

SYMPLECTIC DEGENERATE FLAG VARIETIES

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ABSTRACT. Let $Sp\mathcal{F}_\lambda^a$ be the degenerate symplectic flag variety. These are projective singular irreducible \mathbb{G}_a^M degenerations of the classical flag varieties for symplectic group Sp_{2n} . We give an explicit construction for the varieties $Sp\mathcal{F}_\lambda^a$ and construct their desingularizations, similar to the Bott-Samelson resolutions in the classical case. We prove that $Sp\mathcal{F}_\lambda^a$ are normal locally complete intersections with terminal and rational singularities. We also show that these varieties are Frobenius split. Using the above mentioned results, we prove an analogue of the Borel-Weil-Bott theorem and obtain a q -character formula for the characters of irreducible Sp_{2n} -modules via the Atiyah-Bott-Lefschetz fixed points formula.

1. INTRODUCTION

Let G be a complex simple algebraic group and \mathfrak{g} be the corresponding Lie algebra. Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ be the Cartan decomposition, where \mathfrak{n}^- is a nilpotent subalgebra in \mathfrak{g} and \mathfrak{b} is the opposite Borel subalgebra. Let B and N^- be the subgroups in G corresponding to \mathfrak{b} and \mathfrak{n}^- . The Lie algebra \mathfrak{g} has a degeneration $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is the abelian Lie algebra with the underlying vector space \mathfrak{n}^- (see [Fe1], [Fe2]). Here $(\mathfrak{n}^-)^a$ is an abelian ideal in \mathfrak{g}^a and \mathfrak{b} acts on $(\mathfrak{n}^-)^a$ via the adjoint action on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$. The corresponding Lie group G^a is isomorphic to the semi-direct product $B \times \mathbb{G}_a^M$, where \mathbb{G}_a is the additive group of the field and $M = \dim \mathfrak{n}^-$.

For a dominant integral weight λ let V_λ be the corresponding irreducible highest weight \mathfrak{g} -module. Let $v_\lambda \in V_\lambda$ be a highest weight vector, recall that $V_\lambda = U(\mathfrak{n}^-)v_\lambda$. The (generalized) flag varieties G/P (P being a parabolic subgroup of G) are known to be embedded into the projective spaces $\mathbb{P}(V_\lambda)$ with λ chosen in such a way that P is the stabilizer of $\mathbb{C}v_\lambda$. Explicitly, the image is given by the G -orbit through the highest weight line $\mathbb{C}v_\lambda \in \mathbb{P}(V_\lambda)$. We denote the corresponding orbit by \mathcal{F}_λ .

Let V_λ^a be the degeneration of V_λ into a \mathfrak{g}^a -module. More precisely, V_λ^a is the associated graded space with respect to the PBW filtration on V_λ (see e.g. [FFL1], [FFL2]). Denote by the same symbol v_λ its image in V_λ^a , then $V_\lambda^a = \mathbf{S}^\bullet(\mathfrak{n}^-)v_\lambda$. The degenerate flag variety \mathcal{F}_λ^a is defined as the closure of the orbit $G^a(\mathbb{C}v_\lambda)$ inside $\mathbb{P}(V_\lambda^a)$. In contrast with the classical situation, the orbit itself is not closed, it is only an open cell inside \mathcal{F}_λ^a .

The varieties \mathcal{F}_λ^a for $\mathfrak{g} = \mathfrak{sl}_n$ were studied in [Fe1], [Fe2], [FF]. It was shown that in this case the \mathcal{F}_λ^a are singular projective algebraic varieties, which are flat degenerations of the classical \mathcal{F}_λ . The varieties \mathcal{F}_λ^a enjoy several nice properties: as in the classical case, \mathcal{F}_λ^a depends only on the class of regularity of λ ; they are irreducible normal Frobenius split locally complete intersections; they have a nice crepant desingularization isomorphic to a tower of successive \mathbb{P}^1 fibrations; the singularities of \mathcal{F}_λ^a are rational. In addition, the analogue of the classical Borel-Weil-Bott theorem still holds in the degenerate case.

In this paper we study the case of the symplectic group $G = Sp_{2n}$. Since we often use the connection between SL_{2n} flag varieties and the Sp_{2n} flag varieties, we use the notation $Sp\mathcal{F}_\lambda^a$ to distinguish it from the type A case. In the introduction we describe only the case of the complete degenerate symplectic flag variety, which we denote by $Sp\mathcal{F}_{2n}^a$. (In this case the highest weight λ has to be regular, but, as in the classical case, the orbit closure does not depend on a regular λ). However, in the main body of the paper we work out the general case of degenerate parabolic (partial) flag varieties as well.

Let W be a $2n$ -dimensional vector space with a basis w_1, \dots, w_{2n} . Let us equip W with a symplectic form which pairs non-trivially w_i and w_{2n+1-i} . We also denote by $pr_k : W \rightarrow W$, $k = 1, \dots, 2n$ the projection operators along w_k to the span of all the rest basis vectors. Our first theorem is as follows:

Theorem 1.1. *$Sp\mathcal{F}_{2n}^a$ can be realized inside the product $\prod_{k=1}^n Gr_k(W)$ of Grassmannians as a subvariety of collections (V_1, \dots, V_n) , $V_k \in Gr_k(W)$ satisfying the following conditions:*

$$pr_{k+1}V_k \subset V_{k+1}, \quad k = 1, \dots, n-1, \quad V_n \text{ is Lagrangian.}$$

These varieties are flat degenerations of the classical flag varieties Sp_{2n}/B .

Our next task is to study the singularities of $Sp\mathcal{F}_{2n}^a$. As in the type A case, we construct the desingularization SpR_{2n} of $Sp\mathcal{F}_{2n}^a$. We prove that SpR_{2n} is a Bott-Samelson type variety, i.e. it is isomorphic to a tower of successive \mathbb{P}^1 -fibrations. Using this desingularization, we prove our next theorem:

Theorem 1.2. *The varieties $Sp\mathcal{F}_{2n}^a$ are normal locally complete intersections and thus Cohen-Macaulay and Gorenstein. The singularities of $Sp\mathcal{F}_{2n}^a$ are terminal and rational. The varieties $Sp\mathcal{F}_{2n}^a$ are Frobenius split over $\overline{\mathbb{F}}_p$ for all primes p .*

We note that an important difference with the type A case is that the resolution of singularities $SpR_{2n} \rightarrow Sp\mathcal{F}_{2n}^a$ is no longer crepant, though it seems likely the resolution is still semismall.

Finally, we make a connection between the geometry of $Sp\mathcal{F}_{2n}^a$ and the representations V_λ^a . Namely, we prove an analogue of the Borel-Weil-Bott theorem (see e.g. [K]). Let us denote by ι_λ the natural map $Sp\mathcal{F}_{2n}^a \rightarrow \mathbb{P}(V_\lambda^a)$. Then we have:

Theorem 1.3. *$H^0(Sp\mathcal{F}_{2n}^a, \iota_\lambda^* \mathcal{O}(1))^* = V_\lambda^a$, and $H^k(Sp\mathcal{F}_{2n}^a, \iota_\lambda^* \mathcal{O}(1))^* = 0$ for $k > 0$.*

Using the rationality of the singularities of $Sp\mathcal{F}_{2n}^a$ and the Atiyah-Bott-Lefschetz fixed points formula for SpR_{2n} we derive a formula for the q -character of V_λ^a (see [FFL2] for the combinatorial formula).

Our paper is organized as follows:

In Section 2 we introduce notation, recall definitions and collect main results to be used in the main body of the paper.

In Section 3 we establish a connection between the PBW filtrations on \mathfrak{sp}_{2n} -modules and on \mathfrak{sl}_{2n} -modules.

In Section 4 we derive an explicit description for the symplectic Grassmannians and, more generally, for the symplectic degenerate flag varieties. We also prove that $Sp\mathcal{F}_\lambda^a$ are flat degeneration of their classical analogues $Sp\mathcal{F}_\lambda$.

In Section 5 we construct a resolution of singularities of $Sp\mathcal{F}_\lambda^a$.

In Section 6 we prove that all symplectic degenerate flag varieties are normal locally complete intersections and thus Cohen-Macaulay and Gorenstein.

In Section 7 we show that the singularities of $Sp\mathcal{F}_\lambda^a$ are terminal and thus canonical and rational.

In Section 8 we prove that the varieties $Sp\mathcal{F}_\lambda^a$ are Frobenius split. This allows to prove the degenerate analogue of the Borel-Weil-Bott theorem. We also derive an Atiyah-Bott-Lefschetz type formula for the graded characters of V_λ^a .

2. DEGENERATIONS: DEFINITIONS AND THE TYPE A CASE

In this section we fix the notation and recall the main results on the degenerate representations and degenerate flag varieties. We also collect important constructions and theorems from algebraic geometry, which we use in the paper.

2.1. Notation. Let \mathfrak{g} be a simple Lie algebra. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be the corresponding Borel subalgebra. We denote by α_i , $i = 1, \dots, \text{rk}(\mathfrak{g})$ the simple roots of \mathfrak{g} and by ω_i the corresponding fundamental weights. Let Φ^+ be the set of positive roots for \mathfrak{g} . For a root α we often write $\alpha > 0$ instead of $\alpha \in \Phi^+$. In particular, for $\mathfrak{g} = \mathfrak{sl}_n$ the simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and all positive roots are of the form

$$(2.1) \quad \alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j, 1 \leq i \leq j \leq n-1.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ the simple roots are $\alpha_1, \dots, \alpha_n$ and all positive roots are of the form

$$(2.2) \quad \alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j, 1 \leq i \leq j \leq n,$$

$$(2.3) \quad \alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_n + \alpha_{n-1} + \dots + \alpha_{2n-j}, 1 \leq i \leq n < j, i+j \leq 2n.$$

For a positive root α we fix a weight $-\alpha$ element $f_\alpha \in \mathfrak{n}^-$ and a weight α element $e_\alpha \in \mathfrak{n}$. Then the f_α , $\alpha \in \Phi^+$ form a basis in \mathfrak{n}^- and so do the root vectors e_α in \mathfrak{n} . We use the shorthand notations $f_{i,j} = f_{\alpha_{i,j}}$.

Let $P^+ = \sum_{i=1}^{\text{rk}(\mathfrak{g})} \mathbb{Z}_{\geq 0} \omega_i$ be the submonoid of the weight lattice for \mathfrak{g} generated by the fundamental weights, the elements of P^+ are the dominant integral weights. For $\lambda \in P^+$ we denote by V_λ the \mathfrak{g} -module of highest weight λ . We also fix a highest weight vector $v_\lambda \in V_\lambda$. In particular, one has

$$nv_\lambda = 0, hv_\lambda = \lambda(h)v_\lambda, h \in \mathfrak{h}, V_\lambda = U(\mathfrak{n}^-)v_\lambda.$$

Let G be a simple complex algebraic group with the Lie algebra \mathfrak{g} . We denote by B, N, T, N^- the subgroups in G corresponding to the Lie subalgebras $\mathfrak{b}, \mathfrak{n}, \mathfrak{h}, \mathfrak{n}^-$. Let P be a parabolic subgroup in G , the corresponding generalized flag variety is the quotient G/P . These varieties can be also realized as follows: for a dominant integral weight λ the group G acts naturally on the projective space $\mathbb{P}(V_\lambda)$. Assume that $(\lambda, \omega_i) = 0$ if and only if f_{α_i} belongs to the Lie algebra \mathfrak{p} of P . Then G/P is isomorphic to the G -orbit $G(\mathbb{C}v_\lambda) \subset \mathbb{P}(V_\lambda)$.

In what follows we denote the orbit $G(\mathbb{C}v_\lambda)$ by \mathcal{F}_λ . Note that \mathcal{F}_λ depends only on the class of regularity of λ , i.e. $\mathcal{F}_\lambda \simeq \mathcal{F}_\mu$ if and only if the supports of λ and μ coincide, i.e. $(\lambda, \omega_i) \neq 0$ if and only if $(\mu, \omega_i) \neq 0$.

For example, for $G = SL_n$ the Grassmann variety $Gr_d(n)$ of d -dimensional subspaces in n -dimensional space can be realized either as the quotient of SL_n by a maximal parabolic subgroup or as an orbit in the projective space of the fundamental representation V_{ω_d} . In general, all the quotients SL_n/P , where $P \supseteq B$ is a parabolic subgroup, are partial flag varieties: let $1 \leq d_1 < \dots < d_k \leq n$ be the integers such that $f_{\alpha_d} \in \mathfrak{p}$ if and only if $d = d_i$ for some i . Then G/P is known to coincide with the variety of collections $(V_i)_{i=1}^k$ of subspaces V_i in an n -dimensional space such that $\dim V_i = d_i$ and $V_i \subset V_{i+1}$. We note also that the same partial flag variety sits inside $\mathbb{P}(V_{\omega_{d_1+\dots+\omega_{d_k}}})$ as the orbit of the highest weight line. We denote this variety by $\mathcal{F}_{\mathbf{d}}$, $\mathbf{d} = (d_1, \dots, d_k)$.

Another important example for us is $G = Sp_{2n}$. The symplectic Grassmannians $SpGr_d(2n)$ (quotients of Sp_{2n} by maximal parabolic subgroups) are known to coincide with the variety of isotropic d -dimensional subspaces in a $2n$ -dimensional vector space equipped with a non-degenerate symplectic form. In general, as in the SL_{2n} case, let $1 \leq d_1 < \dots < d_k \leq n$ be the numbers such that $f_{\alpha_d} \in \mathfrak{p}$ if and only if $d = d_i$ for some i (\mathfrak{p} is the Lie algebra of a parabolic subgroup $P \subset Sp_{2n}$). Then Sp_{2n}/P is known to coincide with the variety of collections $(V_i)_{i=1}^k$ of subspaces $V_i \in SpGr_{d_i}(2n)$ such that $V_i \subset V_{i+1}$. In addition, the same partial flag variety sits inside $\mathbb{P}(V_{\omega_{d_1+\dots+\omega_{d_k}}})$ as the orbit of the highest weight line. We denote this variety by $Sp\mathcal{F}_{\mathbf{d}}$, $\mathbf{d} = (d_1, \dots, d_k)$.

2.2. Algebraic geometry. We recall the definition of a Frobenius split variety (see [BK]). Let X be an algebraic variety over an algebraically closed field of characteristics $p > 0$. Let $F : X \rightarrow X$ be the Frobenius morphism, i.e. an identity map on the underlying space X and the p -th power map on the space of functions. Then X is called Frobenius split if there exists a projection $F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ such that the composition $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ is the identity map. The Frobenius split varieties enjoy the following important property (see. e.g. Proposition 1 of [MR]):

Proposition 2.1. *Let X be a Frobenius split projective variety with a line bundle \mathcal{L} such that for some i and all large enough m $H^i(X, \mathcal{L}^m) = 0$. Then $H^i(X, \mathcal{L}) = 0$.*

The following two statements are proved in [MR], Proposition 4 and Proposition 8:

Proposition 2.2. *Let $f : Z \rightarrow X$ be a proper morphism of algebraic varieties such that $f_*\mathcal{O}_Z = \mathcal{O}_X$. If Z is Frobenius split, then X is also Frobenius split.*

Theorem 2.3. *Let Z be a smooth projective variety of dimension M and let Z_1, \dots, Z_M be codimension one subvarieties satisfying the following conditions:*

- (i) $\forall I \subset \{1, \dots, M\}$: *the intersection $\cap_{i \in I} Z_i$ is smooth of codimension $\#I$.*
- (ii) *There exists a global section s of the anti-canonical bundle ω^{-1} on Z such that the zero divisor of s equals $\mathcal{O}(Z_1 + \dots + Z_M + D)$ for some effective divisor D with $\cap_{i=1}^M Z_i \notin \text{supp} D$.*

Then Z is Frobenius split and for any subset $I \subset \{1, \dots, M\}$ the intersection $Z_I = \cap_{i \in I} Z_i$ is Frobenius split as well.

Now we recall some results on the singularities of algebraic varieties. Let X be a projective algebraic variety. If X is smooth, then ω_X denotes its canonical line bundle of top degree forms on X . For a singular X one can define a dualizing complex DX , which for Cohen-Macaulay varieties reduces to a (cohomologically shifted) sheaf. If X

is Gorenstein, then this sheaf is a line bundle denoted by ω_X . For example if X is a locally complete intersection, then it is known that X is Gorenstein and thus ω_X is a line bundle.

Let $\pi : Y \rightarrow X$ be a resolution of singularities of X . The singularities of X are called rational if $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^k\pi_*\mathcal{O}_Y = 0$ for $k > 0$ (if this holds for some desingularization, then it holds for all). An important property is that in this case for any line bundle \mathcal{L} on X and any k one has $H^k(X, \mathcal{L}) = H^k(Y, \pi^*\mathcal{L})$.

