

# CREPANT RESOLUTIONS AND BRANE TILINGS I: TORIC REALIZATION

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ABSTRACT. Given a brane tiling, that is, a bipartite graph on a torus, we can associate with it a singular 3-Calabi-Yau variety. In this paper we study its commutative and non-commutative crepant resolutions. We give an explicit toric description of all its commutative crepant resolutions. We also explain how the McKay correspondence in dimension 3 can be interpreted using brane tilings.

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## 1. INTRODUCTION

The main goal of this paper is to give an explicit toric description of all possible crepant resolutions of singular 3-Calabi-Yau varieties arising from brane tilings (see Section 2.1). All these crepant resolutions can be constructed as moduli spaces of representations of some quiver with relations [15, Theorem 15.1]. It follows from the construction of these moduli spaces that they are toric varieties. We give an explicit description of the corresponding fan.

With any brane tiling we can associate a quiver potential  $(Q, W)$  (see Section 2.1). It turns out that under certain consistency conditions on the brane tiling the corresponding quiver potential algebra  $\mathbb{C}Q/(\partial W)$  is a 3-Calabi-Yau algebra [19, 7]. The singular Calabi-Yau variety mentioned above is isomorphic to

the spectrum of the center of  $\mathbb{C}Q/(\partial W)$ . We will show that this variety is a normal Gorenstein toric variety and that  $\mathbb{C}Q/(\partial W)$  is its non-commutative crepant resolution [27]. Related questions are studied in [4].

Using this fact, we can apply the result of Van den Bergh [27, Theorem 6.3.1], which says that certain moduli spaces of representations of a non-commutative crepant resolution give rise to (commutative) crepant resolutions and, moreover, the derived categories of commutative and non-commutative crepant resolutions are equivalent. As Van den Bergh mentions, his result is a generalization of the well-known approach of [6] to the McKay correspondence. In the case of brane tilings, the moduli spaces are  $\mathcal{M}_\theta = \mathcal{M}_\theta(\mathbb{C}Q/(\partial W), \alpha)$  – the moduli spaces of  $\theta$ -semistable  $\mathbb{C}Q/(\partial W)$ -representations of dimension  $\alpha = (1, \dots, 1) \in \mathbb{Z}^{Q_0}$ , where  $\theta \in \mathbb{Z}^{Q_0}$  is  $\alpha$ -generic. A direct proof of the smoothness of these moduli spaces was given by Ishii and Ueda [16]. They also proved the derived equivalence [15] by using some tricky modifications of brane tilings.

The moduli space  $\mathcal{M}_\theta$  has a natural action of a certain 3-dimensional torus. We will prove that every orbit in  $\mathcal{M}_\theta$  is determined by its cosupport – the subset of arrows in  $Q_1$  inducing zero action on the representations from the orbit. It was proved in [16, Lemma 6.1] that the cosupports of 2-dimensional orbits are perfect matchings. We can give similar descriptions for the cosupports of 0-dimensional and 1-dimensional orbits. It turns out that the cosupports of 1-dimensional orbits are unions of two perfect matchings and the cosupports of 0-dimensional orbits are unions of three perfect matchings. This allows us to reconstruct the toric diagram of the toric 3-Calabi-Yau variety  $\mathcal{M}_\theta$ .

We will show that with any finite abelian group  $G \subset \mathrm{SL}_3(\mathbb{C})$  we can associate a brane tiling. The corresponding quiver potential algebra  $\mathbb{C}Q/(\partial W)$  is a non-commutative crepant resolution of the quotient singularity  $\mathbb{C}^3/G$ . It is known that the Hilbert scheme  $\mathrm{Hilb}^G(\mathbb{C}^3)$  of  $G$ -clusters in  $\mathbb{C}^3$  is a crepant resolution of  $\mathbb{C}^3/G$  [20]. This Hilbert scheme is isomorphic to  $\mathcal{M}_\theta$  for certain  $\theta$  (see Remark 5.9). It was shown by Nakamura [20] that  $\mathrm{Hilb}^G(\mathbb{C}^3)$  is a toric variety. He also described the corresponding fan. According to the results mentioned above, we can describe the toric diagram of  $\mathcal{M}_\theta$  for any generic  $\theta$  by using the perfect matchings of the brane tiling. A different algorithm to determine this toric diagram, by computing the vertices of the polyhedron defining  $\mathcal{M}_\theta$ , was proposed in [10].

The paper is organized as follows: In Section 2 we gather preliminary material on brane tilings, quiver potential algebras, consistency conditions, and Calabi-Yau property. In Section 3 we study some properties of the quiver potential algebra induced by the brane tiling and prove, in particular, that it is a non-commutative resolution. In Section 4 we give a toric description of the moduli spaces  $\mathcal{M}_\theta$ . We study the orbits of  $\mathcal{M}_\theta$  and give an explicit description of its toric diagram. In Section 5 we relate the McKay correspondence for finite abelian groups  $G \subset \mathrm{SL}_3(\mathbb{C})$  with brane tilings.

In the subsequent paper, joint with Martin Bender, we will give a toric description of tilting bundles on the crepant resolutions. This result gives a proof of the conjecture of Hanany, Herzog and Vegh [14] and of a version of the conjecture of Aspinwall [1].

I would like to thank Markus Reineke for many useful discussions. I would also like to thank Igor Burban, Alastair Craw, and Victor Ginzburg for many useful comments.

2. PRELIMINARIES

2.1. Brane tilings.

**Definition 2.1.** A brane tiling is a bipartite graph  $G = (G_0^\pm, G_1)$  together with an embedding of the corresponding CW-complex into the real two-dimensional torus  $T$  so that the complement  $T \setminus G$  consists of simply-connected components. We call the elements of  $G_0^+$  (resp.  $G_0^-$ ) white vertices (resp. black vertices). We identify homotopy equivalent embeddings. The set of connected components of  $T \setminus G$  is denoted by  $G_2$  and is called the set of faces of  $G$ .

We define a quiver  $Q = (Q_0, Q_1)$  dual to the graph  $G$  as follows. The set of vertices  $Q_0$  is  $G_2$ , the set of arrows  $Q_1$  is  $G_1$ . For any arrow  $a \in Q_1$  we define its endpoints to be the polygons in  $G_2$  adjacent to  $a$ . The direction of  $a$  is chosen in such a way that the white vertex is on the right of  $a$ . For any arrow  $a \in Q_1$ , we define  $s(a), t(a) \in Q_0$  to be its source and target. The CW-complex corresponding to  $Q$  is automatically embedded in  $T$ . The set of connected components of the complement, called the set of faces of  $Q$ , will be denoted by  $Q_2$ . It can be identified with  $G_0$ . There is a decomposition  $Q_2 = Q_2^+ \cup Q_2^-$  corresponding to the decomposition  $G_0 = G_0^+ \cup G_0^-$ . It follows from our definition that the arrows of the faces from  $Q_2^+$  go clockwise and the arrows of the faces from  $Q_2^-$  go anti-clockwise.

For any face  $F \in Q_2$ , we denote by  $w_F$  the necklace (equivalence class of cycles in  $Q$  up to shift) obtained by going along the arrows of  $F$ . We define the potential of  $Q$  (see e.g. [13, 5]) by

$$W = \sum_{F \in Q_2^+} w_F - \sum_{F \in Q_2^-} w_F.$$

For any cycle  $u = a_1 \dots a_n$  in  $Q$  and for any arrow  $a \in Q_1$ , we define the differential

$$\frac{\partial u}{\partial a} = \sum_{i: a_i = a} a_{i+1} \dots a_n a_1 \dots a_{i-1} \in \mathbb{C}Q.$$

Extending the differential by linearity, we get  $\partial W / \partial a \in \mathbb{C}Q$ .

**Definition 2.2.** Define a two-sided ideal  $(\partial W) \subset \mathbb{C}Q$  to be generated by  $\partial W / \partial a$ ,  $a \in Q_1$ . Define a quiver potential algebra to be the algebra  $\mathbb{C}Q / (\partial W)$ .

2.2. Some groups related to brane tilings. Consider a complex of abelian groups

$$\mathbb{Z}^{Q_2} \xrightarrow{d_2} \mathbb{Z}^{Q_1} \xrightarrow{d_1} \mathbb{Z}^{Q_0},$$

where  $d_2(F) = \sum_{a \in F} a$ ,  $F \in Q_2$  and  $d_1(a) = t(a) - s(a)$  for any arrow  $a \in Q_1$ . Its homology groups coincide with the homology groups of the 2-dimensional torus containing  $Q$ . Following [19], we define

$$\Lambda = \mathbb{Z}^{Q_1} / \langle d_2(F) - d_2(G) \mid F, G \in Q_2 \rangle.$$

Equivalently,  $\Lambda$  is given by a cocartesian left upper square

$$\begin{array}{ccccc}
 \mathbb{Z}^{Q_2} & \xrightarrow{d_2} & \mathbb{Z}^{Q_1} & \xrightarrow{d_1} & \mathbb{Z}^{Q_0} \\
 \downarrow & & \downarrow \text{wt} & \nearrow d & \\
 \mathbb{Z} & \xrightarrow{\omega_\Lambda} & \Lambda & & \\
 \downarrow \omega_M & \nearrow \iota & & & \\
 M & & & & 
 \end{array}$$

where the left arrow is given by  $F \mapsto 1$ ,  $F \in Q_2$ . It is proved in [19, Lemma 3.3], under the condition that  $G$  has at least one perfect matching, that  $\Lambda$  is a free abelian group and the map  $\omega_\Lambda : \mathbb{Z} \rightarrow \Lambda$  is injective. There exists a unique map  $d : \Lambda \rightarrow \mathbb{Z}^{Q_0}$  making the right triangle commutative. Note that  $d\omega_\Lambda = 0$ . We put  $M = \ker(d)$ . Then there exists a unique map  $\omega_M : \mathbb{Z} \rightarrow M$  making the lower triangle commutative.

