove that the subgroup depth of symmetric groups $S_n < S_{n+1}$ is $2n-1$. An appendix by S. Danz and B. Külshammer determines the subgroup depth of alternating groups $A_n < A_{n+1}$ to be $2(n - \lceil \sqrt{n} \rceil) + 1$.

1. INTRODUCTION

Depth two is an algebraic notion for noncommutative ring extensions with a Galois theory associated to it, and closely related to noncommutative normality as well as normality for subgroups and Hopf subalgebras [13, 14, 16, 12, 6]. A subgroup H in a finite group G is normal if and only if the complex group algebra $\mathbb{C}[G]$ is depth two over $\mathbb{C}[H]$ [12]. For M the matrix of the induction-restriction table for a subalgebra pair of semisimple algebras $B \subseteq A$, the depth two condition is given by a matrix inequality $MM^tM \leq qM$ for some $q \in \mathbb{Z}_+$, an observation in [7] that we build on in this paper. We show in Section 4 that if M is the inclusion matrix of a semisimple algebra pair $B \subseteq A$, the depth two condition is equivalent to “ B is a normal subring in A .” As a consequence, the result in [12] for subgroups extends to normality \Leftrightarrow depth two for Hopf subalgebras of semisimple Hopf algebras as well as other semisimple quantum groupoid subalgebras. In addition, higher depth subgroups or semisimple subalgebras may be described as being normal further along in the Jones tower of iterated endomorphism algebras (Corollary 4.8 below).

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In [15] the notion of depth more than two for a Frobenius extension $B \subseteq A$ was shown to be simplified considerably via a generalization of depth two from ring extensions to towers of three rings; then a tower of three rings should be chosen in the Jones tower above $B \subseteq A$. In [7] this idea was applied to a pair of semisimple complex algebras $B \subseteq A$ with inclusion matrix M : the subalgebra pair is depth n if n is least integer for which M^{n+1} is less than a multiple of M^{n-1} , where powers of an $r \times s$ matrix M are understood by $M^2 = MM^t$, $M^3 = MM^tM$, and so forth. As noted in Section 2 below, depth n is equivalently the point of stabilization of the zero entries of even or odd powers of M , which form a descending chain of subsets.

In [7] the generalized depth two condition on a tower of semisimple algebras $C \subseteq \tilde{B} \subseteq A$ with inclusion matrices N and \tilde{M} , respectively, is given by $N\tilde{M}\tilde{M}^t\tilde{M} \leq qNM$ where q is a positive integer: let N be the identity matrix to recover the depth two condition on a subalgebra pair. Build a tower of algebras above $B \subseteq A$, where $A_1 = \text{End } A_B$ and one iterates the endomorphism ring construction and embeds via left multiplication. Then $A \hookrightarrow A_1$ has inclusion matrix M^t , and subalgebra pair $B \subseteq A$ is depth n if n is the least integer for which the tower $B \subseteq A_{n-3} \subseteq A_{n-2}$ satisfies the generalized depth two condition. For this tower of three algebras the inclusion matrices are $N = M^{n-2}$, $\tilde{M} = M$ or M^t , in the generalized depth two condition, which plugged in and simplified, becomes the depth n condition, $M^{n+1} \leq qM^{n-1}$, on the inclusion matrix M .

The paper is organized as follows. In Section 1 we define a matrix M of non-negative integer coefficients with nonzero rows and columns to be depth $n > 1$ if n is the least integer for which the $n+1$ 'st power of M is less than a multiple of the $n-1$ 'st power of M , where M^2 denotes MM^t , $M^3 = MM^tM$ and so on. For example, if M is the induction-restriction table of irreducible characters of a finite group G and a subgroup H , then M is depth two if and only if H is a normal subgroup of G [12]. If M is the inclusion matrix of a subalgebra pair of multimatrix \mathbb{C} -algebras $B \subseteq A$, the matrix M is depth two if and only if B is a normal subring in A (in the sense of Rieffel [19]) as shown below in Theorem 4.6. We study depth three or more in Section 5, 6 and several appendices by Danz and the third coauthor. We prove that the induction-restriction depth of permutation groups $S_n < S_{n+1}$ is $2n-1$. In Section 3, we make use of a well-known interpretation of the inclusion matrix M as the incidence matrix of a bicolored weighted multigraph of semisimple algebras $B \subseteq A$, showing that odd depth is one plus the diameter of the row corresponding to the simples of B ; even depth is two plus the maximum graphical distance along edges of the graph from an equivalence class of simples of B , under one simple of A , to the simple of B furthest away. In Section 6, we show that a subgroup H of G has depth bounded above by $2n$ if the largest normal subgroup of G contained in H (i.e. the core) is the intersection of n conjugates of H (and bounded above by $2n-1$ if the core is trivial).

Throughout this paper we work over \mathbb{C} , which may be replaced by any algebraically closed field of characteristic zero with the same results. In this case semisimple algebras split into direct products of matrix algebras which we call multimatrix algebras. In this paper then, semisimple algebra should be understood as split semisimple or multimatrix algebra.

2. DEPTH OF AN IRREDUNDANT MATRIX

In this section, we make an introductory study of irredundant matrices, which naturally arise as the induction-restriction table of irreducible \mathbb{C} -characters of a subgroup within a finite group. This type of matrix also occurs more generally as the induction-restriction table of simple modules, equivalently incidence matrices of inclusion diagrams, belonging to a subalgebra pair of semisimple algebras as explained in the next section.

Let $M = (m_{ij})$ be an $r \times s$ matrix of non-negative integer entries $m_{ij} \in \mathbb{Z}_{\geq 0}$, where each column and row vector is nonzero; such a matrix is called *irredundant* matrix in this paper. The matrix M is *positive* if each of its entries is a positive integer. Its (right) *square* will be the order r symmetric matrix $\mathcal{S} := M^2 := MM^t$. Note that the (i, j) -entry s_{ij} of \mathcal{S} is the euclidean inner product of rows i and j in M . In particular, the diagonal $s_{ii} > 0$ since each row in M is nonzero.

Continuing, the cube of M is just $M^3 = MM^tM = \mathcal{S}M$, $M^4 = \mathcal{S}^2$, etc. The odd powers $M^{2n+1} = \mathcal{S}^nM$ are all of size $r \times s$, and the even powers $M^{2n} = \mathcal{S}^n$ are symmetric matrices of order r .

Let $N = (n_{ij})$ be another $r \times s$ irredundant matrix. The matrices N and M are *equivalent up to permutation* if there are permutation matrices $P \in S_r$ and $Q \in S_s$ such that $M = PNQ$. We use the ordering $M \geq N$ if $m_{ij} \geq n_{ij}$ for each $i = 1, \dots, r$ and $j = 1, \dots, s$. If $T \geq 0$ denotes a square irredundant matrix of order r , then $TM \geq TN$ if $M \geq N$; similarly $MT \geq NT$ for an $s \times s$ -matrix $T \geq 0$. Note that if T is positive, i.e. $T > 0$, then also $TM > 0$ since columns in M include positive entries.

The definition below comes from considerations of what higher depth is for Frobenius extensions when restricted to semisimple algebra pairs with inclusion matrix M as outlined in the introduction. (Although not needed in this paper, the interested reader should see [13, 15] for the definition of higher depth Frobenius extensions and see [7] for why semisimple \mathbb{C} -algebra pairs are split separable Frobenius extensions.)

Definition 2.1. *An $r \times s$ matrix M is of depth $n \geq 2$ if n is the least integer for which the following inequality (called a depth n matrix inequality) holds for some $q \in \mathbb{Z}_+$,*

$$(1) \quad M^{n+1} \leq qM^{n-1}.$$

The definition depends only on the equivalence class of M up to permutation. In the case $n = 2$, M is depth two if $\mathcal{S}M \leq qM$ for some positive integer q . Multiplying from the right by M^t , one obtains $\mathcal{S}^2 \leq q\mathcal{S}$ (fix the

notation $\mathcal{S} = MM^t$); thus, M also satisfies a depth three condition. Iterating, we see M satisfies depth $n \geq 2$ conditions, and similarly, a depth n matrix satisfies depth $n + k$ matrix inequality for any positive integer k . However, the depth n matrix inequality for M does not imply a depth $n - 1$ matrix inequality for M . We note at several points in this paper that M necessarily has finite depth. Again, the depth of M is the least positive integer $k \geq 2$ for which M satisfies a depth k inequality.

Given an integer irredundant $r \times s$ matrix $M = (m_{ij})$, let the zero entries be collected in $Z(M) = \{(i, j) | m_{ij} = 0\}$; its complement $A(M) = \{(i, j) | m_{ij} \neq 0\}$ will be useful in the next section. Note that the zero entries of the even or odd powers of an irredundant matrix M form a descending chain of subsets of the rectangular array $\underline{r} \times \underline{s}$ where $\underline{r} = \{1, \dots, r\}$:

$$Z(M^{n-1}) \supseteq Z(M^{n+1}) \supseteq Z(M^{n+3}) \supseteq \dots$$

The proof is simply the following. Let $n \geq 2$ and denote the entries of M^{n+1} by (p_{ij}) and the entries of M^{n-1} by (q_{ij}) . Suppose that $0 = p_{ij} = \sum_{k=1}^r s_{ik}q_{kj}$. Then $q_{ij} = 0$ since the diagonal entries of M^2 are positive, so $s_{ii} > 0$.

From the finite order of the matrix M it is clear that the descending chain of subsets $Z(M^{n+2k+1})$ ($k = \dots, -1, 0, 1, \dots$) must stabilize at some point (i.e. satisfy dcc). The next proposition notes that the point at which these subsets are equal is the depth of the matrix.

Proposition 2.2. *An irredundant matrix M satisfies a depth n inequality if and only if*

$$Z(M^{n-1}) = Z(M^{n+1}).$$

Proof. If M is depth n , then there is a positive integer t such that $M^{n+1} \leq tM^{n-1}$. Suppose the entry $q_{ij} = 0$ in the matrix M^{n-1} . By the matrix inequality, the corresponding entry in M^{n+1} is zero, so $Z(M^{n-1}) \subseteq Z(M^{n+1})$. Together with the opposite inclusion noted above, conclude that $Z(M^{n-1}) = Z(M^{n+1})$.

Conversely, if $Z(M^{n-1}) = Z(M^{n+1})$, we may choose t to be the maximum of the integer entries in M^{n+1} , in which case $M^{n+1} \leq tM^{n-1}$. \square

The nonzero entries in M recorded in the subset $A(M)$ occur in the next section. The subset $A(M)$ is of course the complement of $Z(M)$ and the like powers form an ascending chain,

$$A(M^{n-1}) \subseteq A(M^{n+1}) \subseteq A(M^{n+3}) \subseteq \dots$$

which must end at some point (satisfying an acc condition). The following is an obvious corollary:

Corollary 2.3. *An irredundant matrix M satisfies a depth n inequality if and only if $A(M^{n-1}) = A(M^{n+1})$.*

Another obvious corollary:

Corollary 2.4. *If $M^{n-1} > 0$, then M has depth n or less.*

As noted above, an irredundant matrix has finite depth. In fact,

Corollary 2.5. *An irredundant $r \times s$ matrix M has depth $\leq 2r - 1$.*

Proof. The characteristic polynomial of $\mathcal{S} = MM^t$ gives an equation of the form

$$\mathcal{S}^r + a_1\mathcal{S}^{r-1} + \cdots + a_{r-1}\mathcal{S} + a_r = 0$$

with $a_1, \dots, a_r \in \mathbb{C}$, by the Cayley-Hamilton theorem. Thus

$$A(M^{2r}) = A(\mathcal{S}^r) \subseteq A(\mathcal{S}^{r-1}) = A(M^{2r-2}),$$

so that M has depth $2r - 1$ or less. \square

Example 2.6. We compute the depth of the matrix M below, which is the inclusion matrix, equivalently induction-restriction table, of the subgroup $D_8 < S_4$, the symmetries of the square in the group of all permutations of four letters. The powers M , $M^2 = \mathcal{S}$, M^3 , M^4 and M^5 are given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 4 & 0 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 4 & 0 & 0 & 5 & 0 \\ 0 & 3 & 3 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 4 & 9 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 5 & 4 \\ 4 & 5 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 9 \end{pmatrix}$$

Comparing even powers of M the number of zero entries decreases with increasing powers as do the first two odd powers of M , but M^3 and M^5 have the same number (13) of zero entries, so M has depth 4 (indeed $\mathcal{S}^2 M \leq 5SM$).

2.1. Endomorphism ring theorems for depth. We continue our study of irredundant matrices from the point of view of depth. However, our motivation in this subsection comes from inclusion matrices of semisimple pairs of algebras, which are defined in detail below in Section 3, and quickly as follows. Suppose $B \subseteq A$ is an inclusion of multimatrix algebras. In order to obtain the inclusion matrix M of $B \subseteq A$, we first label the simple A -modules by V_1, \dots, V_s , then the simples of B by W_1, \dots, W_r , and we obtain the k 'th column of M by restricting the k 'th simple A -module V_k to a B -module and expressing it in terms of direct sum of B -simples

$$(2) \quad V_k \downarrow_B \cong \bigoplus_{i=1}^r m_{ik} W_i$$

The matrix M is the $r \times s$ -matrix with non-negative integer entries m_{ij} , where no column is zero; nor any row by Frobenius reciprocity, which gives the following isomorphisms for induced modules,

$$(3) \quad A \otimes_B W_i \cong \bigoplus_{j=1}^s m_{ij} V_j$$

The *depth* of the inclusion $B \subseteq A$ of multimatrix algebras is defined to be the depth of its inclusion matrix M . In the next section we return to provide more detail for this definition.

