# CREPANT RESOLUTIONS AND BRANE TILINGS II: TILTING BUNDLES

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Abstract. Given a brane tiling, that is, a bipartite graph on a torus, we can associate with it a singular 3-Calabi-Yau variety. Using the brane tiling, we can also construct all crepant resolutions of the above variety. We give an explicit toric description of tilting bundles on these crepant resolutions. This result proves the conjecture of Hanany, Herzog and Vegh and a version of the conjecture of Aspinwall.

## 1. Introduction

The goal of this paper is to prove the conjecture of Hanany, Herzog and Vegh [7] on the description of tilting bundles on the crepant resolutions of singular 3-Calabi-Yau varieties arising from brane tilings. All these crepant resolutions can be constructed as moduli spaces of representations of some quiver with relations [8, Theorem 15.1]. These moduli spaces are toric 3-Calabi-Yau varieties. An explicit construction of their toric diagrams was given in [11].

Given a brane tiling, we can associate with it a quiver potential $(Q, W)$ and a quiver potential algebra $\mathbb{C}Q/(\partial W)$. The singular Calabi-Yau variety mentioned above is isomorphic to the spectrum of the center of $\mathbb{C}Q/(\partial W)$. It has a non-commutative crepant resolution $\mathbb{C}Q/(\partial W)$ [3, 11]. Its crepant resolutions are given by the moduli spaces $\mathcal{M}_\theta = \mathcal{M}_\theta(\mathbb{C}Q/(\partial W), \alpha)$ of $\theta$-semistable $\mathbb{C}Q/(\partial W)$-representations of dimension $\alpha = (1, \dots, 1) \in \mathbb{Z}^{Q_0}$, where $\theta \in \mathbb{Z}^{Q_0}$ is $\alpha$-generic. All $\theta$-semistable points in $\mathcal{M}_\theta$ are $\theta$-stable for such $\theta$. Therefore there exists a universal (also called tautological) vector bundle $\mathcal{U}$ over $\mathcal{M}_\theta$, endowed with a structure of a left $\mathbb{C}Q/(\partial W)$-module. It follows from the results of Van den Bergh (see [16]) that $\mathcal{U}$ is a tilting bundle (an alternative proof can be found in [8]). This vector bundle can be decomposed into a sum of $\#Q_0$ line bundles. We will describe the toric Cartier divisors inducing these line bundles. Namely, we fix some $i_0 \in Q_0$ and for every vertex $i \in Q_0$ we choose some path $u_i : i_0 \to i$. Intersecting the path $u_i$ with perfect matchings (they parametrize 2-dimensional orbits of $\mathcal{M}_\theta$, i.e. rays of the corresponding fan, see Section 2), we get a toric Cartier divisor which induces a line bundle $\overline{L}_i$ over $\mathcal{M}_\theta$. We will show that $\mathcal{U}$ is isomorphic to the direct sum of $\overline{L}_i$, $i \in Q_0$. This description of the tilting bundle was conjectured by Hanany, Herzog and Vegh [7, Section 5.2]. Our result proves also a conjecture of Aspinwall [2] on the existence of some "globally defined" collection of line bundles that gives rise to the tilting collection on $\mathcal{M}_\theta$ for arbitrary generic $\theta$. We should note that a similar description of the exceptional collections in the context of toric quiver varieties (these are moduli spaces of quiver representation for quiver without relations) was given by Altmann and Hille [1].

The paper is organized as follows: In Section 2 we gather preliminary material on brane tilings and the induced quiver potential algebras. In Section 3 we recall some results of Thaddeus [15] about toric quotients of toric varieties and prove some facts on the descent of line bundles with respect to such quotients. In Section 4 we give a toric description of the tilting bundle on $\mathcal{M}_\theta$. In Section 5 we give some explicit examples.

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## 2. Preliminaries

Most of the content of this section can be found in [11]. We briefly recall some material for the convenience of the reader.

A brane tiling is a bipartite graph $G = (G_0^{\pm}, G_1)$ together with an embedding of the corresponding CW-complex into the real two-dimensional torus $T$ so that the complement $T\backslash G$ consists of simply-connected components. The set of connected components of $T\backslash G$ is denoted by $G_2$ and is called the set of faces of $G$.

With any brane tiling we can associate a quiver $Q = (Q_0, Q_1)$ embedded in a torus $T$ and a potential $W$ (linear combination of cycles in $Q$), see [12]. The set $Q_2$ of connected components of $T\backslash Q$ is called the set of faces of $Q$. The summands of $W$ are the cycles along the faces of $Q$ taken with appropriate signs. With this data, we associate a quiver potential algebra $A = \mathbb{C}Q/(\partial W)$, see [12].

For any arrow $a \in Q_1$, we define $s(a), t(a) \in Q_0$ to be its source and target (also called tail and head). Consider a complex of abelian groups

$$\mathbb{Z}^{Q_2} \xrightarrow{d_2} \mathbb{Z}^{Q_1} \xrightarrow{d_1} \mathbb{Z}^{Q_0},$$

where $d_2(F) = \sum_{a\in F} a$, $F \in Q_2$ and $d_1(a) = t(a) - s(a)$ for any arrow $a \in Q$. Its homology groups are isomorphic to the homology groups of the 2-dimensional torus containing $Q$. We define an abelian group $\Lambda$ by a cocartesian left upper square of the following diagram

$$\begin{tikzcd}
\mathbb{Z}^{Q_2} \arrow[r, "d_2"] \arrow[d] & \mathbb{Z}^{Q_1} \arrow[r, "d_1"] \arrow[d, "\mathrm{wt}"'] & \mathbb{Z}^{Q_0} \\
\mathbb{Z} \arrow[r, "\omega_\Lambda"] \arrow[d, dotted, "\omega_M"'] & \Lambda \arrow[ur, dotted, "d"'] & \\
M \arrow[ur, "i"'] & &
\end{tikzcd}$$

where the left arrow of the square is given by $F \mapsto 1,\ F \in Q_2$. There exists a unique map $d : \Lambda \to \mathbb{Z}^{Q_0}$ making the right triangle commutative. Let $M = \ker(d)$. There exists a unique map $\omega_M : \mathbb{Z} \to M$ making the lower triangle commutative. If $G$ has at least one perfect matching then $\Lambda$ is a free abelian group and the map $\omega_\Lambda : \mathbb{Z} \to \Lambda$ is injective (see [12, Lemma 3.3]).

We define a weak path in $Q$ to be a path consisting of arrows of $Q$ and their inverses (for any arrow $a$, we identify $aa^{-1}$ and $a^{-1}a$ with trivial paths). For any weak path $u$, we define its content $|u| \in \mathbb{Z}^{Q_1}$ by counting the arrows of $u$ with appropriate signs. We define the weight of $u$ to be $\mathrm{wt}(u) = \mathrm{wt}(|u|) \in \Lambda$. We define $\overline{\omega} \in \Lambda$ to be the weight of any cycle along some face of $Q$. Note that $\overline{\omega} = \omega_\Lambda(1)$ and that $\overline{\omega} \in M$.