Let  $B = \ker(\mathbb{Z}^{Q_0} \rightarrow \mathbb{Z})$ , where the map is given by  $i \mapsto 1$ ,  $i \in Q_0$ . This group is generated by the elements of the form  $j - i$ , where  $i, j \in Q_0$ . This implies that  $B = \text{im } d_1 = \text{im } d$ , as the quiver is connected. We have then an exact sequence

$$0 \rightarrow M \xrightarrow{\iota} \Lambda \xrightarrow{d} B \rightarrow 0.$$

Let us compute the ranks of the groups in this exact sequence. It is clear that  $\text{rk } B = \#Q_0 - 1$ . We have

$$\text{rk } \Lambda = \text{rk}(\text{coker } d_2) + 1 = \text{rk}(\ker d_1) + 3 = \text{rk } \mathbb{Z}^{Q_0} - \text{rk}(\text{coker } d_1) + 3 = \#Q_0 + 2.$$

This implies that  $\text{rk } M = 3$ .

**2.3. Weights and equivalence relations.** We define a weak path in  $Q$  to be a path consisting of arrows of  $Q$  and their inverses (for any arrow  $a$  we identify  $aa^{-1}$  and  $a^{-1}a$  with trivial paths). For any weak path  $u$ , we define its content  $|u| \in \mathbb{Z}^{Q_1}$  by counting every arrow of  $u$  with appropriate sign. We define the weight of  $u$  by  $\text{wt}(u) = \text{wt}(|u|) \in \Lambda$ .

Let  $A = \mathbb{C}Q/(\partial W)$  and let  $A'$  be obtained from  $A$  by inverting all arrows. We say that two paths in  $Q$  are equivalent if they are equal in  $A$ . We say that two weak paths are weakly equivalent if they are equal in  $A'$  (cf. [19, Section 4]). It is proved in [19, Prop. 4.8] under the condition that  $G$  has at least one perfect matching

**Proposition 2.3.** *Two weak paths in  $Q$  having the same start points are weakly equivalent if and only if they have the same weights.*

For any face  $F \in Q_2$  and for any vertex  $i \in F$  we define  $\omega_{i,F}$  to be a cycle starting at  $i$  and going along  $F$ . It is proved in [19] that  $\omega_{i,F} \sim \omega_{i,G}$  if  $i \in F \cap G$ . We denote the corresponding equivalence class by  $\omega_i$ . We denote its weight by  $\bar{\omega}$ .

Let  $\pi : \tilde{T} \rightarrow T$  be the universal cover of the torus and let  $\tilde{Q}$  be the preimage of  $Q$ . Then  $\tilde{Q}$  is a periodic quiver. We can define equivalence (resp. weak equivalence) relation on the set of paths (resp. weak paths) of  $\tilde{Q}$  in the same way as above (see [19]). For any weak paths  $u$  in  $\tilde{Q}$  we can define a weak path  $\pi(u)$  in  $Q$ . We define then the weight  $\text{wt}(u) \in \Lambda$  to be the weight of  $\pi(u)$ . Similarly to Proposition 2.3 we can prove that two weak paths in  $\tilde{Q}$  having the same start points are weakly equivalent if and only if they have the same weights.

**2.4. Consistency conditions.** Let  $G$  be a brane tiling and let  $(Q, W)$  be the corresponding quiver potential.

**Definition 2.4.** A brane tiling  $G$  is called consistent (resp. geometrically consistent) if there exists a map  $R : Q_1 \rightarrow (0, 1]$  (resp.  $R : Q_1 \rightarrow (0, 1)$ ), called an R-charge, that satisfies

$$(1) \quad \sum_{a \in F} R_a = 2, \quad F \in Q_2,$$

$$(2) \quad \sum_{a \ni i} (1 - R_a) = 2, \quad i \in Q_0.$$

**Remark 2.5.** By the Birkhoff-von Neumann Theorem (see e.g. [24, Corollary 8.6a]) consistency condition implies that the bipartite graph  $G$  is non-degenerate, that is, every edge of  $G$  is contained in some perfect matching.

**Theorem 2.6** ([4, Theorem 8.15]). *A brane tiling is consistent if and only if any two weakly equivalent paths in  $Q$  are equivalent.*

**Remark 2.7.** It was proved earlier in [14, Lemma 5.3.1] that geometric consistency implies that any two paths in  $Q$  having the same start points and the same weight are equivalent. This implies that weakly equivalent paths are equivalent.

**Remark 2.8.** We do not discuss in this paper consistency conditions on brane tilings involving zig-zag paths (see [7, Section 3.4.2], [15, Def. 5.2], [4, Theorem 8.12]). These consistency conditions are equivalent to Definition 2.4 by [4, Theorem 8.12] and [15, Section 5].

**Definition 2.9.** We say that a path  $u : i \rightarrow j$  in  $Q$  is minimal if it is not equivalent to  $v\omega_i$  for any path  $v : i \rightarrow j$ .

**Proposition 2.10** (see [12, Lemma 7.3]). *Assume that the brane tiling is consistent. Then for any minimal path  $u : i \rightarrow j$  in  $Q$  there exists an arrow  $a \in Q_1$  such that  $s(a) = j$  and  $au$  is still minimal.*

**Remark 2.11.** This property together with a consistency condition was used in [19] (see also [7], [12]) to show that the quiver potential algebra is a 3-Calabi-Yau algebra (see Section 2.5).

Let  $\mathcal{A}$  be the set of perfect matchings on  $G$ . Any perfect matching  $I \in \mathcal{A}$  can be considered as a subset of  $Q_1$ , so we can define a linear map  $\chi_I : \mathbb{Z}^{Q_1} \rightarrow \mathbb{Z}$

$$\chi_I(a) = \begin{cases} 1, & a \in I, \\ 0, & a \notin I. \end{cases}$$

Note that  $\chi_I(d_2(F)) = 1$  for any face  $F \in Q_2$ , so we can factor  $\chi_I : \mathbb{Z}^{Q_1} \rightarrow \mathbb{Z}$  through  $\Lambda$  and get  $\chi_I : \Lambda \rightarrow \mathbb{Z}$ . Thus  $\chi_I \in \Lambda^\vee$  and we can consider  $\bar{\chi}_I := i^* \chi_I \in M^\vee$ . We define a cone  $\sigma \subset M_\mathbb{Q}^\vee$  to be generated by  $\bar{\chi}_I, I \in \mathcal{A}$ . The following result is proved in [15, Prop. 6.5]

**Proposition 2.12.** *Let  $I$  be some perfect matching in the consistent brane tiling. Then the following conditions are equivalent*

- (1) *The ray in  $M_\mathbb{Q}^\vee$  generated by  $\bar{\chi}_I$  is an extremal ray of  $\sigma$ .*
- (2) *For any  $J \in \mathcal{A}, J \neq I$  we have  $\bar{\chi}_I \neq \bar{\chi}_J$ .*

- (3) The quiver  $Q \setminus I = (Q_0, Q_1 \setminus I)$  is strongly connected, i.e. for any vertices  $i, j \in Q_0$  there exists a path from  $i$  to  $j$  in  $Q \setminus I$ .

A perfect matching satisfying these conditions is called an extremal (or corner, or external) perfect matching.

In this paper we will work only with geometrically consistent brane tilings because we will need the following important result proved by Broomhead [7, Prop. 6.2]

**Proposition 2.13.** *Assume that the brane tiling is geometrically consistent. Then, for any vertices  $i, j \in \tilde{Q}$ , there exists a path  $u : i \rightarrow j$  such that  $\chi_I(u) = 0$  for some extremal perfect matching.*

**Remark 2.14.** It is conjectured that the analogous statement also holds for consistent brane tilings. All the results of our paper can be then proved in this generality.

**2.5. Calabi-Yau property.** In this section  $A$  will be a (left and right) noetherian algebra, finitely generated over a field  $k$ . We define its enveloping algebra  $A^e = A \otimes_k A^{\text{op}}$ . Then  $A$  is a module over  $A^e$  in a natural way.

**Definition 2.15.** We say that  $A$  has finite Hochschild dimension if  $A$  has a finite projective resolution as an  $A^e$ -module. We say that  $A$  is homologically smooth if, moreover, this resolution can be chosen to consist only of finitely generated  $A^e$ -modules.

**Remark 2.16.** If  $A$  has finite Hochschild dimension then  $A$  and  $A^{\text{op}}$  have finite global dimension [9, Ch.9, Prop. 7.6]. In particular,  $A$  has finite injective dimension as a module over  $A$  and over  $A^{\text{op}}$  (we say that  $A$  is Gorenstein in this case).

**Definition 2.17.** An algebra  $A$  is called a Calabi-Yau algebra of dimension  $d$  if  $A$  is homologically smooth and

$$\text{RHom}_{A^e}(A, A \otimes A) \simeq A[-d]$$

in the category  $D^b(A^e)$ .

**Definition 2.18** ([26, Def. 8.1], [28, Def. 5.1]). An object  $K \in D^b(A^e)$  is called a rigid dualizing complex if

- (1)  $K$  has finite injective dimension over  $A$  and  $A^{\text{op}}$ .
- (2) The cohomologies of  $K$  are finitely generated over  $A$  and  $A^{\text{op}}$ .
- (3) Canonical morphisms  $A \rightarrow \text{RHom}_A(K, K)$  and  $A \rightarrow \text{RHom}_{A^{\text{op}}}(K, K)$  are isomorphisms in  $D^b(A^e)$ .
- (4) (Rigidity)  $\text{RHom}_{A^e}(A, K \otimes K) \simeq K$  in  $D^b(A^e)$ .

**Proposition 2.19** ([26, Prop. 8.2]). *Any two rigid dualizing complexes in  $D^b(A^e)$  are isomorphic.*

We will denote the dualizing complex of  $A$  (if it exists) by  $K_A$ . The following result gives an explicit description of  $K_A$  under certain conditions.