If M is the inclusion matrix of a subalgebra pair of semisimple algebras $B \subseteq A$, then its transpose irredundant matrix M^t is the inclusion matrix of $A \hookrightarrow \text{End } A_B$ (via $a \mapsto \lambda_a$ where $\lambda_a(x) = ax$ for every $a, x \in A$) by an argument that goes as follows. It is clear that the natural module A_B is finitely generated projective; it is indeed also a generator since the ground field has characteristic zero. Thus B and $E := \text{End } A_B$ are Morita equivalent algebras with context bimodules ${}_E A_B$ and $A^* := {}_B \text{Hom}(A_B, B_B)_E$; the E -simples are then $A \otimes_B W_i$ ($i = 1, \dots, r$). Restricting the E -simples down to A and using Eq. (3), the columns of the inclusion matrix of $A \hookrightarrow E$ are the rows of M . We conclude that the inclusion matrix of $A \hookrightarrow E$ is M^t .

Thus, it is interesting to compare the depths of M and M^t in the next purely matrix-theoretic theorem.

Theorem 2.7. *If an irredundant matrix M has depth n , then M^t has depth $\leq n + 1$. If n is even, then M^t is moreover of depth n .*

Proof. If $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$, we multiply from the left by M^t to obtain $(M^t)^{n+2} \leq q(M^t)^n$, which shows that M^t has depth $\leq n + 1$.

If n is even then the transpose of the inequality $M^{n+1} \leq qM^{n-1}$ is the inequality $(M^t)^{n+1} \leq q(M^t)^{n-1}$. \square

In the next corollary the depth of a subalgebra pair of semisimple algebras $B \subseteq A$ is as noted the depth of its inclusion matrix. Let E denote $\text{End } A_B$ and embed A in E via the mapping λ defined above.

Corollary 2.8. *The subalgebra pair of semisimple algebras $B \subseteq A$ is of depth $2n$ if and only if its endomorphism algebra extension $A \hookrightarrow E$ is of depth $2n$. If $B \subseteq A$ is of depth $2n - 1$, then $A \hookrightarrow E$ is of depth $\leq 2n$.*

This corollary is consistent with several general ‘endomorphism ring theorems’ in [15, 16] and is an improvement in the semisimple case.

Example 2.9. The matrix M below, obtained from the inclusion matrix of $S_2 < S_3$ ([7]), has depth three while its transpose has depth four:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad MM^t = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad M^t M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

This is easier to see graphically; we return to this example in the next section.

It is also easy to see from the precise definition in the next section that if $C \subseteq B$ and $B \subseteq A$ are successive subalgebra pairs of semisimple algebras with inclusion matrices M and N , respectively, then the inclusion matrix of the composite subalgebra pair $C \subseteq A$ is of course MN . As a simple consequence we may note an improved version of the embedding theorem

[15, 8.6]. We prove that any depth n subalgebra pair may be embedded in a depth two extension, depth two being an improvement from the point of view of Galois theory (see [13, 12, 15, 16] and papers in their bibliographies). We set up the theorem by introducing the Jones tower above the subalgebra pair and its endomorphism ring, $B \subseteq A \hookrightarrow E_1 := E = \text{End } A_B$. The Jones tower is just obtained via iteration of the right endomorphism ring construct:

$$(4) \quad B \subseteq A \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \dots$$

I.e., $E_2 = \text{End } E_A$, and iterate with respect to $\lambda : E_1 \hookrightarrow E_2$ to form E_3 , then continuing like this. Note that E_1 is Morita equivalent to B , E_2 is Morita equivalent to A (the details are brought together in [7, 2.2]), so all E_m 's are themselves semisimple algebras. Then if $B \subseteq A$ has inclusion matrix M , $A \hookrightarrow E_1$ has inclusion matrix M^t , and $E_1 \hookrightarrow E_2$ has again inclusion matrix M , and so on in alternating fashion.

Theorem 2.10. *A depth n subalgebra pair of semisimple algebras $B \subseteq A$ is embedded in the depth two subalgebra pair of semisimple algebras $B \hookrightarrow E_{n-2}$.*

Proof. If the inclusion matrix of $B \subseteq A$ is M , then $M^{n+1} \leq qM^{n-1}$ for some $q \in \mathbb{Z}_+$. Since $n \geq 2$, we have $3n - 3 \geq n + 1$ and M also satisfies a depth $3n - 3$ matrix inequality. Then there is $r \in \mathbb{Z}_+$ such that $M^{3n-3} \leq rM^{n-1}$. In other words, by checking odd and even case, this is the same as

$$M^{n-1}(M^{n-1})^t M^{n-1} \leq rM^{n-1},$$

which of course is the depth two condition for the matrix $M^{n-1} = MM^tM \dots$ ($n - 1$ times M and M^t alternately). But M^{n-1} is the inclusion matrix of the composite subalgebra pair $B \hookrightarrow E_{n-2}$. \square

2.2. Depth of tensor product of matrices. Given two irredundant matrices, an $r \times s$ matrix $M = (m_{ij})$ and a $p \times q$ matrix $N = (n_{ij})$, we form the tensor product $M \otimes N$ corresponding to the tensor product of linear mappings between vector spaces. In terms of block matrix representation, $M \otimes N$ is the $rp \times sq$ matrix $(m_{ij}N)$, or equivalently up to permutation (Mn_{ij}) . Our interest in determining the depth of $M \otimes N$ knowing the depths of M and of N comes from the following situation in group theory. Given a subgroup $H_1 < G_1$ with inclusion matrix M and another subgroup $H_2 < G_2$ with inclusion matrix N , the inclusion matrix of $H_1 \times H_2 < G_1 \times G_2$ is none other than $M \otimes N$.

Proposition 2.11. *Suppose irredundant matrix M has depth n and irredundant matrix N has depth m . Then $M \otimes N$ has depth at most $\max\{n, m\}$.*

Proof. Suppose $n \geq m$. Then we show that $(M \otimes N)^{n+1} \leq q(M \otimes N)^{n-1}$ for some $q \in \mathbb{Z}_+$. Note that $(M \otimes N)^t = M^t \otimes N^t$, then $(M \otimes N)^m = M^m \otimes N^m$ in the meaning of power of non-square matrices given above. But we are given that $M^{n+1} \leq q_1 M^{n-1}$ for some $q_1 \in \mathbb{Z}_+$, and since N satisfies a depth m matrix inequality $\Rightarrow N$ satisfies a depth n matrix inequality, there is

$q_2 \in \mathbb{Z}_+$ such that also $N^{n+1} \leq q_2 N^{n-1}$. It follows from the definition of tensor product that $(M \otimes N)^{n+1} \leq q(M \otimes N)^{n-1}$ with $q = q_1 q_2$. \square

3. ON MULTIMATRIX ALGEBRA INCLUSIONS

Suppose $B \subseteq A$ is an inclusion of multimatrix algebras. Label the simple A -modules by V_1, \dots, V_s and the simples of B by W_1, \dots, W_r . Restrict the k 'th simple A -module V_k to a B -module and express in terms of direct sum of simples

$$(5) \quad V_k \downarrow_B \cong \bigoplus_{i=1}^r m_{ik} W_i$$

We let M be the $r \times s$ -matrix, or table, with entries m_{ij} : $M = (m_{ij})$. By a well-known generalization of Frobenius reciprocity, the rows give induction of the B -simples:

$$(6) \quad W_i \uparrow^A = W_i^A = \bigoplus_{j=1}^s m_{ij} V_j$$

since $W_j^A = W_j \otimes_B A$, $V_k \downarrow_B \cong \text{Hom}(A_B, V_k)$ and $\text{Hom}(W_i, W_j) = \{0\}$ if $i \neq j$; i.e., if $[W_j^A, V_k]$ denotes the number of constituents in W_j^A isomorphic to V_k , Frobenius reciprocity is given by

$$(7) \quad [W_i^A, V_k] = m_{ik} = [W_i, V_k \downarrow_B]$$

The matrix M is known as the *inclusion matrix* of B in A [9]. As seen above it corresponds to the induction-restriction table (as it is known in group theory) for simples or their irreducible characters. It may also be viewed as a matrix representation of a linear mapping in K -theory, between the groups $\mathbb{Z}^r \cong K_0(B) \rightarrow K_0(A) \cong \mathbb{Z}^s$.

For group algebras A and B corresponding to a finite group G with subgroup H , we will in practice obtain M as follows. Both character tables of G and H will be assumed known. Restrict each of the s irreducible characters of G to H , then express the restricted character of G as a non-negative integer coefficient linear combination of the r irreducibles of H by using inner products of characters.

3.1. The relation and the equivalence relation. We say that $W_i \sim W_j$ if and only if there is a simple A -module V such that W_i and W_j are both constituents of $V \downarrow_B$. The relation \sim is reflexive and symmetric but not transitive in general.

Its transitive closure is denoted by \approx . Thus we say that $W_i \approx W_j$ if and only if there is $m \geq 1$ and a sequence $W_{i_0}, W_{i_1}, \dots, W_{i_{m-1}}, W_{i_m}$ of simple B -modules such that $W_i = W_{i_0} \sim W_{i_1} \sim W_{i_2} \sim \dots \sim W_{i_{m-1}} \sim W_{i_m} = W_j$.

Notation: We denote the above equivalence relation by d_B^A . (This equivalence relation is considered before in [19, Rieffel]; this section and the next may be viewed as a continuation of results in this article.)

3.2. On the equivalence relation. Let B be a multimatrix subalgebra of A . For an irreducible character $\alpha \in \text{Irr}(B)$ let f_α be the central idempotent corresponding to α . Similarly if $\chi \in \text{Irr}(A)$ then e_χ is the corresponding central idempotent in A . Consider the commutative algebra $\mathcal{Z}(A) \cap B$ as a subalgebra of $\mathcal{Z}(A)$ and $\mathcal{Z}(B)$. Then there are partitions of characters $\text{Irr}(A) = \bigsqcup_{i=1}^t \mathcal{A}_i$ and $\text{Irr}(B) = \bigsqcup_{i=1}^t \mathcal{B}_i$ such that a basis of primitive idempotents for $\mathcal{Z}(A) \cap B$ is given by

$$(8) \quad m_i = \sum_{\chi \in \mathcal{A}_i} e_\chi = \sum_{\alpha \in \mathcal{B}_i} f_\alpha.$$

Proposition 3.1. *Suppose that $B \subset A$ is an inclusion of multimatrix algebras with A free as left B -module. With the above notations it follows that*

$$(9) \quad \left(\sum_{\chi \in \mathcal{A}_i} \chi(1)\chi \right) \downarrow_B^A = \frac{\dim A}{\dim B} \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha.$$

$$(10) \quad \left(\sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha \right) \uparrow_B^A = \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi.$$

Proof. For each partition set \mathcal{A}_i let $\chi_i = \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi$. The regular character ρ_A of A is given by $\rho_A = \sum_{i=1}^t \chi_i$. Since A is free as left B -module it follows that $\rho_A \downarrow_B^A = \frac{\dim A}{\dim B} \rho_B$ where ρ_B is the regular character of B . Similarly for each partition set \mathcal{B}_i let $\alpha_i = \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha$. Therefore $\rho_B = \sum_{i=1}^t \alpha_i$. From Eq. (8) it follows that if $i \neq j$ then $\chi_i(m_j) = 0$. This implies that $\chi_i(f_\alpha) = 0$ for all $j \neq i$ and $\alpha \in \mathcal{B}_j$. Since $\chi_i(f_\alpha) = \alpha(1)m_B(\chi_i \downarrow_B^A, \alpha)$ one gets Eq. (9). Since $\rho_B \uparrow_B^A = \rho_A$, Eq. (10) follows using Frobenius reciprocity and Eq. (9). \square

3.3. Define a relation on $\text{Irr}(A)$ by $\chi \sim \mu$ if $\chi \downarrow_B^A$ and $\mu \downarrow_B^A$ have a common constituent. This is not transitive but we can take the transitive closure. Thus $\chi \approx \mu$ if and only if there are irreducible A -characters μ_0, \dots, μ_r such that $\chi = \mu_0 \sim \mu_1 \sim \dots \sim \mu_r = \mu$. This is an equivalence relation denoted by u_B^A .

3.4. Let α be an irreducible B -character. Then all the irreducible constituents of $\alpha \uparrow_B^A$ are in the same equivalence class of u_B^A (they are \sim related). Let $\alpha \sim \beta$ be two irreducible characters of B . This means that there is an irreducible χ of $\text{Irr}(A)$ such that α, β are irreducible constituents of $\chi \downarrow_B^A$. By Frobenius reciprocity χ is common constituent for $\alpha \uparrow_B^A$ and $\beta \uparrow_B^A$. Thus all the irreducible constituents of $\alpha \uparrow_B^A$ and $\beta \uparrow_B^A$ are u_B^A -equivalent. This implies by transitivity that if $\alpha \approx \beta$ then all the irreducible constituents of $\alpha \uparrow_B^A$ and those of $\beta \uparrow_B^A$ are equivalent.

3.5. A similar argument to the one just given shows that characters equivalent by u_B^A are under restriction of characters equivalent by d_B^A .

Proposition 3.2. *Suppose that $B \subset A$ is an inclusion of multimatrix algebras with A free as left B -module. Then the equivalence classes of u_B^A are the sets \mathcal{A}_i . The equivalence classes of d_B^A are the sets \mathcal{B}_i .*

Proof. The previous two subsections imply that an equivalence class of d_B^A induced to A has all the constituents inside an equivalence class of u_B^A . Conversely, any equivalence class on u_B^A restricted to B has all the constituents inside an equivalence class of d_B^A . Since the regular character of A is a sum of copies of the regular character of B , it follows that the above restriction covers entirely the equivalence class. This implies that for any equivalence class \mathcal{C} of d_B^A the idempotent $\sum_{\alpha \in \mathcal{C}} f_\alpha$ is central in A . This finishes the proof. \square

Corollary 3.3. *Suppose that $B \subset A$ is an inclusion of multimatrix algebras with A free as left B -module. Then the number of equivalence classes of u_B^A is the same as the number of equivalence classes of d_B^A and it equals $\dim(\mathcal{Z}(A) \cap B)$.*

3.6. Distance between modules (characters). We say that the distance $d(W_i, W_j)$ between two modules is m if m is the smallest number such that there are $m - 1$ intermediate simple B -modules with $W_i = W_{i_0} \sim W_{i_1} \sim \dots \sim W_{i_m} = W_j$. Thus $d(W_i, W_j) = 1$ if and only if $W_i \sim W_j$. We put $d(W_i, W_j) = -\infty$ if W_i and W_j are not equivalent under \approx and $d(W_i, W_i) = 0$ for all $1 \leq i \leq r$. Note that the distance defined here is half of the *graphical distance* between the black points corresponding to W_i and W_j in the Bratteli diagram.