Let $B = \ker(\mathbb{Z}^{Q_0} \to \mathbb{Z})$, where the map is given by $i \mapsto 1$, $i \in Q_0$. This group is generated by the elements of the form $i - j$, where $i, j \in Q_0$. As $Q$ is connected, we conclude that $B = \operatorname{im} d$. There is a short exact sequence

$$0 \to M \xrightarrow{i} \Lambda \xrightarrow{d} B \to 0.$$

One can easily see that $\operatorname{rk} B = \#Q_0 - 1$, $\operatorname{rk} \Lambda = \#Q_0 + 2$, and $\operatorname{rk} M = 3$.

Let $\Lambda^+ \subset \Lambda$ be a semigroup generated by the weights of the arrows. Let $P \subset \Lambda_{\mathbb{Q}}$ be a cone generated by $\Lambda^+$

$$P = \{\sum a_i\lambda_i \mid a_i \in \mathbb{Q}_{\geq 0},\ \lambda_i \in \Lambda^+ \text{ for all } i\}.$$

We define $M^+ = \Lambda^+ \cap M$ and $P_M = P \cap M_{\mathbb{Q}}$. We do not have $\Lambda^+ = P \cap \Lambda$ in general. But we have $M^+ = P_M \cap M$ [11]. This implies that $\operatorname{Spec} \mathbb{C}[M^+]$ is a normal toric variety. If the brane tiling is consistent (see e.g. [11]) then the quiver potential algebra $\mathbb{C}Q/(\partial W)$ is a 3-Calabi-Yau algebra (see [12, 4, 5]) and is a non-commutative crepant resolution of $\operatorname{Spec} \mathbb{C}[M^+]$ (see [3, 11]). In this paper we will always assume that our brane tiling is consistent.

Let $\mathcal{A}$ be the set of all perfect matchings of the bipartite graph $G$. Any perfect matching $I \in \mathcal{A}$ can be considered as a subset of $Q_1$. We define a characteristic function $\chi_I : \mathbb{Z}^{Q_1} \to \mathbb{Z}$ by the rule (for $a \in Q_1$)

$$\chi_I(a) = \begin{cases} 1, & a \in I, \\ 0, & a \notin I. \end{cases}$$

For any face $F \in Q_2$ we have $\chi_I(d_2(F)) = 1$. Therefore we can consider $\chi_I$ as a linear map $\Lambda \to \mathbb{Z}$, i.e. as an element $\chi_I \in \Lambda^\vee$. We define $\overline{\chi}_I = i^*\chi_I \in M^\vee$. The family of all $\chi_I : \Lambda \to \mathbb{Z}$ (resp. $\overline{\chi}_i : M \to \mathbb{Z}$) defines a linear map $\chi : \Lambda \to \mathbb{Z}^{\mathcal{A}}$ (resp. $\overline{\chi} : M \to \mathbb{Z}^{\mathcal{A}}$).

The dual cone $P^\vee \in \Lambda^\vee_{\mathbb{Q}}$ is generated by $\chi_I$, $I \in \Lambda$, and all the corresponding rays are extremal in $P^\vee$ [4, Lemma 2.3.4]. Analogously, the dual cone $P_M^\vee \in M_{\mathbb{Q}}^\vee$ is generated by $\overline{\chi}_I$, $I \in \mathcal{A}$. We denote by $\mathcal{A}^e \subset \mathcal{A}$ the set of perfect matchings $I$, such that $\overline{\chi}_I$ generates an extremal ray in $P_M^\vee$. These perfect matchings are called extremal.

All the crepant resolutions of $\operatorname{Spec} \mathbb{C}[M^+]$ can be described as moduli spaces of stable representations of $A = \mathbb{C}Q/(\partial W)$ in the sense of King [10]. Let $\alpha = (1, \ldots, 1) \in \mathbb{Z}^{Q_0}$ and let $\theta \in B$ (i.e. $\theta \in \mathbb{Z}^{Q_0}$ is such that $\theta \cdot \alpha = 0$). Any $A$-module $X$ can be described by the set of vector spaces $(X_i)_{i \in Q_0}$ and linear maps $X_a : X_i \to X_j$ for arrows $a : i \to j$. We define $\dim X = (\dim X_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$. An $A$-module $X$ of dimension $\alpha$ is called $\theta$-semistable (resp. $\theta$-stable) if for any proper $A$-submodule $0 \neq Y \subset X$ we have $\theta \cdot \dim Y \geq 0$ (resp. $\theta \cdot \dim Y > 0$). We say that $\theta$ is $\alpha$-generic if for any $0 < \beta < \alpha$ we have $\theta \cdot \beta \neq 0$. In this case all $\theta$-semistable modules of dimension $\alpha$ are automatically stable. One can construct the moduli space $\mathcal{M}_\theta = \mathcal{M}_\theta(A, \alpha)$ of $\theta$-semistable $A$-modules of dimension $\alpha$ [10]. It is shown in [11] that, for $\alpha$-generic $\theta \in B$, this moduli space is a toric variety (with a dense subtorus $T_M = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, see Section 3)

$$\mathcal{M}_\theta = \operatorname{Spec} \mathbb{C}[\Lambda^+] /\!\!/_\theta T_B = \operatorname{Spec} \mathbb{C}[P \cap \Lambda] /\!\!/_\theta T_B,$$

where $T_B = \operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{C}^*)$. In this case $\mathcal{M}_\theta$ is smooth and is a crepant resolution of $\operatorname{Spec} \mathbb{C}[M^+]$ (see [9, 11]).

An explicit description of a fan of $\mathcal{M}_\theta$ was given in [11]. For any $A$-module $X = ((X_i)_{i\in Q_0}, (X_a)_{a\in Q_1})$, we define its cosupport $I_X = \{a \in Q_1 \mid X_a = 0\}$. It was shown in [11] that every $T_M$-orbit of $\mathcal{M}_\theta$ is uniquely determined by the cosupport of its modules. Any such cosupport $I$ can be considered as a subgraph of the bipartite graph $G$. It was shown in [11] that this subgraph can have at most one connected component containing more than one edge (we call it a big component of $I$).

**Proposition 2.1** (see [11]). *Let $X \in \mathcal{M}_\theta$ and let $O_X \subset \mathcal{M}_\theta$ be its $T_M$-orbit. Then*

(1) $\dim O_X = 3$ *if and only if* $I_X = \emptyset$.
(2) $\dim O_X = 2$ *if and only if $I_X$ is a perfect matching.*
(3) $\dim O_X = 1$ *if and only if $I_X$ contains a big component which is a cycle. In this case $I_X$ is a union of two perfect matchings.*
(4) $\dim O_X = 0$ *if and only if $I_X$ contains a big component which has two trivalent vertices of different colors and all other vertices of valence* 2. *In this case $I_X$ is a union of three perfect matchings.*

For any subset $I \subset Q_1$, we define a $\mathbb{C}Q$-representation $X_I = (X_{I,a})_{a\in Q_1}$ of dimension $\alpha$ by the rule (for $a \in Q_1$)

$$X_{I,a} = \begin{cases} 0, & a \in I, \\ 1, & a \notin I. \end{cases}$$

We say that $I$ is $W$-compatible if $X_I$ is an $A$-representation. For example, all perfect matchings and an empty set are $W$-compatible. We say that $I$ is $\theta$-stable if $I_X$ is $\theta$-stable. The elements of the fan of $\mathcal{M}_\theta$ are in bijection with $W$-compatible $\theta$-stable subsets of $Q_1$. The rays of the fan of $\mathcal{M}_\theta$ are parametrized by $\theta$-stable perfect matchings. All elements $\overline{\chi}_I \in M^\vee$, $I \in \mathcal{A}$, are contained in the hyperplane

$$\{y \in M^\vee_{\mathbb{Q}} \mid \omega^*_M(y) = 1\},$$

where $\omega_M : \mathbb{Z} \to M$ was defined earlier. This implies that $\mathcal{M}_\theta$ is a toric 3-Calabi-Yau variety. The above proposition gives an algorithm to construct its toric diagram (this is an intersection of cones of the fan of $\mathcal{M}_\theta$ with the above hyperplane).