**Proposition 2.20** ([28, Prop. 5.13]). *Assume that  $A$  is Gorenstein and has a rigid dualizing complex  $K_A$ . Then  $K_A \simeq \text{RHom}_A(\text{RHom}_{A^e}(A, A^e), A)$ .*

**Lemma 2.21.** *Assume that  $A$  is a  $d$ -CY algebra. Then  $A[d]$  is a rigid dualizing complex.*

*Proof.*  $A$  has finite injective dimension over  $A$  and  $A^{\text{op}}$  by Remark 2.17. From the Calabi-Yau property we get

$$\text{RHom}_{A^e}(A, A[d] \otimes A[d]) = \text{RHom}_{A^e}(A, A \otimes A)[2d] \simeq A[-d][2d] = A[d].$$

It follows that  $A[d]$  satisfies all the conditions on the rigid dualizing complex.  $\square$

**Proposition 2.22** (see [28, Prop. 5.9]). *If  $A$  is finite over its center and is finitely generated over  $k$  then  $A$  has a rigid dualizing complex. More precisely, if  $S \rightarrow A$  is a finite central morphism, where  $S$  is a commutative smooth algebra of dimension  $n$ , then  $K_A := \text{RHom}_S(A, \Omega_{S/k}^n[n])$  is a rigid dualizing complex over  $A$ .*

**Remark 2.23.** If  $S$  a commutative smooth algebra of dimension  $n$  over  $k$  then  $S$  has a rigid dualizing complex  $K_S = \Omega_{S/k}^d[n]$ . If  $R$  is a commutative algebra of finite type over  $k$ , then we can always find a finite morphism  $S \rightarrow R$  with  $S$  smooth. Then  $R$  has a rigid dualizing complex  $K_R = \text{RHom}_S(R, K_S)$ . If  $R$  is a Cohen-Macaulay algebra of dimension  $d$  then  $K_R$  is concentrated in degree  $-d$  and the module  $\omega_R := K_R[-d]$  is a canonical module of  $R$  (see [8, Def. 3.3.16]). Note that the canonical module of  $R$  is defined only up to tensoring with a projective  $R$ -module of rank 1 (i.e. invertible sheaf on  $\text{Spec } R$ ). The rigid dualizing complex  $K_R$  of  $R$  is, in contrast, uniquely determined. Thus  $K_R[-d]$  gives a canonical choice of a canonical module.

**Theorem 2.24** (see [13, Theorem 7.2.14]). *Let  $A$  have finite Hochschild dimension and let  $R \subset A$  be a central subalgebra, such that  $A$  is finitely generated as a module over  $R$  and  $R$  is a Cohen-Macaulay domain, equidimensional of dimension  $d$ . Then the following conditions are equivalent*

- (1)  $A$  is a  $d$ -CY algebra.
- (2)  $A$  is a maximal Cohen-Macaulay module over  $R$  and  $A \simeq \text{Hom}_R(A, K_R[-d])$ , where  $K_R$  is a rigid dualizing complex of  $R$ .
- (3) For any  $X \in D^b(\text{mod } A)$ ,  $Y \in D^-(\text{mod } A)$ , we have (functorially)

$$\text{RHom}_A(X, Y[d]) \simeq D \text{RHom}_A(Y, X),$$

where  $D : D^+(\text{mod } R) \rightarrow D^-(\text{mod } R)$  is defined by  $Z \mapsto \text{RHom}_R(Z, K_R)$ .

**Remark 2.25.** The algebra  $R$  is finitely generated over  $k$  under the conditions of the theorem. This follows from [2, Prop. 7.8] if  $A$  is commutative. For the non-commutative  $A$  the proof goes through the same lines.

**Lemma 2.26.** *Let  $R$  be a commutative algebra of finite type over an algebraically closed field  $k$ . Assume that  $R$  is Gorenstein and is equidimensional of dimension  $d$ . Let  $\text{mod}_{\text{fl}} R$  be the category of finite length  $R$ -modules. Then the following contravariant endofunctors on  $\text{mod}_{\text{fl}} R$  are isomorphic.*

- (1)  $M \mapsto \text{Hom}_R(M, E_R)$ , where  $E_R = \bigoplus_{m \in \text{Specm } R} E(R/m)$ .
- (2)  $M \mapsto \text{Hom}_k(M, k)$ .
- (3)  $M \mapsto \text{Ext}_R^d(M, R)$ .
- (4)  $M \mapsto \text{RHom}_R(M, K_R)$ .

*Proof.* According to [22, Prop. 1.1], there exists a unique (up to equivalence) contravariant, exact functor  $D : \text{mod}_{\text{fl}} R \rightarrow \text{mod}_{\text{fl}} R$  such that  $D(R/m) \simeq R/m$  for every  $m \in \text{Specm } R$ . The first and the second functors obviously satisfy these conditions (we use here the assumption that  $k$  is algebraically closed). It follows from

our assumptions that  $\omega_R := K_R[-d]$  is an invertible sheaf on  $\text{Spec } R$ . Recall that  $R$  has an injective resolution [3, §1]

$$0 \rightarrow R \rightarrow \bigoplus_{\text{ht } p=0} E(R/p) \rightarrow \bigoplus_{\text{ht } p=1} E(R/p) \rightarrow \dots \rightarrow \bigoplus_{\text{ht } p=d} E(R/p) = E_R \rightarrow 0.$$

This implies that  $\text{Ext}_R^d(M, R) = \text{Hom}_R(M, E)$  for any  $M \in \text{mod}_{\text{fl}} R$  and so the first and the third functors are equivalent. This also implies

$$\text{RHom}(M, K_R) = \text{RHom}(M, \omega_R[d]) = \text{RHom}(M, A[d]) \otimes \omega_R = \text{Hom}(M, E_R) \otimes \omega_R.$$

Therefore the fourth functor also satisfies the above conditions.  $\square$

**Corollary 2.27.** *Under the conditions of the theorem, assume that  $R$  is Gorenstein and  $k$  is algebraically closed. Then for any finite dimensional  $A$ -modules  $X, Y$  we have (functorially)*

$$\text{Ext}^{d-i}(X, Y) \simeq \text{Hom}_k(\text{Ext}^i(Y, X), k), \quad 0 \leq i \leq d.$$

### 3. NON-COMMUTATIVE CREPANT RESOLUTION

Let  $A = \mathbb{C}Q/(\partial W)$  be the quiver potential algebra associated to a geometrically consistent brane tiling. In this section we will show that its center  $R = Z(A)$  is a normal Gorenstein domain and  $A$  is a non-commutative crepant resolution of  $R$  in the sense of Van den Bergh. Related questions are studied in [4].

Let  $\Lambda^+ \subset \Lambda$  be a semigroup generated by the weights of arrows. Define a cone  $P \subset \Lambda_{\mathbb{Q}}$  by

$$P = \left\{ \sum a_i \lambda_i \mid a_i \in \mathbb{Q}_{\geq 0}, \lambda_i \in \Lambda^+ \text{ for all } i \right\}.$$

The saturation of  $\Lambda^+$  is given by  $\overline{\Lambda^+} = P \cap \Lambda$ . Recall that in Section 2.2 we have constructed an exact sequence of free abelian groups

$$0 \rightarrow M \xrightarrow{i} \Lambda \xrightarrow{d} B \rightarrow 0.$$

For any  $i, j \in Q_0$ , we define

$$\Lambda_{ij} = d^{-1}(j - i) \subset \Lambda, \quad \Lambda_{ij}^+ = \Lambda_{ij} \cap \Lambda^+.$$

We define  $M^+ = M \cap \Lambda^+$ . The following result is a consequence of the Birkhoff-von Neumann Theorem (see e.g. [24, Corollary 8.6a])

**Proposition 3.1.** *The dual cone  $P^\vee \subset \Lambda_{\mathbb{Q}}^\vee$  is generated by  $\chi_I$ ,  $I \in \mathcal{A}$  (see Section 2.4).*

**Remark 3.2.** It is proved in [7, Lemma 2.3.4] that the semigroup  $P^\vee \cap \Lambda^\vee$  is generated by  $\chi_I$ ,  $I \in \mathcal{A}$ . Moreover, every  $\chi_I$  generates a 1-dimensional face of  $P^\vee$ .

**Corollary 3.3.** *The cone  $P$  equals*

$$P = \{\lambda \in \Lambda_{\mathbb{Q}} \mid \chi_I(\lambda) \geq 0 \forall I \in \mathcal{A}\}.$$

*The saturation of  $\Lambda^+$  equals*

$$\overline{\Lambda^+} = P \cap \Lambda = \{\lambda \in \Lambda \mid \chi_I(\lambda) \geq 0 \forall I \in \mathcal{A}\}.$$

**Lemma 3.4.** *If  $u$  is a weak path in  $Q$  such that  $\chi_I(u) \geq 0$  for any extremal perfect matching then  $u$  is weakly equivalent to a strict path.*



*Proof.* We consider  $u$  as a weak path from  $i$  to  $j$  in the periodic quiver  $\tilde{Q}$ . Let  $v : i \rightarrow j$  be a strict path such that  $\chi_I(v) = 0$  for some perfect matching  $I$  (see Prop. 2.13). Then  $u = v\omega^k$  for some  $k \in \mathbb{Z}$  (see [19, Lemma 4.6]) and we have  $0 \leq \chi_I(u) = \chi_I(v) + k = k$ . This implies that  $v\omega^k$  is a strict path.  $\square$

**Corollary 3.5.** *A path  $u$  in  $\tilde{Q}$  is minimal (see Def. 2.9) if and only if  $\chi_I(u) = 0$  for some extremal perfect matching.*

*Proof.* Assume that  $\chi_I(u) = 0$  for some (not necessarily extremal) perfect matching. If  $u = v\omega^k$  for some path  $v$  and some  $k \geq 0$  then  $0 = \chi_I(u) = \chi_I(v) + k \geq k$ , so  $k = 0$ . This implies that  $u$  is minimal.

Assume that  $u$  is minimal. Then  $u\omega^{-1}$  is not equivalent to any strict path. It follows from Lemma 3.4 that  $\chi_I(u\omega^{-1}) < 0$  for some extremal perfect matching. This implies  $\chi_I(u) = 0$ .  $\square$

**Corollary 3.6.** *For any  $i, j \in Q_0$ , we have  $\Lambda_{ij}^+ = \Lambda_{ij} \cap P$ . The semigroup  $M^+$  is saturated.*

*Proof.* We have from Corollary 3.3

$$\Lambda_{ij} \cap P = \{\lambda \in \Lambda_{ij} \mid \chi_I(\lambda) \geq 0 \forall I \in \mathcal{A}\}.$$

For any  $\lambda$  as above, we can find a weak path  $u : i \rightarrow j$  such that  $\text{wt}(u) = \lambda$ . Then  $\chi_I(u) \geq 0 \forall I \in \mathcal{A}$  and it follows from Lemma 3.4 that  $u$  is weakly equivalent to a strict path, so  $\lambda = \text{wt}(u) \in \Lambda^+$ . This proves the first statement. The second statement follows from the first for  $i = j$ .  $\square$

**Remark 3.7.** The semigroup  $\Lambda^+$  is not saturated (i.e.  $\Lambda^+ \neq \Lambda \cap P$ ) in general. So the above corollary can be quite confusing, as it says that  $\Lambda_{ij} \cap \Lambda^+ = \Lambda_{ij} \cap P$ . Note that  $\Lambda$  is not the union of  $\Lambda_{ij}$ ,  $i, j \in Q_0$ .