Let $Z \subset Y$ be an irreducible divisor. Then Z is said to be exceptional if $\dim \pi(Z) < \dim Z$. Assume that

$$\omega_Y = \pi^*\omega_X \otimes \bigotimes_{i=1}^N \mathcal{O}(a_i Z_i),$$

where Z_1, \dots, Z_N is the set of all exceptional divisors in Y . Then the singularities of X are called canonical, if $a_i \geq 0$ for all i . If moreover all a_i are positive then the singularities are called terminal. We will use the following theorem from [E], Theorem 1 (see also [F], 1.3):

Theorem 2.4. *Let X be a Gorenstein variety and $f : Y \rightarrow X$ be a desingularization such that the natural inclusion $f_*\omega_Y \hookrightarrow \omega_X$ is an isomorphism. Then X has rational singularities.*

2.3. Degenerate representations. Consider the PBW filtration $U(\mathfrak{n}^-)_s$ on the universal enveloping algebra of $U(\mathfrak{n}^-)$:

$$U(\mathfrak{n}^-)_s = \text{span}(x_1 \dots x_l : x_i \in \mathfrak{n}^-, l \leq s).$$

The associated graded algebra is isomorphic to the symmetric algebra $S^\bullet(\mathfrak{n}^-)$. For any $\lambda \in P^+$ consider the induced PBW filtration $F_s = U(\mathfrak{n}^-)_s v_\lambda$ on the space V_λ . We denote the associated graded space by V_λ^a , so V_λ^a carries additional grading:

$$V_\lambda = \bigoplus_{s \geq 0} V_\lambda(s) = \bigoplus_{s \geq 0} F_s / F_{s-1}.$$

Note that the operators from \mathfrak{n}^- acting on V_λ induce an action of the abelian algebra $(\mathfrak{n}^-)^a$ on V_λ^a , where $(\mathfrak{n}^-)^a$ is isomorphic to \mathfrak{n}^- as vector spaces. Thus $V_\lambda^a = S^\bullet(\mathfrak{n}^-)^a v_\lambda$, where (slightly abusing notations) we denote by $v_\lambda \in V_\lambda^a$ the image of the highest weight vector in V_λ . More concretely, $V_\lambda^a \simeq \mathbb{C}[f_\alpha]_{\alpha > 0} / I_\lambda$, where I_λ is an ideal.

We note that $(\mathfrak{n}^-)^a$ comes equipped with the natural structure of \mathfrak{b} -module. In fact, we have isomorphism of vector spaces $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ and \mathfrak{b} acts on the right hand side via the adjoint action. This gives the structure of \mathfrak{b} -module on $S^\bullet(\mathfrak{n}^-)$. We denote this \mathfrak{b} -action by \circ . By construction, the ideals I_λ are \mathfrak{b} -invariant and thus \mathfrak{b} acts on all modules V_λ^a . It is convenient to combine the actions of $(\mathfrak{n}^-)^a$ and \mathfrak{b} together. Namely, consider the Lie algebra $\mathfrak{g}^a \simeq \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian ideal and the Borel subalgebra \mathfrak{b} acts on $(\mathfrak{n}^-)^a$ via the induced adjoint action on the quotient space $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$. Then each V_λ^a carries the structure of a \mathfrak{g}^a -module (see [Fe1], [Fe2]). In what follows we will need explicit description of the ideals I_λ for $\mathfrak{g} = \mathfrak{sl}_n$ (see [FFL1]), and $\mathfrak{g} = \mathfrak{sp}_{2n}$ (see [FFL2]).

Theorem 2.5. *Let $\mathfrak{g} = \mathfrak{sl}_n$, $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$. Then*

$$I_\lambda = S(\mathfrak{n}^-) \left(U(\mathfrak{b}) \circ f_{\alpha_{i,j}}^{m_i + \dots + m_j + 1} \mid 1 \leq i \leq j \leq n-1 \right).$$

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\lambda = \sum_{i=1}^n m_i \omega_i$. Then

$$I_\lambda = S(\mathfrak{n}^-) \left(\begin{array}{l} U(\mathfrak{b}) \circ f_{\alpha_{i,j}}^{m_i + \dots + m_j + 1} \mid 1 \leq i \leq j < n, \\ U(\mathfrak{b}) \circ f_{\alpha_{i,2n-i}}^{m_i + \dots + m_n + 1} \mid 1 \leq i \leq n. \end{array} \right)$$

We will also use monomial bases in V_λ^a (see [FFL1], [FFL2], [V]). Namely, given an element $\mathbf{s} = (s_\alpha)_{\alpha > 0}$, $s_\alpha \in \mathbb{Z}_{\geq 0}$, we define $f^{\mathbf{s}} = \prod_{\alpha > 0} f_\alpha^{s_\alpha}$. We need the definition of a Dyck path inside the set of positive roots for \mathfrak{sl}_n and \mathfrak{sp}_{2n} .

Definition 2.6. We call a sequence $\mathbf{p} = (\beta(0), \beta(1), \dots, \beta(k))$ of positive roots of \mathfrak{sl}_n a Dyck path if $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ for some $1 \leq i \leq j \leq n-1$ and

$$\text{if } \beta(s) = \alpha_{p,q} \quad \text{then} \quad \beta(s+1) = \alpha_{p,q+1} \text{ or } \beta(s+1) = \alpha_{p+1,q}.$$

We call a sequence $\mathbf{p} = (\beta(0), \beta(1), \dots, \beta(k))$ of positive roots of \mathfrak{sp}_{2n} a (symplectic) Dyck path if $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ or $\beta(k) = \alpha_{j,2n-j}$ for some $1 \leq i \leq j \leq n-1$ and

$$\text{if } \beta(s) = \alpha_{p,q} \quad \text{then} \quad \beta(s+1) = \alpha_{p,q+1} \text{ or } \beta(s+1) = \alpha_{p+1,q}.$$

For a dominant \mathfrak{sl}_n -weight $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$ let $P_{\mathfrak{sl}_n}(\lambda) \subset \mathbb{R}_{\geq 0}^{\frac{1}{2}n(n-1)}$ be the polytope consisting of collections $(r_\alpha)_{\alpha \in \Phi_{\mathfrak{sl}_n}^+}$ such that for any \mathfrak{sl}_n Dyck path \mathbf{p} with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ one has

$$r_{\beta(0)} + \dots + r_{\beta(k)} \leq m_i + \dots + m_j.$$

Let $S_{\mathfrak{sl}_n}(\lambda)$ be the set of integral points in $P_{\mathfrak{sl}_n}(\lambda)$.

Similarly, for a dominant \mathfrak{sp}_{2n} -weight $\lambda = \sum_{i=1}^n m_i \omega_i$ let $P(\lambda) \subset \mathbb{R}_{\geq 0}^{n^2}$ be the polytope consisting of collections $(r_\alpha)_{\alpha \in \Phi_{\mathfrak{sp}_{2n}}^+}$ such that for any \mathfrak{sp}_{2n} Dyck path \mathbf{p} with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_j$ one has

$$r_{\beta(0)} + \dots + r_{\beta(k)} \leq m_i + \dots + m_j$$

and for any \mathfrak{sp}_{2n} Dyck path \mathbf{p} with $\beta(0) = \alpha_i$, $\beta(k) = \alpha_{j,2n-j}$ one has

$$r_{\beta(0)} + \dots + r_{\beta(k)} \leq m_i + \dots + m_n.$$

Let $S_{\mathfrak{sp}_{2n}}(\lambda)$ be the set of integral points in $P_{\mathfrak{sp}_{2n}}(\lambda)$.

Theorem 2.7. *For a dominant \mathfrak{sl}_n -weight λ the monomials $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S_{\mathfrak{sl}_n}(\lambda)$ form a basis of V_λ^a . For a dominant \mathfrak{sp}_{2n} -weight λ the monomials $f^{\mathbf{s}} v_\lambda$, $\mathbf{s} \in S_{\mathfrak{sp}_{2n}}(\lambda)$ form a basis of V_λ^a .*

Finally, for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$ the representations V_λ^a have the following important property (similar to the classical situation): for two dominant weights λ and μ the representation $V_{\lambda+\mu}^a$ is embedded into the tensor product $V_\lambda^a \otimes V_\mu^a$ as the highest component, i.e. there exists a homomorphism of \mathfrak{g}^a -modules:

$$(2.4) \quad V_{\lambda+\mu}^a \hookrightarrow V_\lambda^a \otimes V_\mu^a, \quad v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu.$$

2.4. Degenerate flag varieties. Here we follow [Fe1], [Fe2] and [FF]. Let G^a be the Lie group of the Lie algebra \mathfrak{g}^a . Explicitly, G^a is a semi-direct product of the Borel subgroup B and the normal abelian subgroup $(N^-)^a$, which is isomorphic to the product of $\dim \mathfrak{n}^-$ copies of the additive group of the field \mathbb{G}_a . Let λ be a dominant integral weight. Then G^a naturally acts on $\mathbb{P}(V_\lambda^a)$. By definition, the degenerate flag variety \mathcal{F}_λ^a is the orbit closure of the highest weight line, i.e.

$$\mathcal{F}_\lambda^a = \overline{G^a(\mathbb{C}v_\lambda)} \subset \mathbb{P}(V_\lambda^a).$$

We note that one has $\mathcal{F}_\lambda^a = \overline{(N^-)^a \cdot (\mathbb{C}v_\lambda)}$, so that the varieties \mathcal{F}_λ^a are so-called $\mathbb{G}_a^{\dim \mathfrak{n}^-}$ -varieties (i.e. the group acts on \mathcal{F}_λ^a with the open orbit, see [HT], [A]). We also note that in contrast with the classical situation, the orbit is properly contained in its closure.

The existence of the embeddings (2.4) implies two important properties of the varieties \mathcal{F}_λ^a in types A and C. First, for two dominant weights λ and μ one has the embedding of varieties $\mathcal{F}_{\lambda+\mu}^a \hookrightarrow \mathcal{F}_\lambda^a \times \mathcal{F}_\mu^a$ sending the highest weight line $\mathbb{C}v_{\lambda+\mu}$ to the product $\mathbb{C}v_\lambda \times \mathbb{C}v_\mu$. Secondly, $\mathcal{F}_\lambda^a \simeq \mathcal{F}_\mu^a$ provided $(\lambda, \omega_i) = 0$ if and only if $(\mu, \omega_i) = 0$, i.e. the supports of λ and μ coincide.

By definition, the group G^a acts on the variety \mathcal{F}_λ^a . The group G^a has a natural one-dimensional extension still acting on \mathcal{F}_λ^a . Consider the Lie algebra $\mathfrak{g}^a \oplus \mathbb{C}d$, where $[d, \mathfrak{b}] = 0$ and $[d, f_\alpha] = f_\alpha$ for any $\alpha > 0$. Then the extended algebra acts on V_λ^a and d acts as “a counting the PBW-degree operator”, i.e. on $V_\lambda(s)$ the operator d acts as a scalar s . Let $G^a \rtimes \mathbb{C}^*$ be the corresponding extended Lie group. Note that the dimension of the torus of the extended group is increased by one.

Assume now $G = SL_n$. Then the varieties \mathcal{F}_λ^a are known to be flat degenerations of the corresponding classical flag varieties. They have analogues of the Plücker embeddings defined by the degenerate Plücker relations. The varieties \mathcal{F}_λ^a share some important properties with their classical analogues. In particular, $\mathcal{F}_\lambda^a \simeq \mathcal{F}_\mu^a$ if and only if $(\lambda, \omega_i) = 0$ implies $(\mu, \omega_i) = 0$ and vice versa. They also enjoy the following explicit description. Let W be an n -dimensional vector space with a basis w_1, \dots, w_n . We define the projection operators $pr_k : W \rightarrow W$, $k = 1, \dots, n$ by the formula $pr_k(\sum_{i=1}^n c_i w_i) = \sum_{i \neq k} c_i w_i$. Fix a collection of numbers $1 \leq d_1 < \dots < d_k \leq n$. Then $\mathcal{F}_{\omega_{d_1} + \dots + \omega_{d_k}}^a$ is isomorphic to the variety of collections $(V_i)_{i=1}^k$ of subspaces $V_i \subset W$ satisfying for $i = 1, \dots, k-1$

$$\dim V_i = d_i, \quad pr_{d_{i+1}} \dots pr_{d_k} V_i \subset V_{i+1}.$$

In what follows, we sometimes denote this variety by $\mathcal{F}_{(d_1, \dots, d_k)}^a$ or simply by $\mathcal{F}_{\mathbf{d}}^a$ with $\mathbf{d} = (d_1, \dots, d_k)$. For instance, $\mathcal{F}_{(d)}^a \simeq \mathcal{F}_{\omega_d} \simeq Gr_d(n)$.

The varieties $\mathcal{F}_{\mathbf{d}}^a$ are singular projective algebraic varieties with rational singularities. They come equipped with line bundles \mathcal{L}_μ , where $(\mu, \omega_d) = 0$ unless $d \in \mathbf{d}$ and \mathcal{L}_μ is the pullback of the line bundle $\mathcal{O}(1)$ under the map $\mathcal{F}_{\mathbf{d}}^a \rightarrow \mathbb{P}(V_\mu^a)$. One of the main tools for the study of algebro-geometric properties of the degenerate flag varieties is an explicit construction of the resolution of singularities of \mathcal{F}_λ^a . Namely, given a collection $\mathbf{d} = (d_1, \dots, d_n)$ we denote by $P_{\mathbf{d}}$ the set of positive roots in the radical of the parabolic subalgebra of \mathfrak{g} containing (exactly) the simple roots $\alpha_{d_1}, \dots, \alpha_{d_k}$. We define $R_{\mathbf{d}}$ as the variety of collections of subspaces $V_{i,j}$, $1 \leq i \leq j \leq n-1$, $\alpha_{i,j} \in P_{\mathbf{d}}$, satisfying the

following conditions:

$$(2.5) \quad \dim V_{i,j} = i, \quad V_{i,j} \subset \text{span}(w_1, \dots, w_i, w_{j+1}, \dots, w_n),$$

$$(2.6) \quad V_{i,j} \subset V_{i+1,j}, \quad V_{i,j} \subset V_{i,j+1} \oplus \mathbb{C}w_{j+1}.$$

Using this desingularization, the following theorem was proved in [FF].

Theorem 2.8. (i) *The resolution $R_{\mathbf{d}} \rightarrow \mathcal{F}_{\mathbf{d}}^a$ is crepant.*

(ii) *The varieties $\mathcal{F}_{\mathbf{d}}^a$ are normal locally complete intersections (thus Cohen-Macaulay and Gorenstein).*

(iii) *The varieties $\mathcal{F}_{\mathbf{d}}^a$ have rational singularities and are Frobenius split.*

(iv) *The cohomology groups $H^m(\mathcal{F}_{\mathbf{d}}^a, \mathcal{L}_{\mu})$ vanish unless $m = 0$ and the zero cohomology is isomorphic to $(V_{\mu}^a)^*$.*

3. FILTRATIONS: \mathfrak{sp}_{2n} VS \mathfrak{sl}_{2n}

Let w_1, \dots, w_{2n} be a basis of a $2n$ -dimensional vector space W . We fix a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ defined by the conditions $\langle w_i, w_{2n+1-i} \rangle = 1$ for $1 \leq i \leq n$ and $\langle w_i, w_j \rangle = 0$ for all $1 \leq i, j \leq 2n$, $j \neq 2n+1-i$. We realize the symplectic group Sp_{2n} as the group of automorphisms of W leaving the form invariant. The diagonal matrices

$$T = \left\{ t = \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \mid t_1, \dots, t_n \in \mathbb{C}^* \right\}$$

form a maximal torus $T \subset Sp_{2n}$, and the subgroup $B \subset Sp_{2n}$ of upper triangular matrices is a Borel subgroup for Sp_{2n} . In such a realization the root vectors of $\mathfrak{sp}_{2n} = \text{Lie } Sp_{2n}$ are explicitly given by the formulas:

$$(3.1) \quad f_{i,j} = f_{\alpha_{i,j}} = \begin{cases} E_{j+1,i} - E_{2n+1-i,2n-j}, & 1 \leq i \leq j < n, \\ E_{j+1,i} + E_{2n+1-i,2n-j}, & j \geq n, i + j < 2n, \\ E_{2n+1-i,i}, & 1 \leq i \leq n. \end{cases}$$

As usual, $E_{j,i}$ is the matrix having zero entries everywhere except for the entry 1 in the j -th row, i -th column.