## 3. Toric quotients

In this section we will recall some facts from [15] about toric quotients of toric varieties and give further information on the line bundles on such quotients. We refer to [6] and [14] for the relevant definitions and properties of toric varieties.

Consider a pair $(\Lambda, P)$, where $\Lambda$ is a lattice (i.e. a free abelian group of finite rank) and $P \subset \Lambda_{\mathbb{Q}}$ is a polyhedral cone. We associate with it a scheme

$$X_P = X(\Lambda, P) := \operatorname{Spec} \mathbb{C}[P \cap \Lambda].$$

More generally, given a pair $(\Lambda, P)$, where $\Lambda$ is a lattice and $P \subset \Lambda_{\mathbb{Q}}$ is a polyhedron, we associate with it a scheme $X(\Lambda, P)$ in the following way. Let $C(P) \subset \mathbb{Q} \times \Lambda_{\mathbb{Q}}$ be a cone which is a closure of

$$\{\lambda(1, x) \mid \lambda \in \mathbb{Q}_{\geq 0}, x \in P\}.$$

We endow $\mathbb{C}[C(P) \cap (\mathbb{Z} \times \Lambda)]$ with a $\mathbb{Z}$-grading induced by the first coordinate and define

$$X_P = X(\Lambda, P) := \operatorname{Proj} \mathbb{C}[C(P) \cap (\mathbb{Z} \times \Lambda)].$$

Let $T_\Lambda = \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*)$. There is a canonical $T_\Lambda$-action on $X_P$ and a canonical $T_\Lambda$-linearization of the canonical line bundle $\mathcal{O}(1)$ on $X_P$. If $P$ is a cone then

$C(P) = \mathbb{Q}_{\geq 0} \times P$ and $C(P) \cap (\mathbb{Z} \times \Lambda) = \mathbb{N} \times (P \cap \Lambda)$. So our new definition of $X_P$ is compatible with the old one.

The scheme $X_P$ can be described as a toric variety associated to a fan. For any $y \in \Lambda_{\mathbb{Q}}^{\vee}$ define

$$\operatorname{face}_y(P) := \{x \in P \mid \langle x, y\rangle = \min_P \langle -, y\rangle\}.$$

All faces of $P$ have this form for some $y \in \Lambda_{\mathbb{Q}}^{\vee}$. For any face $F \subset P$, define its normal cone

$$N_F = N_F P := \{y \in \Lambda_{\mathbb{Q}}^{\vee} \mid \operatorname{face}_y(P) \supset F\} = \{y \in \Lambda_{\mathbb{Q}}^{\vee} \mid \langle F, y\rangle \leq \langle P, y\rangle\}.$$

For any faces $F, G \subset P$, we have $F \subset G$ if and only if $N_G P \subset N_F P$. The set of cones

$$N(P) = \{N_F P \mid F \text{ face of } P\}$$

is a fan in $\Lambda^{\vee}$ and the associated toric variety is isomorphic to $X_P$ (cf. [15, Prop. 2.17]).

**Lemma 3.1.** *Let $F \subset P$ be some face and let $\langle F\rangle \subset \Lambda_{\mathbb{Q}}$ be a vector space generated by the differences $x - y$, for $x, y \in F$. Then $\langle F\rangle = N_F^{\perp}$.*

*Proof.* We can suppose that $0 \in F$. Then $y \in N_F$ if and only if

$$\langle F, y\rangle = 0, \qquad \langle P, y\rangle \geq 0.$$

This implies that $\langle F\rangle \subset N_F^{\perp}$. This vetor spaces have equal dimension as $\dim F + \dim N_F = \dim \Lambda_{\mathbb{Q}}$. □

For any face $F \subset P$, let $O_F$ denote the $T_\Lambda$-orbit corresponding to $N_F$.

**Lemma 3.2** (see [15, Prop. 2.13])**.** *Let $F \subset P$ be some face. Then*

1. $\dim O_F = \dim F$.
2. *The character group of the stabilizer of $O_F$ in $T_\Lambda$ equals* $\operatorname{coker}(N_F^{\perp} \cap \Lambda \to \Lambda)$.
3. *The closure of $O_F$ equals $X(\Lambda, F)$.*
4. *For any two faces $F, G$ we have $F \subset G$ if and only if $O_F \subset O_G$.*

Consider an exact sequence of lattices

$$0 \to M \xrightarrow{i} \Lambda \xrightarrow{d} B \to 0.$$

For any face $F \subset P$, we define $F_M = F \cap M_{\mathbb{Q}}$. There is an inclusion $T_B \subset T_\Lambda$ that induces an action of $T_B$ on $X_P$ and on the line bundle $\mathcal{O}(1)$. It is shown in [15, Prop. 3.2] that the corresponding GIT quotient is given by

$$X(\Lambda, P)/\!\!/T_B = X(M, P_M).$$

**Lemma 3.3** ([15, Lemma 3.3])**.** *Let $F \subset P$ be a face. Then*

1. *$O_F$ is $T_B$-semistable if and only if $F \cap M_{\mathbb{Q}} \neq \emptyset$.*
2. *$O_F$ is $T_B$-stable if and only if $M_{\mathbb{Q}}$ intersects* $\operatorname{inn}(F)$ *transversally.*
3. *The image of a $T_B$-semistable orbit $O_F$ in $X_P/\!\!/T_B$ is $O_{F_M}$.*

**Remark 3.4.** Condition that $M_{\mathbb{Q}}$ intersects $\operatorname{inn}(F)$ transversally means that $\operatorname{inn}(F) \cap M_{\mathbb{Q}} \neq \emptyset$ and $\langle F\rangle + M_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}$. We say that $F$ is $M$-stable in this case. We denote the subscheme of stable points of $X_P$ by $X_P^s$.

**Remark 3.5.** There is a bijection between the faces of $P_M$ and the faces $F \subset P$ such that $\operatorname{inn}(F) \cap M_{\mathbb{Q}} \neq \emptyset$.