**Remark 3.8.** For any  $i, j \in Q_0$  we can identify  $e_j A e_i$  with a vector space  $\mathbb{C}[\Lambda_{ij}^+]$  (we denote its basis elements by  $e^\lambda$ ,  $\lambda \in \Lambda_{ij}^+$ ). This is the content of the algebraic consistency condition [7, Definition 4.4.2] proved in [7]. For any  $\lambda \in \Lambda_{ij}$ , we denote by  $u_{ij}^\lambda$  an (equivalence class of a) path from  $i$  to  $j$  having weight  $\lambda$ . The above identification is given by  $u_{ij}^\lambda \mapsto e^\lambda$ ,  $\lambda \in \Lambda_{ij}^+$ .

**Remark 3.9.** Let  $P_M = P \cap M_{\mathbb{Q}}$ . Then the dual cone  $P_M^\vee \subset M_{\mathbb{Q}}^\vee$  is generated by  $\bar{\chi}_I$  for extremal perfect matchings  $I$  (see Proposition 2.12). The elements  $\bar{\chi}_I$  are contained in the hyperplane

$$\{y \in M_{\mathbb{Q}}^\vee \mid \omega_M^*(y) = 1\},$$

where  $\omega_M : \mathbb{Z} \rightarrow M$  was defined in Section 2.2. This implies that  $\text{Spec } \mathbb{C}[M^+] = \text{Spec } \mathbb{C}[P_M \cap M]$  is a toric Calabi-Yau variety. Its toric diagram is defined as an intersection of the cone  $P_M^\vee$  with the above hyperplane.

**Lemma 3.10** (see [7, Lemma 4.3.1]). *The center of the quiver potential algebra  $A = \mathbb{C}Q/(\partial W)$  is isomorphic to  $\mathbb{C}[M^+]$ .*

Let  $R = \mathbb{C}[M^+]$ . The inclusion  $R \rightarrow A$  from the above lemma is given by  $e^\lambda \mapsto \sum_{i \in Q_0} u_{ii}^\lambda$ . Note that every  $e_j A e_i$  is automatically an  $R$ -module.

**Lemma 3.11.** *For any  $i, j, k \in Q_0$ , there is a canonical  $R$ -module isomorphism*

$$\text{Hom}_R(e_j A e_i, e_k A e_i) = e_k A e_j.$$

*Proof.* There is a natural embedding of  $e_k Ae_j$  in  $\text{Hom}_R(e_j Ae_i, e_k Ae_i)$ . We just have to prove that it is a bijection. Let  $f \in \text{Hom}_R(e_j Ae_i, e_k Ae_i)$ . We can assume that there exists some weak path  $u : j \rightarrow k$  such that  $f(v) = uv$  for any path  $v : i \rightarrow j$ . We have to show that  $u$  is equivalent to a strict path. Let  $u = \omega^k u'$  for the minimal path  $u'$  and  $k \in \mathbb{Z}$ . By Corollary 3.5 there exists an external perfect matching  $I$  such that  $\chi_I(u') = 0$ . We know that the quiver  $Q \setminus I$  is strongly connected (see Proposition 2.12), so there exists a strict path  $u'' : i \rightarrow j$  such that  $\chi_I(u'') = 0$ . Then the path  $f(u'') = \omega^k u' u''$  is strict. Therefore  $0 \leq \chi_I(\omega^k u' u'') = k + \chi_I(u') + \chi_I(u'') = k$ . This implies that  $u' \omega^k$  is a strict path.  $\square$

**Corollary 3.12.** *For any  $i \in Q_0$ , we have  $\text{Hom}_R(Ae_i, Ae_i) = A$ .*

**Proposition 3.13.** *The algebra  $R$  is a normal Gorenstein domain and the algebra  $A = \mathbb{C}Q/(\partial W)$  is its non-commutative crepant resolution (see [27, Definition 4.1]).*

*Proof.* The algebra  $R = \mathbb{C}[M^+]$  is normal, as  $M^+$  is saturated. Let

$$P_M = P \cap M_{\mathbb{Q}}.$$

Then  $M^+ = P_M \cap M$ . According to [21, p. 126], the algebra  $\mathbb{C}[M^+]$  is Gorenstein if and only if

$$\text{inn}(P_M) \cap M = m + (P_M \cap M)$$

for some  $m \in M$ . Let  $\mathcal{A}^e \subset \mathcal{A}$  be the set of extremal perfect matchings. We have

$$\text{inn}(P_M) \cap M = \{\lambda \in P \cap M \mid \chi_I(\lambda) > 0 \forall I \in \mathcal{A}^e\}.$$

This implies that, for any  $\lambda \in \text{inn}(P_M) \cap M$ , we have  $\chi_I(\lambda - \bar{\omega}) = \chi_I(\lambda) - 1 \geq 0$ ,  $I \in \mathcal{A}^e$ , so  $\lambda - \bar{\omega} \in P_M \cap M$ . It follows that

$$\text{inn}(P_M) \cap M = \bar{\omega} + (P_M \cap M)$$

and the algebra  $\mathbb{C}[M^+]$  is Gorenstein.

Let  $i \in Q_0$ . We have seen that  $A = \text{End}_R(Ae_i)$ . The  $R$ -module  $Ae_i$  is reflexive. Indeed, we have

$$\text{Hom}_R(e_j Ae_i, R) = \text{Hom}_R(e_j Ae_i, e_i Ae_i) = e_i Ae_j.$$

Taking again the dual, we see that  $e_j Ae_i$  is reflexive and therefore also  $Ae_i$  is reflexive. According to [27, Lemma 4.2], we just have to show that  $A$  has finite global dimension and  $A$  is a CM-module over  $R$ . It is proved in [19] that  $A$  is a 3-Calabi-Yau algebra. This together with Remark 2.16 implies that  $A$  has finite global dimension. To show that  $A$  is a CM-module over  $R$  we will apply Theorem 2.24. To do this we have to show that  $A$  is a finitely generated module over  $R$ . It is enough to show that  $e_j Ae_i$  is finitely generated over  $R$  for every  $i, j \in Q_0$ . Let us choose some path  $v : j \rightarrow i$ . Then the map  $e_j Ae_i \rightarrow e_i Ae_i \simeq R$ ,  $u \mapsto vu$  is injective. It follows that  $e_j Ae_i$  is finitely generated over  $R$ , as  $R = \mathbb{C}[M^+]$  is noetherian.  $\square$

#### 4. CONSTRUCTION OF CREPANT RESOLUTIONS

Let  $(Q, W)$  be the quiver potential associated with a geometrically consistent brane tiling and let  $A = \mathbb{C}Q/(\partial W)$  be the corresponding quiver potential algebra. We denote by  $\text{mod } A$  the category of finitely generated left  $A$ -modules.

Recall that  $\Lambda^+ \subset \Lambda$  is a semigroup generated by the weights of the arrows from  $Q_1$ . It was shown in [19] that  $\Lambda^+ \cap (-\Lambda^+) = \{0\}$ . The semigroup  $\Lambda^+$  is not saturated and  $\text{Spec } \mathbb{C}[\Lambda^+]$  is not normal in general.

We will consider the moduli spaces of representations of  $A$  having dimension vector  $\alpha = (1, \dots, 1) \in \mathbb{Z}^{Q_0}$ . The space of  $\mathbb{C}Q$ -representations of dimension  $\alpha$  is given by

$$R(\mathbb{C}Q, \alpha)(\mathbb{C}) = \mathbb{C}^{Q_1}.$$

Let  $R(A, \alpha) \subset R(\mathbb{C}Q, \alpha)$  be the subvariety of those representations that satisfy relations induced by the potential  $W$ . The structure ring of  $R(\mathbb{C}Q, \alpha)$  is a polynomial algebra  $\mathbb{C}[\mathbb{N}^{Q_1}]$ . We will denote its natural basis by  $(e^\lambda)_{\lambda \in \mathbb{N}^{Q_1}}$ . For any arrow  $a \in Q_1$ , let  $F_a^\pm \in Q_2$  be the faces containing  $a$ . Let  $u_a^\pm$  be a path in  $Q$  such that  $au_a^\pm$  is a cycle along  $F_a^\pm$ . Then

$$\frac{\partial W}{\partial a} = u_a^+ - u_a^-.$$

This implies that the structure ring of  $R(A, \alpha)$  is given by

$$\mathbb{C}[\mathbb{N}^{Q_1}] / (e^{|u_a^+|} - e^{|u_a^-|} \mid a \in Q_1),$$

where, for any path  $u$ , the vector  $|u| \in \mathbb{Z}^{Q_1}$  denotes its content (see Section 2.3). The natural surjective map  $\mathbb{C}[\mathbb{N}^{Q_1}] \rightarrow \mathbb{C}[\Lambda^+]$  can be factored

$$\mathbb{C}[\mathbb{N}^{Q_1}] / (e^{|u_a^+|} - e^{|u_a^-|} \mid a \in Q_1) \rightarrow \mathbb{C}[\Lambda^+].$$

This implies that there is a closed embedding

$$\text{Spec } \mathbb{C}[\Lambda^+] \rightarrow R(A, \alpha).$$

**Remark 4.1.** This map need not be an isomorphism. We thank Alastair Craw for this remark.

**Proposition 4.2.** *Variety  $\text{Spec } \mathbb{C}[\Lambda^+]$  is an irreducible component of  $R(A, \alpha)$*

*Proof.* Recall that we have a map  $\text{wt} : \mathbb{Z}^{Q_1} \rightarrow \Lambda$  (see Section 2.2). We define ideals  $I, J \subset \mathbb{C}[\mathbb{N}^{Q_1}]$  by the rule

$$I = \left( e^{\lambda^+} - e^{\lambda^-} \mid \lambda^\pm \in \mathbb{N}^{Q_1}, \lambda^+ - \lambda^- \in \ker(\text{wt}) \right),$$

$$J = \left( e^{|u_a^+|} - e^{|u_a^-|} \mid a \in Q_1 \right).$$

Note that the elements  $|u_a^+| - |u_a^-|$ ,  $a \in Q_1$ , generate the group  $\ker(\text{wt})$  ( $\Lambda$  was defined as a factor group of  $\mathbb{Z}^{Q_1}$  by the subgroup generated by the above elements, see Section 2.2). Ideal  $I$  defines variety  $\text{Spec } \mathbb{C}[\Lambda^+]$  and ideal  $J$  defines variety  $R(A, \alpha)$ . The proof now literally repeats the proof of [10, Theorem 3.10].  $\square$