Recall the inclusion matrix $M = (m_{ij})$ for the inclusion of multimatrix algebras $B \subseteq A$ and define its symmetric ‘square’ $\mathcal{S} = MM^t = (s_{ij})$. The entries of \mathcal{S} and its powers, or indeed the powers $\mathcal{S}^m M$ of M in the sense of the previous section, will be denoted by $(\mathcal{S}^m M^k)_{ij}$ where $k = 0, 1$.

Remark 3.4. Let $1 \leq i \leq r$ and $1 \leq u \leq s$. Then $m_{iu} > 0$ if and only if W_i is a constituent of $V_u \downarrow_B$.

Proposition 3.5. *Suppose $i \neq j$. One has $(\mathcal{S}^m)_{ij} > 0$ if and only if $0 < d(W_i, W_j) \leq m$. This is equivalent to the existence of a path of length $2m$ between W_i and W_j .*

Proof. Observe first that $s_{ij} > 0$ if and only if $W_i \sim W_j$ or equivalently $d(W_i, W_j) = 1$. Indeed $s_{ij} = \sum_{u=1}^s m_{iu} m_{ju}$. Thus $s_{ij} > 0$ if and only if there is u such that $m_{iu} > 0$ and $m_{ju} > 0$. That means that W_i and W_j are constituents of $V_u \downarrow_B$ and therefore $W_i \sim W_j$.

For $m > 1$ note that $(\mathcal{S}^m)_{ij} = \sum_{l_1, \dots, l_{m-1}} s_{il_1} s_{l_1 l_2} \dots s_{l_{m-1} j}$. Thus $(\mathcal{S}^m)_{ij} > 0$ if and only if there are $1 \leq l_1, \dots, l_{m-1} \leq r$ such that $W_i \sim V_{l_1} \sim \dots \sim V_{l_{m-1}} \sim W_j$, i.e. $d(W_i, W_j) \leq m$. \square

Remark 3.6. For $M \in \mathcal{M}_{rs}(\mathbb{R}_{\geq 0})$ let $A(M)$ be the set of ordered pairs (i, u) with $m_{iu} > 0$. Recall that given two matrices $M, N \in \mathcal{M}_{rs}(\mathbb{R}_{\geq 0})$ then there is $q \in \mathbb{Z}_+$ such that $M \leq qN$ if and only if $A(M) \subseteq A(N)$.

Remark 3.7. We recall a few things from Section 2. Note that $\mathcal{S}_{ii} > 0$ and therefore $(\mathcal{S}^p)_{ii} > 0$ for all $p > 0$. This implies that if $(\mathcal{S}^m)_{ij} > 0$ then also $(\mathcal{S}^{m+p})_{ij} > 0$ since $(\mathcal{S}^{m+p})_{ij} = \sum_{l=1}^r (\mathcal{S}^m)_{il} (\mathcal{S}^p)_{lj}$ and for $l = j$ both terms are positive. In terms of distance in the Bratteli diagram this is equivalent to the fact that if there is a path between W_i and W_j of length $2m$ then there is also a path of length $2(m+p)$ between the same points. For example, one can travel p -times back and forth along the last edge of the path of length $2m$. Using the notations from the previous remark this implies that $A(\mathcal{S}^m) \subseteq A(\mathcal{S}^{m+1})$ for all $m > 0$.

Definition 3.8. *The depth of a multimatrix algebra inclusion $B \subseteq A$ is defined to be the depth of its inclusion matrix M (in terms of Section 2, which also notes the definition is independent of the ordering in the basis of simples). A subgroup H of a finite group G is said to be depth n if the corresponding group algebras over \mathbb{C} have multimatrix algebra inclusion (via Maschke and Wedderburn theory) of depth n .*

The background for this definition is given in [7, 13, 14, 12, 15, 16] and their bibliographies; the definition coincides with the definition of depth introduced briefly in [7, Burciu-Kadison]. For example, given group algebras with inclusion matrix M , where $B = \mathbb{C}[H]$, $A = \mathbb{C}[G]$, subgroup H of finite group G , we easily see that the (i, j) -entry in the depth two condition $M^3 \leq qM$ is the same as the condition for depth two in [12, Section 3] as follows. Suppose the bases of irreducible characters are given by $\text{Irr}(G) = \{\chi_1, \dots, \chi_s\}$ and $\text{Irr}(H) = \{\psi_1, \dots, \psi_r\}$. Then $m_{ij} = \langle \psi_i | (\chi_j)_H \rangle$ and by Frobenius reciprocity $m_{ij} = \langle \psi_i^G | \chi_j \rangle$. The entries of \mathcal{S} are then $s_{ij} = \langle \psi_i^G | \psi_j^G \rangle$, since this entry is the inner product of the rows i and j of M . Then we apply reciprocity and orthogonal expansion:

$$\begin{aligned}
 (\mathcal{S}M)_{ij} &= \sum_k \langle \psi_i^G | \psi_k^G \rangle \langle \psi_k | (\chi_j)_H \rangle \\
 &= \sum_k \langle (\psi_i^G)_H | \psi_k \rangle \langle \psi_k | (\chi_j)_H \rangle \\
 (11) \qquad &= \langle (\psi_i^G)_H | (\chi_j)_H \rangle = \langle ((\psi_i^G)_H)^G | \chi_j \rangle
 \end{aligned}$$

Theorem 3.9. *The inclusion matrix of $B \subseteq A$ satisfies a depth $2m + 1$ inequality ($m \geq 1$) if and only if the distance between any two simple B -modules is at most m .*

Proof. By definition, (cf. [7, Theorem 2.5]) the inclusion matrix M of $B \subseteq A$ satisfies a depth $2m + 1$ inequality iff $\mathcal{S}^{m+1} \leq q\mathcal{S}^m$ for some $q > 0$ and $\mathcal{S} = MM^t$.

Suppose that $B \subseteq A$ is of depth $2m + 1$. By Corollary 2.3, $A(\mathcal{S}^m) = A(\mathcal{S}^{m+p})$ for all $p > 0$. If $(\mathcal{S}^{m+p})_{ij} > 0$, then $(\mathcal{S}^m)_{ij} > 0$, so $d(W_i, W_j) \leq m$

by Prop. 3.5. It follows that $d(W_i, W_j) \leq m$ for all pairs of simples W_i, W_j over B (the distance is $-\infty$ if two simples are not in the same connected component of inclusion diagram).

Conversely, suppose that the distance between any two simple B -modules is at most m . We have to show that $\mathcal{S}^{m+1} \leq q\mathcal{S}^m$ for some $q > 0$ which by Remark 3.6 is equivalent to $A(\mathcal{S}^{m+1}) \subseteq A(\mathcal{S}^m)$. If $\mathcal{S}_{ij}^{m+1} > 0$ then in the Bratteli diagram there is a path of length $2m$ between W_i and W_j . Therefore the distance between these two points is positive (not $-\infty$) and by the assumption it should be less or equal to m . Thus $\mathcal{S}_{ij}^m > 0$ by Proposition 3.5. \square

If we define *diameter* of a row of simples in an inclusion diagram to be the greatest graphical distance between simples (an even number), the theorem says that the minimum odd depth inequality satisfied by the inclusion matrix of $B \subseteq A$ is one plus the diameter of the simples of B in its inclusion diagram.

Corollary 3.10. *The inclusion $B \subseteq A$ is of depth ≤ 3 if and only if \sim is an equivalence relation.*

Proof. Suppose that $B \subseteq A$ is of depth ≤ 3 . By the above theorem the distance between any two modules is less or equal to 1. Thus this distance is either $-\infty$ or 1. If $W_i \sim W_j$ and $W_j \sim W_k$ then $0 < d(W_i, W_k) \leq 2$. The assumption implies $d(W_i, W_k) = 1$, i.e. $W_i \sim W_k$. This proves that \sim is transitive.

Conversely suppose \sim is transitive. Then this implies that the distance between any two modules is less or equal to 1. It is 1 if the modules are equivalent under \sim and $-\infty$ if they are not equivalent. The above theorem implies that $B \subseteq A$ is of depth ≤ 3 . \square

Let V_u be a simple A -module with $1 \leq u \leq s$. The irreducible constituents of $V_u \downarrow_B$ are all inside of one equivalence class of \approx . Denote the set of these constituents by \mathcal{V}_u . The distance between a simple B -module W_i and the set \mathcal{V}_u is defined as usually, by the minimal distance between W_i and any element of the set \mathcal{V}_u . Thus

$$(12) \quad d(W_i, \mathcal{V}_u) = \min_{W_j \in \mathcal{V}_u} d(W_i, W_j).$$

Definition 3.11. *We define $m(V_u)$ to be the maximal distance between any simple B -module W_i and the set \mathcal{V}_u .*

Note that remark 3.4 can now be written as $m_{iu} > 0$ if and only if $W_i \in \mathcal{V}_u$.

Proposition 3.12. *Let $m \geq 1$. Then $(\mathcal{S}^m M)_{iu} > 0$ if and only if $0 < d(W_i, \mathcal{V}_u) \leq m$.*

Proof. Suppose that $(\mathcal{S}^m M)_{iu} > 0$. Since $(\mathcal{S}^m M)_{iu} = \sum_{l=1}^r (\mathcal{S}^m)_{il} m_{lu}$ there is an $1 \leq l \leq r$ such that $(\mathcal{S}^m)_{il} > 0$ and $m_{lu} > 0$. Proposition 3.5 and Remark 3.4 imply that $d(W_i, W_l) \leq m$ and $W_l \in \mathcal{V}_u$. Thus $0 < d(W_i, \mathcal{V}_u) \leq m$.

Conversely, if $0 < d(W_i, \mathcal{V}_u) \leq m$ then there is an $1 \leq l \leq r$ such that $d(W_i, W_l) \leq m$ and $W_l \in \mathcal{V}_u$. This implies that $(\mathcal{S}^m)_{il} > 0$ and $m_{lu} > 0$ which together give that $(\mathcal{S}^m M)_{iu} > 0$. \square

Theorem 3.13. *The inclusion matrix of $B \subseteq A$ satisfies a depth $2m$ inequality (with $m \geq 2$) if and only if $m(V_u) \leq m - 1$ for any simple A -module V_u .*

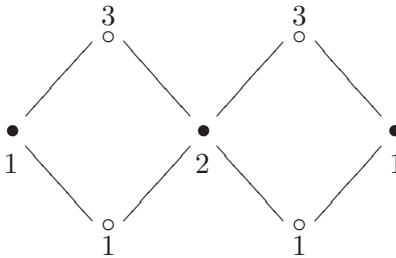
Proof. Recall that the inclusion matrix M satisfies a depth $2m$ inequality iff $\mathcal{S}^m M \leq q\mathcal{S}^{m-1}M$.

Suppose that the inclusion is depth $2m$. Therefore $\mathcal{S}^m M \leq q\mathcal{S}^{m-1}M$ for some $q > 0$. By induction one can prove that $\mathcal{S}^{m+p}M \leq q^{p+1}\mathcal{S}^{m-1}M$ (multiplying with \mathcal{S} to the left). Suppose that $m(V_u) = m + p$ with $p \geq 0$ and some u . This implies that there is W_i a simple B -module such that its distance to \mathcal{V}_u is $m + p$. Proposition 3.12 implies that $(\mathcal{S}^{m+p}M)_{iu} > 0$. From remark 3.6 one has $A(\mathcal{S}^{m+p}M) \subseteq q^{p+1}A(\mathcal{S}^{m-1}M)$. Thus $(\mathcal{S}^{m-1}M)_{iu} > 0$ and Proposition 3.12 implies that $d(W_i, \mathcal{V}_u) \leq m - 1$. This is a contradiction and the proof in one direction is completed.

Conversely suppose that $m(V_u) \leq m - 1$ for any simple A -module V_u . We have to show that $\mathcal{S}^m M \leq q\mathcal{S}^{m-1}M$ which by Remark 3.6 is equivalent to $A(\mathcal{S}^m M) \subseteq A(\mathcal{S}^{m-1}M)$. Suppose $(i, u) \in A(\mathcal{S}^m M)$. Then $(\mathcal{S}^m M)_{iu} > 0$ and Proposition 3.12 implies that $0 < d(W_i, \mathcal{V}_u) \leq m$. The assumption of the theorem implies that this distance should be less or equal to $m - 1$. Then Proposition 3.12 implies that $(\mathcal{S}^{m-1}M)_{iu} > 0$. Thus $(i, u) \in A(\mathcal{S}^{m-1}M)$. \square

We note that in terms of graphical distance, the minimal even depth matrix inequality satisfied by the inclusion diagram of B in A is two plus the largest graphical distance of a B -simple from an equivalence class of B -simples under one A -simple.

Example 3.14. The inclusion (or Bratteli) diagram starting with $B = \mathbb{C}[S_2] \subset A = \mathbb{C}[S_3]$ at the bottom level, and proceeding to its semisimple pair $A \leftrightarrow \text{End } A_B$ via λ at the top level is shown below:



Notice that the inclusion diagram of $A \hookrightarrow E$ is the reflection of the diagram of $B \subseteq A$ about the middle row, true in general by Morita theory [9]. Applying Theorem 3.9, we see from the bottom graph that the graphical distance between simples is 2, so depth of subgroup $S_2 < S_3$ is three. Applying Theorem 3.13, we see from the top graph that the maximal distance

from a simple away from a set of two simples in \mathcal{V}_u on the middle line has graphical distance 2, so that the depth of $A \hookrightarrow E$ is four.