Given a dominant weight $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ for the group Sp_{2n} , we can consider this also as dominant weight for the larger group $SL_{2n} \supset Sp_{2n}$. In fact, let $V_{\lambda, \mathfrak{sl}_{2n}}$ be the corresponding irreducible \mathfrak{sl}_{2n} -representation and fix a highest weight vector $v_{\mathfrak{sl}_{2n}} \in V_{\lambda, \mathfrak{sl}_{2n}}$. We identify $V_{\lambda, \mathfrak{sp}_{2n}}$ with the irreducible \mathfrak{sp}_{2n} -submodule of $V_{\lambda, \mathfrak{sl}_{2n}}$ generated by $v_{\mathfrak{sl}_{2n}}$.

Consider now the action of \mathfrak{sl}_{2n} on $V_{\lambda, \mathfrak{sl}_{2n}}$. Let $\mathfrak{n}_{\mathfrak{sl}_{2n}}^+, \mathfrak{n}_{\mathfrak{sl}_{2n}}^- \subset \mathfrak{sl}_{2n}$ respectively be the Lie algebra of the unipotent radical of the Borel subgroup of upper triangular matrices $B_{SL_{2n}} \subset SL_{2n}$ and of the opposite Borel subgroup $B_{\overline{SL_{2n}}}$. Recall the group $SL_{2n}^a = B_{SL_{2n}} \times N_{\overline{SL_{2n}}}^-$ and the degenerate flag varieties $\mathcal{F}_{\lambda}^a := \overline{SL_{2n}^a \cdot v_{\mathfrak{sl}_{2n}}} \subset \mathbb{P}(V_{\lambda, \mathfrak{sl}_{2n}}^a)$.

The inclusions $Sp_{2n} \subset SL_{2n}$, $B \subset B_{SL_{2n}}$ and $\mathfrak{n}^- \subset \mathfrak{n}_{\mathfrak{sl}_{2n}}^-$ give rise to an action of $S(\mathfrak{n}^-)$ on $V_{\lambda, \mathfrak{sl}_{2n}}^a$. We want to compare this action with the action on $V_{\lambda, \mathfrak{sp}_{2n}}^a$. Let $\mathfrak{C} \subset V_{\lambda, \mathfrak{sl}_{2n}}^a$ be the cyclic module $\mathfrak{C}_\lambda := S(\mathfrak{n}^-).v_{\mathfrak{sl}_{2n}} \subset V_{\lambda, \mathfrak{sl}_{2n}}^a$.

Proposition 3.1. $\mathfrak{C}_\lambda \simeq V_{\lambda, \mathfrak{sp}_{2n}}^a$ as $S(\mathfrak{n}^-)$ -module.

Proof. Let $\Phi_{\mathfrak{sl}_{2n}}^+$, $\Phi_{\mathfrak{sp}_{2n}}^+$ be the sets of positive roots for \mathfrak{sl}_{2n} respectively \mathfrak{sp}_{2n} , see (2.1), (2.2), (2.3). We define an injective map $\varphi : \Phi_{\mathfrak{sp}_{2n}}^+ \rightarrow \Phi_{\mathfrak{sl}_{2n}}^+$ between the sets of positive roots by the formula $\alpha_{i,j} \mapsto \alpha_{i,j}$. For $\alpha = \alpha_{i,j} \in \Phi_{\mathfrak{sl}_{2n}}^+$ we write F_α for the root vector associated to $-\alpha$. Note that $F_\alpha = E_{j,i}$. For $\alpha \in \Phi_{\mathfrak{sp}_{2n}}^+$ we keep the usual notation f_α for the root vector in \mathfrak{n}^- associated to $-\alpha$. The matrix corresponding to $f_{\alpha_{i,j}}$ is given in (3.1). We use the shorthand notations $F_{i,j} = F_{\alpha_{i,j}}$, $f_{i,j} = f_{\alpha_{i,j}}$. Note that since $E_{j,i}$ with $i > n$ acts trivially on $V_{\lambda, \mathfrak{sl}_{2n}}^a$ ($(\lambda, \omega_k) = 0$ for $k > n$), the image of f_α in $\text{End } V_{\lambda, \mathfrak{sl}_{2n}}^a$ is equal to the image of $F_{\varphi(\alpha)}$.

The cyclic $\mathfrak{n}_{\mathfrak{sl}_{2n}}^-$ -module $V_{\lambda, \mathfrak{sl}_{2n}}^a$ can be described in terms of generators and relations, see Theorem 2.5. Let us write \mathfrak{C}_λ as a quotient $S(\mathfrak{n}^-)/I_{\mathfrak{C}_\lambda}$. Then the ideal $I_{\mathfrak{C}_\lambda}$ contains the ideal:

$$I_\lambda = \left\langle \begin{array}{l} U(\mathfrak{n}^+) \circ f_{i,j}^{a_i + \dots + a_j + 1} \mid 1 \leq i \leq j \leq n-1, \\ U(\mathfrak{n}^+) \circ f_{i, 2n-i}^{a_i + \dots + a_n + 1} \mid 1 \leq i \leq n. \end{array} \right\rangle$$

Theorem 2.5 implies that one has a natural $S(\mathfrak{n}^-)$ -equivariant surjective map $V_{\lambda, \mathfrak{sp}_{2n}}^a \rightarrow \mathfrak{C}_\lambda$. To prove that the map is an isomorphism, it suffices to show that both modules have the same dimension. For the proof we use the bases of $V_{\lambda, \mathfrak{sp}_{2n}}^a$ and $V_{\lambda, \mathfrak{sl}_{2n}}^a$, see Theorem 2.7. We define an injective map

$$\varphi : P_{\mathfrak{sp}_{2n}}(\lambda) \rightarrow P_{\mathfrak{sl}_{2n}}(\lambda), \quad (c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} \mapsto (d_\gamma)_{\gamma \in \Phi_{\mathfrak{sl}_{2n}}^+},$$

where $(d_\gamma)_{\gamma \in \Phi_{\mathfrak{sl}_{2n}}^+} := \varphi((c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+})$ is defined by the rule:

$$d_\gamma := \begin{cases} c_\beta & \text{if } \gamma = \varphi(\beta) \text{ for some } \beta \in \Phi_{\mathfrak{sp}_{2n}}^+, \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to see that

$$(3.2) \quad (d_\gamma)_{\gamma \in \Phi_{\mathfrak{sl}_{2n}}^+} = \varphi((c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+}) \in P_{\mathfrak{sl}_{2n}}(\lambda)$$

is an element in the associated polytope for $\mathfrak{g} = \mathfrak{sl}_{2n}$. We know that the vectors

$$\left\{ \prod_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} f_\beta^{c_\beta} v_{\mathfrak{sp}_{2n}} \mid (c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} \in P_{\mathfrak{sp}_{2n}}(\lambda) \right\} \subset V_{\lambda, \mathfrak{sp}_{2n}}^a$$

form a basis. We want to show that the vectors

$$\left\{ \prod_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} f_\beta^{c_\beta} v_{\mathfrak{sl}_{2n}} \mid (c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} \in P_{\mathfrak{sp}_{2n}}(\lambda) \right\} \subset \mathfrak{C}_\lambda \subset V_{\lambda, \mathfrak{sl}_{2n}}^a$$

are linearly independent. Recall that f_β acts on $V_{\lambda, \mathfrak{sl}_{2n}}^a$ in the same way as $F_{\varphi(\beta)}$ for the roots $\beta = \alpha_{i,j}$, $1 \leq i \leq j < n$ and $\beta = \alpha_{i, 2n-i}$, $1 \leq i \leq n$, and for $\beta = \alpha_{i,j}$ with $j \geq n$,

$i + j < 2n$ f_β acts as $F_{i,j} + F_{2n-j,2n-i}$. Hence for $(d_\gamma)_{\gamma \in \Phi_{\mathfrak{sl}_{2n}}^+} = \varphi((c_\beta)_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+})$ we get

$$(3.3) \quad \begin{aligned} b_{(c_\beta)} &:= \prod_{\beta \in \Phi_{\mathfrak{sp}_{2n}}^+} f_\beta^{c_\beta} v_{\mathfrak{sl}_{2n}} \\ &= \left(\prod_{i+j < 2n, j > n} (F_{i,j} + F_{2n-j,2n-i})^{d_{\alpha_{i,j}}} \right) \left(\prod_{1 \leq i \leq j < n} F_{i,j}^{d_{\alpha_{i,j}}} \right) \left(\prod_{1 \leq i \leq n} F_{i,2n-i}^{d_{\alpha_{i,2n-i}}} \right) v_{\mathfrak{sl}_{2n}}. \end{aligned}$$

Let $\alpha_1, \dots, \alpha_{2n-1}$ be the set of simple roots for the root system $\Phi_{\mathfrak{sl}_{2n}}$ (Bourbaki enumeration). We associate to a vector $v_{(\ell_\beta)} = \prod_{\beta \in \Phi_{\mathfrak{sl}_{2n}}^+} F_\beta^{\ell_\beta} v_{\mathfrak{sl}_{2n}}$ the collection

$$\zeta(v_{(\ell_\beta)}) = (r_1, \dots, r_{2n-1}) \quad \text{such that} \quad \sum_{\beta \in \Phi_{\mathfrak{sl}_{2n}}^+} \ell_\beta \beta = \sum_{i=1}^{2n-1} r_i \alpha_i,$$

called the *root weight*. We define a partial order on these vectors by:

$$v_{(\ell_\beta)} > v_{(k_\beta)} \quad \text{if} \quad \zeta(v_{(\ell_\beta)}) > \zeta(v_{(k_\beta)})$$

with respect to the induced lexicographic order on the $(2n-1)$ -tuples. It follows for the vector $b_{(c_\beta)}$ in (3.3):

$$(3.4) \quad b_{(c_\beta)} = v_{(d_\alpha)} + \text{a sum} \sum x_{(\ell_\beta)} v_{(\ell_\beta)} \text{ of smaller terms,}$$

because if one chooses in a factor in (3.3) the factor $F_{2n-j,2n-i}$ instead of $F_{i,j}$, then the corresponding $(2n-1)$ -tuple is strictly smaller with respect to the partial order above. It follows in particular that $b_{(c_\beta)} \neq 0$ because $v_{(d_\alpha)}$ is a basis vector for $V_{\lambda, \mathfrak{sl}_{2n}}^a$ by (3.2) and the smaller summands are weight vectors of different weights.

The linear independence of the vectors $\{b_{(c_\beta)} \mid (c_\beta) \in P_{\mathfrak{sp}_{2n}}(\lambda)\}$, follows along the same lines: given a linear dependence relation for the $b_{(c_\beta)}$, choose a maximal element (p_1, \dots, p_{2n-1}) among the $\zeta(v_{(\varphi(c_\beta))})$. By weight reasons the linear independence of the $b_{(c_\beta)}$ implies a linear independence relation between the summands of associated root weight (p_1, \dots, p_{2n-1}) . By maximality, a summand of $b_{(c_\beta)}$ as in (3.4) has root weight (p_1, \dots, p_{2n-1}) if and only if $v_{(d_\alpha)}$ has root weight (p_1, \dots, p_{2n-1}) , so we obtain a linear dependence relation between the $v_{(d_\alpha)}$, which is not possible.

It follows: $\dim \mathfrak{C}_\lambda = \dim V_{\lambda, \mathfrak{sp}_{2n}}^a$ and the canonical map $V_{\lambda, \mathfrak{sp}_{2n}}^{ab} \rightarrow \mathfrak{C}_\lambda$ is an isomorphism. \square

Corollary 3.2. $Sp\mathcal{F}_\lambda \subset \mathcal{F}_\lambda$ for any λ of the form $\sum_{i=1}^n a_i \omega_i$.

Proof. By Proposition 3.1, we can identify $Sp\mathcal{F}_\lambda^a := \overline{Sp_{2n}^a \cdot v_{Sp_{2n}}} \subset \mathbb{P}(V_{\lambda, Sp_{2n}}^a)$ with $\overline{Sp_{2n}^a \cdot v_{\mathfrak{sl}_{2n}}} \subset \mathbb{P}(\mathfrak{C}_\lambda)$ and hence

$$Sp\mathcal{F}_\lambda^a = \overline{Sp_{2n}^a \cdot v_{\mathfrak{sl}_{2n}}} \subset \overline{SL_{2n}^a \cdot v_{\mathfrak{sl}_{2n}}} = \mathcal{F}_\lambda^a \subset \mathbb{P}(V_{\lambda, \mathfrak{sl}_{2n}}^a). \quad \square$$

4. EXPLICIT DESCRIPTION

In this section we give an explicit description of the symplectic degenerate flag varieties in the linear algebra terms. We start with the case of symplectic Grassmannians.

4.1. The degenerate symplectic Grassmann variety. Let V_{ω_k} be the irreducible fundamental Sp_{2n} -representation of highest weight ω_k (Bourbaki enumeration). In particular, $V_{\omega_1} \simeq W$ is the standard vector representation. Let $V_{\omega_k} \hookrightarrow \Lambda^k W$ be the canonical embedding defined by mapping a fixed highest weight vector to $w_1 \wedge \dots \wedge w_k$. In the following we will identify V_{ω_k} with the image. By definition, the symplectic degenerate Grassmann variety $SpGr_k^a(W) \subset \mathbb{P}(V_{\omega_k}^a)$ is equal to $Sp\mathcal{F}_{\omega_k}^a$.

Let $P \subset Sp_{2n}$ be the maximal parabolic subgroup associated to ω_k . Set $\mathfrak{p} = \text{Lie } P$ and let $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ be a decomposition such that \mathfrak{l} is the Lie algebra of the Levi subgroup of P containing T , $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, and \mathfrak{u}^- is the Lie algebra of the unipotent radical of the parabolic subgroup P^- opposite to P . The Lie algebra \mathfrak{u}^- consists of matrices of the form

$$(4.1) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ C & B^{tn} & -A^{nt} & 0 \end{pmatrix},$$

where A, B are $(n-k) \times k$ matrices, X^{nt} denotes the transposed matrix with respect to the skew diagonal, and C is a $k \times k$ matrix such that $C^{nt} = C$.

We write $W = W_{k,1} \oplus W_{k,2} \oplus W_{k,3}$, where

$$W_{k,1} = \text{span}(w_1, \dots, w_k), \quad W_{k,2} = \text{span}(w_{k+1}, \dots, w_{2n-k}), \quad W_{k,3} = \text{span}(w_{2n-k+1}, \dots, w_{2n}).$$

Denote by $p_{1,3}$ the projection $p_{1,3} : W \rightarrow W_{k,1} \oplus W_{k,3}$, i.e.,

$$p_{1,3}(x_1, \dots, x_{2n}) = (x_1, \dots, x_k, 0, \dots, 0, x_{2n-k+1}, \dots, x_{2n}).$$

Proposition 4.1.

$$SpGr_k^a(W) = \{U \in Gr_k(W) \mid p_{1,3}(U) \text{ is isotropic}\}.$$

Denote by Z_k the subvariety $\{U \in Gr_k(W) \mid p_{1,3}(U) \text{ is isotropic}\}$. The first simple observation in the proof of the proposition is:

Lemma 4.2. $Gr_k(W_{k,1} \oplus W_{k,2}) \subset Z_k$.

Proof. Since $p_{1,3}(U) \subset W_{k,1}$ for all $U \in Gr_k(W_{k,1} \oplus W_{k,2})$, we have $p_{1,3}(U)$ isotropic and hence $U \in Z_k$. \square

Denote by $Sp_{2k} \subset Sp_{2n}$ the symplectic subgroup acting only on the first and last k coordinates. The matrix for an element of this subgroup looks like

$$\begin{pmatrix} A & 0 & 0 & B \\ 0 & \mathbb{I} & 0 & 0 \\ 0 & 0 & \mathbb{I} & 0 \\ C & 0 & 0 & D \end{pmatrix},$$

where A, B, C, D are $k \times k$ matrices,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a $2k \times 2k$ -symplectic matrix, and \mathbb{I} is an $(n-k) \times (n-k)$ identity matrix. Further, let

$$P_k := \{g \in Sp_{2k} \mid C = 0\},$$

then P_k is the maximal parabolic subgroup of Sp_{2k} associated to the long simple root. The next simple observation is:

Lemma 4.3. $Z_k = Sp_{2k} \cdot Gr_k(W_{k,1} \oplus W_{k,2})$.