In order to construct the moduli spaces of left  $A$ -modules of dimension  $\alpha$ , we have to identify isomorphic representations from  $R(A, \alpha)$  with each other. This is achieved by taking GIT quotients with respect to the natural action of the group

$$\text{GL}_\alpha(\mathbb{C}) = \prod_{i \in Q_0} \text{GL}_{\alpha_i}(\mathbb{C})$$

on  $R(A, \alpha)$  [18]. One can see that two representations from  $R(A, \alpha)$  are isomorphic if and only if they are contained in the same orbit. In our case

$$\text{GL}_\alpha(\mathbb{C}) = (\mathbb{C}^*)^{Q_0} = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{C}^*).$$

The diagonal  $\mathbb{C}^* \subset \text{GL}_\alpha(\mathbb{C})$  acts trivially on  $R(A, \alpha)$ , so our action factors through

$$T_B = \text{Hom}_{\mathbb{Z}}(B, \mathbb{C}^*),$$

where  $B = \ker(\mathbb{Z}^{Q_0} \rightarrow \mathbb{Z})$  was defined in Section 2.2. Given an element  $\theta \in B$ , we say that a representation  $X \in R(A, \alpha)$  is stable (resp. semistable) if for any proper nonzero subrepresentation  $Y \subset X$ , we have  $\theta \cdot \dim Y > 0$  (resp.  $\theta \cdot \dim Y \geq 0$ ). According to [18], the moduli space of  $\theta$ -semistable left  $A$ -modules of dimension  $\alpha$  is given by the GIT quotient

$$\mathcal{M}_\theta(A, \alpha) = R(A, \alpha) //_\theta T_B.$$

**Definition 4.3.** An element  $\theta \in B$  is called  $\alpha$ -generic if, for any  $0 < \beta < \alpha$  we have  $\theta \cdot \beta \neq 0$ .

If  $\theta \in B$  is  $\alpha$ -generic, then all  $\theta$ -semistable  $A$ -modules of dimension  $\alpha$  are stable.

**Proposition 4.4.** *Assume that  $\theta \in B$  is  $\alpha$ -generic. Then we have*

$$\mathcal{M}_\theta(A, \alpha) = \text{Spec } \mathbb{C}[\Lambda^+] //_\theta T_B = \text{Spec } \mathbb{C}[P \cap \Lambda] //_\theta T_B.$$

*Proof.* We know from [16, Prop. 5.1] that  $\mathcal{M}_\theta$  is irreducible. We can find some irreducible component  $Z \subset R(A, \alpha)$  such that  $Z //_\theta T_B$  equals  $\mathcal{M}_\theta = \mathcal{M}_\theta(A, \alpha)$ . As all points of  $T_\Lambda \subset R(A, \alpha)$  are  $\theta$ -stable (they correspond to simple modules), we deduce that  $T_\Lambda \subset Z$ . Variety  $T_\Lambda$  is dense in  $\text{Spec } \mathbb{C}[\Lambda^+]$ , so  $\text{Spec } \mathbb{C}[\Lambda^+] \subset Z$ . By the previous proposition this inclusion is an isomorphism. This implies  $\text{Spec } \mathbb{C}[\Lambda^+] //_\theta T_B = \mathcal{M}_\theta$ . To prove the second equality, we note that  $\text{Spec } \mathbb{C}[P \cap \Lambda] //_\theta T_B$  is a normalization of  $\text{Spec } \mathbb{C}[\Lambda^+] //_\theta T_B = \mathcal{M}_\theta$  and  $\mathcal{M}_\theta$  is smooth by [16, Prop. 5.1].  $\square$

If  $\theta \in B$  is  $\alpha$ -generic then there exists a universal vector bundle  $\mathcal{U}$  on  $\mathcal{M}_\theta = \mathcal{M}_\theta(A, \alpha)$ .

**Theorem 4.5.** *Let  $\theta \in B$  be  $\alpha$ -generic. Then  $\mathcal{M}_\theta$  is smooth, the natural map  $\mathcal{M}_\theta \rightarrow \text{Spec } Z(A)$  is a crepant resolution, and there is a pair of inverse equivalences of derived categories*

$$\begin{aligned} \Phi : D^b(\text{coh } \mathcal{M}_\theta) &\rightarrow D^b(\text{mod } A^{\text{op}}), & F &\mapsto R\Gamma(F \otimes_{\mathcal{M}_\theta}^L \mathcal{U}^*), \\ \Psi : D^b(\text{mod } A^{\text{op}}) &\rightarrow D^b(\text{coh } \mathcal{M}_\theta), & M &\mapsto M \otimes_A^L \mathcal{U}. \end{aligned}$$

*Proof.* This follows from [27, Theorem 6.3.1] and the fact that  $A$  is a non-commutative crepant resolution of  $Z(A)$  (see Proposition 3.13). We should just note that Van den Bergh considers the moduli spaces of right  $A$ -modules while we consider the moduli spaces of left  $A$ -modules.  $\square$

**Remark 4.6.** This result was proved directly by Ishii and Ueda [16, 15].

According to [25], variety

$$\mathcal{M}_\theta(A, \alpha) = \text{Spec } \mathbb{C}[P \cap \Lambda] //_\theta T_B$$

is a toric variety endowed with an action of the torus  $T_M = T_\Lambda / T_B$ . We are going to describe  $T_M$ -orbits of  $\mathcal{M}_\theta(A, \alpha)$ . To do this we will describe the  $T_\Lambda$ -orbits of  $R(A, \alpha)$ . It turns out that orbits corresponding to indecomposable modules are parametrized by their supports.

**Definition 4.7.** For any subset  $I \subset Q_1$  we define representation  $x_I = (x_{I,a})_{a \in Q_1} \in R(\mathbb{C}Q, \alpha)$  by the rule

$$x_{I,a} = \begin{cases} 0, & a \in I \\ 1, & a \notin I \end{cases}$$

We say that  $I$  is  $W$ -compatible if  $x_I$  is contained in  $R(A, \alpha)$ . We say that  $I$  is  $\theta$ -stable (resp. semistable) for  $\theta \in B$ , if  $x_I$  is  $\theta$ -stable (resp. semistable). We say that  $I$  is indecomposable if  $x_I$  is indecomposable.

**Remark 4.8.** Note that any perfect matching  $I \subset Q_1$  is  $W$ -compatible. If  $I$  is an extremal perfect matching then  $Q \setminus I$  is a strongly connected quiver. This implies that  $x_I$  is a simple representation and, in particular,  $\theta$ -stable for any  $\theta \in B$ .

**Definition 4.9.** For any representation  $x = (x_a)_{a \in Q_1} \in R(\mathbb{C}Q, \alpha)$  we define its cosupport

$$I_x = \{a \in Q_1 \mid x_a = 0\}.$$

All representations in the same  $(\mathbb{C}^*)^{Q_1}$ -orbit of  $R(\mathbb{C}Q, \alpha)$  have the same cosupport.

**Lemma 4.10.** *Let  $x \in R(A, \alpha)$ . Then  $I_x$  is  $W$ -compatible. Representation  $x$  is  $\theta$ -stable (resp.  $\theta$ -semistable, resp. indecomposable) if and only if  $I_x$  is  $\theta$ -stable (resp.  $\theta$ -semistable, resp. indecomposable).*

**Definition 4.11.** Let  $I \subset Q_1$ . We consider it as a subgraph of the bipartite graph  $G = (G_0^\pm, G_1)$  dual to  $Q$  in the two-dimensional torus  $T$ . A connected component of  $I$  is called a big component of  $I$  if it contains more than one edge. Otherwise it is called a small component of  $I$ .

**Proposition 4.12.** *Two indecomposable representations  $x, y \in R(A, \alpha)$  are contained in the same  $T_\Lambda$ -orbit if and only if they have the same support.*

*Proof.* It is clear that any two representations in the same  $T_\Lambda$ -orbit have equal supports. Let us prove the converse. So let  $x, y \in R(A, \alpha)$  be such that  $I_x = I_y =: I$ . We define a new  $Q$ -representation  $z$  by the rule

$$z_a = \begin{cases} 0, & a \in I \\ x_a^{-1}y_a, & a \notin I \end{cases}$$

It is clear that  $z$  is also an  $A$ -representation. It is also indecomposable, as this is a property of the support. We claim that we can extend  $(z_a)_{a \in Q_1 \setminus I}$  to the element in  $(\mathbb{C}^*)^{Q_1}$  that is still an  $A$ -representation. Such an element will be automatically contained in  $T_\Lambda$  and will map  $x$  to  $y$ , so that both elements will be in the same  $T_\Lambda$ -orbit.

We may suppose that  $I \neq \emptyset$ , as otherwise our claim is automatically satisfied. By the  $W$ -relations the elements  $\prod_{a \in F} z_a$  are independent of  $F \in Q_2$  and therefore are all zero. It follows that every face intersects  $I$  non-trivially. If every face intersects  $I$  in precisely one element, then  $I$  is a perfect matching. In this case  $z$  can be obviously extended to  $(\mathbb{C}^*)^{Q_1}$  and we are done.

So let us assume that  $I$  is not a perfect matching. We consider  $I$  as a subgraph of the bipartite graph  $G = (G_0^\pm, G_1)$  dual to  $Q$  in the two-dimensional torus  $T$ . Graph  $I$  can have many connected components. Our assumptions imply that the set of vertices of  $I$  equals the set of vertices of  $G$ , that every vertex (i.e. face in  $Q$ ) has valence at least one, and that there is at least one vertex having valence  $\geq 2$ . It follows from  $W$ -relations that any vertex connected to a vertex of valence  $\geq 2$  also has valence  $\geq 2$ . So, if a connected component of  $I$  is big then all its vertices have valence  $\geq 2$ . We claim that there is precisely one big component.