By simply adding dots and the same pattern of edges to the right of the diagram, we create diagrams (Dynkin diagrams of type A_n) for multimatrix algebra inclusions of arbitrary odd or even depth. In terms of explicit inclusion mappings, the following inclusion $B := \mathbb{C}^n \rightarrow A := \mathbb{C} \times M_2(\mathbb{C})^{n-1} \times \mathbb{C}$ has depth $2n - 1$: ($\lambda_i \in \mathbb{C}$, $n \geq 2$)

$$(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix}, \dots, \begin{pmatrix} \lambda_{n-1} & 0 \\ 0 & \lambda_n \end{pmatrix}, \lambda_n)$$

while its endomorphism algebra extension $A \hookrightarrow E = M_3(\mathbb{C}) \times M_4(\mathbb{C})^{n-2} \times M_3(\mathbb{C})$ has depth $2n$: ($M_i \in M_2(\mathbb{C})$)

$$(\lambda_1, M_1, \dots, M_{n-1}, \lambda_n) \mapsto \left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & M_1 \end{pmatrix}, \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \dots, \begin{pmatrix} M_{n-1} & 0 \\ 0 & \lambda_n \end{pmatrix} \right)$$

4. DEPTH TWO AND NORMALITY

Let $S \subset R$ an inclusion of multimatrix algebras. We define the subring S to be *normal* in R if the restriction of every maximal ideal I (in R) to S is R -invariant, meaning that $(I \cap S)R = R(I \cap S)$ as subsets of R . This definition of normal subrings is first given in [19, Rieffel] and used to provide a ring-theoretic setting for Clifford theory. It is also closely related historically to the $HK^+ = K^+H$ condition of normality of a Hopf subalgebra K in a Hopf algebra H .

Notation: Let \hat{R} the set of maximal two sided ideals of R . Similarly define \hat{S} . Any $I \in \hat{R}$ determines up to isomorphism a unique simple R -module denoted by V_I and a minimal (primitive) central idempotent f_I of R . Similarly any $J \in \hat{S}$ determines up to isomorphism a unique simple S -module denoted by W_J and a minimal central idempotent q_J of S .

Proposition 4.1. *W_J is a constituent of $V_I \downarrow_S$ if and only if $I \cap S \subset J$ if and only if $q_J f_I \neq 0$.*

Proof. W_J is a constituent of $V_I \downarrow_S$ if and only if $q_J(Rf_I) \neq 0$. Since f_I is central this is equivalent with $q_J f_I \neq 0$. The first statement results taking annihilators in S . \square

Proposition 4.2. 1) *For any idempotent $e \in S$ one has*

$$e = e \left(\sum_{\{I \in \hat{R} \mid e f_I \neq 0\}} f_I \right).$$

2) *For any idempotent $f \in R$ one has*

$$f = f \left(\sum_{\{J \in \hat{S} \mid f q_J \neq 0\}} q_J \right).$$

The next proposition makes an improvement of [19, Prop. 2.10].

Proposition 4.3. *Assume that for any simple A -module V the irreducible constituents of V form an entire equivalence class of \approx . Then \sim is transitive and B is normal in A .*

Proof. Clearly \sim is transitive. We follow the reasoning from Rieffel's proof of Proposition 2.10. Fix $J_0 \in \hat{S}$. Construct $e_{J_0} = \sum_{I \in X_0} f_I$ where $X_0 = \{I \in \hat{R} \mid q_{J_0} f_I \neq 0\}$. Then $q_{J_0} e_{J_0} = q_{J_0}$ by the first item of the above Proposition. Suppose that $q_J e_{J_0} \neq 0$ for some other $J \in \hat{S}$. Then $q_J f_I \neq 0$ for at least one $I \in X_0$. Then $W_{J_0} \sim W_J$ by the above remark. The assumption on the restriction of modules implies that $q_J f_I \neq 0$ for all $I \in X_0$ and $q_J f_I = 0$ for all $I \notin X_0$. The first item of the above Proposition shows that $q_J = q_J e_{J_0}$. The second item of the same Proposition implies that

$$(13) \quad e_{J_0} = \sum_{\{J \in \hat{S} \mid e_{J_0} q_J \neq 0\}} e_{J_0} q_J = \sum_{\{J \in \hat{S} \mid e_{J_0} q_J \neq 0\}} q_J.$$

Thus $q_{J_0} \in S$. Then the rest of the proof is the same as in Rieffel's paper. Indeed, if $I \in \hat{R}$ such that $I \cap S \subseteq J_0$ then $1 - e_{J_0} \in I \cap S$. On the other hand if $J \in \hat{S}$ and $I \cap S \not\subseteq J$ then $q_J(1 - e_{J_0}) = q_J$. Thus $1 - e_{J_0}$ is the identity element of $I \cap S$. Since it is central in R , Proposition 2.6 of [19, Rieffel] implies that $I \cap S$ is R -invariant. Thus S is normal in R . \square

Lemma 4.4. *$SM \leq nM$ for some $n > 0$ if and only if for any simple A -module V the irreducible constituents of $V \downarrow_B$ form an entire equivalence class of \approx .*

Remark 4.5. With the previous notations the statement of the lemma can be rephrased as $SM \leq nM$ for some $n > 0$ if and only if \mathcal{V}_u coincides with an entire equivalence class of \sim for any simple A -module V_u .

Proof. Suppose that $SM \leq nM$ for some $n > 0$. Clearly the nonzero-entry subsets satisfy $A(SM) \subset A(M)$. Suppose that $W_i \sim W_j$ and $W_j \in \mathcal{V}_u$. Then Proposition 3.5 and Remark 3.4 imply that $s_{ij} > 0$ and $m_{ju} > 0$. Thus $(SM)_{iu} > 0$. This means $(i, u) \in A(SM) \subset A(M)$. Thus $m_{iu} > 0$, i.e. $W_i \in \mathcal{V}_u$.

Conversely, suppose that \mathcal{V}_u coincides with an entire equivalence class of \sim . We need to show $SM \leq nM$ for some $n > 0$ or $A(SM) \subset A(M)$. Let $\{(i, u), (u, i)\} \in A(SM)$. Thus there is $1 \leq l \leq r$ such that $\mathcal{S}_{il} > 0$ and $m_{lu} > 0$. This means $W_i \sim W_l$ and $W_l \in \mathcal{V}_u$. Since \mathcal{V}_u coincides with an entire equivalence class of \sim it follows that $W_i \in \mathcal{V}_u$. Thus $m_{iu} > 0$ and $(i, u) \in A(M)$. \square

Theorem 4.6. *The inclusion $B \subseteq A$ is of depth 2 if and only if B is normal in A .*

Proof. (\Rightarrow) By Proposition 2.2 of [7, Burciu-Kadison], $B \subseteq A$ is depth 2 if and only if $SM \leq nM$ for some n . If $B \subseteq A$ is normal then from Proposition 2.8 in [19] it follows that \sim is an equivalence relation and for

any simple A -module V then the irreducible constituents of $V \downarrow_B$ form an entire equivalence class. Lemma 4.4 implies that the multimatrix algebra inclusion is depth two.

(\Leftarrow) The converse follows from Lemma 4.4 and Proposition 4.3. \square

This theorem provides a third proof of [6, main theorem, 5.1]. Recall that a Hopf subalgebra K of a Hopf algebra H is *normal* if $HK^+ = K^+H$ for K^+ the kernel of the counit $\varepsilon : K \rightarrow \mathbb{C}$.

Corollary 4.7. *A depth two Hopf subalgebra of a semisimple Hopf algebra is normal.*

Proof. A Hopf subalgebra of a semisimple Hopf algebra is known to be semisimple. Since it is depth two, it is normal as defined by Rieffel. But $H^+ \cap K = K^+$ for the kernel of the counit $\varepsilon : H \rightarrow \mathbb{C}$, a maximal ideal in H . This ideal in K is then H -invariant, so K is a normal Hopf subalgebra. \square

The following corollary puts the last theorem together with Theorem 2.10 in a previous section. Recall that given a subalgebra $B \subseteq A$, the first endomorphism ring $E_1 = \text{End } A_B$, the second $E_2 = \text{End } (E_1)_A$, and so forth (with $A = E_0$ and $B = E_{-1}$). We embed E_n in E_{n+1} as before via left multiplication in what we call the *Jones tower* above $B \subseteq A$.

Corollary 4.8. *Let B be a semisimple subalgebra in a semisimple algebra A . If B is depth n then B is normal in E_{n-2} .*

The converse may be shown as well by a theorem in [8] and its generalization to higher depth. We have then seen depth two subalgebra to be normal in the overalgebra, a depth three subalgebra to be normal in E_1 and a higher depth subalgebra to be normal further along in the Jones tower.

5. INCLUSIONS OF SEMISIMPLE HOPF ALGEBRAS

Let $K \subset H$ be an inclusion of semisimple Hopf subalgebras. Let $C(H)$ and $C(K)$ be the character rings of H and K respectively. These are commutative rings if H and K are quasitriangular or cocommutative Hopf algebras such as group algebras.

If M and N are two H -modules with characters χ and μ respectively, then $m_H(M, N) := \dim \text{Hom}_H(M, N)$. The same quantity is also denoted by $m_H(\chi, \mu)$. In this manner one obtains a nondegenerate symmetric bilinear form $m_H(\ , \)$ on the character ring $C(H)$ of H . The following result is Proposition 2 of [4]. It shows that the image of the induction map is a two sided ideal in $C(H)$. A different proof that also works in the nonsemisimple case is given below.

Lemma 5.1. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H . Let M be an H -module and V a K -module. Then*

$$M \otimes V \uparrow_K^H = (M \downarrow_K^H \otimes V) \uparrow_K^H$$

and

$$V \uparrow_K^H \otimes M = (V \otimes M \downarrow_K^H) \uparrow_K^H$$

Proof. Define

$$\phi : M \otimes (H \otimes_K V) \rightarrow H \otimes_K (M \otimes V)$$

by $m \otimes h \otimes_K v \mapsto h_{(2)} \otimes_K (S^{-1}(h_{(1)})m \otimes v)$. It can be checked that ϕ is a well-defined morphism of H -modules. Moreover ϕ is bijective since $h \otimes_K m \otimes v \mapsto (h_{(1)}m \otimes h_{(2)} \otimes_K v)$ is its inverse map. \square

In terms of characters the first formula can be written as $\chi \text{Ind}(\alpha) = \text{Ind}(\text{Res}(\chi)\alpha)$ or

$$(14) \quad \chi \alpha \uparrow = (\chi \downarrow \alpha) \uparrow.$$

In terms of characters the second formula can be written as $\text{Ind}(\alpha)\chi = \text{Ind}(\alpha \text{Res}(\chi))$ or

$$(15) \quad \alpha \uparrow \chi = (\alpha \chi \downarrow) \uparrow.$$

Let $T : C(K) \rightarrow C(K)$ be given by $T(\alpha) = \text{Res}(\text{Ind}(\alpha))$. Thus $T(\alpha) = \alpha \uparrow \downarrow$. Note that the matrix of the operator T with respect to the basis of $C(K)$ given by the irreducible characters of K is the matrix \mathcal{S} defined in the previous section.

Lemma 5.2. *With the above notations one has*

$$(16) \quad T(\alpha T(\beta)) = T(\alpha)T(\beta) = T(T(\alpha)\beta)$$

for all $\alpha, \beta \in C(K)$.

Proof. One has

$$T(\alpha T(\beta)) = (\alpha(\beta \uparrow \downarrow)) \uparrow \downarrow = (\alpha((\beta \uparrow) \downarrow) \uparrow) \downarrow = (\alpha \uparrow \beta \uparrow) \downarrow = \alpha \uparrow \downarrow \beta \uparrow \downarrow$$

We have applied relation 15 for the fourth equality and the fact that Res is an algebra map in the last equality. So the first equation in the lemma is proved, and the other is obtained in a similar way. \square

Proposition 5.3. *With the above notations one has*

- 1) $T^n(\varepsilon) = T(\varepsilon)^n$ for all $n \geq 1$.
- 2) $T^n(\alpha) = T(\alpha)T(\varepsilon)^{n-1}$ for all $n \geq 1$.

Proof. The first part is a special case of the second one. We prove the second by induction on n . The case $n = 1$ is trivial. Suppose that $T^n(\alpha) = T(\alpha)T(\varepsilon)^{n-1}$ for some n . Then

$$\begin{aligned} T^{n+1}(\alpha) &= T(T^n(\alpha)) = T(T(\alpha)T(\varepsilon)^{n-1}) = T(T(\alpha))T(\varepsilon)^{n-1} \\ &= T(T(\alpha)\varepsilon)T(\varepsilon)^{n-1} = T(\alpha)T(\varepsilon)^n, \end{aligned}$$

by induction and the preceding lemma. \square

Lemma 5.4. *Let α and β be two irreducible characters of K . Then $0 < d(\alpha, \beta) \leq m$ if and only if $m_K(\alpha, T^m(\beta)) > 0$.*

Proof. This follows from Proposition 3.5 since as noted before the matrix of the operator T with respect to the basis of $C(K)$ given by the irreducible characters of K is the matrix \mathcal{S} defined in the previous section. \square

5.1. The operator U . Define $U : C(H) \rightarrow C(H)$ such that $U(\chi) = \chi \downarrow_K^H \uparrow_K^H$. It follows from Eq. (15) for $\alpha = \varepsilon_H$ that $U^m(\chi) = \chi(\varepsilon \uparrow_K^H)^m$ for all $m \geq 0$. Recall the equivalence relation u_K^H on $\text{Irr}(H)$ from Section 3. One has $\chi \sim \mu$ if and only if $\chi \downarrow_K^H$ and $\mu \downarrow_K^H$ have a common constituent. Then u_K^H is the equivalence relation obtained by taking the transitive closure of \sim .

Remark 5.5. Note that $\chi \sim \mu$ if and only if $m_H(\chi, U(\mu)) > 0$. Inductively it can be shown that $\chi u_K^H \mu$ if and only if there is $l > 0$ such that $m(\chi, U^l(\mu)) > 0$.