Proof. Note first that $Y = Gr_k(W_{k,1} \oplus W_{k,2})$ is stable under the action of P_k , so $Sp_{2k} \cdot Y$ is closed. Since Z_k is Sp_{2k} -stable, by Lemma 4.2 we have $Sp_{2k} \cdot Y \subset Z_k$. Now assume $U \in Z_k$ and let $\bar{U} = pr_{1,3}(U)$. This subspace of $W_{k,1} \oplus W_{k,3}$ is isotropic by assumption, so there exists a $g \in Sp_{2k}$ such that $g \cdot \bar{U} \subset W_{k,1}$, and hence $g \cdot U \subset W_{k,1} \oplus W_{k,2}$. It follows $g \cdot U \in Gr_k(W_{k,1} \oplus W_{k,2})$ and hence $Z_k = Sp_{2k} \cdot Gr_k(W_{k,1} \oplus W_{k,2})$. \square

As an immediate consequence one sees that Z_k is the image of the canonical product map

$$\tilde{\pi} : Sp_{2k} \times Gr_k(W_{k,1} \oplus W_{k,2}) \rightarrow Z_k \subset Gr_k(W), \quad (g, U) \mapsto g \cdot U,$$

and hence:

Corollary 4.4. Z_k is irreducible.

Proof of Proposition 4.1. Recall the description of the Lie algebra \mathfrak{u}^- in (4.1) and the inclusion of the abelianized action described in Proposition 3.1. It follows that if $\gamma \in \mathfrak{u}^-$ is a matrix as in (4.1), then in $V_{\omega_k}^a$ we have $\exp \gamma \cdot [w_1 \wedge \dots \wedge w_k]$ is the k -dimensional subspace having as basis the column vectors of the matrix

$$(4.2) \quad \begin{pmatrix} \mathbb{I} \\ A \\ B \\ C \end{pmatrix}.$$

The symmetry condition of the matrix C implies that these subspaces lie in Z_k . To prove equality, for $\underline{i} = (i_1, \dots, i_k)$, $1 \leq i_1 < \dots < i_k \leq 2n$, let $w_{\underline{i}} = w_{i_1} \wedge \dots \wedge w_{i_k} \in \Lambda^k W$ and denote by $p_{\underline{i}}$ the corresponding Plücker coordinate, i.e. $p_{\underline{i}}(w_{\underline{j}}) = \delta_{\underline{i}, \underline{j}}$.

Consider in Z_k the open affine set

$$Z_{k(1,2,\dots,k)} = \{U \in Z_k \mid p_{(1,2,\dots,k)}(U) \neq 0\}.$$

The spaces having a basis as in (4.2) lie in $Z_{k(1,2,\dots,k)}$, so this set is non-empty and (since Z_k is irreducible) dense. Now given an element in $Z_{k(1,2,\dots,k)}$, one can find a basis corresponding to the columns of a matrix of the form

$$(4.3) \quad \begin{pmatrix} \mathbb{I} \\ A \\ B \\ C \end{pmatrix},$$

where A, B are $(n-k) \times k$ matrices, and the condition $pr_{1,3}(U)$ is isotropic implies that $C = C^{nt}$. It follows that $Z_{k(1,2,\dots,k)} = Sp_{2n}^a \cdot [w_1 \wedge \dots \wedge w_k]$, and hence $Z_k = SpGr_k^a(W)$. \square

Remark 4.5. a) It is easy to check that the canonical map

$$\pi : Sp_{2k} \times_{P_k} Gr_k(W_{k,1} \oplus W_{k,2}) \rightarrow SpGr_k^a(W)$$

is a desingularization.

b) Denote by Z_j , $0 \leq j \leq k$, the subset

$$Z_j := \{U \in SpGr_k^a(W) \mid \dim pr_{1,3}(U) = j\}$$

and define

$$Gr_k^j(W_{k,1} \oplus W_{k,2}) := \{U \in Gr_k(W_{k,1} \oplus W_{k,2}) \mid \dim pr_{1,3}(U) = j\}.$$

We have obviously a partition:

$$SpGr_k(W) := \bigcup_{j=0,\dots,k} Z_j.$$

One can show: Z_j is smooth, $\overline{Z_j} = \bigcup_{i=0,\dots,j} Z_i$, the desingularization map above is compatible with the partition and induces maps

$$\pi^j : Sp_{2k} \times_{P_k} Gr_k^j(W_{k,1} \oplus W_{k,2}) \rightarrow Z_j$$

such that the fibres of π^j are all of dimension $\frac{1}{2}(k-j)(k-j+1)$. In particular, the map π is semismall but not small (failure for $j = k-1$).

4.2. Symplectic degenerate flag varieties. Based on Proposition 4.1 we derive explicit description for all symplectic degenerate flag varieties. For a regular dominant weight λ we denote the complete symplectic degenerate flag variety \mathcal{F}_λ^a by $Sp\mathcal{F}_{2n}^a$.

Recall the basis w_1, \dots, w_{2n} of W . We denote by $pr_i : w \rightarrow W$ the projections along the w_i , i.e. $pr_i(\sum_{j=1}^{2n} c_j w_j) = \sum_{j \neq i} c_j w_j$.

Theorem 4.6. *The degenerate symplectic flag variety $Sp\mathcal{F}_{2n}^a$ is naturally embedded into the product $\prod_{i=1}^n SpGr_i^a(2n)$ of degenerate symplectic Grassmannians. The image of the embedding is equal to the set of collections $(V_i)_{i=1}^n$, $V_i \in SpGr_i^a(2n)$ satisfying the conditions*

$$pr_{i+1}V_i \subset V_{i+1}, i = 1, \dots, n-1.$$

Proof. According to the proof of Proposition 4.1 we have

$$Sp\mathcal{F}_{2n}^a \cap \prod_{i=1}^n Z_{i(1,\dots,i)} = \{(V_1, \dots, V_n) : V_i \in Z_{i(1,\dots,i)}, pr_{i+1}V_i \subset V_{i+1}\}.$$

Moreover this subvariety of $Sp\mathcal{F}_{2n}^a$ coincides with the Sp_{2n}^a -orbit of the highest weight line. Thus we only need to show that the variety defined above is irreducible. This is proved in Corollary 5.7, using the desingularization for $Sp\mathcal{F}_{2n}^a$. \square

For a subspace $V \subset W$ we denote by $V^\perp \subset W$ the orthogonal complement to V . Define an order two automorphism $\sigma \in Aut(\prod_{i=1}^{2n-1} Gr_i(2n))$ by the formula

$$\sigma(V_i)_{i=1}^{2n-1} = (V_{2n-1}^\perp, V_{2n-2}^\perp, \dots, V_1^\perp).$$

Proposition 4.7. *The automorphism σ defines an order two automorphism of the complete degenerate flag variety \mathcal{F}_{2n}^a . The set of σ -fixed points $(\mathcal{F}_{2n}^a)^\sigma$ is isomorphic to the complete symplectic degenerate flag variety $Sp\mathcal{F}_{2n}^a$.*

Proof. Follows from the definition. \square

Let $\mathbf{d} = (d_1, \dots, d_k)$ be a collection of integers such that $1 \leq d_1 < \dots < d_k \leq n$. Let $Sp\mathcal{F}_{\mathbf{d}}^a$ be the parabolic degenerate flag variety, corresponding to the highest weight $\sum_{i=1}^k \omega_{d_i}$.

Theorem 4.8. *The parabolic degenerate symplectic flag variety $Sp\mathcal{F}_{\mathbf{d}}^a$ is naturally embedded into the product $\prod_{i=1}^k SpGr_{d_i}^a(2n)$ of degenerate symplectic Grassmannians. The image of the embedding is equal to the variety of collections $(V_i)_{i=1}^k$, $V_i \in SpGr_{d_i}^a(2n)$ satisfying the conditions*

$$pr_{d_i+1} \dots pr_{d_{i+1}} V_{d_i} \subset V_{d_{i+1}}, i = 1, \dots, k-1.$$

Proof. As in above, we only need to show the irreducibility of $Sp\mathcal{F}_{\mathbf{d}}^a$. This is proved in Corollary 5.12. \square

Let us fix a collection $\mathbf{d} = (d_1, \dots, d_k)$, $1 \leq d_1 < \dots < d_k \leq n$. Assume $d_k < n$. Then we define an extended collection \mathbf{D} by the formula

$$\mathbf{D} = (d_1, \dots, d_k, 2n - d_k, \dots, 2n - d_1).$$

If $d_k = n$, then we set $\mathbf{D} = (d_1, \dots, d_k, 2n - d_{k-1}, \dots, 2n - d_1)$. We define an order two automorphism $\sigma_{\mathbf{d}} \in Aut(\prod_{d \in \mathbf{D}} Gr_d(2n))$ by the formula

$$\sigma_{\mathbf{d}}(V_i)_{i=1}^l = (V_l^\perp, V_{l-1}^\perp, \dots, V_1^\perp),$$

where $l = 2k$ if $d_k < n$ and $l = 2k - 1$ otherwise.

Proposition 4.9. *The automorphism $\sigma_{\mathbf{d}}$ defines an order two automorphism of the SL_{2n} parabolic degenerate flag variety $\mathcal{F}_{\mathbf{D}}^a$. The set of $\sigma_{\mathbf{d}}$ -fixed points $(\mathcal{F}_{\mathbf{D}}^a)^{\sigma_{\mathbf{d}}}$ is isomorphic to the parabolic symplectic degenerate flag variety $Sp\mathcal{F}_{\mathbf{D}}^a$.*

4.3. The degeneration. In this subsection we prove that the varieties $Sp\mathcal{F}_{\mathbf{d}}^a$ are flat degenerations of their classical analogues. Let J_s be the $2n \times 2n$ -matrix

$$\begin{pmatrix} 0 & 0 & 0 & I_k \\ 0 & 0 & sI_{n-k} & 0 \\ 0 & -sI_{n-k} & 0 & 0 \\ -I_k & 0 & 0 & 0 \end{pmatrix}, \quad \text{where } I_l \text{ is a } l \times l \text{ matrix } \begin{pmatrix} 0 & 0 & 1 \\ 0 & \dots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix defines a non-degenerate symplectic form for $s \neq 0$, the group Sp_{2n} is the one leaving invariant the form for $s = 1$, and

$$SpGr_k(2n) = \{U \in Gr_k(\mathbb{C}^{2n}) \mid U \text{ is isotropic with respect to } J_0\}.$$

Denote by η the following one-parameter subgroup:

$$\eta : \mathbb{C}^* \rightarrow D = \text{diagonal matrices in } GL_{2n}, \quad s \mapsto \begin{pmatrix} \mathbb{1}_k & 0 & 0 \\ 0 & s\mathbb{1}_{2(n-k)} & 0 \\ 0 & 0 & \mathbb{1}_k \end{pmatrix}.$$

then $\eta(s)^t J_1 \eta(s) = J_{s^2}$, and it follows that if $U \in SpGr_k(\mathbb{C}^{2n})$ is a subspace isotropic with respect to J_1 , then $\eta(s^{-1})(U)$ is isotropic with respect to J_{s^2} .

Recall that Sp_{2n}/P_k is sitting in the Grassmann variety as the set of isotropic subspaces. Consider

$$(4.4) \quad Y := \overline{\{(\eta(s^{-1})(U), s) \mid s \in \mathbb{C}^*, U \in Sp_{2n}/P_k\}} \subset Gr_k(\mathbb{C}^{2n}) \times \mathbb{C}$$

together with the natural projection $\phi : Y \rightarrow \mathbb{C}$ onto the second factor.

Proposition 4.10. *The projection map ϕ is flat, $\phi^{-1}(s) \simeq Sp_{2n}/P_k$ for all $s \neq 0$ and $\phi^{-1}(0) = SpGr_k^a(\mathbb{C}^{2n})$.*

Proof. The map is flat because Y is irreducible and ϕ is dominant ([H], Chap. III, Proposition 9.7). It remains to show that the fibres are the spaces described above. We assume without loss of generality: $k \geq 2$. Consider the map

$$\begin{aligned} \Psi : \Lambda^k \mathbb{C}^{2n} \times \mathbb{C} &\rightarrow \Lambda^{k-2} \mathbb{C}^{2n}, \\ (v_1 \wedge \dots \wedge v_k, s) &\mapsto \sum_{\ell < m} Q_s(v_\ell, v_m) (-1)^{m+\ell-1} v_1 \wedge \dots \wedge \hat{v}_\ell \wedge \dots \wedge \hat{v}_m \wedge \dots \wedge v_k \end{aligned}$$

where $Q_s(v, w) = v^t J_s w$. The map is homogeneous with respect to $\Lambda^k \mathbb{C}^{2n}$, the preimage $\Psi^{-1}(0)$ defines hence a closed subset of $\mathbb{P}(\Lambda^k \mathbb{C}^{2n}) \times \mathbb{C}$, and the intersection with the Grassmann variety defines a closed subset $\tilde{Y} \subset Gr_k(\mathbb{C}^{2n}) \times \mathbb{C}$. Let $\sigma : \tilde{Y} \rightarrow \mathbb{C}$ be the projection onto the second component. By the definition of Ψ one has $(U, s) \in \sigma^{-1}(s) \subset \tilde{Y}$ if and only if $\Psi(v_1 \wedge \dots \wedge v_k, s) = 0$ for a basis $\{v_1, \dots, v_k\}$ of U , that is iff U is an isotropic subspace with respect to the form Q_s . It follows that \tilde{Y} and Y have in common the subset

$$Y' = \{(\eta(s^{-1})(U), s) \mid s \in \mathbb{C}^*, U \in Sp_{2n}/P_k\} \subset Y.$$

so $Y = \overline{Y'}$ is an irreducible component of \tilde{Y} . Since $\sigma^{-1}(0) = SpGr_k^a(\mathbb{C}^{2n})$, it follows that σ has equidimensional and irreducible fibers, and hence \tilde{Y} is irreducible, which implies that $Y = \tilde{Y}$ and $\sigma = \phi$. \square

Let $\mathbf{d} = (d_1, \dots, d_m)$.

Theorem 4.11. *The varieties $Sp\mathcal{F}_{\mathbf{d}}^a$ are flat degenerations of their classical analogues $Sp\mathcal{F}_{\mathbf{d}}$.*

Proof. Let us denote by Y_k the variety defined by (4.4). Recall the projections $\phi_k : Y_k \rightarrow \mathbb{C}$. Let $Y_{\mathbf{d}}$ be the fibered product of all Y_{d_i} over \mathbb{C} , i.e.

$$Y_{\mathbf{d}} = \{(y_1, \dots, y_m) : y_i \in Y_{d_i}, \phi_{d_1}(y_1) = \dots = \phi_{d_m}(y_m)\}.$$

Recall (see [Fe1]) that there exists flat degeneration $\psi_{\mathbf{d}} : M_{\mathbf{d}} \rightarrow \mathbb{C}$ of the SL_{2n} flag variety $\mathcal{F}_{\mathbf{d}}$ into the degenerate version $\mathcal{F}_{\mathbf{d}}^a$, so $\psi_{\mathbf{d}}^{-1}(0) \simeq \mathcal{F}_{\mathbf{d}}^a$ and the general fiber is isomorphic to $\mathcal{F}_{\mathbf{d}}$. We note that both $M_{\mathbf{d}}$ and $Y_{\mathbf{d}}$ are subvarieties in the product $\mathbb{C} \times \prod_{i=1}^m Gr_{d_i}(\mathbb{C}^{2n})$. Consider the intersection $M_{\mathbf{d}} \cap Y_{\mathbf{d}}$ and the natural projection $\varphi_{\mathbf{d}} : M_{\mathbf{d}} \cap Y_{\mathbf{d}} \rightarrow \mathbb{C}$. Then the general fiber is isomorphic to the classical symplectic flag variety $Sp\mathcal{F}_{\mathbf{d}}$ and $\varphi_{\mathbf{d}}^{-1}(0) \simeq Sp\mathcal{F}_{\mathbf{d}}^a$. Since $M_{\mathbf{d}} \cap Y_{\mathbf{d}} = \overline{\varphi_{\mathbf{d}}^{-1}(\mathbb{C}^*)}$, the left hand side is irreducible. Now [H], Chap. III, Proposition 9.7 implies the theorem. \square

5. RESOLUTION OF SINGULARITIES

In this section we construct varieties $SpR_{\mathbf{d}}$ which serve as desingularizations for symplectic degenerate flag varieties $Sp\mathcal{F}_{\mathbf{d}}^a$. We start with the case of complete flags.

5.1. Complete flag varieties. Recall the $2n$ -dimensional vector space W with a basis w_i , $i = 1, \dots, 2n$. Let $W_{i,j}$ be the linear span of the vectors $w_1, \dots, w_i, w_{j+1}, \dots, w_{2n}$.

We define SpR_{2n} as the variety of collections of subspaces

$$V_{i,j}, \quad 1 \leq i \leq j \leq 2n, \quad i + j \leq 2n$$

subject to the conditions

$$(5.1) \quad \dim V_{i,j} = i, \quad V_{i,j} \subset W_{i,j},$$

$$(5.2) \quad V_{i,j} \subset V_{i+1,j}, \quad pr_{j+1} V_{i,j} \subset V_{i,j+1},$$

$$(5.3) \quad V_{i,2n-i} \text{ are isotropic for } i = 1, \dots, n.$$

Remark 5.1. It is often convenient to view the spaces $V_{i,j}$ as being attached to the positive roots $\alpha_{i,j}$ of \mathfrak{sp}_{2n} (see (2.2), (2.3)). So in what follows we sometimes denote $V_{i,j}$ by $V_{\alpha_{i,j}}$.