**Proposition 3.6.** *The set of cones*

$$N^s(P_M) = \{N_{F_M} P_M \subset M_{\mathbb{Q}}^\vee \mid F \subset P \text{ is } M\text{-stable}\}$$

*forms a fan in $M^\vee$. The corresponding toric variety is isomorphic to $X_P^s /\!\!/ T_B$.*

*Proof.* The set $N^s(P_M)$ is a subset of the fan $N(P_M)$. To show that $N^s(P_M)$ is a fan, we just have to show that any face of the cone from $N^s(P_M)$ is contained in $N^s(P_M)$. Let $F \subset P$ be $M$-stable. Let $\tau' \subset N_{F_M} P_M$ be some face. We will show that $\tau' \in N^s(P_M)$. We can find some face $G' \subset P_M$ such that $\tau' = N_{G'} P_M$. We choose a minimal face $G \subset P$ such that $G_M = G'$. The minimality property implies that $\operatorname{inn} G \cap M_{\mathbb{Q}} \neq \emptyset$. Moreover, $N_{G_M} P_M \subset N_{F_M} P_M$, so $F_M \subset G_M$ and therefore $F \subset G$. In particular, $\langle G \rangle + M_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}$ and therefore $G$ is $M$-stable. This implies that $\tau' \subset N^s(P_M)$. □

**Lemma 3.7.** *Let $F \subset P$ be an $M$-stable face. Consider the normal cones $N_F = N_F P \subset \Lambda_{\mathbb{Q}}^\vee$ and $N_{F_M} = N_{F_M} P_M \subset M_{\mathbb{Q}}^\vee$. Then the map $i^* : \Lambda_{\mathbb{Q}}^\vee \to M_{\mathbb{Q}}^\vee$ restricts to a bijection*

$$i_F^* : N_F \to N_{F_M}.$$

*Proof.* Without loss of generality we may assume that $0 \in \operatorname{inn}(F)$. Then for any $y \in N_F$, we have

$$\langle F, y \rangle = 0, \qquad \langle P, y \rangle \geq 0.$$

This implies that

$$\langle F_M, i^*(y) \rangle = 0, \qquad \langle P_M, i^*(y) \rangle \geq 0$$

and therefore $i^*(y) \in N_{F_M}$.

It follows from our assumption that the vector space $\langle F \rangle$ intersects $M_{\mathbb{Q}}$ transversally. This implies that the homomorphism of vector spaces

$$F^\perp \to F_M^\perp$$

is an isomorphism. Therefore the map $i^* : N_F \to N_{F_M}$ is injective, as $N_F \subset F^\perp$.

Let us prove the surjectivity. Consider $y' \in N_{F_M} \subset F_M^\perp$. We can find $y \in F^\perp$ such that $i^*(y) = y'$. We know that $\langle P_M, y \rangle \geq 0$ and we have to show that $\langle P, y \rangle \geq 0$, as this will imply $y \in N_F$. Assume that there exists $x_0 \in P$ such that $\langle x_0, y \rangle < 0$. Without loss of generality we may assume that $\Lambda_{\mathbb{Q}}$ is generated by $x_0$ and $F$, and that $P$ is a convex hull of $x_0$ and $F$. Then $y^\perp = \langle F \rangle$. It follows from the transversality of the intersection of $M_{\mathbb{Q}}$ and $F$ that $M_{\mathbb{Q}}$ intersects $P \backslash F$. But for any point $x$ in this intersection we have $\langle x, y \rangle < 0$ and $x \in P_M$. This contradicts our assumption $\langle P_M, y \rangle \geq 0$. □

**Corollary 3.8.** *For any $M$-stable face $F \subset P$, we have*

$$N_{F_M}^\vee = N_F^\vee \cap M_{\mathbb{Q}}.$$

*Proof.* We have

$$N_F^\vee \cap M_{\mathbb{Q}} = \{x \in M_{\mathbb{Q}} \mid \langle x, N_F \rangle \geq 0\} = \{x \in M_{\mathbb{Q}} \mid \langle x, N_{F_M} \rangle \geq 0\} = N_{F_M}^\vee.$$

□

**Corollary 3.9.** *Let $F \subset P$ be an $M$-stable face. Let $\sigma = N_F P$, $\sigma' = N_{F_M} P_M$, $U_\sigma = \mathbb{C}[\sigma^\vee \cap \Lambda]$, and $U_{\sigma'} = \mathbb{C}[(\sigma')^\vee \cap M]$. Then the map $X_P^s \to X_P^s /\!\!/ T_B$ is given over $U_{\sigma'}$ by*

$$U_\sigma = \operatorname{Spec} \mathbb{C}[N_F^\vee \cap \Lambda] \to \operatorname{Spec} \mathbb{C}[N_F^\vee \cap M] = U_{\sigma'}.$$

Recall that with any $N(P)$-linear support function $h : |N(P)| \to \mathbb{Q}$ (see e.g. [14, Section 2.1]) we can associate a $T_\Lambda$-equivariant line bundle $L_h$ over $X_P$ (see [14, Prop. 2.1]). If $T_B$ acts freely on $X_P^s$ then this line bundle descends to a $T_M$-equivariant line bundle on $X_P^s /\!\!/ T_B$ (this follows from [13, Prop. 0.9] and descent theory). We are going to describe an $N^s(P_M)$-linear support function that gives this line bundle.

**Theorem 3.10.** *Assume that $T_B$ acts freely on $X_P^s$. Let $h : |N(P)| \to \mathbb{Q}$ be an $N(P)$-linear support function. Define an $N^s(P_M)$-linear support function $h' : |N^s(P_M)| \to \mathbb{Q}$ by the rule*

$$h'(y) = h((i_F^*)^{-1}(y)), \qquad y \in N_{F_M},$$

*where $F \subset P$ is an $M$-stable face and $i_F^* : N_F \to N_{F_M}$ is a bijection defined earlier. Then the descend of $L_h$ to $X_P^s /\!\!/ T_B$ is isomorphic to $L_{h'}$ as a $T_M$-equivariant line bundle.*

*Proof.* Let us recall the construction of a $T_\Lambda$-equivariant line bundle $L_h$ on $X_P$ associated to the support function $h : |N(P)| \to \mathbb{Q}$ (see [14, Prop. 2.1]). For any commutative semigroup $\Gamma$, we denote the canonical basis of the semigroup algebra $\mathbb{C}[\Gamma]$ by $(e^\gamma)_{\gamma\in\Gamma}$. We can find a system of elements $(l_\sigma \in \Lambda)_{\sigma\in N(P)}$ such that $h|_\sigma = l_\sigma|_\sigma$ for any $\sigma \in N(P)$. The line bundle $L_h$ is defined by gluing the line bundles $U_\sigma \times \mathbb{C}$ over $U_\sigma$, $\sigma \in N(P)$ using the gluing functions

$$g_{\tau\sigma} : (U_\sigma \times \mathbb{C})|_{U_\tau} \to U_\tau \times \mathbb{C}, \quad (x, c) \mapsto (x, e^{l_\sigma - l_\tau}(x)c)$$

for $\tau < \sigma$. The action of $T_\Lambda$ on $U_\sigma \times \mathbb{C}$ is given by

$$t(x, c) = (tx, e^{-l_\sigma}(t)c), \quad t \in T_\Lambda.$$

Let now $\sigma = N_F$ and $\sigma' = N_{F_M}$, for some $M$-stable face $F \subset P$. Let $\pi : U_\sigma \to U_{\sigma'} = U_\sigma /\!\!/ T_B$ be a canonical projection We give an explicit description of the descend line bundle $(U_\sigma \times \mathbb{C}) /\!\!/ T_B$ over $U_\sigma /\!\!/ T_B = U_{\sigma'}$.