Proposition 5.6. *A Hopf subalgebra K is normal in H if and only if ε_K by itself forms an equivalence class of d_K^H .*

Proof. From the decomposition of $\mathcal{Z}(H) \cap K$ it follows that the integral element Λ_K is central in H [18]. \square

Remark 5.7. Suppose $H = \mathbb{C}G$ and $K = \mathbb{C}H$ for finite groups $H \subset G$. Then $\mathcal{Z}(\mathbb{C}G) \cap \mathbb{C}H \subset \mathcal{Z}(\mathbb{C}N)$ where N is the core of H in G . This follows since a basis for $\mathcal{Z}(\mathbb{C}G)$ is given by $\sum_{g \in \mathcal{C}} g$ where \mathcal{C} runs through all conjugacy classes of G .

6. INCLUSION OF GROUP ALGEBRAS

Let $H \subset G$ be an inclusion of finite groups and let $N := \text{core}_G(H)$ be the core of H in G , i.e., the largest normal subgroup in G contained in H . We will use the short notations u_H^G and d_H^G for the equivalence relations $u_{\mathbb{C}H}^{\mathbb{C}G}$ and $d_{\mathbb{C}H}^{\mathbb{C}G}$ defined in subsections 3.3 and 3.1, respectively.

Proposition 6.1. *For all $n \geq 1$ one has that $\ker_G(U^n(\varepsilon)) = N$.*

Proof. Clearly $\ker_G(U^n(\varepsilon)) \subset \ker_G(U(\varepsilon))$ since $U(\varepsilon)$ has all the constituents inside $U^n(\varepsilon)$. On the other hand $\ker_G(\chi^n) \supset \ker_G(\chi)$ for any character χ . Thus $\ker_G(U^n(\varepsilon)) = \ker_G(U(\varepsilon))$ for all $n \geq 1$. It remains to show $\ker_G(U(\varepsilon)) = N$. If $n \in N$ then one has that $n(g \otimes_H 1) = g(g^{-1}ng) \otimes_H 1 = g \otimes_H 1$ since $gng^{-1} \subset N \subset H$. Thus $N \subset \ker_G(U(\varepsilon))$. On the other hand $xg \otimes_H 1 = g \otimes_H 1$ implies that $xgH = gH$ and therefore $x \in gHg^{-1}$. Thus if $x \in \ker_G(U(\varepsilon))$ then $x \in \bigcap_{g \in G} gHg^{-1} = N$. \square

Corollary 6.2. *For all $n \geq 1$ one has that $\ker_H(T^n(\varepsilon)) = N$.*

Proof. It follows from the previous proposition since $U^n(\varepsilon) \downarrow_H^G = T^n(\varepsilon)$. \square

Corollary 6.3. *Let $H \subset G$ be a group inclusion and N be the core of H in G . Consider the equivalence relation d_H^G on the irreducible characters of H as above. Then the equivalence class of ε_H is $\text{Irr}(H/N)$.*

Proof. If $\alpha \in \text{Irr}(H/N)$ then α is a constituent of $\varepsilon_N \uparrow_N^H$. Therefore $\alpha \uparrow_H^G$ is a constituent of $\varepsilon_N \uparrow_N^G = U^m(\varepsilon_H)$. But $U^m(\varepsilon_H) \downarrow_H^G$ has all the constituents inside the equivalence class of ε_H . Since α is a constituent of $T(\alpha)$ it follows that $\alpha \approx \varepsilon$. Thus all the irreducible characters of $\text{Irr}(H/N)$ are equivalent to ε_H .

Conversely suppose that $\beta \sim \varepsilon$. Then β is a constituent of $T(\varepsilon)$ and therefore by Corollary 6.2 its restriction to N contains $\beta(1)$ copies of the trivial character of N . Thus $\beta \in \text{Irr}(H/N)$. \square

Let m be minimal such that $T^m(\varepsilon)$ has all the possible constituents of all powers $T^n(\varepsilon)$ with $n \geq 0$. Such m exists since there are only finitely many characters that can appear and if they appear in a certain power of T then they appear in any greater power of T .

Remark 6.4. Since $N \triangleleft G$ it is well known that $\chi u_N^G \mu$ if and only if χ and μ have exactly the same irreducible constituents viewed as N -characters by restriction. It also follows from [5] that $U_N(\chi) = \chi \varepsilon \uparrow_N^G$ where $U_N : C(G) \rightarrow C(G)$ is given by $U_N(\chi) = \chi \downarrow_N^G \uparrow_N^G$. For any irreducible character $\alpha \in \text{Irr}(N)$ the constituents of $\alpha \uparrow_N^G$ form an entire equivalence class under d_N^G . Similarly, for any irreducible character $\chi \in \text{Irr}(G)$ the constituents of $\chi \downarrow_N^G$ form an entire equivalence class under d_N^G . Thus, by Clifford theory, the equivalence classes of $\text{Irr}(N)$ under d_N^G are just the G -orbits on $\text{Irr}(N)$ (under the conjugation action).

The following result from [5] will be used in the sequel:

Proposition 6.5. *Let G be a group, χ a character of G and $N = \ker(\chi)$. Then ε_N^G has as irreducible constituents all the possible irreducible constituents of all the powers of χ .*

Corollary 6.6. *The equivalence relation u_H^G is the same as the equivalence relation u_N^G coming from $N \trianglelefteq G$. Thus the equivalence classes of $\text{Irr}(G)$ under u_H^G are in natural bijection with the G -orbits on $\text{Irr}(N)$.*

Proof. Write as above $\chi \sim \mu$ if and only if $\chi \downarrow_H^G$ and $\mu \downarrow_H^G$ have a common constituent. If $\chi \sim \mu$ then clearly $\chi u_N^G \mu$. This implies that if $\chi u_H^G \mu$ then $\chi u_N^G \mu$. Conversely by Remark 5.5 we see that $\chi u_N^G \mu$ if and only if $m_G(\chi, U_N(\mu)) > 0$. Remark 6.4 implies that $U_N(\mu) = \mu \varepsilon_N \uparrow_N^G$. On the other hand, using the previous proposition it follows that $m_G(\chi, U_N(\mu)) = m_G(\chi, \mu \varepsilon_N^G) = (\varepsilon_N \uparrow_N^G, \mu \chi^*) > 0$ if and only if $m_G(U^m(\varepsilon), \chi \mu^*) = m_G(\chi, U^m(\mu)) > 0$. \square

Corollary 6.7. *One has that the relation \sim on $\text{Irr}(G)$ coming from the inclusion $H \subset G$ is an equivalence relation if and only if $\varepsilon \uparrow_N^G$ and $\varepsilon \uparrow_H^G$ have the same constituents.*

Proof. \sim is an equivalence relation if and only if $\varepsilon \uparrow_H^G$ and $(\varepsilon \uparrow_H^G)^m$ have the same constituents. But $(\varepsilon \uparrow_H^G)^m$ has the same constituents as $\varepsilon_N \uparrow_N^G$. \square

Proposition 6.8. *Let $N \subset H \subset G$ with $N \triangleleft G$. The depth of H/N inside G/N is less than or equal to the depth of H in G . If H has depth three or less in G then H/N has depth three or less in G/N .*

Proof. Let $\bar{T} : C(H/N) \rightarrow C(H/N)$ be the operator T defined as above but for the inclusion $H/N \subset G/N$. Since $\text{Rep}(G/N) \subset \text{Rep}(G)$ and $\text{Rep}(H/N) \subset \text{Rep}(H)$ it is easy to check that \bar{T} is the restriction of T to $C(H/N)$. Indeed both restriction and induction for the inclusion $H/N \subset G/N$ come from the restriction and induction for the inclusion $H \subset G$.

Then the proposition follows from Theorem 3.9 and Theorem 3.13. \square

For example, with $G = S_4$, $H = D_8$ and

$$N = \{(1), (12)(34), (13)(24), (14)(23)\} = V_4,$$

depth of $H < G$ is four (computed in Section 2), while depth of $H/N \cong S_2 < G/N \cong S_3$ is three (computed graphically in Section 3).

Again let G be a finite group, H a proper subgroup of G , and $N := \text{core}_G(H)$ denote the core of H in G . We say $\chi \in \text{Irr}(G)$ and $\psi \in \text{Irr}(H)$ are *linked* if

$$(17) \quad 0 \neq \langle \psi \uparrow^G | \chi \rangle_G = \langle \psi | \chi \downarrow_H \rangle_H.$$

This defines a bipartite graph Γ with vertices $\text{Irr}(G) \cup \text{Irr}(H)$ (the inclusion diagram of the corresponding group algebras are a weighted multigraph variant of this). As usual, we denote by $\text{Irr}(G|\kappa)$ the set of all $\chi \in \text{Irr}(G)$ such that $\langle \chi \downarrow_N, \kappa \rangle \neq 0$, for $\kappa \in \text{Irr}(N)$.

Proposition 6.6 implies that the connected components of Γ are in bijection with the orbits of G on $\text{Irr}(N)$.

6.1. A theorem with examples. Recall that the *core* $\text{core}_G(H)$ of a subgroup $H < G$ is the largest normal subgroup of G contained in H . It is also defined by $\text{core}_G(H) = \bigcap_{x \in G} {}^x H$ where ${}^x H$ denotes the subgroup xHx^{-1} conjugate to H .

Theorem 6.9. *Let $H \subseteq G$ be an inclusion of finite groups, and suppose that $N := \text{core}_G(H)$ is the intersection of m conjugates of H . Then H has depth $\leq 2m$ in G . Moreover, if $N \subseteq \mathcal{Z}(G)$ then H has depth $\leq 2m - 1$ in G .*

Proof. Let $\alpha \in \text{Irr}(H)$, and let $x \in G$. Then Mackey decomposition shows that $\text{Ind}_{H \cap xHx^{-1}}^H(\text{Res}_{H \cap xHx^{-1}}^{xHx^{-1}}({}^x \alpha))$ is a summand of $T(\alpha) = \text{Res}_H^G(\text{Ind}_H^G(\alpha))$. Thus $\text{Res}_H^G(\text{Ind}_{H \cap xHx^{-1}}^G(\text{Res}_{H \cap xHx^{-1}}^{xHx^{-1}}({}^x \alpha)))$ is a summand of

$$T^2(\alpha) = \text{Res}_H^G(\text{Ind}_H^G(T(\alpha))).$$

Let $y \in G$. Then, by Mackey decomposition again,

$$\text{Ind}_{H \cap yHy^{-1} \cap yxHx^{-1}y^{-1}}^H(\text{Res}_{H \cap yHy^{-1} \cap yxHx^{-1}y^{-1}}^{yxHx^{-1}y^{-1}}({}^{yx} \alpha))$$

is a summand of $T^2(\alpha)$. Continuing in this fashion, we see that, for $x_1 := 1, x_2, \dots, x_m \in G$,

$$\text{Ind}_{x_1 H x_1^{-1} \cap \dots \cap x_m H x_m^{-1}}^H (\text{Res}_{x_1 H x_1^{-1} \cap \dots \cap x_m H x_m^{-1}}^{x_m H x_m^{-1}} ({}^{x_m} \alpha))$$

is a summand of $T^{m-1}(\alpha)$. We can choose $x_1 = 1, x_2, \dots, x_m =: z$ in such a way that $x_1 H x_1^{-1} \cap \dots \cap x_m H x_m^{-1} = N$. Then $T^{m-1}(\alpha)$ has a summand of the form

$$\text{Ind}_N^H (\text{Res}_N^{z H z^{-1}} (z \alpha)) = \text{Ind}_N^H ({}^z \text{Res}_N^H (\alpha)).$$

Let β be an irreducible constituent of $\text{Res}_N^H(\alpha)$. Then $\text{Ind}_N^G(z\beta) = \text{Ind}_N^G(\beta)$ is a summand of $\text{Ind}_H^G(T^{m-1}(\alpha))$. But the irreducible constituents of $\text{Ind}_N^G(\beta)$ form a complete equivalence class of $\text{Irr}(G)$ under u_H^G , by Corollary 6.6. Thus α has graphical distance at most $2m - 1$ to any $\chi \in \text{Irr}(G)$. So α has graphical distance at most $2m - 2$ to any set of irreducible constituents of $\text{Res}_H^G(\chi)$, for any $\chi \in \text{Irr}(G)$, and the first part is proved.

Now suppose that $N \subseteq \mathcal{Z}(G)$. Then, in the notation above, $\text{Ind}_N^H(z\beta) = \text{Ind}_N^H(\beta)$ is a summand of $T^{m-1}(\alpha)$. But now the irreducible constituents of $\text{Ind}_N^H(\beta)$ form a complete equivalence class of $\text{Irr}(H)$ under d_H^G . This shows that any two irreducible characters of H have graphical distance at most $2m - 2$, so that H has depth at most $2m - 1$ in G . \square

We illustrate the theorem with three examples.

Example 6.10. (1) Let $G = S_4$ and $H = D_8$, so that $N = V_4$ is the intersection of $m = 2$ conjugates of H . By the theorem, D_8 has depth ≤ 4 in S_4 ; indeed the depth is four by our earlier computations. In the appendix, the depth of D_{2n} in S_n is shown to be three for $n > 5$.

(2) Let $G = S_{n+1}$ and $H = S_n$ for some n . Then $N = 1$, which is the intersection of $m = n$ conjugates of H :

$$\{1\} = S_n \cap S_n^{(1\ n+1)} \cap \dots \cap S_n^{(n-1\ n+1)}.$$

By the theorem, S_n has depth at most $2n - 1$ in S_{n+1} . We will see later that $2n - 1$ is precisely the depth of $S_n < S_{n+1}$.

(3) Let $G = A_6$ and $H = A_5$, so that $N = 1$ again. A computation with character tables shows that A_5 has depth 5 in A_6 . However, in this case, N is *not* the intersection of 3 conjugates of H , so the bound in the theorem is not sharp here. The depth of the inclusion of alternating groups $A_n \subseteq A_{n+1}$ will be computed in the appendix.