We note that there is a natural embedding $SpR_{2n} \subset \prod Gr_i(W_{i,j})$, where the product is taken over $i \leq j$, $i + j \leq 2n$.

Let us order the set of positive roots of \mathfrak{sp}_{2n} into a sequence $\beta_1, \beta_2, \dots, \beta_{n^2}$ as follows:

$$\beta_1 = \alpha_{1,2n-1},$$

$$\beta_2 = \alpha_{1,2n-2},$$

$$\beta_3 = \alpha_{2,2n-2}, \quad \beta_4 = \alpha_{1,2n-3},$$

$$\beta_5 = \alpha_{2,2n-3}, \quad \beta_6 = \alpha_{1,2n-4},$$

$$\beta_7 = \alpha_{3,2n-3}, \quad \beta_8 = \alpha_{2,2n-4}, \quad \beta_9 = \alpha_{1,2n-5},$$

and, finally,

$$\beta_{n^2-n+1} = \alpha_{n,n}, \quad \beta_{n^2-n+2} = \alpha_{n-1,n-1}, \dots, \beta_{n^2} = \alpha_{1,1}.$$

For $l = 1, \dots, n^2$ we define the step l variety $SpR_{2n}(l)$ as the image of projection

$$(V_{\beta_k})_{k=1}^{n^2} \mapsto (V_{\beta_k})_{k=1}^l.$$

Lemma 5.2. *The variety $SpR_{2n}(l)$ consists of collections $(V_{\beta_k})_{k=1}^l$ satisfying conditions (5.1), (5.2), (5.3) whenever the pairs of indices (i,j) appearing in the conditions satisfy $\alpha_{i,j} = \beta_m$ with $m \leq l$.*

For example, $SpR_{2n}(n^2) = SpR_{2n}$. We also set $SpR_{2n}(0) = pt$.

Lemma 5.3. *For all $l = 1, \dots, n^2$ the map $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ is a \mathbb{P}^1 -fibration.*

Proof. Fix a point $(V_{\beta_k})_{k=1}^{l-1} \in SpR_{2n}(l-1)$. We need to show that the possible choices of V_{β_l} are parametrized by points of \mathbb{P}^1 . Let $\beta_l = \alpha_{i,j}$. First, let $i + j = 2n$. If $i = 1$, then we are looking for a one-dimensional subspace in $\text{span}(w_1, w_{2n})$ (it is automatically isotropic). This gives the isomorphism $SpR_{2n}(1) \simeq \mathbb{P}^1$. Now assume $i > 1$ and still $i + j = 2n$. Then we have the following conditions for the space $V_{i,2n-i}$:

$$V_{i-1,2n-i} \subset V_{i,2n-i} \subset W_{i,2n-i}, \quad V_{i,2n-i} \text{ is isotropic.}$$

We note that $V_{i-1,2n-i} \subset \text{span}(w_1, \dots, w_{i-1}, w_{2n-i+1}, \dots, w_{2n})$ is an $(i-1)$ -dimensional vector space such that $pr_{2n-i+1} V_{i-1,2n-i}$ is isotropic. Since $V_{i,2n-i}$ has to be isotropic, the datum of an i -dimensional $V_{i,2n-i} \subset W_{i,j}$ is equivalent to the datum of a line in

the 2-dimensional space $W_{i,2n-i}/V_{i-1,2n-i}^\perp$. Therefore, the fiber of the map $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ is \mathbb{P}^1 .

Now assume $i+j < 2n$ and $\beta_l = \alpha_{i,j}$. Then the conditions for $V_{i,j}$ are

$$V_{i-1,j} \subset V_{i,j} \subset W_{i,j} \cap (V_{i,j+1} \oplus \mathbb{C}w_{j+1}).$$

It is easy to see that the variety of choices of such a $V_{i,j}$ is isomorphic to \mathbb{P}^1 (actually, the same situation appears in the \mathfrak{sl}_{2n} case). \square

Corollary 5.4. *The variety SpR_{2n} is a tower of successive \mathbb{P}^1 -fibrations.*

We now construct sections of the maps $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ similar to the ones in type A (see [FF]). A section $s_l : SpR_{2n}(l-1) \rightarrow SpR_{2n}(l)$ is determined by the β_l -th component of the image $s_l(\mathbf{V})$ for $\mathbf{V} \in SpR_{2n}(l-1)$. So let $\beta_l = \alpha_{i,j}$. Then we set

$$(s_l(\mathbf{V}))_{i,j} = \begin{cases} \mathbb{C}w_{j+1}, & \text{if } i = 1, \\ V_{i-1,j+1} \oplus \mathbb{C}w_{j+1}, & \text{if } i > 1. \end{cases}$$

We sometimes denote s_l by $s_{i,j}$. Let us denote by $Z_{i,j} \subset SpR_{2n}$ the divisor corresponding to the above section, i.e.

$$Z_{i,j} = \{\mathbf{V} \in SpR_{2n} : V_{i,j} \in (\text{Im } s_{i,j})_{i,j}\}.$$

We also denote by $Z_{i,j}^\circ \subset Z_{i,j}$ an open part of $Z_{i,j}$ which is the complement to the intersection of $Z_{i,j}$ with all other divisors:

$$(5.4) \quad Z_{i,j}^\circ = Z_{i,j} \setminus \bigcup_{(i_0,j_0) \neq (i,j)} Z_{i_0,j_0}.$$

Let us denote by $SpR_{2n}^\circ \subset SpR_{2n}$ the complement in SpR_{2n} to the union of the divisors $Z_{i,j}$.

Lemma 5.5. *SpR_{2n}° is an open cell in SpR_{2n} enjoying the following explicit description:*

$$SpR_{2n}^\circ = \{\mathbf{V} \in SpR_{2n} : p_{1,\dots,i} V_{i,j} \neq 0 \text{ for all } i, j\},$$

where $p_{1,\dots,i}$ is the corresponding Plücker coordinate.

Proof. First we prove that for any $\mathbf{V} \in SpR_{2n}^\circ$ one has

$$(5.5) \quad \dim pr_{i+1} \dots pr_{2n-i} V_{i,i} = i, \quad i = 1, \dots, n.$$

In fact, let i be the minimal number with the property that there exists $i+1 \leq j \leq 2n-i$ such that $\dim pr_{i+1} \dots pr_j V_{i,i} = i-1$. For such an i let us take minimal j with the property as above. Then since $pr_{i+1} \dots pr_j V_{i,i} \subset V_{i,j}$ we obtain $w_j \in V_{i,j-1}$. Because of the assumption of the minimality of i , $w_j \notin V_{i-1,j-1}$. Hence, $\dim pr_j V_{i-1,j-1} = i-1$ and $V_{i-1,j} = pr_j V_{i-1,j-1}$. In addition, $V_{i,j-1} = V_{i-1,j} \oplus \mathbb{C}w_j$. This implies $V_{i,j-1} = V_{i-1,j} \oplus \mathbb{C}w_j$, which means $\mathbf{V} \in Z_{i,j-1}$.

Second, we claim that (5.5) implies

$$(5.6) \quad \dim pr_{i+1} \dots pr_{2n} V_{i,i} = i, \quad i = 1, \dots, n.$$

In fact, since $\dim pr_{i+1} \dots pr_{2n-i} V_{i,i} = i$, we obtain

$$V_{i,2n-i} = pr_{i+1} \dots pr_{2n-i} V_{i,i}.$$

Recall that $V_{i,2n-i}$ is an i -dimensional isotropic subspace in $W_{i,2n-i}$. This implies the claim.

Finally, since for any i, j we have $V_{i,j} = pr_{i+1} \dots pr_j V_{i,i}$ and (5.6) holds, we obtain $pr_{1,\dots,i} V_{i,j} \neq 0$. \square

We have a natural map $\pi_{2n} : SpR_{2n} \rightarrow Sp\mathcal{F}_{2n}^a$

$$\pi_{2n}(V_{i,j})_{i,j} = (V_{i,i})_{i=1}^n$$

forgetting the off-diagonal elements.

Lemma 5.6. *The map π_{2n} is a birational isomorphism.*

Proof. We prove that π_{2n} is surjective and one-to-one on SpR_{2n}^o . The second statement is simple: since $\dim pr_{i+1} \dots pr_j V_i = i$ for any $j = i+1, \dots, 2n-i$ we have $V_{i,j} = pr_{i+1} \dots pr_j V_i$ (recall $\dim V_{i,j} = i$). Let us prove the surjectivity.

Given a point $(V_i)_{i=1}^n \in Sp\mathcal{F}_{2n}^a$, we need to construct a point $\mathbf{V} \in SpR_{2n}$ such that $\pi_{2n} \mathbf{V} = (V_i)_{i=1}^n$. We construct the entries $V_{i,j}$ of \mathbf{V} by increasing induction on j . In the course of the construction, we always check that $pr_{j+1} \dots pr_{2n-i} V_{i,j}$ is isotropic (this is a necessary condition for $\mathbf{V} \in SpR_{2n}$). For $j=1$ we have the only space $V_{1,1}$, which has to be equal to V_1 .

Now assume $j \leq n$. We need to define $V_{i,j}$ with $i = 1, \dots, j$. We do this by decreasing induction on i . For $i=j$ we have $V_{j,j} = V_j$. Now assume $V_{i+1,j}$ is already defined. Then the conditions for $V_{i,j}$ are as follows:

$$pr_j V_{i,j-1} \subset V_{i,j} \subset W_{i,j} \cap V_{i+1,j}, \quad pr_{j+1} \dots pr_{2n-i} V_{i,j} \text{ is isotropic.}$$

If $\dim pr_j V_{i,j-1} = i$, then we have $V_{i,j} = pr_j V_{i,j-1}$. Assume that the dimension drops, i.e. $w_j \in V_{i,j-1}$. Then, by induction $pr_j V_{i,j-1} \subset W_{i,j} \cap V_{i+1,j}$. In addition $\dim(W_{i,j} \cap V_{i+1,j}) \geq i$. Therefore there exists an i -dimensional subspace in between $pr_j V_{i,j-1}$ and $W_{i,j} \cap V_{i+1,j}$. Now since $pr_{j+1} \dots pr_{2n-i-1} V_{i+1,j}$ is isotropic and $V_{i,j} \subset W_{i,j} \cap V_{i+1,j}$, we obtain that $pr_{j+1} \dots pr_{2n-i} V_{i,j}$ is also isotropic.

Finally, assume $j > n$. We need to define $V_{i,j}$ with $i = 1, \dots, 2n-j$. Again, we do this by decreasing induction on i . Let us start with $i = 2n-j$. Then the conditions we have are

$$pr_{j+1} V_{2n-j,j-1} \subset V_{2n-j,j} \subset W_{2n-j,j}, \quad V_{2n-j,j} \text{ is isotropic.}$$

If $\dim pr_{j+1} V_{2n-j,j-1} = 2n-j$, then we are done, since $pr_{j+1} V_{2n-j,j-1}$ is isotropic by induction. If $\dim pr_{j+1} V_{2n-j,j-1} = 2n-j-1$, then we need to show that $(pr_{j+1} V_{2n-j,j-1})^\perp \cap W_{2n-j,j}$ is not zero. But the first space is $(2n-j-1)$ -dimensional and the second one is $2(2n-j)$ -dimensional and the restriction of the symplectic form to $W_{2n-j,j}$ is non-degenerate. Now assume that we have fixed $V_{i+1,j}$. Using the same arguments as above one can show that there exists $V_{i,j}$ satisfying

$$pr_j V_{i,j-1} \subset V_{i,j} \subset V_{i+1,j} \cap W_{i,j}, \quad pr_{j+1} \dots pr_{2n-i} V_{i,j} \text{ is isotropic.}$$

\square

Corollary 5.7. *The variety $Sp\mathcal{F}_{2n}^a$ is irreducible.*

We now establish a connection between the SL_{2n} resolution R_{2n} and its symplectic analogue SpR_{2n} . Define an involution $\sigma : R_{2n} \rightarrow R_{2n}$ by the following formula: let $\mathbf{V} = (V_{i,j})_{1 \leq i \leq j}$ be a point in R_{2n} . Then

$$(5.7) \quad (\sigma \mathbf{V})_{i,j} = V_{2n-j, 2n-i}^\perp \cap W_{i,j}.$$

Proposition 5.8. *Formula (5.7) defines an order two automorphism of R_{2n} . The variety of σ fixed points R_{2n}^σ is isomorphic to SpR_{2n} . For the resolution map $\pi_{2n} : SpR_{2n} \rightarrow Sp\mathcal{F}_{2n}^a$ one has $\pi_{2n}\sigma = \sigma\pi_{2n}$ (i.e. π_{2n} is σ -equivariant).*

Proof. First, we need to show that $\dim(V_{2n-j,2n-i}^\perp \cap W_{i,j}) = i$. We consider the case $i + j > 2n$ (the opposite case is very similar). We note that $\dim V_{2n-j,2n-i} = 2n - j$ and thus $\dim V_{i,j}^\perp = j$. Since $V_{2n-j,2n-i} \subset W_{2n-j,2n-i}$ we have

$$V_{2n-j,2n-i}^\perp \supset W_{2n-j,2n-i}^\perp \supset \{w_{i+1}, \dots, w_j\}.$$

Since $W_{i,j} = \text{span}(w_{i+1}, \dots, w_j)$, we arrive at $\dim(V_{2n-j,2n-i}^\perp \cap W_{i,j}) = j - (j - i) = i$.

Second, we need to prove that all other conditions from the definition of R_{2n} are satisfied. This is a direct verification. \square

Recall that for a resolution of singularities $\pi : Y \rightarrow X$ an irreducible divisor $Z \subset Y$ is called exceptional if $\dim \pi(Z) < \dim Z$.

Proposition 5.9. *A divisor $Z_{i,j}$ is exceptional if and only if $j \geq n$ and $i + j < 2n$.*

Proof. Fix a pair $i < j$ such that $j \geq n$ and $i + j < 2n$. We prove that the divisor $Z_{i,j}$ is exceptional. Fix a point $(V_i)_{i=1}^n \in Sp\mathcal{F}_{2n}^a$ and a point $\mathbf{V} \in Z_{i,j}$ such that $\pi_{2n}\mathbf{V} = (V_i)_{i=1}^n$, i.e. $V_{i,i} = V_i$. We prove that the preimage $\pi_{2n}^{-1}(V_i)_{i=1}^n$ is at least one-dimensional. Let us construct a one-dimensional family of elements $\mathbf{U} \in Z_{i,j}$ such that $\pi_{2n}\mathbf{U} = \pi_{2n}\mathbf{V}$. We set $U_{k,l} = V_{k,l}$ if $l \leq j$. Now (as in the proof of Lemma 5.6) let us try to extend already fixed components of \mathbf{U} to some element of $Z_{i,j}$. We start with $U_{2n-j-1,j+1}$. In order to guarantee that $\mathbf{U} \in SpR_{2n}$, we need the following conditions:

$$(5.8) \quad pr_{j+1}U_{2n-j-1,j} \subset U_{2n-j-1,j+1} \subset W_{2n-j-1,j+1}, \quad U_{2n-j-1,j+1} \text{ is isotropic.}$$

Since $V_{i,j} \supset w_{j+1}$ ($\mathbf{V} \in Z_{i,j}$) we have $\dim pr_{j+1}U_{2n-j-1,j} = 2n - j - 2$. The restriction of the symplectic form to $W_{2n-j-1,j+1}$ is non-degenerate, thus the set of solutions of (5.8) is isomorphic to \mathbb{P}^1 . Now using the same arguments as in Lemma 5.6 we define successively $U_{2n-j-2,j+1}, \dots, U_{i,j+1}$. Now let us try to define $U_{i-1,j+1}$. We want $U_{i-1,j+1}$ to be contained in the (already defined) $U_{i,j+1}$. We note that $U_{i,j} = V_{i,j} = V_{i-1,j+1} \oplus \mathbb{C}w_{j+1}$ and $pr_{j+1}U_{i,j} \subset U_{i,j+1}$ (by induction). Therefore, we can set $U_{i-1,j+1} = V_{i-1,j+1}$, which leads to $U_{i,j} = V_{i,j} = U_{i-1,j+1} \oplus \mathbb{C}w_{j+1}$. We note that since we want $\mathbf{U} \in Z_{i,j}$ such a relation has to hold. Now according to the proof of Lemma 5.6, we can define the remaining $U_{k,l}$ (by increasing induction on l and decreasing induction on k) in such a way that $\mathbf{U} \in SpR_{2n}$. Moreover, by the construction as above, $\mathbf{U} \in Z_{i,j}$.