First, let us note that the complement of  $I$  in the torus is connected as  $z$  is indecomposable. One can easily see that the complement of two non-intersecting

loops in the torus always has at least two connected components. Every big component of  $I$  has loops, so the existence of two big components would imply that the complement of  $I$  is not connected. This proves our claim that there exists just one big component in  $I$ .

We will extend  $(z_a)_{a \in Q_1 \setminus I}$  to the element in  $(\mathbb{C}^*)^{Q_1}$  in such a way that products along the faces (elements in  $Q_2$ ) are all equal one. The choice for the arrows of small components is clear. Let  $J \subset I$  denotes the big component. For any  $F \in G_0 = Q_2$  we define

$$z_F = \prod_{a \in F \setminus I} z_a^{\varepsilon(F)},$$

where  $\varepsilon(F) = \pm 1$  for  $F \in G_0^\pm$ . It follows from the  $W$ -relations that

$$\prod_{F \in J_0} z_F = 1,$$

where  $J_0 \subset G_0$  (resp.  $J_1 \subset G_1$ ) denotes the set of vertices of  $J$  (resp. the set of edges of  $J$ ). More generally, we consider an arbitrary abelian group  $\Gamma$  and a sequence

$$\Gamma^{J_1} \xrightarrow{d} \Gamma^{J_0} \xrightarrow{p} \Gamma,$$

where  $d(\gamma a) = \gamma F_a^+ - \gamma F_a^-$  with  $F_a^\pm \in J_0^\pm$  incident with  $a \in J_1$ , and  $p(\gamma F) = \gamma$  for  $F \in J_0$ ,  $\gamma \in \Gamma$ . The kernel of  $p$  is generated by the elements of the form  $\gamma F_a^+ - \gamma F_a^-$ ,  $a \in J_1$ ,  $\gamma \in \Gamma$ , as  $J$  is connected. So the above sequence is exact. In our situation this means that there exists  $(t_a)_{a \in J_1} \in (\mathbb{C}^*)^{J_1}$  such that, for any  $F \in J_0 \subset Q_2$ , we have

$$\prod_{a \in F \cap J_1} t_a^{\varepsilon(F)} = z_F.$$

The elements  $t_a^{-1}$ ,  $a \in J_1$  give then the desired extension.  $\square$

**Remark 4.13.** We have shown that the cosupport of an indecomposable  $A$ -module of dimension  $\alpha$  can have at most one big component. Every vertex of a big component has valence at least 2.

**Corollary 4.14.** *There is a bijection between the  $T_\Lambda$ -orbits of indecomposable representations in  $R(A, \alpha)$  and  $W$ -compatible, indecomposable subsets in  $Q_1$ .*

Let now  $\theta \in B$  be  $\alpha$ -generic. Then all representations from  $\mathcal{M}_\theta = \mathcal{M}_\theta(A, \alpha)$  are stable and in particular indecomposable. It follows that  $T_B$ -orbits of  $\mathcal{M}_\theta$  are parametrized by  $W$ -compatible,  $\theta$ -stable subsets in  $Q_1$ . We are going to give a precise description of these subsets according to the dimension of the corresponding orbits. For any  $W$ -compatible,  $\theta$ -stable set  $I \subset Q_1$ , we denote by  $\sigma_I$  the cone of the fan of  $\mathcal{M}_\theta$  corresponding to the orbit defined by  $I$ .

**Proposition 4.15.** *Let  $x \in \mathcal{M}_\theta$  and let  $O_x \subset \mathcal{M}_\theta$  be its  $T_M$ -orbit. Then*

- (1)  $\dim O_x = 3$  if and only if  $I_x = \emptyset$ .
- (2)  $\dim O_x = 2$  if and only if  $I_x$  is a perfect matching.
- (3)  $\dim O_x = 1$  if and only if the big component of  $I_x$  is a cycle.
- (4)  $\dim O_x = 0$  if and only if the big component of  $I_x$  has two trivalent vertices of different colors and all other vertices of valence 2.

*Proof.* The first statement is trivial. The last statement is just a translation of [16, Lemma 4.5] to our language in the case of a consistent brane tiling. It is proved there also that if  $\sigma$  is a cone corresponding to the fixed point  $x$  then  $U_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap M]$

is isomorphic to  $\mathbb{C}^3$ . This isomorphism can be described in the following way. Let  $v^\pm$  be the white and black vertices of valence 3 in the big component of  $I_x$ . Let  $a_i^\pm, i = 1, 2, 3$  be the edges in the big component incident with  $v^\pm$ . Then for any choice of values  $t_i \in \mathbb{C}, i = 1, 2, 3$  for the edges  $a_i^+, i = 1, 2, 3$  there exists the unique choice of the values for the rest of the edges in  $I_x$  such that the corresponding quiver representation satisfies  $W$ -relations (representation  $x$  corresponds to zero values on the edges  $a_i^+, i = 1, 2, 3$ ). This gives the required isomorphism  $U_\sigma \simeq \mathbb{C}^3$ .

For any one-dimensional orbit  $O_y$  there exists a fixed point  $x$  in its closure. Let  $\sigma$  be the cone corresponding to the point  $x$ . Then  $O_y \subset U_\sigma$  and using the above identification  $U_\sigma \simeq \mathbb{C}^3$ , we can write without loss of generality

$$O_y = \{(t, 0, 0) \mid t \in \mathbb{C}^*\}.$$

Let

$$(a_{i,1}, \dots, a_{i,2k_i+1}), \quad k_i \geq 0, \quad i = 1, 2, 3$$

be the chains of edges in the big component of  $I_x$  that connect  $v^+$  and  $v^-$ . We assume that  $a_{i,1} = a_i^+, i = 1, 2, 3$ . Then representation  $y$  satisfies

$$\begin{aligned} y_{a_{1,2k+1}} &\neq 0, & 0 \leq k \leq k_1 \\ y_{a_{1,2k}} &= 0, & 1 \leq k \leq k_1 \\ y_{a_{i,k}} &= 0, & 1 \leq k \leq 2k_i + 1, \quad i = 2, 3. \end{aligned}$$

This implies that the big component of  $I_y$  consists of the edges  $a_{i,k}, 1 \leq k \leq 2k_i + 1, i = 2, 3$ . This is a cycle.

For any two-dimensional orbit  $O_y$  we again consider a fixed point  $x$  in its closure and the cone  $\sigma$  corresponding to  $x$ . We can write without loss of generality

$$O_y = \{(t_1, t_2, 0) \mid t_1, t_2 \in \mathbb{C}^*\}.$$

We use the same notation as above for the chains connecting  $v_+$  and  $v_-$ . Then representation  $y$  satisfies

$$\begin{aligned} y_{a_{i,2k+1}} &\neq 0, & 0 \leq k \leq k_i, \quad i = 1, 2 \\ y_{a_{i,2k}} &= 0, & 1 \leq k \leq k_i, \quad i = 1, 2 \\ y_{a_{3,2k+1}} &= 0, & 0 \leq k \leq k_3 \\ y_{a_{3,2k}} &\neq 0, & 1 \leq k \leq k_3. \end{aligned}$$

This implies that  $I_y$  is a perfect matching. □

**Remark 4.16.** The one-dimensional cones of the fan of  $\mathcal{M}_\theta$  are generated by  $\bar{\chi}_I \in M^\vee$ , where  $I \in \mathcal{A}$  are  $\theta$ -stable. All these vectors are contained in the hyperplane

$$\{y \in M_\mathbb{Q}^\vee \mid \omega_M^*(y) = 1\},$$

where  $\omega_M : \mathbb{Z} \rightarrow M$  was defined in Section 2.2. This implies that  $\mathcal{M}_\theta$  is a toric Calabi-Yau variety (cf. Remark 3.9). Its toric diagram is defined as an intersection of the cones of the fan of  $\mathcal{M}_\theta$  with the above hyperplane. This toric diagram is a triangulation of the toric diagram of  $\mathbb{C}[M^+]$  (see Remark 3.9).

**Remark 4.17.** For any non-extremal point of the toric diagram there exists more than one perfect matching that is mapped to it (see Proposition 2.12). However, only one of these perfect matchings is  $\theta$ -stable.

**Corollary 4.18.** *Let  $x \in \mathcal{M}_\theta$  be a fixed point and let*

$$(a_{i,1}, \dots, a_{i,2k_i+1}), \quad i = 1, 2, 3$$

*be the chains of edges connecting the trivalent points of the big component of  $I_x$ . Then there are precisely three 2-dimensional orbits containing  $x$  in their closures. The corresponding perfect matchings are given by (for  $i = 1, 2, 3$ )*

$$I_x \setminus \{a_{j,2k+1} \mid 0 \leq k \leq k_j, j \neq i\} \setminus \{a_{i,2k} \mid 1 \leq k \leq k_i\}.$$

*These are the only perfect matchings contained in  $I_x$ .  $I_x$  is a union of these perfect matchings.*

**Corollary 4.19.** *Let  $O_x \subset \mathcal{M}_\theta$  be a one-dimensional orbit and let*

$$(a_1, \dots, a_{2k})$$

*be the big component of  $I_x$  which is a cycle. Then there are precisely two 2-dimensional orbits containing  $O_x$  in their closures. The corresponding perfect matchings are given by*

$$I_x \setminus \{a_1, a_3, \dots, a_{2k-1}\}, \quad I_x \setminus \{a_2, a_4, \dots, a_{2k}\}.$$

*These are the only perfect matchings contained in  $I_x$ .  $I_x$  is a union of these perfect matchings.*

We have now a simple algorithm to construct a fan of  $\mathcal{M}_\theta$  for  $\alpha$ -generic  $\theta \in B$ . Construction of such fan is equivalent to the triangulation of the toric diagram. We find first those perfect matchings that are  $\theta$ -stable. Then we test for all the pairs of these perfect matchings if their union is  $\theta$ -stable (note that the union of any two perfect matchings is automatically  $W$ -compatible). In this way we get all 1-dimensional and 2-dimensional cones of the required fan. The 3-dimensional cones are constructed then automatically. The triangulation of the toric diagram is uniquely determined from this data.