We obtain a corollary by recalling that G acts on the set of subgroups of G by conjugation. Let $N_G(H)$ be the normalizer of H in G , which is the stabilizer subgroup of H under conjugation. The proof is a simple application of the orbit counting theorem:

Corollary 6.11. *The depth of a subgroup H of a finite group G is bounded above by $2[G : N_G(H)]$.*

The following are examples of depth three or more subgroups from the literature on group theory.

Example 6.12. Brodkey's theorem (cf. Theorem 1.37 in [10, Isaacs]) states that if a finite group G has an abelian Sylow p -subgroup H , then the largest normal p -subgroup $\mathcal{O}_p(G) = N$ of G is the intersection of two conjugates of H . In other terms then, H is a depth four or less subgroup in G ; depth three or less if $N = \{1_G\}$.

Example 6.13. If G is p -solvable, where p is odd and not a Mersenne prime, then the largest normal p -subgroup N of G is an intersection of two Sylow p -subgroups. If p is even or a Mersenne prime, then N is an intersection of three Sylow p -subgroups [3, Brewster-Hauck]. This in our terms implies that the Sylow p -subgroup has depth ≤ 4 or 6 , respectively. If $N = 1$ then these numbers can be improved to 3 and 5 , respectively.

Example 6.14. The theorem above implies that a subgroup H of a finite group G has at most depth three if $H \cap xHx^{-1} = 1$ for some $x \in G$. For example, a Sylow p -subgroup of $\mathrm{GL}(n, q)$ has depth three, as well as certain Borel and Weyl subgroups (for specific values of n and $q = p^r$, left as an exercise to the interested reader) [1].

The results of this paper are suited for creating a program using GAP to calculate the depth of subgroups of suitably small groups. We thank Susanne Danz for implementing such a program at the University of Jena.

In this paper we have found subgroups of depth at each odd number (the symmetric group series), at depth four (the dihedral group in S_4 with some additional examples) and a search with this program yields a subgroup of depth 6 (the 108 -element normalizer subgroup of the Sylow 3 -subgroup of the 432 -element affine group $\mathrm{AGL}(3, 2)$). We found no subgroups of depth an even number greater than 6 .

Remark 6.15. Suppose $K < H < G$ is a tower of finite groups, where the subgroup $H < G$ is corefree and m conjugates of H have trivial intersection. Then the depth of the subgroup $K < G$ is bounded above by $2m - 1$. This follows from the same theorem since K satisfies the same core hypothesis. For example, by the results of one of the examples above, any subgroup K of S_n has depth less than or equal to $2n - 1$ in S_{n+1} .

6.2. Computations for the operator T . Suppose that H is a subgroup of a finite group G . We denote by $\mathrm{Cl}(G)$ the set of conjugacy classes of G and by $\mathrm{CF}(G)$ the ring of complex class functions on G . For a union X of conjugacy classes of G , we denote by $\gamma_{G,X}$ the characteristic function of X in $\mathrm{CF}(G)$. Then

$$\gamma_{G,X} \downarrow_H = \gamma_{H,H \cap X}.$$

Similarly, if C and D denote the conjugacy classes in G and H , respectively, of an element in H then an easy computation shows that

$$\gamma_{H,D} \uparrow^G = \frac{|G|}{|H|} \cdot \frac{|D|}{|C|} \gamma_{G,C}.$$

This implies that the eigenvectors of the linear map

$$T : \text{CF}(H) \longrightarrow \text{CF}(H), \quad \chi \longmapsto \chi \uparrow^G \downarrow_H,$$

corresponding to nonzero eigenvalues are precisely the class functions $\gamma_{H,C \cap H}$ ($C \in \text{Cl}(G)$, $C \cap H \neq \emptyset$). Moreover, the eigenvalue of T corresponding to an eigenvector $\gamma_{C,C \cap H}$ is clearly

$$\frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|}.$$

We denote by

$$t := |\{ \frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|} : C \in \text{Cl}(G), C \cap H \neq \emptyset \}|$$

the number of distinct nonzero eigenvalues of T . Then the minimum polynomial of T has degree t or $t + 1$. Since \mathcal{S} is the matrix of T with respect to the basis $\text{Irr}(H)$ of $\text{CF}(H)$, we get an equation

$$\mathcal{S}^{t+1} + a_1 \mathcal{S}^t + \cdots + a_t \mathcal{S} + a_{t+1} = 0$$

where $a_1, \dots, a_{t+1} \in \mathbb{C}$. Thus $A(\mathcal{S}^{t+1}) \subseteq A(\mathcal{S}^t)$, so that H has depth $\leq 2t + 1$ in G .

We also note that all eigenvalues of T are nonzero if and only if T is surjective. This is equivalent to the condition that two elements in H are conjugate in G if and only if they are already conjugate in H . In this case, the minimum polynomial of T has degree t . So, arguing as above, we conclude that H has depth $\leq 2t - 1$ in G . We summarize:

Theorem 6.16. (i) *The nonzero eigenvalues of \mathcal{S} are the numbers*

$$\frac{|G|}{|H|} \cdot \frac{|C \cap H|}{|C|} \quad (C \in \text{Cl}(G), C \cap H \neq \emptyset).$$

(ii) *The subgroup H of G has depth $\leq 2t + 1$ in G where t denotes the number of distinct nonzero eigenvalues of \mathcal{S} .*

(iii) *All eigenvalues of \mathcal{S} are nonzero if and only if any two elements in H which are conjugate in G are already conjugate in H . In this case, H has depth $\leq 2t - 1$ in G .*

Example 6.17. The inclusion matrix (or induction-restriction table) of the alternating groups $A_4 < A_5$ may be checked from character tables to be the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

up to permutation, with its symmetric square matrix computed beside. The conjugacy classes of A_5 are $\mathcal{C}_{(1)}$, $\mathcal{C}_{(12)(34)}$ of 15 elements, $\mathcal{C}_{(123)}$ of 20 elements and two more classes of five-cycles. The intersection of the three designated conjugacy classes of A_5 with A_4 have respectively 1, 3, 8 elements each. Thus the nonzero eigenvalues of \mathcal{S} are 5, 1, 2. Indeed, the minimum polynomial of \mathcal{S} is $X(X-1)(X-2)(X-5)$. By the theorem above, A_4 has depth ≤ 7 in A_5 . Computing the powers of M , one sees that the subgroup $A_4 < A_5$ has depth five. The depth of the inclusion $A_n \subseteq A_{n+1}$ for arbitrary n is computed in the appendix.

Example 6.18. Consider the standard subgroup $S_3 < S_4$ of permutation groups. It is well-known that the conjugacy classes of S_n are given strictly by the cycle decomposition and do not split into smaller conjugacy classes for subgroups $S_m < S_n$. From the character tables we compute the inclusion matrix M and its square \mathcal{S} :

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

with minimum polynomial $m(X) = X^3 - 7X^2 + 14X - 8 = (X-4)(X-2)(X-1)$. The nonzero eigenvalues of \mathcal{S} are 1, 2, 4. By the theorem above, the depth of $S_3 < S_4$ is at most five, which we see is the precise depth from Section 2.

6.3. Depth of inclusions of symmetric groups. In this subsection we will prove the following:

Theorem 6.19. *For any $n \geq 2$ the standard inclusion $S_n \subseteq S_{n+1}$ has depth $2n - 1$.*

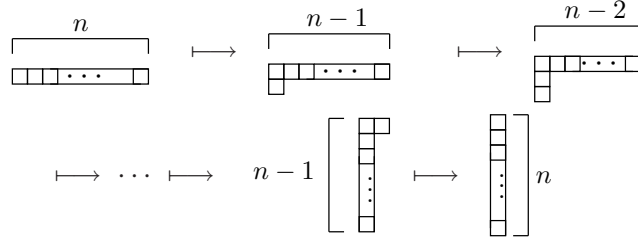
In order to prove the theorem, we recall that the irreducible characters of S_n are in bijection with partitions of n . Moreover, partitions of n can be visualized by their Young diagrams. For example, the trivial character of S_n corresponds to the trivial partition (n) of n , and the Young diagram of (n) is a row of n boxes. Similarly, the sign character of S_n corresponds to the partition $(1^n) = (1, \dots, 1)$, and the Young diagram of (1^n) is a column of n boxes.

By the branching rules, restricting an irreducible character of S_{n+1} to S_n means removing a box from the corresponding Young diagram, and inducing an irreducible character of S_n to S_{n+1} means adding a box to the corresponding Young diagram.

By Theorem 6.9 above, the inclusion $S_n \subseteq S_{n+1}$ has depth $\leq 2n - 1$. It is easy to give an alternative proof of this, based on the combinatorics of Young diagrams. These ideas are explained in more detail in the appendix where they are also used to determine the depth of the inclusion of alternating groups $A_n \subseteq A_{n+1}$.

It only remains to show that the inclusion matrix of $S_n \subseteq S_{n+1}$ does not satisfy a depth $2n - 2$ inequality. For this we may argue as follows:

The sign character of S_{n+1} , denoted by V_u , restricts to the sign character σ of S_n . Thus, in the notation of Section 3, the set \mathcal{V}_u consists of σ alone. It is easy to see that $d(\varepsilon, \sigma) = n - 1$:



It follows that $m(V_u) = n - 1$. Thus Theorem 3.13 shows that the inclusion matrix of S_n in S_{n+1} cannot satisfy a depth $2n - 2$ inequality.

This result also applies to the semisimple Hecke algebras: $H(q, n)$ is depth $2n - 1$ in $H(q, n + 1)$, since they share the same representation theory with the permutation groups $S_n < S_{n+1}$ (see [9]).

Appendix: Depth of subgroups – some examples

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APPENDIX A. INCLUSIONS OF SYMMETRIC AND ALTERNATING GROUPS

Throughout this section, let $n \geq 1$, let \mathfrak{S}_n denote the symmetric group of degree n , and let \mathfrak{A}_n denote the alternating group of degree n . Moreover, let \mathcal{P}_n be the set of all partitions of n . By Theorem 6.19, we know that for $n \geq 2$ the ring extension $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathfrak{S}_{n+1}$ is of depth $2n - 1$. We now aim to determine the depth of the ring extension $\mathbb{C}\mathfrak{A}_n \subset \mathbb{C}\mathfrak{A}_{n+1}$. Moreover, we will give a combinatorial proof of Theorem 6.19. Before stating the results, we fix some further notation.

Remark A.1. (a) For $\lambda \in \mathcal{P}_n$, we denote the conjugate partition by λ' . That is, the Young diagram of λ' is obtained by transposing the Young diagram of λ . For $\lambda \in \mathcal{P}_n$, let χ^λ be the corresponding ordinary irreducible \mathfrak{S}_n -character. If $\lambda = \lambda'$ then $\chi^\lambda \downarrow_{\mathfrak{A}_n} = \chi_+^\lambda + \chi_-^\lambda$, for irreducible \mathfrak{A}_n -characters $\chi_+^\lambda \neq \chi_-^\lambda$. We choose our labelling in accordance with [11], Sec. 2.5. With this convention, for $\alpha' = \alpha \in \mathcal{P}_{n+1}$ and $\lambda' = \lambda \in \mathcal{P}_n$ such that $\langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0$, we have $\langle \chi_+^\alpha \downarrow_{\mathfrak{A}_n}, \chi_+^\lambda \rangle \neq 0 = \langle \chi_+^\alpha \downarrow_{\mathfrak{A}_n}, \chi_-^\lambda \rangle$ and $\langle \chi_-^\alpha \downarrow_{\mathfrak{A}_n}, \chi_-^\lambda \rangle \neq 0 = \langle \chi_-^\alpha \downarrow_{\mathfrak{A}_n}, \chi_+^\lambda \rangle$ (see [11], Thm. 2.5.13, and [2]). If $\lambda \neq \lambda'$ then $\chi^\lambda \downarrow_{\mathfrak{A}_n} = \chi^{\lambda'} \downarrow_{\mathfrak{A}_n}$ is irreducible. We may then suppose that $\lambda > \lambda'$,

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and set $\chi_0^\lambda := \chi^\lambda \downarrow_{\mathfrak{A}_n}$. Here “ \geq ” denotes the usual lexicographic ordering on partitions.