The character group of the stabilizer of $O_F$ in $T_B$ is given by $\operatorname{coker}(\sigma^\perp \cap \Lambda \to B)$ (see e.g. [15, Prop. 2.6]). By our assumptions this stabilizer is trivial, so

$$(\sigma^\perp \cap \Lambda) + M = \Lambda.$$

This means that we can find some $m_\sigma \in M$ such that $l_\sigma - m_\sigma \in \sigma^\perp$. Consider the action of $T_M$ on $U_{\sigma'} \times \mathbb{C}$ given by

$$t(x, c) = (tx, e^{-m_\sigma}(t)c), \quad t \in T_M.$$

The map

$$\overline{\pi} : U_\sigma \times \mathbb{C} \to U_{\sigma'} \times \mathbb{C}, \qquad (x, c) \mapsto (\pi(x), e^{l_\sigma - m_\sigma}(x)c),$$

is $T_\Lambda$-equivariant. Indeed, for any $t \in T_\Lambda$ we have

$$\overline{\pi}(t(x, c)) = \overline{\pi}(tx, e^{-l_\sigma}(t)c) = (\pi(x), e^{l_\sigma - m_\sigma}(tx)e^{-l_\sigma}(t)c) = (\pi(x), e^{-m_\sigma}(t)e^{l_\sigma - m_\sigma}(x)c).$$

On the other hand

$$t\overline{\pi}((x, c)) = t(\pi(x), e^{l_\sigma - m_\sigma}(x)c) = (\pi(x), e^{-m_\sigma}(t)e^{l_\sigma - m_\sigma}(x)c).$$

This shows that $\overline{\pi} : U_\sigma \times \mathbb{C} \to U_{\sigma'} \times \mathbb{C}$ is a quotient with respect to the action of $T_B$.

The gluing of line bundles $(U_\sigma /\!\!/ T_B)\times\mathbb{C}$ is induced by the gluing of line bundles $U_\sigma\times\mathbb{C}$ and is given by the formula

$$U_{\tau'}\times\mathbb{C}\to(U_{\sigma'}\times\mathbb{C})|_{U_{\tau'}},\quad (x,c)\mapsto(x,e^{m_\sigma-m_\tau}(x)c),$$

where $\sigma=N_F$, $\sigma'=N_{F_M}$, $\tau=N_G$, $\tau'=N_{G_M}$ for $M$-stable faces $F\subset G$ of $P$. The corresponding support function $h':|N(P_M)|\to\mathbb{Q}$ is given on $y'\in\sigma'$ by

$$h'(y')=m_\sigma(y')=m_\sigma((i_F^*)^{-1}(y'))=l_\sigma((i_F^*)^{-1}(y'))=h((i_F^*)^{-1}(y')),$$

as $l_\sigma-m_\sigma\in\sigma^\perp$. □

Any element $\theta\in B$ can be considered as a character $\theta:T_B\to\mathbb{C}^*$. We can tensor the action of $T_B$ on $\mathcal{O}(1)$ with this character. The stable (resp. semistable) points of $X_P$ with respect to this linearization are called $\theta$-stable (resp. $\theta$-semistable). The corresponding GIT quotient is denoted by $X_P /\!\!/_\theta T_B$. We have (see [15, 2.16])

$$X_P /\!\!/_\theta T_B\simeq X(\Lambda,P^\theta)/\!\!/T_B=X(M,P^\theta\cap M_\mathbb{Q}),$$

where $P^\theta=P-\lambda$ for some $\lambda\in\Lambda$ with $d(\lambda)=\theta$.

## 4. Tilting bundles

Let $(Q,W)$ be a quiver potential associated to some consistent brane tiling, let $A=\mathbb{C}Q/(\partial W)$, and let $\alpha=(1,\dots,1)\in\mathbb{Z}^{Q_0}$. The goal of this section is to give a toric description of the tilting bundles on the moduli spaces $\mathcal{M}_\theta(A,\alpha)$.

**Definition 4.1.** Let $X$ be an algebraic variety. A coherent sheaf $T\in\operatorname{Coh}X$ is called a tilting sheaf if $\operatorname{Ext}^n(T,T)=0$ for $n>0$ and the triangulated category $D^b(X)=D^b(\operatorname{Coh}X)$ is generated by the summands of $T$. A collection of coherent sheaves $(T_i)_{i\in I}$ is called a tilting collection if $\operatorname{Ext}^n(T_i,T_j)=0$ for $n>0$ and $i,j\in I$, and the triangulated category $D^b(X)$ is generated by the objects $T_i$, $i\in I$.

We use notation from Section 2. In particular, we have defined an exact sequence of free abelian groups

$$0\to M\xrightarrow{i}\Lambda\xrightarrow{d}B\to0$$

and a cone $P\subset\Lambda_\mathbb{Q}$ there. Let $\theta\in B$ be $\alpha$-generic. We have seen that the moduli space $\mathcal{M}_\theta=\mathcal{M}_\theta(A,\alpha)$ is a toric quotient

$$\mathcal{M}_\theta=\operatorname{Spec}\mathbb{C}[P\cap\Lambda]/\!\!/_\theta T_B=X_P/\!\!/_\theta T_B=X(\Lambda,P^\theta)/\!\!/T_B=X(M,P^\theta\cap M_\mathbb{Q}),$$

where $P^\theta=P-\lambda$ for some $\lambda\in\Lambda$ with $d(\lambda)=\theta$.

We know already how to parametrize the set of $T_M$-orbits of $\mathcal{M}_\theta$, or equivalently, the fan of $\mathcal{M}_\theta$. The set of rays of the fan of $\mathcal{M}_\theta$ is in bijection with $\theta$-stable perfect matchings. It is also in bijection with the the set of facets (codimension 1 faces) of $P^\theta_M=P^\theta\cap M_\mathbb{Q}$.

For any weak path $u$ and for any perfect matching $I\in\mathcal{A}$, we define $\chi_I(u)=\chi_I(\operatorname{wt}(u))$. The extremal rays of $P^\vee$ are parametrized by the perfect matchings (see [4, Lemma 2.3.4]). This implies that for any weak path $u$ the system of integers $(\chi_I(u))_{I\in\mathcal{A}}$ determines a $T_\Lambda$-Cartier divisor and therefore a $T_\Lambda$-equivariant line bundle over $X_P$ which we denote by $L(u)$ (forgetting the $T_\Lambda$-action, we get just a trivial line bundle). If we restrict this system of integers to $\theta$-stable perfect matchings, we get a $T_M$-Cartier divisor and a $T_M$-equivariant line bundle over $\mathcal{M}_\theta$ which we denote by $\overline{L}(u)$.

Let us fix some vertex $i_0 \in Q_0$. For any vertex $i \in Q$ we fix some weak path $u_i : i_0 \to i$. The following result proves a conjecture of Hanany, Herzog and Vegh [7, Section 5.2]

**Theorem 4.2.** *For any $\alpha$-generic $\theta \in B$, the line bundles $\overline{L}(u_i)$, $i \in Q_0$, form a tilting collection on $\mathcal{M}_\theta(A, \alpha)$.*

*Proof.* We know from [16, Theorem 6.3.1] and the fact that $A = \mathbb{C}Q/(\partial W)$ is a non-commutative crepant resolution of its center [11, 3] that there is an equivalence of categories

$$\Psi : D^b(\operatorname{mod} A^{\mathrm{op}}) \to D^b(\operatorname{Coh} \mathcal{M}_\theta), \quad M \mapsto M \otimes^L_A \mathcal{U},$$

where $\mathcal{U}$ is a universal vector bundle on $\mathcal{M}_\theta$ (see also [8]). This implies, in particular, that the vector bundle $\mathcal{U} = \Psi(A)$ is a tilting sheaf. We will give its toric description. Let us recall the construction of the universal vector bundle from [10, Prop. 5.3].