Now let us fix $j < n$. We prove that the divisor $Z_{i,j}$ with $i \leq j$ is non-exceptional. More precisely, we prove that π_{2n} restricted to $Z_{i,j}^\circ$ (see (5.4)) is one-to-one. So let $\mathbf{V} \in Z_{i,j}^\circ$.

First, we claim that if $w_{l+1} \in V_{k,l}$ then $l = j$ and $i \leq k \leq j$. In fact, assume that for a given l the number k is minimal with the property $w_{l+1} \in V_{k,l}$. Then $k \neq 1$ (unless $i = 1$) since $w_{l+1} \in V_{1,l}$ means $\mathbf{V} \in Z_{1,l}$. We know that $w_{l+1} \notin V_{k-1,l}$. Hence $pr_{l+1}V_{k-1,l} = V_{k-1,l+1}$ and $V_{k,l} = V_{k-1,l} \oplus \mathbb{C}w_{l+1}$. Therefore, $V_{k,l} = V_{k-1,l+1} \oplus \mathbb{C}w_{l+1}$, implying $\mathbf{V} \in Z_{k,l}$. Since $\mathbf{V} \in Z_{i,j}^\circ$, we arrive at $(k, l) = (i, j)$.

Second, we note that the above claim implies that for any pair (k, l) except for $l = j$, $i \leq k \leq j$ we have

$$(5.9) \quad \dim pr_{l+1} V_{k,l} = V_{k,l+1}.$$

In fact, for such pairs (k, l) the left and right hand sides have the same dimensions and the left hand side is contained in the right hand side. From equation (5.9) we obtain that if a point $(V_i)_{i=1}^n \in Sp\mathcal{F}_{2n}^a$ is fixed, the spaces $V_{k,l}$ with $k < i$ or $k > j$ or $l \leq j$ are fixed as well. Also, if one proves that the spaces $V_{k,j+1}$ are also uniquely fixed by the choice of $(V_i)_{i=1}^n$, this would imply that the whole collection $\mathbf{V} \in Z_{i,j}^\circ$ with the property $\pi_{2n} \mathbf{V} = (V_i)_{i=1}^n$ is unique.

Third, we note that $\mathbf{V} \in Z_{i,j}^\circ$ implies that for $l > j$ one has $p_{1,\dots,k} V_{k,l} \neq 0$ (where $p_{1,\dots,l}$ is the corresponding Plücker coordinate). In fact, first $V_{1,2n-1}$ is a one-dimensional space, not equal to $\mathbb{C}w_{2n}$ (since $\mathbf{V} \in Z_{i,j}^\circ$). Therefore, $p_1(V_{1,2n-1}) \neq 0$. Now the statement can be verified directly by decreasing induction on l and increasing induction on k .

Finally, we are left to show that the values of V_i and of $V_{k,l}$ with $l \leq j$ fix $V_{k,j+1}$. First, $V_{j+1,j+1} = V_{j+1}$ is fixed. Now, we use the decreasing induction on k . The space $V_{k,j+1}$ has to satisfy

$$V_{k,j+1} \subset V_{k+1,j+1} \cap W_{k,j+1}.$$

We claim that the right hand side is (at most) k -dimensional. In fact, we have $p_{1,\dots,k+1}(V_{k+1,j+1}) \neq 0$. Since $w_{k+1} \notin W_{k,j+1}$, the $\dim(V_{k+1,j+1} \cap W_{k,j+1}) \leq k$. Since $\dim V_{k,j+1} = k$, the space $V_{k,j+1}$ is uniquely fixed.

The last statement of the proposition is that the divisors $Z_{i,2n-i}$ are non-exceptional. This can be proved along the same lines as the above case. \square

5.2. Parabolic case. We define a desingularization $SpR_{\mathbf{d}}$ of $Sp\mathcal{F}_{\mathbf{d}}^a$ in the following way. Let $P_{\mathbf{d}}$ be the subset of the set of positive roots for \mathfrak{sp}_{2n} corresponding to the radical of the parabolic subalgebra defined by the roots $\alpha_{d_1}, \dots, \alpha_{d_k}$. Explicitly, $\alpha \in P_{\mathbf{d}}$ if and only if there exists an $i = 1, \dots, k$ such that $(\alpha, \omega_{d_i}) > 0$. We sometimes write $(i, j) \in P_{\mathbf{d}}$ instead of $\alpha_{i,j} \in P_{\mathbf{d}}$. Now, a point in $SpR_{\mathbf{d}}$ is a collection of subspaces $V_{i,j} \subset W$, $(i, j) \in P_{\mathbf{d}}$ subject to the conditions

$$\begin{aligned} \dim V_{i,j} &= i, \quad V_{i,j} \subset W_{i,j}, \\ V_{i,j} &\subset V_{i+1,j}, \quad pr_{j+1} V_{i,j} \subset V_{i,j+1}, \\ V_{i,2n-i} &\text{ are isotropic for } i = 1, \dots, n. \end{aligned}$$

We note that there is a natural embedding $SpR_{\mathbf{d}} \subset \prod_{(i,j) \in P_{\mathbf{d}}} Gr_i(W_{i,j})$.

Proposition 5.10. *$SpR_{\mathbf{d}}$ is a tower of successive \mathbb{P}^1 -fibrations.*

Proof. The proof is very similar to the proof of Lemma 5.3 and we omit it. \square

Proposition 5.11. *A map $(V_{i,j})_{(i,j) \in P_{\mathbf{d}}} \mapsto (V_{d_i, d_i})_{i=1}^k$ is a birational surjective map from $SpR_{\mathbf{d}}$ to $Sp\mathcal{F}_{\mathbf{d}}^a$.*

Proof. The proof is very similar to the proof of Lemma 5.6 and we omit it. \square

Corollary 5.12. *The varieties $Sp\mathcal{F}_{\mathbf{d}}^a$ are irreducible.*

In what follows we denote the desingularization map $SpR_{\mathbf{d}} \rightarrow Sp\mathcal{F}_{\mathbf{d}}^a$ by $\pi_{\mathbf{d}}$. As in the case of the complete flag varieties, for any pair $(i, j) \in P_{\mathbf{d}}$, we define the corresponding divisor $Z_{i,j} \subset SpR_{\mathbf{d}}$.

Proposition 5.13. *The non-exceptional divisors are exactly the $Z_{i,j}$ with (i, j) from the following list:*

$$(1, d_i - 1), i = 2, \dots, k; \quad (d_i + 1, d_j - 1), i \leq j - 2, j \leq k;$$

$$(1, 2n - 1); \quad (d_i + 1, 2n - d_i - 1), i = 1, \dots, k - 1.$$

Proof. The proof is similar to the proof of Proposition 5.9 and we omit it. □

6. NORMAL LOCALLY COMPLETE INTERSECTIONS

Our next goal is to prove that all symplectic degenerate flag varieties are normal locally complete intersections (and thus Cohen-Macaulay and Gorenstein). We start with the case of the complete flags.

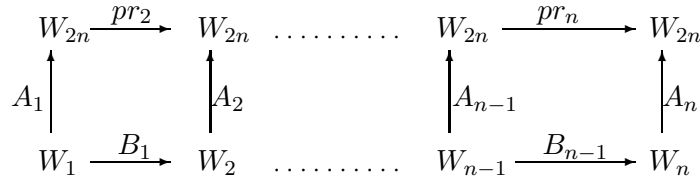
6.1. Complete flag varieties. As in [FF], we first define an affine scheme Q_{2n} . Let W_1, \dots, W_n be a collection of vector spaces with $\dim W_i = i$. Also recall the space W_{2n} with a basis w_1, \dots, w_{2n} , a non-degenerate symplectic form and the projections pr_k along w_k . We now construct an affine scheme Q_{2n} as follows. A point of Q_{2n} is a collection of linear maps

$$A_i : W_i \rightarrow W_{2n}, \quad i = 1, \dots, n, \quad B_j : W_j \rightarrow W_{j+1}, \quad j = 1, \dots, n - 1$$

subject to the relations

$$(6.1) \quad A_{i+1}B_i = pr_{i+1}A_i, \quad i = 1, \dots, n - 1,$$

and such that the image $A_n(W_n)$ is isotropic. The following picture illustrates the construction:



We also consider an open part $Q_{2n}^\circ \subset Q_{2n}$ consisting of collections (A_i, B_j) such that $\ker A_i = 0$ for all i . The group $\Gamma = \prod_{i=1}^n GL(W_i)$ acts freely on Q_{2n}° via the change of bases. Consider the map

$$Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a, \quad (A_i, B_j) \mapsto (\text{Im}A_1, \dots, \text{Im}A_n).$$

Lemma 6.1. *The map $Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a$ is a locally trivial Γ -torsor in the Zariski topology. The dimension of Q_{2n}° is equal to $n^2 + 1^2 + 2^2 + \dots + n^2$.*

Proof. Consider the embedding $Sp\mathcal{F}_{2n}^a \hookrightarrow \prod_{d=1}^n SpGr_d(2n)$. For a point $p \in Sp\mathcal{F}_{2n}^a$ let $U \ni p$ be an open part of $\prod_{d=1}^n SpGr_d(2n)$ such that all tautological bundles on Grassmannians are trivial on U . Let $U' = U \cap Sp\mathcal{F}_{2n}^a$. Then on U' the map $Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a$ has a section. Now using the Γ action on Q_{2n} we obtain that $Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a$ is Γ -torsor. In particular, $\dim Q_{2n}^\circ = \dim Sp\mathcal{F}_{2n}^a + \dim \Gamma$. □

We note that Q_{2n} is a subscheme in the affine space

$$(6.2) \quad \prod_{i=1}^n \operatorname{Hom}(W_i, W_{2n}) \times \prod_{i=1}^{n-1} \operatorname{Hom}(W_i, W_{i+1}).$$

Lemma 6.2. Q_{2n}° is a locally complete intersection.

Proof. We note that the condition $A_{i+1}B_i = pr_{i+1}A_i$ produces $2n \times i$ equations (the number of equations is equal to $\dim \operatorname{Hom}(W_i, W_{2n})$). Also the condition that $A_n(W_n)$ is Lagrangian produces another $n(n-1)/2$ equations. Now our lemma follows from the equality

$$\dim Q_{2n}^\circ = \sum_{i=1}^n 2ni + \sum_{i=1}^{n-1} i(i+1) - \sum_{i=1}^{n-1} 2ni - \frac{n(n-1)}{2}.$$

□

Theorem 6.3. The degenerate flag varieties $Sp\mathcal{F}_{2n}^a$ are normal locally complete intersections (in particular, Cohen-Macaulay and even Gorenstein).

Proof. Since $Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a$ is a torsor, it suffices to prove that Q_{2n}° is a normal reduced scheme (i.e. a variety). Since Q_{2n}° is a locally complete intersection, the property of being reduced (resp. normality) of Q_{2n}° follows from the fact that the singularities of Q_{2n}° are contained in the subvariety of codimension at least two by the virtue of Proposition 5.8.5 (resp. Theorem 5.8.6) of [EGA]. Using again that $Q_{2n}^\circ \rightarrow Sp\mathcal{F}_{2n}^a$ is a torsor, it suffices to prove that the codimension of the variety of singular points of $Sp\mathcal{F}_{2n}^a$ is at least two. We prove this statement in a separate lemma. □

Lemma 6.4. $Sp\mathcal{F}_{2n}^a$ is smooth off codimension two.

Proof. We use the desingularization $\pi_{2n} : SpR_{2n} \rightarrow Sp\mathcal{F}_{2n}^a$. Since SpR_{2n} is smooth, it suffices to show that the map π_{2n} is one-to-one on $\pi_{2n}^{-1}M$, where $M \subset Sp\mathcal{F}_{2n}^a$ is an open part such that the codimension of the complement of M is at least two. We set M to be the union of $\pi_{2n}SpR_{2n}^\circ$ (U is the open cell in SpR_{2n}) and $\pi_{2n}Z_{i,j}^\circ$ for all non-exceptional divisors $Z_{i,j}$. □

The following theorem can be proved along the same lines as Theorem 6.3.

Theorem 6.5. The degenerate flag varieties $Sp\mathcal{F}_d^a$ are normal locally complete intersections (in particular, Cohen-Macaulay and Gorenstein).

7. THE SINGULARITIES OF $Sp\mathcal{F}_d^a$ ARE RATIONAL.

7.1. The complete flag varieties. In this section we prove that the singularities of $Sp\mathcal{F}_{2n}^a$ are terminal, canonical and rational. We use the sections $s_{i,j}$ as above in order to compute the line bundle $\omega_{SpR_{2n}} \otimes \pi_{2n}^* \omega_{Sp\mathcal{F}_{2n}^a}^{-1}$.

We use the notation $\Omega_{i,j}$ for the determinant of the i -dimensional bundle on SpR_{2n} , whose fiber at a point \mathbf{V} equals $\Lambda^i(V_{i,j})^*$. Also, we denote by Ω_i the line bundle on $Sp\mathcal{F}_{2n}^a$, whose fiber at a point (V_1, \dots, V_n) is equal to $\Lambda^i(V_i^*)$.

Consider the following general situation. Let $\rho : E \rightarrow B$ be a \mathbb{P}^1 -fibration with a section $s : B \rightarrow E$. Let $Z = s(B) \subset E$ be the corresponding divisor. Then for any line

bundle \mathcal{F} on E such that the restriction of \mathcal{F} to a fiber of ρ is isomorphic to $\mathcal{O}(k)$ one has

$$(7.1) \quad \mathcal{F} = \mathcal{O}(kZ) \otimes \rho^*(\mathcal{F}|_Z \otimes \mathcal{O}(-kZ)|_Z).$$

Assume that there exists a two-dimensional bundle \mathcal{L}_2 on B and a line subbundle $\mathcal{L}_1 \subset \mathcal{L}_2$ on B such that $E = \mathbb{P}(\mathcal{L}_2)$ and $Z = \mathbb{P}(\mathcal{L}_1)$. Let Ω be a line bundle such that the restriction of Ω to a fiber of ρ is isomorphic to $\mathcal{O}(1)$.

Lemma 7.1. $\omega_E^{-1} = \mathcal{O}(Z) \otimes \Omega \otimes \rho^*(\Omega^{-1}|_Z \otimes \omega_B^{-1})$.

Proof. First, (7.1) with $\mathcal{F} = \Omega$ gives

$$(7.2) \quad \Omega = \mathcal{O}(Z) \otimes \rho^*(\Omega|_Z \otimes \mathcal{O}(-Z)|_Z).$$

Second, (7.1) with $\mathcal{F} = \omega_E^{-1}$ gives

$$(7.3) \quad \omega_E^{-1} = \mathcal{O}(2Z) \otimes \rho^*(\omega_E^{-1}|_Z \otimes \mathcal{O}(-2Z)|_Z).$$

We note that

$$\rho^*(\Omega^{-1}|_Z \otimes \mathcal{O}(-2Z)|_Z) = \rho^*(\omega_B^{-1}|_Z \otimes \mathcal{O}(-Z)|_Z).$$

Therefore, combining (7.2) and (7.3), we arrive at

$$\omega_E^{-1} = \mathcal{O}(Z) \otimes \Omega \otimes \rho^*(\Omega^{-1}|_Z \otimes \omega_B^{-1}).$$

□

We now consider the \mathbb{P}^1 -fibration $\rho_l : SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ and the corresponding divisors $Z_{i,j}$.

Lemma 7.2.

$$\mathcal{O}(Z_{i,j}) = \begin{cases} \Omega_{1,j} \otimes \Omega_{1,j+1}^*, & \text{if } i = 1, \\ \Omega_{i,j} \otimes \Omega_{i-1,j}^* \otimes \Omega_{i,j+1}^* \otimes \Omega_{i-1,j+1}, & \text{if } i > 1 \text{ and } i + j < 2n, \\ \Omega_{i,j} \otimes (\Omega_{i-1,j}^*)^{\otimes 2} \otimes \Omega_{i-1,j+1}, & \text{if } i > 1 \text{ and } i + j = 2n. \end{cases}$$

Proof. The first two cases are worked out in [FF]. The only new case is the last one $i + j = 2n$. Consider the space $V_{i,2n-i}$. We know that

$$V_{i-1,2n-i} \subset V_{i,2n-i} \subset V_{i-1,2n-i}^\perp \cap W_{i,2n-i}.$$

Let $\Omega_{i,j}^\perp$ be the line bundle on SpR_{2n} , whose fiber at a point \mathbf{V} is equal to the top wedge power of $(V_{i,j}^\perp)^*$. Then one shows that $\det \Omega_{i,j}^\perp = \det \Omega_{i,j}^*$. Therefore, using (7.1) we arrive at

$$\Omega_{i,2n-i} = \mathcal{O}(Z_{i,2n-i}) \otimes \Omega_{i-1,2n-i+1}^* \otimes \Omega_{i-1,2n-i}^{\otimes 2}.$$

□

Theorem 7.3. $\omega_{SpR_{2n}} = \bigotimes_{i=1}^n (\Omega_{i,i}^*)^{\otimes 2} \otimes \prod_{i+j < 2n, j \geq n} \mathcal{O}(Z_{i,j})$.