## 5. ORBIFOLDS AND BRANE TILINGS

In this section, for any finite abelian subgroup  $G \subset \mathrm{SL}_3(\mathbb{C})$ , we construct certain brane tiling. The underlying quiver will coincide with the McKay quiver of the  $G$ -representation  $\mathbb{C}^3$ . It turns out that the corresponding quiver potential algebra is Morita-equivalent (and even isomorphic) to the skew product  $\mathbb{C}[x, y, z] \rtimes G$  and it is therefore the most natural candidate for the non-commutative crepant resolution of the orbifold singularity  $\mathbb{C}^3/G$ .

**5.1. McKay quiver.** Let  $G$  be an arbitrary finite group and let  $V$  be its finite-dimensional representation over  $\mathbb{C}$ . We denote by  $\hat{G}$  the set of all irreducible  $G$ -representations.

**Remark 5.1.** If  $G$  is abelian, then  $\hat{G}$  has a group structure induced by tensor products and  $\hat{G}$  can be identified with  $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$ . It is called the group of characters of  $G$ . This group is non-canonically isomorphic to  $G$ .

Recall the definition of the McKay quiver  $Q$  of a  $G$ -representation  $V$ . Its set of vertices is given by  $Q_0 = \hat{G}$ . The set of arrows from  $\sigma \in \hat{G}$  to  $\rho \in \hat{G}$  is given by some fixed basis of

$$\mathrm{Hom}_G(\sigma, \rho \otimes V).$$



This quiver can be used to describe left modules over a skew-group algebra  $S(V^\vee) \rtimes G$ . To define such module structure on a vector space  $M$  it is enough to endow  $M$  with a structure of a  $G$ -representation and to give a  $G$ -equivariant linear map  $V^\vee \otimes M \rightarrow M$  (corresponding to the multiplication) that satisfies certain axioms (see e.g. [17, Section 3]). Let us show that this data defines a representation of the McKay quiver. We decompose  $M = \bigoplus_{\rho \in \hat{G}} M_\rho \otimes \rho$ , where  $M_\rho$  are some vector spaces. We put this vector spaces at the vertices of the McKay quiver (recall that  $Q_0 = \hat{G}$ ). We have

$$\begin{aligned} \mathrm{Hom}_G(V^\vee \otimes M, M) &= \bigoplus_{\sigma, \rho} \mathrm{Hom}_G(V^\vee \otimes \sigma, \rho) \otimes \mathrm{Hom}(M_\sigma, M_\rho) \\ &= \bigoplus_{\sigma, \rho} \mathrm{Hom}_G(\sigma, \rho \otimes V) \otimes \mathrm{Hom}(M_\sigma, M_\rho). \end{aligned}$$

This means that for any arrow from  $\sigma \in \hat{G}$  to  $\rho \in \hat{G}$ , we have a linear map  $M_\sigma \rightarrow M_\rho$ . This gives us a required quiver representation. In this way we get a full and faithful functor

$$\mathrm{mod}(S(V^\vee) \rtimes G) \rightarrow \mathrm{mod} \mathbb{C}Q.$$

To get an equivalence of categories, we have to impose certain relations in the path algebra  $\mathbb{C}Q$  [17, Section 3].

Let now  $G$  be a finite abelian subgroup  $G \subset \mathrm{SL}_3(\mathbb{C})$ . For any character  $\rho \in \hat{G}$  we denote the corresponding one-dimensional  $G$ -representation also by  $\rho$ . We can decompose the  $G$ -representation  $V = \mathbb{C}^3$  (induced by the inclusion  $G \subset \mathrm{SL}(\mathbb{C}^3)$ )

$$V = \rho_1 \oplus \rho_2 \oplus \rho_3,$$

where  $\rho_1, \rho_2, \rho_3 \in \hat{G}$ . Note that  $\rho_1 \rho_2 \rho_3 = 1$ , as  $G \subset \mathrm{SL}_3(\mathbb{C})$ . Then

$$\rho \otimes V \simeq \rho \rho_1 \oplus \rho \rho_2 \oplus \rho \rho_3$$

and  $\mathrm{Hom}_G(\sigma, \rho \otimes V)$  can be nonzero only if  $\sigma = \rho \rho_i$  for some  $i = 1, 2, 3$ . It follows that any vertex  $\rho \in Q_0 = \hat{G}$  has three ingoing arrows

$$a_i^\rho : \rho \rho_i \rightarrow \rho, \quad i = 1, 2, 3.$$

Sometimes we will omit the upper index if the ingoing or outgoing vertex is known. Now we describe the set of faces  $Q_2$  corresponding to some brane tiling. All faces will contain just three arrows. For any vertex  $\rho \in \hat{G}$  and any permutation  $\pi \in S_3$ , we consider the face

$$\rho \xrightarrow{a_{\pi(3)}} \rho \rho_{\pi(1)} \rho_{\pi(2)} \xrightarrow{a_{\pi(2)}} \rho \rho_{\pi(1)} \xrightarrow{a_{\pi(1)}} \rho.$$

It is clear that every arrow is contained in precisely two faces. This implies that if we glue the faces along the common arrows we obtain some oriented compact surface. The number of arrows equals  $3 \cdot \#G$  and the number of faces equals  $2 \cdot \#G$ . This implies that the Euler number of our surface is zero and therefore the surface is homotopic to a torus. So we obtain a brane tiling. Let  $W$  be the corresponding potential.

**Remark 5.2.** We have used the fact that  $Q$  is a connected quiver. This follows from the fact that  $\rho_1, \rho_2, \rho_3$  generate the whole group  $\hat{G}$ . Indeed, assume that they generate some proper subgroup  $\hat{H} \subset \hat{G}$ . Then the map  $G \hookrightarrow \mathrm{SL}_3(\mathbb{C})$  can be factored through  $G \rightarrow H \rightarrow \mathrm{SL}_3(\mathbb{C})$ , where  $H = \mathrm{Hom}_{\mathbb{Z}}(\hat{H}, \mathbb{C}^*)$ . This would imply that  $G \rightarrow \mathrm{SL}_3(\mathbb{C})$  is not injective.

The next result follows from [23, Section 5.2] or [17, Section 3] or [10, Section 2]

**Proposition 5.3.** *There is an equivalence of categories*

$$\text{mod}(\mathbb{C}[x, y, z] \rtimes G) \rightarrow \text{mod } \mathbb{C}Q/(\partial W)$$

**Remark 5.4.** It is proved in [11, Proposition 2.8] that the above algebras are actually isomorphic.

**Remark 5.5.** It was shown in [13, Section 4.4] that for any finite subgroup  $G \subset \text{SL}_3(\mathbb{C})$  one can endow the corresponding McKay quiver with a potential (depending on some choices) in such a way that the quiver potential algebra is Morita equivalent to the skew group algebra. The coefficients of the cycles in that potential are not always  $\pm 1$ , so it can not correspond to some brane tiling. However one can show that if  $G$  is abelian then one can make such choices that the coefficients of the cycles of the potential are  $\pm 1$  and this potential is induced by the brane tiling constructed above.

**Remark 5.6.** The periodic quiver of the above brane tiling coincides with the periodic quiver corresponding to the brane tiling of  $\mathbb{C}^3$ . This implies that the constructed brane tiling is always geometrically consistent.

**5.2. Toric realization of orbifolds.** Let  $G \subset \text{SL}_3(\mathbb{C})$  be a finite abelian group. We have associated a quiver potential  $(Q, W)$  and a triple of characters  $\rho_1, \rho_2, \rho_3 \in \hat{G}$  with such a group. These characters generate  $\hat{G}$ . So, we get a surjective map

$$p : M_0 = \mathbb{Z}^3 \rightarrow \hat{G}.$$

There is a map  $\pi : \mathbb{Z}^{Q_1} \rightarrow M_0$ , that maps an arrow  $a_i^\rho$  to the  $i$ -th canonical basis vector of  $M_0 = \mathbb{Z}^3$ . It can be factored through  $\pi : \Lambda \rightarrow M_0$ . Recall that we have defined  $B = \ker(\mathbb{Z}^{Q_0} \rightarrow \mathbb{Z})$ . We define a map  $\nu : B \rightarrow \hat{G}$  to be the composition  $B \hookrightarrow \mathbb{Z}^{Q_0} = \mathbb{Z}^{\hat{G}} \rightarrow \hat{G}$ . It follows from [23, Lemma 10.5]

**Lemma 5.7.** *The following diagram is cartesian and cocartesian*

$$\begin{array}{ccc} \Lambda & \xrightarrow{d} & B \\ \pi \downarrow & & \downarrow \nu \\ M_0 & \xrightarrow{p} & \hat{G} \end{array}$$

**Corollary 5.8.** *We have a commutative diagram with short exact sequences in the rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \Lambda & \xrightarrow{d} & B \longrightarrow 0 \\ & & \parallel & & \pi \downarrow & & \downarrow \nu \\ 0 & \longrightarrow & M & \longrightarrow & M_0 & \xrightarrow{p} & \hat{G} \longrightarrow 0 \end{array}$$

This corollary allows us to interpret all the results of [23], [20] or [10] in the context of brane tilings. For example, let  $P_0 \subset (M_0)_{\mathbb{Q}} = \mathbb{Q}^3$  be a cone generated by the basis vectors. Then  $\mathbb{C}^3 = \text{Spec } \mathbb{C}[P_0 \cap M_0]$  and  $\mathbb{C}^3/G = \text{Spec } \mathbb{C}[P_0 \cap M]$  by the general theory of toric quotients [25, Prop. 3.1] (see also [20, Section 1]). But the last scheme coincides with  $\text{Spec } Z(A) = \text{Spec } \mathbb{C}[P \cap M]$ , where  $A = \mathbb{C}Q/(\partial W)$  and  $P \subset \Lambda_{\mathbb{Q}}$  is a cone defined in Section 3.

**Remark 5.9.** It is proved in [20] that  $\mathbb{C}^3/G$  has a crepant resolution  $\text{Hilb}^G(\mathbb{C}^3)$  – the Hilbert scheme of  $G$ -clusters in  $\mathbb{C}^3$  (see [20, 10]). It follows from [10, Prop. 5.2] that  $\text{Hilb}^G(\mathbb{C}^3)$  can be identified with  $\mathcal{M}_\theta(A, \alpha)$ , where  $A = \mathbb{C}Q/(\partial W)$ ,  $\alpha = (1, \dots, 1) \in \mathbb{Z}^{Q_0}$  and  $\theta \in B$  is such that  $\theta_{\rho_0} < 0$  and  $\theta_\rho > 0$  for  $\rho \neq \rho_0$  ( $\rho_0 \in \hat{G}$  is a trivial representation).