Let $(e_i)_{i \in Q_0}$ be the canonical basis of $\mathbb{Z}^{Q_0}$ and let $T_0 := \operatorname{Hom}(\mathbb{Z}^{Q_0}, \mathbb{C}^*) = \mathrm{GL}_\alpha(\mathbb{C})$. Let $R = R(A, \alpha)$ and let $R^s \subset R$ be the subvariety of $\theta$-stable representations. The diagonal $\Delta = \mathbb{C}^* \subset T_0$ acts trivially on $R$. We have $T_B = T_0/\Delta$ and $\mathcal{M}_\theta = R/\!\!/_\theta T_B = R^s/T_B$.

For any $i \in Q_0$, we define a $T_0$-equivariant line bundle $L_i$ over $R$ to be $R \times \mathbb{C}$ with an action on the second factor induced by $e_i$. Explicitly, the action is given by

$$t(x, c) = (tx, t_i c), \qquad t = (t_i)_{i \in Q_0} \in T_0,\ (x, c) \in R \times \mathbb{C}.$$

The action of $T_B$ on $L_i$ is not well defined as $\Delta$ acts nontrivially on $L_i$. Namely, it acts with weight 1 on the second factor. To overcome this problem, we can multiply the action of $T_0$ with an action going through some homomorphism $T_0 \to \Delta$ such that the new action restricted to $\Delta$ is trivial (see [10, Prop. 5.3], note that the $T_0$-orbits will not change). The homomorphism $\psi : T_0 \to \Delta$ is a character of $T_0$, that is, an element $\psi \in \mathbb{Z}^{Q_0}$. The triviality condition means that $\psi \cdot \alpha = -1$. We make the choice $\psi = e_{i_0}$. Then the action of $T_B = T_0/\Delta$ on $L_i$ is given by the character $e_i - e_{i_0} \in B$. Let the $T_B$-equivariant line bundle $L_i$ on $R$ descend to the line bundle $\overline{L}_i$ on $\mathcal{M}_\theta = R^s/T_B$. It is shown in [10, Prop. 5.3] that $\oplus_{i \in Q_0} \overline{L}_i$ is a universal vector bundle on $\mathcal{M}_\theta$.

There is a natural action of $T_\Lambda$ on $R$. In order to extend it to an action on $L_i = R \times \mathbb{C}$ compatible with an action of $T_B$, we have to choose some $\lambda_i \in \Lambda$ such that $d(\lambda) = e_i - e_0$. We choose $\lambda_i = \mathrm{wt}(u_i) \in \Lambda$. The inverse image of $L_i$ with respect to the natural map $X_P \to R$ (this is a normalization of some irreducible component of $R$, see [11]) is given by $L(u_i)$. This implies that the descent line bundle of $L_i$ with respect to $R^s \to \mathcal{M}_\theta$ is isomorphic to the descent line bundle of $L(u_i)$ with respect to $X_P^s \to R^s \to \mathcal{M}_\theta$. According to Theorem 3.10, the descent line bundle of $L(u_i)$ is $\overline{L}(u_i)$. This means that $\overline{L}_i \simeq \overline{L}(u_i)$. Therefore $\mathcal{U} \simeq \oplus_{i \in Q_0} \overline{L}(u_i)$. □

**Remark 4.3.** If $u, v : i \to j$ are two weak paths then $uv^{-1}$ is a weak cycle. This implies that $\mathrm{wt}(u) - \mathrm{wt}(v) \in M$ and therefore $\overline{L}(u)$ and $\overline{L}(v)$ are isomorphic line bundles (see [6, Section 3.4]). If we substitute the vertex $i_0$ by some vertex $i'_0$, then the line bundles $L(u_i)$, $i \in Q_0$ should be tensored with a line bundle $L(u)$, where $u : i'_0 \to i_0$ is any weak path. This ambiguity corresponds to the ambiguity of the universal vector bundle over $\mathcal{M}_\theta$. The universal vector bundle is defined only up to tensoring with a line bundle.

**Remark 4.4.** The conjecture of [7, Section 5.2] states actually that the collection $\overline{L}(u_i)$, $i \in Q_0$, is an exceptional collection. But this is certainly false, as $\operatorname{Hom}_{\mathcal{M}_\theta}(\overline{L}(u_i), \overline{L}(u_j)) = e_j A e_i$ (see Corollary 4.6) is always nonzero.

**Remark 4.5.** The collection of line bundles $L(u_i)$, $i \in Q_0$ on $X_P$ has a property that for any $\alpha$-generic $\theta \in B$ it descends to a tilting collection on $\mathcal{M}_\theta$. The existence of such "globally defined" collection was conjectured by Aspinwall [2].

**Corollary 4.6.** *For any weak path $u : i \to j$ in $Q$, we have*

(1) $H^n(\mathcal{M}_\theta, \overline{L}(u)) = 0$*,* $n > 0$*.*
(2) $H^0(\mathcal{M}_\theta, \overline{L}(u)) = e_j A e_i$*, where* $A = \mathbb{Q}/(\partial W)$*.*

*Proof.* Let $A = \mathbb{C}Q/(\partial W)$. By the proof of the above theorem, the vector bundle

$$\mathcal{U} = \oplus_{k \in Q_0} \overline{L}(u_k)$$

can be endowed with a structure of a universal vector bundle. The map $\Psi : D^b(\operatorname{mod} A^{\mathrm{op}}) \to D^b(\operatorname{Coh} \mathcal{M}_\theta)$ maps the right $A$-module $e_k A$ to the summand $\overline{L}(u_k)$ of $\mathcal{U}$. This implies, for $n > 0$,

$$\begin{aligned}\operatorname{Ext}^n_{A^{\mathrm{op}}}(e_i A, e_j A) = 0 = \operatorname{Ext}^n_{\mathcal{M}_\theta}(\overline{L}(u_i), \overline{L}(u_j)) \\ = \operatorname{Ext}^n_{\mathcal{M}_\theta}(\mathcal{O}, \overline{L}(u_j u_i^{-1})) = H^n(\mathcal{M}_\theta, \overline{L}(u)).\end{aligned}$$

For the Hom-space we get

$$\begin{aligned}\operatorname{Hom}_{A^{\mathrm{op}}}(e_i A, e_j A) = e_j A e_i = \operatorname{Hom}_{\mathcal{M}_\theta}(\overline{L}(u_i), \overline{L}(u_j)) \\ = \operatorname{Hom}_{\mathcal{M}_\theta}(\mathcal{O}, \overline{L}(u_j u_i^{-1})) = H^0(\mathcal{M}_\theta, \overline{L}(u)).\end{aligned}$$

□

## 5. Examples

In this section we will consider two examples: the suspended pinch point and the quotient singularity $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In the first example we will study all possible generic stabilities and in the second example we will study only three particular stabilities.

### 5.1. **Suspended pinch point.**

Here we consider a brane tiling called a suspended pinch point. The corresponding periodic quiver with a fundamental domain is given in Figure 1.

Let $Q = (Q_0, Q_1, Q_2)$ be the corresponding quiver embedded in a torus. We will denote the arrow from a vertex $i \in Q_0$ to a vertex $j \in Q_0$ by $ij$. The list of all perfect matchings of the brane tiling is given in Table 1. Every perfect matching is described there as a subset of $Q_1$.