Proof. From Lemma 7.1 we obtain

$$\omega_{SpR_{2n}}^{-1} = \bigotimes_{i,j} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{i,j} \Omega_{i,j} \otimes \bigotimes_{j-i > 2} \Omega_{i,j}^*.$$

Now using Lemma 7.2 we are left to show that

$$\bigotimes_{i,j} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{i=1}^n \Omega_{i,i} \otimes \bigotimes_{i=1}^{n-1} \Omega_{i,i+1} = \bigotimes_{i=1}^n \Omega_{i,i}^{\otimes 2} \otimes \bigotimes_{j \geq n, i+j < 2n} \mathcal{O}(-Z_{i,j}).$$

This equality follows from Lemma 7.2 again by writing both sides in terms of $\Omega_{i,j}$ only. \square

Corollary 7.4. $\omega_{Sp\mathcal{F}_{2n}^a} = \bigotimes_{i=1}^n (\Omega_i^*)^{\otimes 2}$.

Proof. Recall the open cell $SpR_{2n}^\circ \subset SpR_{2n}$ with the property that the restriction of π_{2n} to SpR_{2n}° is one-to-one. Consider a subvariety $M \subset SpR_{2n}$ defined by

$$M = SpR_{2n}^\circ \cup \bigcup_{j < n} Z_{i,j}^\circ \cup \bigcup_{i+j=2n} Z_{i,j}^\circ.$$

Then the restriction of π_{2n} to $SpR_{2n} \setminus M$ is one-to-one and $\pi_{2n}(SpR_{2n} \setminus M)$ has codimension at least two in $Sp\mathcal{F}_{2n}^a$. Because of Theorem 7.3, the canonical line bundle ω on M is equal to $\bigotimes_{i=1}^n (\Omega_{i,i}^*)^{\otimes 2}$. Since we also have $\pi_{2n}^* \omega_{Sp\mathcal{F}_{2n}^a} = \omega_{SpR_{2n}}$ on the image of M , we obtain $\omega_{Sp\mathcal{F}_{2n}^a} = \bigotimes_{i=1}^n (\Omega_i^*)^{\otimes 2}$. \square

Theorem 7.5. *The singularities of $Sp\mathcal{F}_{2n}^a$ are terminal and hence rational.*

Proof. Using Corollary 7.4 and Theorem 7.3 we obtain

$$\omega_{SpR_{2n}} = \pi_{2n}^* \omega_{Sp\mathcal{F}_{2n}^a} \otimes \prod_{i+j < 2n, j \geq n} \mathcal{O}(Z_{i,j}).$$

Thus the singularities are terminal (and hence canonical) by Theorem 7.3 (all exceptional divisors appear in the discrepancy with coefficient one). Now rationality follows from [E], Theorem 1. In fact, the varieties $Sp\mathcal{F}_{2n}^a$ are locally complete intersections and so Gorenstein. Now, by the projection formula and Theorem 7.3, one gets

$$(\pi_{2n})_* \omega_{SpR_{2n}} = \bigotimes_{i=1}^n (\Omega_{i,i}^*)^{\otimes 2} \otimes (\pi_{2n})_* \prod_{i+j < 2n, j \geq n} \mathcal{O}(Z_{i,j}).$$

Since $\mathcal{O}(Z_{i,j}) \supset \mathcal{O}$ and $Sp\mathcal{F}_{2n}^a$ is normal (thus $(\pi_{2n})_* \mathcal{O} = \mathcal{O}$), we obtain

$$(\pi_{2n})_* \omega_{SpR_{2n}} \supset \omega_{Sp\mathcal{F}_{2n}^a}.$$

Using the natural opposite inclusion, we arrive at the isomorphism. \square

7.2. Rationality for parabolic flag varieties. In this subsection we work out the case of partial degenerate flag varieties. Our goal is to compute $\omega_{SpR_{\mathbf{d}}} \otimes \pi_{\mathbf{d}}^* \omega_{Sp\mathcal{F}_{\mathbf{d}}}^{-1}$ (note that $Sp\mathcal{F}_{\mathbf{d}}$ is singular but Gorenstein, thus its dualizing complex is a line bundle). We will prove that the expression for $\omega_{Sp\mathcal{F}_{\mathbf{d}}}$ coincides with the one in the classical situation. So let $Sp\mathcal{F}_{\mathbf{d}}$ be the classical parabolic flag variety.

Lemma 7.6. $\omega_{Sp\mathcal{F}_{\mathbf{d}}}^{-1} = \Omega_{d_1}^{\otimes d_2} \otimes \prod_{i=2}^{k-1} \Omega_{d_i}^{\otimes (d_{i+1} - d_{i-1})} \otimes \Omega_{d_k}^{\otimes (2n+1 - d_k - d_{k-1})}$.

Proof. Since the canonical line bundle is the top wedge power of the cotangent bundle, its weight is given by $-2\rho_{Sp_{2n}} + 2\rho_{L_{\mathbf{d}}}$, where $2\rho_{Sp_{2n}}$ is the sum of all positive roots of \mathfrak{sp}_{2n} and $2\rho_{L_{\mathbf{d}}}$ is the sum of all positive roots of the Levi subalgebra corresponding to \mathbf{d} . Now direct computation gives the desired formula. \square

Now our strategy is as follows. We define a line bundle $\tilde{\omega}$ on $Sp\mathcal{F}_{\mathbf{d}}^a$ by the formula

$$\tilde{\omega}^{-1} = \Omega_{d_1}^{\otimes d_2} \otimes \prod_{i=2}^{k-1} \Omega_{d_i}^{\otimes (d_{i+1} - d_{i-1})} \otimes \Omega_{d_k}^{\otimes (2n+1 - d_k - d_{k-1})}.$$

We will show that the discrepancy $\omega_{SpR_{\mathbf{d}}} \otimes \pi_{\mathbf{d}}^*(\tilde{\omega}^{-1})$ is given by the product of exceptional divisors with positive coefficients. Since the images of the exceptional divisors have codimension at least two, this will prove that in the degenerate case the expression for the canonical class coincides with the one in the classical situation.

We start with the following lemma.

Lemma 7.7. *We have the equality*

$$\omega_{SpR_{\mathbf{d}}}^{-1} = \bigotimes_{(i,j) \in P_{\mathbf{d}}} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{(i,j) \in B_{\mathbf{d}}} \Omega_{i,j},$$

where $B_{\mathbf{d}} \subset P_{\mathbf{d}}$ is the subset of pairs from the following list:

$$\begin{aligned} & (1, d_1), (2, d_1), \dots, (d_1, d_1), \\ & (d_1, d_1 + 1), (d_1, d_1 + 2), \dots, (d_1, d_2), \\ & (d_1 + 1, d_2), (d_1 + 2, d_2), \dots, (d_2, d_2), \\ & \dots \\ & (d_{k-1} + 1, d_k), (d_{k-1} + 2, d_k), \dots, (d_k, d_k), \\ & (d_k, d_k + 1), (d_k, d_k + 2), \dots, (d_k, 2n - d_k). \end{aligned}$$

Proof. Follows from Lemma 7.1. In fact the sections $s_{i,j}$ are constructed in such a way that the space $V_{i,j}$ at a point of the section is given by $V_{i-1,j+1}$ plus some constant vector. The set $B_{\mathbf{d}}$ is exactly the set of pairs (i_0, j_0) which can not be written as $i_0 = i - 1$, $j_0 = j + 1$ for some $(i, j) \in P_{\mathbf{d}}$. \square

We now prove the main theorem of this section.

Theorem 7.8. *There exist non-negative integers $a_{i,j}$, $(i, j) \in P_{\mathbf{d}}$ such that*

$$\omega_{SpR_{\mathbf{d}}} \otimes \pi_{\mathbf{d}}^*(\tilde{\omega}^{-1}) = \bigotimes_{(i,j) \in P_{\mathbf{d}}} \mathcal{O}(a_{i,j} Z_{i,j}).$$

In addition, $a_{i,j} = 0$ if and only if $Z_{i,j}$ is not exceptional.

Proof. We use the formula from Lemma 7.2:

$$(7.4) \quad \mathcal{O}(Z_{i,j}) = \begin{cases} \Omega_{1,j} \otimes \Omega_{1,j+1}^*, & \text{if } i = 1, \\ \Omega_{i,j} \otimes \Omega_{i-1,j}^* \otimes \Omega_{i,j+1}^* \otimes \Omega_{i-1,j+1}, & \text{if } i > 1 \text{ and } i + j < 2n, \\ \Omega_{i,j} \otimes (\Omega_{i-1,j}^*)^{\otimes 2} \otimes \Omega_{i-1,j+1}, & \text{if } i > 1 \text{ and } i + j = 2n. \end{cases}$$

Using Lemma 7.7 we obtain

$$\omega_{SpR_{\mathbf{d}}} \otimes \pi_{\mathbf{d}}^*(\tilde{\omega}^{-1}) = \bigotimes_{(i,j) \in P_{\mathbf{d}}} \mathcal{O}(-Z_{i,j}) \otimes \bigotimes_{(i,j) \in B_{\mathbf{d}}} \Omega_{i,j}^{-1} \otimes \pi_{\mathbf{d}}^*(\tilde{\omega}^{-1}).$$

Our goal is to find strictly positive integers $b_{i,j} = a_{i,j} + 1$ such that

$$(7.5) \quad \bigotimes_{(i,j) \in B_{\mathbf{d}}} \Omega_{i,j}^{-1} \otimes \Omega_{d_1}^{\otimes d_2} \otimes \prod_{i=2}^{k-1} \Omega_{d_i}^{\otimes (d_{i+1} - d_{i-1})} \otimes \Omega_{d_k}^{\otimes (2n+1-d_k-d_{k-1})} = \bigotimes_{(i,j) \in P_{\mathbf{d}}} \mathcal{O}(b_{i,j} Z_{i,j}).$$

In addition, we want $b_{i,j} = 1$ if and only if $Z_{i,j}$ is non-exceptional.

First, formula (7.5) implies $b_{d_1, d_1} = d_2 - 1$, $b_{d_i, d_i} = d_{i+1} - d_{i-1} - 1$ for $i = 2, \dots, k-1$ and $b_{d_k, d_k} = 2n - d_k - d_{k-1}$. Now formula (7.4) gives unique way to determine other $b_{i,j}$ in the following order:

$$\begin{aligned} & b_{d_1, d_1}, b_{d_1-1, d_1}, \dots, b_{1, d_1}, \\ & b_{d_1, d_1+1}, b_{d_1-1, d_1+1}, \dots, b_{1, d_1+1}, \\ & \dots \\ & b_{d_1, d_2-1}, b_{d_1-1, d_2-1}, \dots, b_{1, d_2-1}, \\ & b_{d_2, d_2}, b_{d_2-1, d_2}, \dots, b_{1, d_2}, \\ & \dots \\ & b_{d_2, d_3-1}, b_{d_2-1, d_3-1}, \dots, b_{1, d_3-1}, \\ & \dots \\ & \dots \\ & b_{d_k, d_k}, b_{d_k-1, d_k}, \dots, b_{1, d_k}, \\ & b_{d_k, d_k+1}, b_{d_k-1, d_k+1}, \dots, b_{1, d_k+1}, \\ & \dots \\ & b_{d_k, 2n-d_k}, b_{d_k-1, 2n-d_k}, \dots, b_{1, 2n-d_k}, \\ & b_{d_k-1, 2n-d_k+1}, b_{d_k-2, 2n-d_k+1}, \dots, b_{1, 2n-d_k+1}, \\ & \dots \\ & b_{2, 2n-2}, b_{1, 2n-2}, \\ & b_{1, 2n-1}. \end{aligned}$$

We now write down the values of the solutions $b_{i,j}$ (we assume $d_0 = 0$):

Let $d_1 \leq j \leq d_2 - 1$ and $1 \leq i \leq d_1$. Then

$$b_{i,j} = d_2 - j + i - 1.$$

Let $d_{s-1} \leq j \leq d_s - 1$ and $d_l + 1 \leq i \leq d_{l+1}$ for $0 \leq l < s < k$. Then

$$b_{i,j} = d_s - d_l - j + i - 1.$$

Let $d_k \leq j < 2n - d_k$ and $d_l + 1 \leq i \leq d_{l+1}$ for $0 \leq l < k$. Then

$$b_{i,j} = 2n - d_k - d_l - j + i.$$

Let $2n - d_s \leq j < 2n - d_{s-1} - 1$ and $i = 2n - j$ for $1 \leq s \leq k$. Then

$$b_{i,j} = 2n - j - d_{s-1}.$$

Let $2n - d_s \leq j \leq 2n - d_{s-1} - 1$ and $d_s - 1 \geq i \geq d_{s-1} + 1$ for $1 \leq s \leq k$. Then

$$b_{i,j} = 2n - 2d_{s-1} - j + i.$$

Let $2n - d_s \leq j \leq 2n - d_{s-1} - 1$ and $d_l \geq i \geq d_{l-1} + 1$ for $1 \leq s \leq k$, $s - 1 \geq l \geq 1$. Then

$$b_{i,j} = 2n - d_{s-1} - d_{l-1} - j + i.$$

In particular, one sees that $b_{i,j} \geq 1$ and $b_{i,j} = 1$ if and only if (i, j) is from the following list

$$(1, d_i - 1), i = 2, \dots, k; \quad (d_i + 1, d_j - 1), i \leq j - 2, j \leq k; \\ (1, 2n - 1); \quad (d_i + 1, 2n - d_i - 1), i = 1, \dots, k - 1.$$

Now Proposition 5.13 gives the desired result. \square

Corollary 7.9. $\omega_{Sp\mathcal{F}_{\mathbf{d}}^a} = \tilde{\omega}$.

Proof. Similar to the proof of Corollary 7.4. \square

Corollary 7.10. $Sp\mathcal{F}_{\mathbf{d}}^a$ has terminal and thus canonical and rational singularities.

8. FROBENIUS SPLITTING, THE BWB-TYPE THEOREM AND GRADED CHARACTER FORMULA

In this section we derive several applications of the results in the previous sections.

8.1. Frobenius splitting. The varieties $Sp\mathcal{F}_{\mathbf{d}}^a$ and $SpR_{\mathbf{d}}$ are defined over any field \mathbf{k} (simply replacing \mathbb{C} by \mathbf{k} in all definitions).

Theorem 8.1. *The varieties $Sp\mathcal{F}_{\mathbf{d}}^a$ and $SpR_{\mathbf{d}}$ over $\overline{\mathbb{F}}_p$ are Frobenius split for all prime p .*

Proof. Lemma 7.7 says that

$$\omega_{SpR_{\mathbf{d}}}^{-1} = \bigotimes_{(i,j) \in P_{\mathbf{d}}} \mathcal{O}(Z_{i,j}) \otimes \bigotimes_{(i,j) \in B_{\mathbf{d}}} \Omega_{i,j}.$$

Now since $\bigotimes_{(i,j) \in B_{\mathbf{d}}} \Omega_{i,j}$ is base point free, the Frobenius splitting for $SpR_{\mathbf{d}}$ and $Sp\mathcal{F}_{\mathbf{d}}^a$ follows from the Mehta-Ramanathan criterion (Proposition 8 of [MR]). \square

8.2. The BWB-type theorem. Let \mathcal{L}_{λ} be a line bundle on $Sp\mathcal{F}_{2n}^a$ which is the pull back of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V_{\lambda}^a)$ for a dominant \mathfrak{sp}_{2n} -weight λ . We prove the analogue of the Borel-Weil-Bott theorem.

Theorem 8.2. *We have*

$$H^0(Sp\mathcal{F}_{2n}^a, \mathcal{L}_{\lambda})^* \simeq H^0(SpR_{2n}, \pi_{2n}^* \mathcal{L}_{\lambda})^* \simeq V_{\lambda}^a, \\ H^{>0}(Sp\mathcal{F}_{2n}^a, \mathcal{L}_{\lambda}) = H^{>0}(SpR_{2n}, \pi_{2n}^* \mathcal{L}_{\lambda}) = 0.$$

Proof. First, we note that since $Sp\mathcal{F}_{2n}^a$ has rational singularities, we have the equalities

$$H^k(Sp\mathcal{F}_{2n}^a, \mathcal{L}_{\lambda}) \simeq H^k(SpR_{2n}, \pi_{2n}^* \mathcal{L}_{\lambda})$$

for all $k \geq 0$.