**Remark 5.10.** It was shown in [20] that  $\text{Hilb}^G(\mathbb{C}^3)$  is a toric variety and it was proposed there a way to compute the corresponding fan. In [23, 10] it was shown that  $\mathcal{M}_\theta(A, \alpha)$  is a toric variety for  $\alpha$ -generic  $\theta \in B$  and in [10] an algorithm was given to compute the corresponding fan. This algorithm consists in computing the vertices of the polyhedron  $P^\theta \cap M_\mathbb{Q}$  (this polyhedron defines a toric variety  $\mathcal{M}_\theta$ , see [25] for the general results on toric varieties defined by polyhedra), where  $P^\theta = P - \lambda$  for some  $\lambda \in \Lambda$  with  $d(\lambda) = \theta$ . The results of the previous sections give us a new way to compute this fan by using  $\theta$ -stable perfect matchings on the brane tiling constructed above.

**5.3. Example.** In this example we will describe the toric diagram of  $\text{Hilb}^G(\mathbb{C}^3)$ , where  $G = \mathbb{Z}_6$  and the action on  $\mathbb{C}^3$  is given by  $\frac{1}{6}(1, 2, 3)$ . We have chosen this example as it was also considered by Nakamura [20] using completely different methods.

Let us make first some general remarks. We have an exact sequence

$$0 \rightarrow M \rightarrow M_0 \rightarrow \hat{G} \rightarrow 0.$$

Applying the functor  $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$  we get an exact sequence

$$0 \rightarrow \text{Hom}(M_0, \mathbb{Z}) \rightarrow \text{Hom}(M, \mathbb{Z}) \rightarrow \text{Ext}^1(\hat{G}, \mathbb{Z}) \rightarrow 0.$$

We claim that  $\text{Ext}^1(\hat{G}, \mathbb{Z}) = G$ . The module  $\mathbb{Z}$  over the ring  $\mathbb{Z}$  has an injective resolution

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Using it we get

$$\text{Ext}^1(\hat{G}, \mathbb{Z}) = \text{Hom}(\hat{G}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\hat{G}, \mathbb{C}^*) = G,$$

where we have used an inclusion  $\mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^*$ ,  $x \mapsto e^{2\pi ix}$ . This implies that there is an exact sequence

$$0 \rightarrow M_0^\vee \rightarrow M^\vee \rightarrow G \rightarrow 0.$$

Let  $n = \#G$ . Then, for any  $f \in M^\vee$ , we have  $nf \in M_0^\vee$ . In particular, for any perfect matching  $I \in \mathcal{A}$ , we have  $n\bar{\chi}_I \in M_0^\vee$ . To determine this function, we have to find its values on the basis elements  $e_i \in M_0$ ,  $i = 1, 2, 3$ . Given an arrow  $a_i^\rho$ ,  $\rho \in \hat{G}$ ,  $i = 1, 2, 3$ , we say that it has type  $i$ . We have  $n\bar{\chi}_I(e_i) = \bar{\chi}_I(ne_i)$ , so to find this value, we have to evaluate  $\chi_I$  on any path consisting of  $n$  arrows of type  $i$ . Such a path will not necessarily contain only pairwise different arrows (there are precisely  $n$  different arrows of type  $i$  in quiver  $Q$ ). But one can easily show that the value of  $\chi_I$  on such a path equals the number of arrows of type  $i$  in the perfect matching  $I$ . This gives us an easy way to determine the elements  $\bar{\chi}_I \in (M_0)^\vee = M_0^\vee$ .

Let us return now to the case  $G = \mathbb{Z}_6$  with an action  $\frac{1}{6}(1, 2, 3)$ . We depict first the periodic quiver and the fundamental domain corresponding to the brane tiling constructed earlier, see Figure 1. The corresponding quiver on the torus is a McKay quiver of the  $G$ -representation  $\mathbb{C}^3$ , see Figure 2

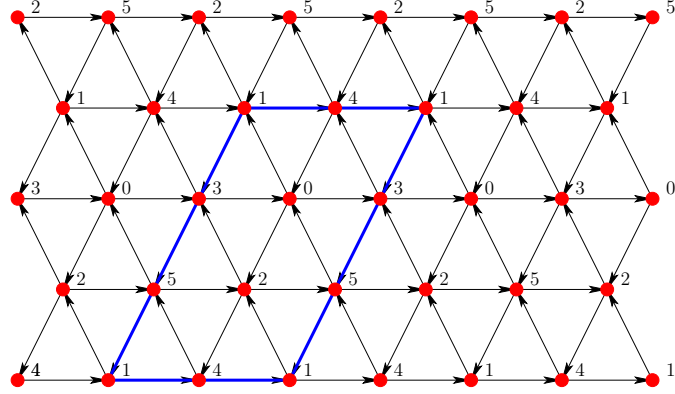


FIGURE 1. Periodic quiver with a fundamental domain

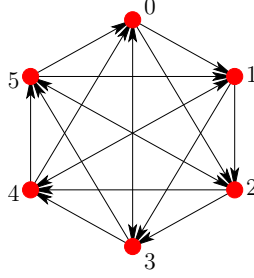


FIGURE 2. McKay quiver

We will denote the arrow from vertex  $i$  to vertex  $j$  by  $ij$ . The type of such arrow equals  $j - i$  if  $j > i$  and  $j - i + 6$  otherwise. The list of all perfect matchings of the brane tiling is given in the Table 1. Every perfect matching is written as a set of arrows from  $Q$ .

**Remark 5.11.** We see that the elements  $\frac{1}{6}(1, 2, 3)$ ,  $\frac{1}{6}(2, 4, 0)$ , etc. are contained in  $M^\vee$ . Let  $g \in G = \mathbb{Z}_6$  be the image of  $\frac{1}{6}(1, 2, 3)$  with respect to the canonical map  $M^\vee \rightarrow G$ . This is a generator of  $G$ . We can see that  $\frac{1}{6}(2, 4, 0) \mapsto g^2$ ,  $\frac{1}{6}(3, 0, 3) \mapsto g^3$ ,  $\frac{1}{6}(4, 2, 0) \mapsto g^4$  (cf. [20]). These elements of  $M^\vee$  will be sometimes denoted by their images in  $G$ . The perfect matchings  $I$  such that  $\bar{\chi}_I = e_i$ ,  $i = 1, 2, 3$ , are precisely the extremal perfect matchings.

We consider now a stability  $\theta \in B$  that corresponds to the resolution  $\text{Hilb}^G(\mathbb{C}^3)$  of  $\mathbb{C}^3/G$ . It should satisfy  $\theta_0 < 0$  and  $\theta_i > 0$  for  $1 \leq i \leq 5$ . A perfect matching  $I \in \mathcal{A}$  is  $\theta$ -stable if and only if one can reach any vertex of  $Q$  from vertex  $0 \in Q_0$  by going only through the arrows of  $Q \setminus I$ . The  $\theta$ -stable perfect matchings are listed in Table 2.

We choose now such pairs of  $\theta$ -stable perfect matchings that their union is still  $\theta$ -stable. All pairs including  $I_{13}$ , except the pair  $\{I_{13}, I_4\}$ , satisfy this condition. This allows us to reconstruct the toric diagram of  $\text{Hilb}^G(\mathbb{C}^3)$ , see Figure 3. This diagram coincides with a toric diagram constructed by Nakamura [20].

$N$	$I$	$6\bar{\chi}_I$
1	34, 01, 02, 35, 24, 51	(2, 4, 0)
2	34, 01, 02, 35, 45, 12	(4, 2, 0)
3	34, 01, 23, 50, 24, 51	(4, 2, 0)
4	34, 01, 23, 50, 45, 12	(6, 0, 0)
5	34, 14, 30, 50, 52, 12	(3, 0, 3)
6	34, 14, 30, 35, 24, 25	(1, 2, 3)
7	13, 40, 02, 35, 24, 51	(0, 6, 0)
8	13, 40, 02, 35, 45, 12	(2, 4, 0)
9	13, 40, 23, 50, 24, 51	(2, 4, 0)
10	13, 40, 23, 50, 45, 12	(4, 2, 0)
11	13, 14, 02, 03, 52, 12	(1, 2, 3)
12	13, 14, 23, 03, 24, 25	(1, 2, 3)
13	41, 40, 30, 50, 52, 51	(1, 2, 3)
14	41, 40, 30, 35, 45, 25	(1, 2, 3)
15	41, 03, 01, 02, 52, 51	(1, 2, 3)
16	41, 03, 01, 23, 45, 25	(3, 0, 3)
17	41, 03, 14, 52, 25, 30	(0, 0, 6)

TABLE 1. Perfect matchings and their coordinates in  $M_{\mathbb{Q}}^{\vee}$

$N$	$I$	$\bar{\chi}_I$
4	34, 01, 23, 50, 45, 12	$e_1 = (1, 0, 0)$
5	34, 14, 30, 50, 52, 12	$g^3 = \frac{1}{6}(3, 0, 3)$
7	13, 40, 02, 35, 24, 51	$e_2 = (0, 1, 0)$
9	13, 40, 23, 50, 24, 51	$g^2 = \frac{1}{6}(2, 4, 0)$
10	13, 40, 23, 50, 45, 12	$g^4 = \frac{1}{6}(4, 2, 0)$
13	41, 40, 30, 50, 52, 51	$g = \frac{1}{6}(1, 2, 3)$
17	41, 03, 14, 52, 25, 30	$e_3 = (0, 0, 1)$

TABLE 2.  $\theta$ -stable perfect matchings

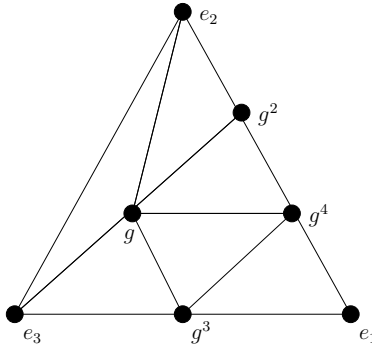


FIGURE 3. Toric diagram of  $\text{Hilb}^{\mathbb{Z}_6}(\mathbb{C}^3)$

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