Recall that we have defined a linear map $\overline{\chi} : M \to \mathbb{Z}^{\mathcal{A}}$ in Section 2. We can choose such basis of $M$ that the map $\overline{\chi}^* : \mathbb{Z}^{\mathcal{A}} \to M^\vee$ is given by the matrix

$$\begin{pmatrix} 0&2&1&0&1&1 \\ 0&0&1&1&0&0 \\ 1&1&1&1&1&1 \end{pmatrix}$$

and the map $\omega_M^* : M^\vee \to \mathbb{Z}$ is given by the matrix $\begin{pmatrix} 0&0&1 \end{pmatrix}$. This gives us the toric diagram of $\operatorname{Spec} \mathbb{C}[M^+]$

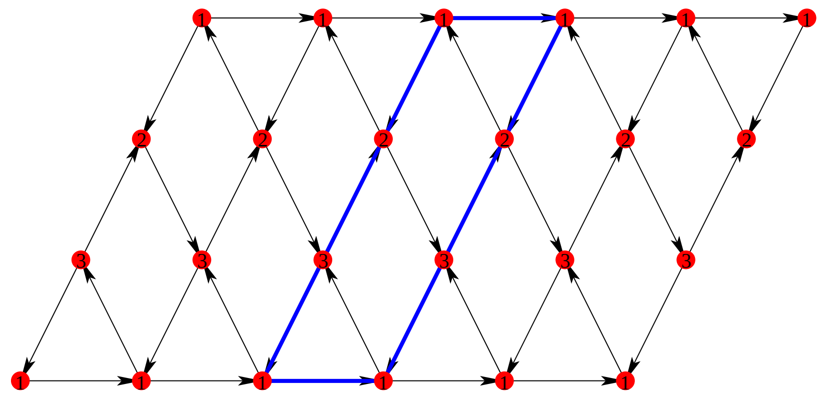
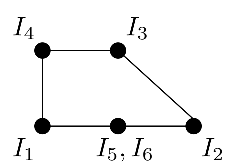


Figure 1: Periodic quiver and a fundamental domain for SPP

| $N$ | $I$ |
|---|---|
| 1 | $12, 31$ |
| 2 | $21, 13$ |
| 3 | $32, 11$ |
| 4 | $23, 11$ |
| 5 | $12, 13$ |
| 6 | $21, 31$ |

Table 1: Perfect matchings of SPP




For any $\alpha = (1,1,1)$-generic $\theta \in B$, the fan $\Sigma_\theta$ of $\mathcal{M}_\theta$ has five rays. The matrix of $\overline{\chi}_\theta^* : \mathbb{Z}^{\Sigma_\theta(1)} \to M^\vee$ equals $\begin{pmatrix} 0&2&1&0&1\\ 0&0&1&1&0\\ 1&1&1&1&1 \end{pmatrix}$ and is independent of the stability $\theta$. The Picard group $\operatorname{Pic}(\mathcal{M}_\theta)$ is isomorphic to the cokernel of $\overline{\chi}_\theta : M \to \mathbb{Z}^{\Sigma_\theta(1)}$ (see [6, Section 3.4]). We can choose a basis of $\operatorname{Pic}(\mathcal{M}_\theta)$ such the matrix of $\mathbb{Z}^{\Sigma_\theta(1)} \to \operatorname{Pic}(\mathcal{M}_\theta)$ is given by

$$\begin{pmatrix} 1&0&1&-1&-1\\ 0&1&-1&1&-1 \end{pmatrix}.$$

There are 6 different chambers of $\alpha$-generic stabilities. Their representatives are given in Table 2.

It is easy to see that the perfect matching $I_5$ is stable with respect to $\theta_2$, $\theta_3$, and $\theta_4$. The perfect matching $I_6$ is stable with respect to $\theta_1$, $\theta_5$, and $\theta_6$. The other perfect matchings are extremal and therefore stable with respect to all $\theta_i$, $i = 1, \ldots, 6$.

We will say that a pair of perfect matchings is $\theta$-stable if their union is $\theta$-stable. One can see that the pair $\{I_1, I_3\}$ is stable only with respect to $\theta_2$ and $\theta_6$. The pair $\{I_2, I_4\}$ is stable only with respect to $\theta_3$ and $\theta_5$. This uniquely determines the triangulation of the toric diagram for any generic stability.

| $N$ | $\theta$ |
|---|---|
| 1 | $(-2,1,1)$ |
| 2 | $(1,-2,1)$ |
| 3 | $(1,1,-2)$ |
| 4 | $(2,-1,-1)$ |
| 5 | $(-1,2,-1)$ |
| 6 | $(-1,-1,2)$ |

Table 2: $\alpha$-generic stabilities

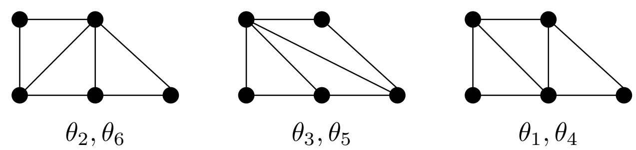



Figure 2: Triangulations of the toric diagram

To construct the tilting bundle $\mathcal{U}_\theta$ on the moduli space $\mathcal{M}_\theta$, we choose paths $u_1 = e_1$ (trivial path), $u_2 = 12$, $u_3 = 13$ and intersect them with $\theta$-stable perfect matchings. This gives us three vectors $\chi_\theta(u_i) \in \mathbb{Z}^{\Sigma_\theta(1)} = \mathbb{Z}^5$. Their images with respect to the map $\mathbb{Z}^{\Sigma_\theta(1)} \to \operatorname{Pic}(\mathcal{M}_\theta)$ (given by the matrix $\left(\begin{smallmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 \end{smallmatrix}\right)$) determine line bundles which are summands of $\mathcal{U}_\theta$. The result is given by the following table

| $\theta$ | $\chi_\theta(u_1)$ | $\chi_\theta(u_2)$ | $\chi_\theta(u_3)$ | $\mathcal{U}_\theta$ |
|---|---|---|---|---|
| $\theta_2, \theta_3, \theta_4$ | 0 | $(1,0,0,0,1)$ | $(0,1,0,0,1)$ | $\mathcal{O} \oplus \mathcal{O}\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) \oplus \mathcal{O}\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$ |
| $\theta_1, \theta_5, \theta_6$ | 0 | $(1,0,0,0,0)$ | $(0,1,0,0,0)$ | $\mathcal{O} \oplus \mathcal{O}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \oplus \mathcal{O}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ |

5.2. **Orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.** Consider a finite abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider an embedding $G \subset \mathrm{SL}_3(\mathbb{C})$, where the action of the first copy of $\mathbb{Z}_2$ is given by $\frac{1}{2}(1,1,0)$ and the action of the second copy of $\mathbb{Z}_2$ is given by $\frac{1}{2}(0,1,1)$. The character group $\hat{G}$ can be identified with $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will use the following notation for the elements of $\hat{G}$ (and sometimes for the elements of $G$): $a = (1,0)$, $b = (0,1)$ and $c = (1,1)$. The above representation of $G$ is isomorphic to $a \oplus b \oplus c$.

We can associate a brane tiling with the above embedding (see [11] for the construction and notation). The corresponding periodic quiver with a fundamental domain is given in Figure 3.