(b) We consider the bipartite graphs $\Gamma(\mathfrak{S}_n)$ and $\Gamma(\mathfrak{A}_n)$. Here $\Gamma(\mathfrak{S}_n)$ has vertices $V := \mathcal{P}_n \cup \mathcal{P}_{n+1}$ and edges

$$E := \{(\alpha, \lambda) \in \mathcal{P}_{n+1} \times \mathcal{P}_n \mid \langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0\}.$$

The graph $\Gamma(\mathfrak{A}_n)$ has vertices $\tilde{V} := V(n) \cup V(n+1)$ and edges \tilde{E} where

$$\begin{aligned} V(n) &:= \{[\lambda, 0] \mid \lambda \in \mathcal{P}_n, \lambda > \lambda'\} \cup \{[\lambda, +], [\lambda, -] \mid \lambda = \lambda' \in \mathcal{P}_n\}, \\ V(n+1) &:= \{[\alpha, 0] \mid \alpha \in \mathcal{P}_{n+1}, \alpha > \alpha'\} \cup \{[\alpha, +], [\alpha, -] \mid \alpha = \alpha' \in \mathcal{P}_{n+1}\}, \\ \tilde{E} &:= \{([\alpha, x], [\lambda, y]) \in V(n+1) \times V(n) \mid \langle \chi_x^\alpha \downarrow_{\mathfrak{A}_n}, \chi_y^\lambda \rangle \neq 0\}. \end{aligned}$$

(c) Let $\lambda, \mu \in \mathcal{P}_n$ with corresponding Young diagrams $[\lambda]$ and $[\mu]$, respectively. We set

$$d(\lambda, \mu) := |[\lambda] \setminus [\mu]| + |[\mu] \setminus [\lambda]| = 2(n - |[\lambda] \cap [\mu]|).$$

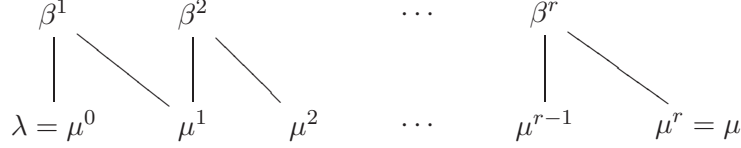
With this notation, we have:

Proposition A.2. *Let $n \geq 2$, and let $\lambda, \mu \in \mathcal{P}_n$. Then $d(\lambda, \mu)$ is the length of a shortest path from λ to μ in $\Gamma(\mathfrak{S}_n)$. In particular, the ring extension $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathfrak{S}_{n+1}$ has depth $2n - 1$.*

Proof. Let $\lambda, \mu \in \mathcal{P}_n$, and set $2m := d(\lambda, \mu)$. We argue with induction on m , in order to show that in $\Gamma(\mathfrak{S}_n)$ there is a path of length $2m$ from λ to μ . For $m = 0$ this is trivially true, so we may now suppose that $m \geq 1$. We construct a partition λ^1 of n such that $d(\lambda, \lambda^1) = 2$, $d(\lambda^1, \mu) = 2m - 2$, and such that there is a path of length 2 from λ to λ^1 . Since $\lambda \neq \mu$, we have $[\lambda] \not\subseteq [\mu] \not\subseteq [\lambda]$. Thus there is some $i \in \mathbb{N}$ such that $(i, \lambda_i) \in [\lambda] \setminus [\mu]$ and $(i+1, \lambda_i) \notin [\lambda]$. That is, (i, λ_i) is a removable node of $[\lambda]$. Analogously, there are some $r, s \in \mathbb{N}$ such that $(r, s) \in [\mu] \setminus [\lambda]$. We may suppose further that $(t, s) \in [\lambda]$, for $1 \leq t \leq r-1$, and $(r, u) \in [\lambda]$, for $1 \leq u \leq s-1$. So (r, s) is an addable node of $[\lambda]$. We define $\alpha \in \mathcal{P}_{n+1}$ with Young diagram $[\alpha] := [\lambda] \cup \{(r, s)\}$. Assume that (i, λ_i) is not a removable node of $[\alpha]$. This can happen only if $(r, s) = (i+1, \lambda_i)$ or $(r, s) = (i, \lambda_i + 1)$. But, since $(r, s) \in [\mu]$, this implies also $(i, \lambda_i) \in [\mu]$, a contradiction. Therefore, $[\lambda^1] := [\alpha] \setminus \{(i, \lambda_i)\}$ is the Young diagram of a partition $\lambda^1 \in \mathcal{P}_n$ with $d(\lambda, \lambda^1) = 2$, $d(\lambda^1, \mu) = 2m - 2$, and $\langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^\lambda \rangle \neq 0 \neq \langle \chi^\alpha \downarrow_{\mathfrak{S}_n}, \chi^{\lambda^1} \rangle$. So there is a path of length 2 from λ to λ^1 in $\Gamma(\mathfrak{S}_n)$. By induction, there is a path of length $2m - 2$ from λ^1 to μ in $\Gamma(\mathfrak{S}_n)$. So we obtain a path



of length $2m$ from λ to μ in $\Gamma(\mathfrak{S}_n)$. Conversely, let



be a shortest path in $\Gamma(\mathfrak{S}_n)$. That is, $2r \leq d(\lambda, \mu)$. We argue with induction on r to show $d(\lambda, \mu) = 2r$. If $r = 0$ then $\lambda = \mu$, and $d(\lambda, \mu) = 0 = 2r$. Next suppose that $r = 1$. Then $\lambda \neq \mu$, and $[\mu]$ is obtained by first adding a node (i, j) to $[\lambda]$, and then removing a node $(r, s) \neq (i, j)$ from $[\lambda] \cup \{(i, j)\} = [\beta^1]$. That is $d(\lambda, \mu) = 2$. Now we may suppose that $r \geq 2$. By induction, $d(\lambda, \mu^{r-1}) = 2(r-1)$ and $d(\mu^{r-1}, \mu) = 2$. So $d(\lambda, \mu) \leq 2r$, and thus $2r = d(\lambda, \mu)$. This proves the first part of the statement.

Since, for any $\lambda, \mu \in \mathcal{P}_n$, we have $(1, 1) \in [\lambda] \cap [\mu]$, it follows that $d(\lambda, \mu) \leq 2(n-1)$. Moreover, for $\lambda := (n)$ and $\mu := (1^n)$, we have $d(\lambda, \mu) = 2(n-1)$, so that, by what we have just shown and Theorem 3.9, the extension $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathfrak{S}_{n+1}$ has depth $2(n-1) + 1$. On the other hand, $\chi^{(n)} = \chi^{(n+1)} \downarrow_{\mathfrak{S}_n}$. Therefore, for $n > 2$, Theorem 3.13 implies that the inclusion matrix of $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathfrak{S}_{n+1}$ cannot satisfy a depth $2(n-1)$ inequality. If $n = 2$ then $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathfrak{S}_{n+1}$ cannot have depth $2(n-1) = 2$, since $\mathfrak{S}_2 \not\trianglelefteq \mathfrak{S}_3$. This completes the proof of the proposition. \square

Lemma A.3. *Let $\lambda, \mu \in \mathcal{P}_n$. Suppose that $\mu' = \mu \neq \lambda = \lambda'$. Then $d(\lambda, \mu) \geq 4$.*

Proof. Let $\mu' = \mu \neq \lambda = \lambda'$. Then there are some $i, j \in \{1, \dots, n\}$ such that $(i, \lambda_i) \notin [\mu]$ and $(j, \mu_j) \notin [\lambda]$. Since both λ and μ are symmetric, also $(\lambda_i, i) \in [\lambda] \setminus [\mu]$ and $(\mu_j, j) \in [\mu] \setminus [\lambda]$. In particular, $i \neq \lambda_i$ or $j \neq \mu_j$, and also $i \neq j$. Hence $d(\lambda, \mu) \geq 3$, and so $d(\lambda, \mu) \geq 4$, since $d(\lambda, \mu)$ is even. \square

Example A.4. Consider, for instance, the symmetric partitions $\lambda = (4, 3, 2, 1)$ and $\mu = (5, 2, 1^3)$ of 10. Then we have $d(\lambda, \mu) = 2(10 - 4 - 2 - 1 - 1) = 4$.

Proposition A.5. *Let $n \geq 3$. Then the ring extension $\mathbb{C}\mathfrak{A}_n \subset \mathbb{C}\mathfrak{A}_{n+1}$ has depth $2(n - \lceil \sqrt{n} \rceil) + 1$.*

Proof. In consequence of Theorem 3.9 and Theorem 3.13, it suffices to show the following:

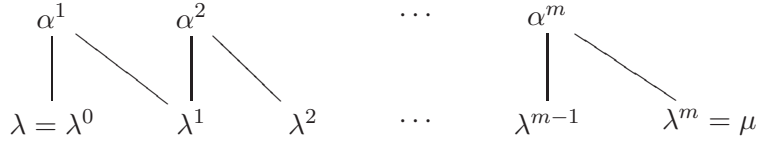
- (1) For any $v, w \in V(n)$, there is a path of length at most $2(n - \lceil \sqrt{n} \rceil)$ from v to w in $\Gamma(\mathfrak{A}_n)$, and
- (2) there is some $v \in V(n)$ such that in $\Gamma(\mathfrak{A}_n)$ there is no path of length less than $2(n - \lceil \sqrt{n} \rceil)$ from v to $[(n), 0]$.

For this, let $v, w \in V(n)$. Suppose first that $\lambda = \lambda' \in \mathcal{P}_n$ and that $v := [\lambda, +]$ and $w := [\lambda, -]$. Let further $\alpha \in \mathcal{P}_{n+1}$ with Young diagram $[\alpha] := [\lambda] \cup \{(1, \lambda_1 + 1)\}$. Then $\alpha > \alpha'$, and $\langle \chi_0^\alpha \downarrow_{\mathfrak{A}_n}, \chi_+^\lambda \rangle = 1 = \langle \chi_0^\alpha \downarrow_{\mathfrak{A}_n}, \chi_-^\lambda \rangle$,

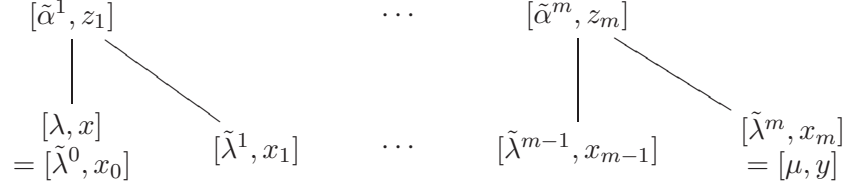
by [2]. Hence in $\Gamma(\mathfrak{A}_n)$ there is a path of length $2 \leq 2(n - \lceil \sqrt{n} \rceil)$ from v to w .

Therefore, from now on, we may suppose that $v = [\lambda, x]$ and $w = [\mu, y]$, for some $\lambda \geq \lambda'$ and $\mu \geq \mu'$ with $\lambda \neq \mu$, and appropriate $x, y \in \{0, +, -\}$. We set $2m := d(\lambda, \mu)$, and show that there is a path from v to w in $\Gamma(\mathfrak{A}_n)$ of length $2m$. Note that, since $\lambda \geq \lambda'$ and $\mu \geq \mu'$, we must have $\lambda_1 \geq \lceil \sqrt{n} \rceil$ and also $\mu_1 \geq \lceil \sqrt{n} \rceil$. So $2m \leq 2(n - \lceil \sqrt{n} \rceil)$, and we then get (1).

First of all, by Proposition A.2, there is a path



of length $2m$ in $\Gamma(\mathfrak{S}_n)$. Here $\lambda^0, \dots, \lambda^m \in \mathcal{P}_n$, and $\alpha^1, \dots, \alpha^m \in \mathcal{P}_{n+1}$. We now construct a path



in $\Gamma(\mathfrak{A}_n)$ as follows. For $i = 1, \dots, m$, we set $\tilde{\lambda}^i := \max\{\lambda^i, (\lambda^i)'\}$ and $\tilde{\alpha}^i := \max\{\alpha^i, (\alpha^i)'\}$ where the maxima are taken with respect to the lexicographic ordering on partitions. We then determine the “signs” $x_0, \dots, x_m, z_1, \dots, z_m$ inductively. Of course, $x_0 = x$. So we may suppose that $i \geq 1$ and that we have already determined x_0, \dots, x_{i-1} and z_1, \dots, z_{i-1} . In order to fix z_i and x_i , we distinguish four cases.

Case 1. $\tilde{\lambda}^{i-1} \neq (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i \neq (\tilde{\alpha}^i)'$. In this case we set $z_i := 0$ and

$$x_i := \begin{cases} +, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y \in \{0, +\}, \\ -, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y = -, \\ 0, & \text{if } \tilde{\lambda}^i \neq (\tilde{\lambda}^i)'. \end{cases}$$

Case 2. $\tilde{\lambda}^{i-1} \neq (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i = (\tilde{\alpha}^i)'$. If $\tilde{\lambda}^i \neq (\tilde{\lambda}^i)'$ then we set $z_i := +$ and $x_i := 0$, otherwise

$$z_i := x_i := \begin{cases} +, & \text{if } y \in \{0, +\}, \\ -, & \text{if } y = -. \end{cases}$$

Case 3. $\tilde{\lambda}^{i-1} = (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i \neq (\tilde{\alpha}^i)'$. We then set $z_i := 0$ and

$$x_i := \begin{cases} +, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y \in \{0, +\}, \\ -, & \text{if } \tilde{\lambda}^i = (\tilde{\lambda}^i)' \text{ and } y = -, \\ 0, & \text{if } \tilde{\lambda}^i \neq (\tilde{\lambda}^i)'. \end{cases}$$

Case 4. $\tilde{\lambda}^{i-1} = (\tilde{\lambda}^{i-1})'$ and $\tilde{\alpha}^i = (\tilde{\alpha}^i)'$. Then, in particular, $\tilde{\lambda}^i \neq (\tilde{\lambda}^i)'$, by Lemma A.3. Thus we may set $z_i := x_{i-1}$ and $x_i := 0$.

Note that this construction ensures that $x_m = y$. Moreover, by [2], for $i = 1, \dots, m$, we get a path of length 2 from $[\tilde{\lambda}^{i-1}, x_{i-1}]$ to $[\tilde{\lambda}^i, x_i]$ in $\Gamma(\mathfrak{A}_n)$, hence a path of length $2m$ from $[\lambda, x]$ to $[\mu, y]$. Note further that these considerations also show the following: in the case where $\mu = \mu'$ there are both a path of length $2m$ from $[\lambda, x]$ to $[\mu, +]$ and a path of length $2m$ from $[\lambda, x]$ to $[\mu, -]$.

In order to prove (2), let conversely $\lambda, \mu \in V(n)$ such that $2 \leq d(\lambda, \mu) =: 2m$, let $x, y \in \{0, +, -\}$ be appropriate signs, and let

$$\begin{array}{ccc} [\beta^1, z_1] & \cdots & [\beta^r, z_r] \\ \begin{array}{l} \downarrow \\ [\lambda, x] \\ = [\lambda^0, x_0] \end{array} & \begin{array}{l} \diagdown \\ [\lambda^1, x_1] \end{array} & \cdots & \begin{array}{l} \downarrow \\ [\lambda^{r-1}, x_{r-1}] \end{array} & \begin{array}{l} \diagdown \\ [\lambda^r, x_r] \\ = [\mu, y] \end{array} \end{array}$$

be a shortest path from $[\lambda, x]$ to $[\mu, y]$ in $\Gamma(\mathfrak{A}_n)$. Then $r \leq m$, by what we have shown above. We also observe that the partitions $\lambda^0, \lambda^1, \dots, \lambda^r$ must be pairwise different. To see this, assume that $\lambda^i = \lambda^j$, for some $0 \leq i < j \leq r$. Then we may suppose that $x_i = +$ and $x_j = -$. As we have seen above, there is a path of length 2 from $[\lambda^i, x_i]$ to $[\lambda^j, x_j]$ so that $j = i + 1$, since the given path is as short as possible. If $j < r$ then $\lambda^{i+2} \neq \lambda^i = \lambda^{i+1}$, by the minimality of r . But, since there is a path of length 2 from $[\lambda^{i+1}, x_{i+1}] = [\lambda^i, x_{i+1}]$ to $[\lambda^{i+2}, x_{i+2}]$, there is also a path of length 2 from $[\lambda^i, x_i]$ to $[\lambda^{i+2}, x_{i+2}]$, as we have proved above. But this contradicts the minimality of r . If $i + 1 = r$ then $i > 0$ and $\lambda^{i-1} \neq \lambda^i = \lambda^{i+1}$. This implies that there is a path of length 2 from $[\lambda^{i-1}, x_{i-1}]$ to $[\lambda^{i+1}, x_{i+1}]$, which is again a contradiction to the minimality of r .