Second, we prove that all non-zero cohomology $H^k(Sp\mathcal{F}_{2n}^a, \mathcal{L}_{\lambda})$ vanish. In fact, first assume λ is regular. Then since the map $Sp\mathcal{F}_{2n}^a \rightarrow \mathbb{P}(V_{\lambda}^a)$ is an embedding, the line bundle

\mathcal{L}_λ is very ample. Therefore, for any k and big enough N one has $H^k(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda^{\otimes N}) = 0$. This implies $H^k(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda) = 0$, because $Sp\mathcal{F}_{2n}^a$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p .

Now consider a non regular λ . Let $Sp\mathcal{F}_{\mathbf{d}}^a$ be the corresponding degenerate parabolic flag variety, which is embedded into $\mathbb{P}(V_\lambda^a)$. Then we have the following commutative diagram of projections:

$$\begin{array}{ccc} SpR_{2n} & \xrightarrow{\eta} & SpR_{\mathbf{d}} \\ \pi_{2n} \downarrow & & \downarrow \pi_{\mathbf{d}} \\ Sp\mathcal{F}_{2n}^a & \xrightarrow{\phi} & Sp\mathcal{F}_{\mathbf{d}}^a \end{array}$$

Let \mathcal{L}'_λ be a line bundle on $Sp\mathcal{F}_{\mathbf{d}}^a$ which is the pull back of the bundle $\mathcal{O}(1)$ on $\mathbb{P}(V_\lambda^a)$. Then $\mathcal{L}_\lambda = \phi^*\mathcal{L}'_\lambda$. Since \mathcal{L}'_λ is very ample, and $Sp\mathcal{F}_{\mathbf{d}}^a$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p , $H^k(Sp\mathcal{F}_{\mathbf{d}}^a, \mathcal{L}'_\lambda) = 0$ (for positive k). Since $Sp\mathcal{F}_{\mathbf{d}}^a$ has rational singularities, $H^k(SpR_{\mathbf{d}}, \pi_{\mathbf{d}}^*\mathcal{L}'_\lambda) = H^k(Sp\mathcal{F}_{\mathbf{d}}^a, \mathcal{L}'_\lambda) (= 0$ for positive k). Now since η is a fibration with the fibers being towers of successive \mathbb{P}^1 -fibrations, we obtain $H^k(SpR_{2n}, \eta^*\pi_{\mathbf{d}}^*\mathcal{L}'_\lambda) = H^k(SpR_{\mathbf{d}}, \pi_{\mathbf{d}}^*\mathcal{L}'_\lambda) (= 0$ for positive k). Finally, since $Sp\mathcal{F}_{2n}^a$ has rational singularities, and $\eta^*\pi_{\mathbf{d}}^*\mathcal{L}'_\lambda = \pi_{2n}^*\mathcal{L}_\lambda$, we arrive at

$$H^k(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda) = H^k(SpR_{2n}, \pi_{2n}^*\mathcal{L}_\lambda) = H^k(SpR_{2n}, \eta^*\pi_{\mathbf{d}}^*\mathcal{L}'_\lambda),$$

which vanishes for $k > 0$.

Third, we note that there exists an embedding $(V_\lambda^a)^* \hookrightarrow H^0(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda)$. In fact take an element $v \in (V_\lambda^a)^* \simeq H^0(\mathbb{P}(V_\lambda^a), \mathcal{O}(1))$. Then restricting to the embedded variety $Sp\mathcal{F}_{2n}^a$ we obtain a section of \mathcal{L}_λ . Assume that it is zero. Then v vanishes on the open cell $(N^-)^a \cdot \mathbb{C}v_\lambda$. But the linear span of the elements of this cell coincides with the whole representation V_λ^a . Therefore, the restriction map $(V_\lambda^a)^* \rightarrow H^0(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda)$ is an embedding.

Finally, we recall that the varieties $Sp\mathcal{F}_{2n}^a$ are flat degenerations of the classical flag varieties. Since the higher cohomology of \mathcal{L}_λ vanish, we arrive at the equality of the dimensions of $H^0(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda)$ and of V_λ^a . Therefore, the embedding $(V_\lambda^a)^* \rightarrow H^0(Sp\mathcal{F}_{2n}^a, \mathcal{L}_\lambda)$ is an isomorphism. \square

Similarly one proves a parabolic version of the BWB-type theorem:

Theorem 8.3. *Let λ be a \mathbf{d} -dominant weight, i.e. $(\lambda, \omega_d) > 0$ implies $d \in \mathbf{d}$. Then there exists a map $\iota_\lambda : Sp\mathcal{F}_{\mathbf{d}}^a \rightarrow \mathbb{P}(V_\lambda^a)$. We have*

$$H^0(Sp\mathcal{F}_{\mathbf{d}}^a, \iota_\lambda^*\mathcal{O}(1))^* \simeq V_\lambda^a, \quad H^{>0}(Sp\mathcal{F}_{\mathbf{d}}^a, \iota_\lambda^*\mathcal{O}(1)) = 0.$$

8.3. The q -character formula. We now compute the q -character (PBW-graded character) of the modules V_λ^a (for combinatorial formula see Theorem 2.7). For this we use the Atiyah-Bott-Lefschetz fixed points formula applied to the variety SpR_{2n} . We first describe the fixed points explicitly.

Lemma 8.4. *The T -fixed points on SpR_{2n} are labeled by the collections $\mathbf{S} = (S_{i,j})$, $1 \leq i \leq j < 2n$, $i + j \leq 2n$, where $S_{i,j}$ are subsets of $\{1, \dots, 2n\}$ satisfying the following properties:*

- $S_{i,j} \subset \{1, \dots, i, j + 1, \dots, n\}$, $\#S_{i,j} = i$,
- $S_{i,j} \subset S_{i+1,j} \subset S_{i+1,j+1} \cup \{j + 1\}$,
- For any $i = 1, \dots, n$ if $k \in S_{i,2n-i}$. then $2n + 1 - i \notin S_{i,2n-i}$.

Proof. Obviously, a collection $\mathbf{V} \in SpR_{2n}$ is a T -fixed point if and only if each $V_{i,j}$ is the linear span of some basis vectors w_l . Now each collection \mathbf{S} as above determines \mathbf{V} by the formula $V_{i,j} = \text{span}(w_l : l \in S_{i,j})$. \square

We call a collection \mathbf{S} satisfying the conditions as above admissible. For an admissible \mathbf{S} let $p(\mathbf{S}) \in SpR_{2n}$ be the corresponding fixed point and let $p(S_{i,j}) \in Gr_i(W_{i,j})$ be its (i, j) -th component.

Recall the extended degenerate group $G^a \rtimes \mathbb{C}^*$.

Lemma 8.5. *The action of the group G^a and its extension $G^a \rtimes \mathbb{C}^*$ on $Sp\mathcal{F}_{2n}^a$ lifts to SpR_{2n} .*

Proof. Recall the embeddings $Sp\mathcal{F}_{2n}^a \hookrightarrow \mathcal{F}_{2n}^a$ and $SpR_{2n} \hookrightarrow R_{2n}$ (see Proposition 4.7). Since the analogue of our Lemma for SL_{2n} holds (see [FF]), we obtain the desired result for Sp_{2n} as well. \square

In order to state the theorem we prepare some notations. Let $\mathbb{C}[e^{\omega_1}, \dots, e^{\omega_n}, e^d]$ be the group algebra of the weight lattice of the extended Lie algebra $\mathfrak{g}^a \oplus \mathbb{C}d$. We sometimes use the notations $z_i = e^{\omega_i}$, $q = e^d$. For an element $\mu = md + \sum_{i=1}^n m_i \omega_i$ we write $e^\mu = q^m \prod_{i=1}^n z_i^{m_i}$. Also for a homogeneous vector $v \in V_\lambda^a$ we denote by $\text{wt}_q(v)$ the extended weight of v .

Recall the Atiyah-Bott-Lefschetz formula (see [AB], [T]): let X be a smooth projective algebraic M -dimensional variety and let \mathcal{L} be a line bundle on X . Let T be an algebraic torus acting on X with a finite set F of fixed points. Assume further that \mathcal{L} is T -equivariant. Then for each $p \in F$ the fiber \mathcal{L}_p is T -stable. We note also that since $p \in F$, the tangent space $T_p X$ carries a natural T -action. Let $\gamma_1^p, \dots, \gamma_M^p$ be the weights of the eigenvectors of T -action on $T_p X$. Then the Atiyah-Bott-Lefschetz formula gives the following expression for the character of the Euler characteristics:

$$(8.1) \quad \sum_{k \geq 0} (-1)^k \text{ch} H^k(X, \mathcal{L}) = \sum_{p \in F} \frac{\text{ch} \mathcal{L}_p}{\prod_{l=1}^M (1 - e^{-\gamma_l^p})}.$$

We apply this formula for $X = SpR_{2n}$, $\mathcal{L} = \pi_{2n}^* \mathcal{L}_\lambda$ with the action of the extended torus $T \cdot \mathbb{C}^*$. Since $H^{>0}(R_n, \pi_n^* \mathcal{L}_\lambda) = 0$, the Euler characteristics coincides with the character of the zeroth cohomology, i.e. with the character of $(V_\lambda^a)^*$. Therefore, for each admissible \mathbf{S} we need to compute the character of $\pi_{2n}^* \mathcal{L}_\lambda$ at $p(\mathbf{S})$ and the eigenvalues of the torus action in $T_{p(\mathbf{S})} SpR_{2n}$.

Let $\iota_\lambda : Sp\mathcal{F}_{2n}^a \rightarrow \mathbb{P}(V_\lambda^a)$ be the standard map (which is an embedding for regular λ). Then $\text{ch}(\pi_{2n}^* \mathcal{L}_\lambda)_{p(\mathbf{S})} = e^{-\text{wt}_q(\iota_\lambda p(\mathbf{S}))}$ (the minus sign comes from the fact that a fiber of $\mathcal{O}(1)$ is a dual line). We note that the weight of $\iota_\lambda p(\mathbf{S})$ depends only on the diagonal entries $S_{i,i}$.

Now let us compute the eigenvalues of the tangent action of the torus at a point $p_{\mathbf{S}}$. For each pair (i, j) , $1 \leq i \leq j < 2n$, $i + j \leq 2n$ define a collection $S'_{i,j}$ as follows.

First, let $i + j < 2n$. Given the sets $S_{i-1,j}$ and $S_{i,j+1}$, let us look at the possible values of $S_{i,j}$ keeping \mathbf{S} admissible. We denote such a possible collections by $\bar{S}_{i,j}$ in order to distinguish it from the already fixed component $S_{i,j}$. The definition of admissibility says that there exist exactly two variants for $\bar{S}_{i,j}$, namely

$$\bar{S}_{i,j} = S_{i-1,j} \cup \{a\} \text{ or } \bar{S}_{i,j} = S_{i-1,j} \cup \{b\},$$

where $\{a, b\} = S_{i,j+1} \cup \{j+1\} \setminus S_{i-1,j}$. Given a collection \mathbf{S} we denote the numbers a, b as above by $a_{i,j}^{\mathbf{S}}$ and $b_{i,j}^{\mathbf{S}}$. We have:

$$S_{i,j} = S_{i-1,j} \cup \{a_{i,j}^{\mathbf{S}}\}, \quad S_{i,j+1} \setminus S_{i-1,j} = \{a_{i,j}^{\mathbf{S}}, b_{i,j}^{\mathbf{S}}\}.$$

We denote by $S'_{i,j}$ the set $S_{i,j} \setminus \{a_{i,j}^{\mathbf{S}}\} \cup \{b_{i,j}^{\mathbf{S}}\}$.

Second, assume $i + j = 2n$. Given the set $S_{i-1,j}$, let us look at the possible values of $\bar{S}_{i,j}$ keeping \mathbf{S} admissible. The definition of admissibility says that there exist exactly two variants for $\bar{S}_{i,j}$, namely

$$\bar{S}_{i,j} = S_{i-1,j} \cup \{a\} \text{ or } \bar{S}_{i,j} = S_{i-1,j} \cup \{b\},$$

where $\{a, b\} = \{1, \dots, i, 2n - i + 1, \dots, 2n\} \setminus \{2n + 1 - l : l \in S_{i-1,j}\}$. We also denote the numbers a, b by $a_{i,j}^{\mathbf{S}}$ and $b_{i,j}^{\mathbf{S}}$. We set $S_{i,j} = S_{i-1,j} \cup \{a_{i,j}^{\mathbf{S}}\}$ and we denote by $S'_{i,j}$ the set $S_{i,j} \setminus \{a_{i,j}^{\mathbf{S}}\} \cup \{b_{i,j}^{\mathbf{S}}\}$.

Recall that the variety $Sp\mathcal{F}_{2n}^a$ sits inside the product of Grassmann varieties $\prod Gr_i(W_{i,j})$. Each $\wedge^i(W_{i,j})$ is acted upon by $\mathfrak{g}^a \oplus \mathbb{C}d$ and therefore each Grassmannian carries a natural action of the group $G^a \rtimes \mathbb{C}^*$ (the additional \mathbb{C}^* part corresponds to the PBW-grading operator). Thus for each collection $S_{i,j}$ we have the corresponding weight $\text{wt}_q p(\mathbf{S}_{i,j})$, which is the weight of the corresponding point in $\wedge^i(W_{i,j})$.

Theorem 8.6. *The q -character of the representation V_{λ}^a is given by the sum over all admissible collections \mathbf{S} of the summands*

$$(8.2) \quad \frac{e^{\text{wt}_q(\iota_{\lambda} p(\mathbf{S}))}}{\prod_{\substack{i+j \leq 2n \\ 1 \leq i \leq j < 2n}} \left(1 - e^{\text{wt}_q p(S'_{i,j}) - \text{wt}_q p(S_{i,j})}\right)}.$$

Proof. Recall that SpR_{2n} can be constructed as a tower of successive \mathbb{P}^1 -fibrations $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$. Fix an admissible \mathbf{S} . Then the surjections $SpR_{2n} \rightarrow SpR_{2n}(l)$ define the T -fixed points $p(\mathbf{S}(l))$ in each $SpR_{2n}(l)$ (note that $\mathbf{S}(l)$ consists of $S_{i,j}$ such that for $\beta_k = \alpha_{i,j}$ one has $k \leq l$). For each $l = 1, \dots, M$ we denote by $v_l \in T_{p(\mathbf{S}(l))} SpR_{2n}(l)$ a tangent vector to the fiber of the map $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ at the point $p(\mathbf{S}(l-1))$. Then it is easy to see that the weights of the eigenvectors of the T action in $T_{p(\mathbf{S})} SpR_{2n}$ are exactly the weights of the vectors v_l , $l = 1, \dots, M$.

So let us fix l , $1 \leq l \leq M$ and i, j with $\alpha_{i,j} = \beta_l$. Let us denote by Y_l the set of all pairs (k, m) such that for the root $\alpha_{k,m} = \beta_r$ one has $r \leq l$. Then the fiber \mathbb{P}^1 of the map $SpR_{2n}(l) \rightarrow SpR_{2n}(l-1)$ at the point $p(\mathbf{S}(l-1))$ consists of all collections $(V_{k,m})$ with $(k, m) \in Y_l$ subject to the following conditions:

- $V_{k,m} = p(S_{k,m})$ if $\alpha_{k,m} \neq \beta_l$,

- $V_{i,j} \supset p(S_{i-1,j})$,
- $V_{i,j} \subset p(S_{i-1,j}) \oplus \mathbb{C}w_{a_{i,j}^{\mathbf{S}}} \oplus \mathbb{C}w_{b_{i,j}^{\mathbf{S}}}$.

Now it is easy to see that the character of the tangent vector to this fiber at the point $p(\mathbf{S}(l-1))$ is equal to $e^{\text{wt}_q p(S'_{i,j}) - \text{wt}_q p(S_{i,j})}$ (recall $a_{i,j}^{\mathbf{S}} \in S_{i,j}$ and $S'_{i,j} = S_{i,j} \setminus \{a_{i,j}^{\mathbf{S}}\} \cup \{b_{i,j}^{\mathbf{S}}\}$). \square

Remark 8.7. We note that the Euler characteristics

$$\sum_{k \geq 0} (-1)^k \text{ch} H^k(SpR_{2n}, \pi_{2n}^* \mathcal{L}_\lambda)$$

is equal to $\text{ch}(V_\lambda^a)^*$. But in each summand (8.2) both numerator and denominator differ from the corresponding summand in the Atiyah-Bott-Lefschetz formula (8.1) by the change of variables $z_i \rightarrow z_i^{-1}$ and $q \rightarrow q^{-1}$. Via this change we pass from the character of $(V_\lambda^a)^*$ to the character of V_λ^a .

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