The toric diagram of $\operatorname{Spec} \mathbb{C}[M^+]$ is given in Figure 4 (see also [11]). The list of all possible perfect matchings is given in Table 3.

We will consider only stabilities

$$\theta_1 = (-3,1,1,1), \quad \theta_2 = (-3,-1,2,2), \quad \theta_3 = (-2,3,1,-2),$$

where the order of the coordinates of $\mathbb{Z}^{Q_0} = \mathbb{Z}^{\hat{G}}$ is given by $0, a, b, c$. Note that $\mathcal{M}_{\theta_1}$ is isomorphic to $\operatorname{Hilb}^G(\mathbb{C}^3)$.

A subset $I \subset Q_1$ is $\theta_1$-stable if and only if there exists a path in $Q \backslash I$ from vertex $0 \in Q_0$ to any other vertex of $Q$. The non-extremal $\theta_1$-stable perfect matchings are

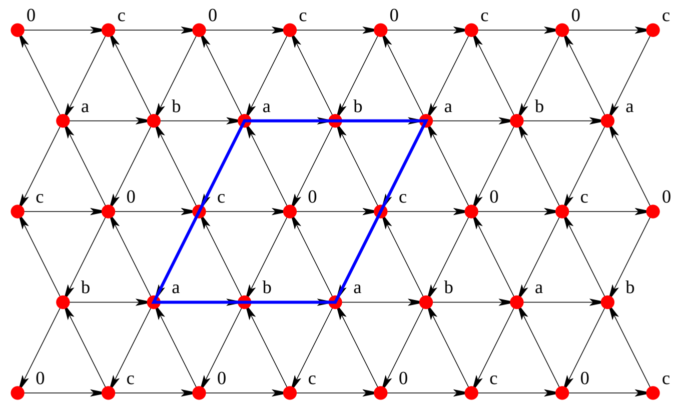



Figure 3: Periodic quiver for $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

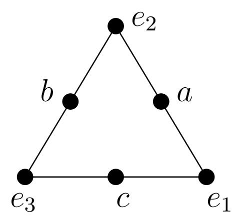



Figure 4: Toric diagram for $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

| $N$ | $I$ | $4\overline{\chi}_I$ | |
|---|---|---|---|
| 1 | $cb, 0a, 0b, ca$ | $(2, 2, 0)$ | $a$ |
| 2 | $cb, 0a, bc, a0$ | $(4, 0, 0)$ | $e_1$ |
| 3 | $cb, ab, c0, a0$ | $(2, 0, 2)$ | $c$ |
| 4 | $ac, b0, 0b, ca$ | $(0, 4, 0)$ | $e_2$ |
| 5 | $ac, b0, bc, a0$ | $(2, 2, 0)$ | $a$ |
| 6 | $ac, ab, 0b, 0c$ | $(0, 2, 2)$ | $b$ |
| 7 | $ba, b0, c0, ca$ | $(0, 2, 2)$ | $b$ |
| 8 | $ba, 0a, bc, 0c$ | $(2, 0, 2)$ | $c$ |
| 9 | $ba, ab, c0, 0c$ | $(0, 0, 4)$ | $e_3$ |

Table 3: Perfect matchings for $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

$I_3, I_5, I_7$. A subset $I \subset Q_1$ is $\theta_2$-stable if and only if there exist paths in $Q\backslash I$ from 0 to $b$ and $c$ and from $a$ to $b$ or $c$. The non-extremal $\theta_2$-stable perfect matchings are the same as for $\theta_1$. A subset $I \subset Q_1$ is $\theta_3$-stable if and only if there exist paths in $Q\backslash I$ from 0 to $a$, from $c$ to $a$, and from 0 or $c$ to $b$. The non-extremal $\theta_3$-stable perfect matchings are $I_3, I_5, I_6$.

To determine the toric diagram of $\mathcal{M}_{\theta_i}$, $i = 1, 2, 3$, we have to find such pairs of $\theta_i$-stable perfect matchings that their union is still $\theta_i$-stable (we call such pairs $\theta_i$-stable). For $\theta_1$, the pairs $\{I_3, I_5\}, \{I_5, I_7\}, \{I_3, I_7\}$ are stable. The toric diagram of $\mathcal{M}_{\theta_1}$ is given in Figure 5. For $\theta_2$, the pair $\{I_3, I_5\}$ (corresponds to the edge $ca$) is non-stable. This uniquely determines the toric diagram of $\mathcal{M}_{\theta_2}$ (see Figure 5). For $\theta_3$, the pair $\{I_3, I_6\}$ (corresponds to the edge $cb$) is non-stable. This uniquely determines the toric diagram of $\mathcal{M}_{\theta_3}$ (see Figure 5).

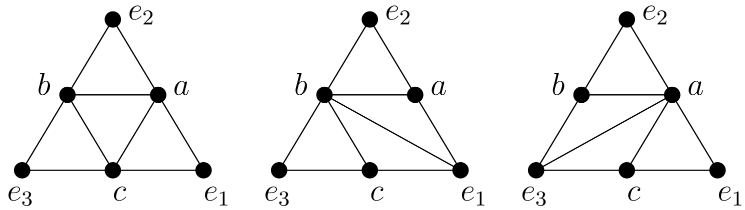



Figure 5: Toric diagrams for $\theta_1, \theta_2, \theta_3$

To determine the tilting bundle $\mathcal{U}_{\theta_i}$ on $\mathcal{M}_{\theta_i}$, $i = 1, 2, 3$, we choose paths from vertex $0 \in Q_0$ to all other vertices of $Q$ and intersect these paths with $\theta_i$-stable perfect matchings. We choose paths $e_0, 0a, 0b, 0c$. The result of intersecting these paths with $\theta_1$-stable perfect matchings is given in Figure 6. In this way we get Cartier divisors for a tilting collection on $\mathcal{M}_{\theta_1}$. The result for $\theta_2$ is the same, as $\theta_1$-stable perfect matchings and $\theta_2$-stable perfect matchings coincide. The result for $\theta_3$ is given in Figure 7. This gives us Cartier divisors for a tilting collection on $\mathcal{M}_{\theta_3}$.

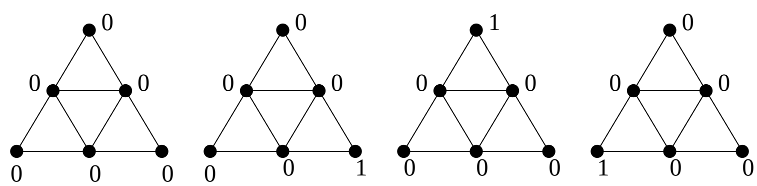


Figure 6: Cartier divisors for a tilting collection on $\mathcal{M}_{\theta_1}$

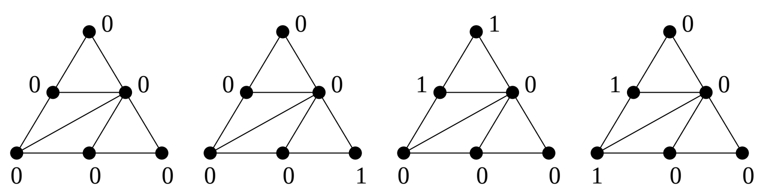


Figure 7: Cartier divisors for a tilting collection on $\mathcal{M}_{\theta_3}$

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