We now set $\tilde{\lambda}^0 := \lambda^0 = \lambda$. Since $\langle \chi_{z_1}^{\beta^1} \downarrow_{\mathfrak{A}_n}, \chi_x^\lambda \rangle \neq 0 \neq \langle \chi_{z_1}^{\beta^1} \downarrow_{\mathfrak{A}_n}, \chi_{x_1}^{\lambda^1} \rangle$, there is some $\tilde{\beta}^1 \in \{\beta^1, (\beta^1)'\}$ such that $\langle \chi_{z_1}^{\tilde{\beta}^1} \downarrow_{\mathfrak{S}_n}, \chi_x^\lambda \rangle \neq 0$. Having fixed $\tilde{\beta}^1$, we can find $\tilde{\lambda}^1 \in \{\lambda^1, (\lambda^1)'\}$ such that also $\langle \chi_{z_1}^{\tilde{\beta}^1} \downarrow_{\mathfrak{S}_n}, \chi_{x_1}^{\tilde{\lambda}^1} \rangle \neq 0$. Let now $i \geq 2$.

We may argue inductively, and suppose that we have a path

$$\begin{array}{ccccccc}
 & \tilde{\beta}^1 & & \tilde{\beta}^2 & & \dots & & \tilde{\beta}^{i-1} \\
 & | & \searrow & | & \searrow & & & | & \searrow \\
 \lambda = \tilde{\lambda}^0 & & & \tilde{\lambda}^1 & & \tilde{\lambda}^2 & \dots & \tilde{\lambda}^{i-2} & & \tilde{\lambda}^{i-1}
 \end{array}$$

in $\Gamma(\mathfrak{S}_n)$, for appropriate $\tilde{\beta}^j \in \{\beta^j, (\beta^j)'\}$, $\tilde{\lambda}^j \in \{\lambda^j, (\lambda^j)'\}$, and all $j = 1, \dots, i-1$. We then choose $\tilde{\beta}^i \in \{\beta^i, (\beta^i)'\}$ and $\tilde{\lambda}^i \in \{\lambda^i, (\lambda^i)'\}$ such that $\langle \chi^{\tilde{\beta}^i} \downarrow_{\mathfrak{S}_n}, \chi^{\tilde{\lambda}^{i-1}} \rangle \neq 0 \neq \langle \chi^{\tilde{\beta}^i} \downarrow_{\mathfrak{S}_n}, \chi^{\tilde{\lambda}^i} \rangle$. In this way we obtain in $\Gamma(\mathfrak{S}_n)$ a path of length $2r$ from λ to μ , or from λ to μ' .

Now let $\lambda := (n)$, and let $\mu \in \mathcal{P}_n$ be such that

$$\begin{aligned}
 \{1, \dots, \lfloor \sqrt{n} \rfloor\} \times \{1, \dots, \lfloor \sqrt{n} \rfloor\} &\subseteq [\mu] \cap [\mu'], \\
 [\mu] \cup [\mu'] &\subseteq \{1, \dots, \lceil \sqrt{n} \rceil\} \times \{1, \dots, \lceil \sqrt{n} \rceil\},
 \end{aligned}$$

and such that $\mu \geq \mu'$. Then $2 \leq d(\lambda, \mu) = 2(n - \lceil \sqrt{n} \rceil)$, and $2(n - \lceil \sqrt{n} \rceil) \leq d(\lambda, \mu') \leq 2(n - \lfloor \sqrt{n} \rfloor) + 2$. Assume that in $\Gamma(\mathfrak{A}_n)$ there is a path of length $2r < 2(n - \lfloor \sqrt{n} \rfloor)$ from $[(n), 0]$ to $[\mu, y]$, for some admissible $y \in \{0, +, -\}$. Then, by the above considerations, in $\Gamma(\mathfrak{S}_n)$ there is a path of length $2r$ from (n) to μ , or from (n) to μ' . But, since $2r < \min\{d((n), \mu), d((n), \mu')\}$, this is impossible, by Proposition A.2. Therefore, we have now also shown (2), and the assertion of the proposition follows. \square

Remark A.6. Note that the ring extensions $\mathbb{C}\mathfrak{S}_1 \subset \mathbb{C}\mathfrak{S}_2$ and $\mathbb{C}\mathfrak{A}_2 \subset \mathbb{C}\mathfrak{A}_3$ are clearly of depth 2, since $\mathfrak{S}_1 \trianglelefteq \mathfrak{S}_2$ and $\mathfrak{A}_2 \trianglelefteq \mathfrak{A}_3$.

APPENDIX B. INCLUSIONS OF DIHEDRAL GROUPS IN SYMMETRIC GROUPS

Lemma B.1. *Let $n \geq 4$, let $G := \mathfrak{S}_n$, and let $H := \langle a \rangle$ where a is an n -cycle in G . Then the ring extension $\mathbb{C}H \subset \mathbb{C}G$ has depth 3.*

Proof. We may suppose that $a = (1, 2, \dots, n)$. Since H is not normal in G , $\mathbb{C}H \subset \mathbb{C}G$ is not of depth 2. We set $g := (2, n-2)(n-1, n)$ and $\tilde{a} := a^g = (1, n-2, 3, 4, \dots, n-3, 2, n, n-1)$. In the case where $n = 4$, this means $g = (3, 4)$ and thus $\tilde{a} = (1, 2, 4, 3)$. It suffices to show that $\langle a \rangle \cap \langle \tilde{a} \rangle = 1$. For $n = 4$ this is obviously true. Let now $n > 4$. Assume, for a contradiction, that $\langle a \rangle \cap \langle \tilde{a} \rangle \neq 1$. Then there are some $1 \neq l \in \mathbb{N}$, some $k \in \mathbb{N}$, and some $i \in \{1, \dots, l-1\}$ such that $n = kl$ and $a^{ki} = \tilde{a}^k \neq 1$. If $k \in \{2, \dots, n-4\}$ then $1 + ki = a^{ki}(1) = \tilde{a}^k(1) = 1 + k$. Thus $i = 1$, and if $k < n-4$ then we have the contradiction $2 + k = \tilde{a}^k(n-2) = a^k(n-2) \equiv n-2+k \pmod{n}$. If $k = n-4$ then $2 = \tilde{a}^k(n-2) = a^k(n-2) \equiv n-2+k \pmod{n}$, hence $k = 4$ and $n = 8$. But then $a^4(n) = 4 \neq 3 = \tilde{a}^4(n)$, a contradiction. If $k = 1$ then $1 + i = a^i(1) = \tilde{a}(1) = n-2$, thus $i = n-3$. But $a^{n-3}(2) = n-1 \neq n = \tilde{a}(2)$, a contradiction. Finally, let $k \in \{n-1, n-2, n-3\}$. Since $n = kl \geq 2k \geq 2(n-3) = 2n-6$, we get $n \leq 6$. Thus $n = 6$, $k = 3$, $l = 2$, $i = 1$, and we have a contradiction. \square

Proposition B.2. *Let $n > 5$, and let $H := D_{2n}$ be the dihedral subgroup of \mathfrak{S}_n of order $2n$, with generators $a := (1, 2, \dots, n)$ and $b := (1, n)(2, n-1)(3, n-2) \cdots (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{2} \rceil)$. Then the ring extension $\mathbb{C}H \subset \mathbb{C}\mathfrak{S}_n$ has depth 3.*

Proof. Since H is not normal in \mathfrak{S}_n , the extension $\mathbb{C}H \subset \mathbb{C}\mathfrak{S}_n$ is not of depth 2. Again we set $g := (2, n-2)(n-1, n)$, $\tilde{a} := a^g = (1, n-2, 3, \dots, n-3, 2, n, n-1)$, and $\tilde{b} := b^g = (1, n-1)(n-2, n)(2, 3)(4, n-3) \cdots (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n+2}{2} \rceil)$. Also here it suffices to show that $H \cap H^g = 1$. For $n = 6$ this is obviously true. Thus, for the remainder of the proof, let $n \geq 7$. Assume, for a contradiction, that there is some $1 \neq x \in H \cap H^g$. Note that $H = \{a^i, a^i b \mid i = 0, \dots, n-1\}$ and $H^g = \{\tilde{a}^i, \tilde{a}^i \tilde{b} \mid i = 0, \dots, n-1\}$. We distinguish between four cases.

Case 1. $x = a^i = \tilde{a}^j$, for some $i, j \in \{1, \dots, n-1\}$. Then the proof of Lemma B.1 leads to a contradiction.

Case 2. $x = a^i = \tilde{a}^j \tilde{b}$, for some $i, j \in \{0, \dots, n-1\}$. That is $a^{2i} = \tilde{a}^j \tilde{b} \tilde{a}^j \tilde{b} = \tilde{a}^j \tilde{a}^{-j} = 1$, hence n is even, and $i = n/2$. In particular,

$$5 \leq 1 + \frac{n}{2} = a^i(1) = \tilde{a}^j(\tilde{b}(1)) = \tilde{a}^j(n-1)$$

which implies $j = 1 + n/2$. So $4 + n/2 = a^i(4) = \tilde{a}^j(\tilde{b}(4)) = \tilde{a}^j(n-3)$. But, on the other hand, $\tilde{a}^j(8-3) = 6$, $\tilde{a}^j(10-3) = 3$, and $\tilde{a}^j(n-3) = (n-4)/2$ for $n > 10$. In any case this is not equal to $4 + n/2$, a contradiction.

Case 3. $x = a^i b = \tilde{a}^j$, for some $i, j \in \{0, \dots, n-1\}$. But then $x^g = a^j = \tilde{a}^i \tilde{b}$ which is impossible, by the considerations in case 2 above.

Case 4. $x = a^i b = \tilde{a}^j \tilde{b}$, for some $i, j \in \{0, \dots, n-1\}$. Therefore, $1+i = a^i(1) = a^i(b(n)) = \tilde{a}^j(\tilde{b}(n)) = \tilde{a}^j(n-2)$. Suppose that $1 \leq j \leq n-5$ so that $1+i = \tilde{a}^j(n-2) = a^j(2) = 2+j$. That is $i = j+1 \in \{2, \dots, n-4\}$. However, if $i = j+1 = 2$ then $\tilde{a}^j(\tilde{b}(1)) = 1 \neq 2 = a^i(b(1))$, if $i = j+1 = 3$ then $\tilde{a}^j(\tilde{b}(1)) = n-2 \neq 3 = a^i(b(1))$, and if $3 < i = j+1 \leq n-4$ then $\tilde{a}^j(\tilde{b}(1)) = j \neq j+1 = i = a^i(b(1))$. Moreover, if $j = 0$ then we have $1+i = n-2$, thus $i = n-3$. But $\tilde{a}^0(\tilde{b}(1)) = n-1 \neq n-3 = a^{n-3}(b(1))$. Consequently, we must have $j \in \{n-4, n-3, n-2, n-1\}$.

Suppose $j = n-1$ so that $a^i b = \tilde{a}^{-1} \tilde{b}$. Then $1+i = a^i(1) = a^i(b(n)) = \tilde{a}^{-1}(\tilde{b}(n)) = \tilde{a}^{-1}(n-2) = 1$, thus $i = 0$ and $b = \tilde{a}^{-1} \tilde{b}$. But $b(n-1) = 2 \neq n-1 = \tilde{a}^{-1}(\tilde{b}(n-1))$, a contradiction.

Next suppose that $j = n-2$ where $a^i b = \tilde{a}^{-2} \tilde{b}$, and $1+i = a^i(b(n)) = \tilde{a}^{-2}(\tilde{b}(n)) = n-1$. Hence $i = n-2$ which gives the contradiction $a^{-2}(b(2)) = a^{-2}(n-1) = n-3 \neq 1 = \tilde{a}^{-2}(3) = \tilde{a}^{-2}(\tilde{b}(2))$.

If $j = n - 3$ then $a^i b = \tilde{a}^{-3} \tilde{b}$, and so $i = n - 1$. But this time we get the contradiction $1 = \tilde{a}^{-3}(4) = \tilde{a}^{-3}(\tilde{b}(n - 3)) = a^{-1}(b(n - 3)) = a^{-1}(4) = 3$.

Lastly, assume that $j = n - 4$ so that $a^i b = \tilde{a}^{-4} \tilde{b}$, and $i = 1$. But, for $n \neq 8$, this implies $n - 6 = \tilde{a}^{-4}(2) = \tilde{a}^{-4}(\tilde{b}(3)) = a(b(3)) = a(n - 2) = n - 1$, and for $n = 8$ we get $6 = \tilde{a}^{-4}(2) = \tilde{a}^{-4}(\tilde{b}(3)) = a(b(3)) = a(6) = 7$. Hence we have again a contradiction.

To summarize, neither of the four cases above can occur, and the assertion of the proposition follows. \square

Remark B.3. (a) By Example 2.6, we know that the ring extension $\mathbb{C}D_8 \subset \mathbb{C}\mathfrak{S}_4$ has depth 4.

(b) The inclusion matrix of the groups $D_{10} < \mathfrak{S}_5$ is

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

which has depth 5. Hence the same applies to the ring extension $\mathbb{C}D_{10} \subset \mathbb{C}\mathfrak{S}_5$.

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