

Borel–Weil Theory for Groups over Commutative Banach Algebras

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Abstract

Let \mathcal{A} be a commutative unital Banach algebra, \mathfrak{g} be a semisimple complex Lie algebra and $G(\mathcal{A})$ be the 1-connected Banach–Lie group with Lie algebra $\mathfrak{g} \otimes \mathcal{A}$. Then there is a natural concept of a parabolic subgroup $P(\mathcal{A})$ of $G(\mathcal{A})$ and we obtain generalizations $X(\mathcal{A}) := G(\mathcal{A})/P(\mathcal{A})$ of the generalized flag manifolds. In this note we provide an explicit description of all homogeneous holomorphic line bundles over $X(\mathcal{A})$ with non-zero holomorphic sections. In particular, we show that all these line bundles are tensor products of pullbacks of line bundles over $X(\mathbb{C})$ by evaluation maps.

For the special case where \mathcal{A} is a C^* -algebra, our results lead to a complete classification of all irreducible involutive holomorphic representations of $G(\mathcal{A})$ on Hilbert spaces.

Keywords: Banach–Lie group, holomorphic vector bundle, holomorphic section, Borel–Weil Theorem

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1 Introduction

If \mathfrak{g} is a finite dimensional complex semisimple Lie algebra and \mathcal{A} is a unital commutative Banach algebra, then $\mathfrak{g}(\mathcal{A}) := \mathfrak{g} \otimes \mathcal{A}$ carries a natural Banach–Lie algebra structure with respect to the \mathcal{A} -bilinear extension of the bracket. As we shall see below, there always exists a (1-connected) Banach–Lie group $G(\mathcal{A})$ with Lie algebra $\mathfrak{g}(\mathcal{A})$. For any parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, we then obtain a connected Banach–Lie subgroup $P(\mathcal{A})$ with Lie algebra $\mathfrak{p}(\mathcal{A}) := \mathfrak{p} \otimes \mathcal{A} \subseteq \mathfrak{g}(\mathcal{A})$, which leads to the complex homogeneous spaces $X(\mathcal{A}) := G(\mathcal{A})/P(\mathcal{A})$ generalizing the finite dimensional complex flag manifolds.

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In [MNS09] we have studied homogeneous vector bundles over a class of Banach manifolds generalizing those of the form $G(\mathcal{A})/P(\mathcal{A})$. Some of the main results of that paper are that for each holomorphic Banach representation $\rho: P(\mathcal{A}) \rightarrow \mathrm{GL}(E)$, the space of holomorphic sections of the associated bundle $G \times_{\rho} E$ always carries a natural Banach space structure turning it into a holomorphic $G(\mathcal{A})$ -module and that every irreducible holomorphic $G(\mathcal{A})$ -module embeds in such a space of holomorphic sections. These results constitute natural extensions of Borel–Weil theory for finite dimensional reductive complex Lie groups.

In this paper we obtain a complete classification of all homogeneous holomorphic line bundles over $X(\mathcal{A})$ with non-zero holomorphic sections. In particular, we show that all these line bundles are tensor products of pullbacks of line bundles over the finite dimensional compact complex manifold $X(\mathbb{C})$ by evaluation maps $\varphi_{\eta}^X: X(\mathcal{A}) \rightarrow X(\mathbb{C})$ induced by unital algebra homomorphisms $\eta: \mathcal{A} \rightarrow \mathbb{C}$.

If, in addition, \mathcal{A} is a C^* -algebra, then the involution of \mathcal{A} and the Cartan involution on \mathfrak{g} can be combined to an involution on the Banach–Lie algebra $\mathfrak{g}(\mathcal{A})$, which leads to an antiholomorphic involution $*$ on the corresponding Lie group $G(\mathcal{A})$. For these groups the natural class of representations are those holomorphic representations $\pi: G(\mathcal{A}) \rightarrow \mathrm{GL}(\mathcal{H})$ on complex Hilbert spaces \mathcal{H} which are compatible with the involution. On the unitary groups

$$\mathrm{U}(G(\mathcal{A})) := \{g \in G(\mathcal{A}): g^* = g^{-1}\}$$

they restrict to norm continuous unitary representation, from which they can be reconstructed by analytic extension (cf. [Ne98]). We show that all irreducible representations of this kind can be realized in holomorphic homogeneous line bundles over $X(\mathcal{A})$, from which we then derive a complete classification. Surprisingly, it turns out that all the irreducible representations are actually finite dimensional and factor through multi-evaluation maps $G(\mathcal{A}) \rightarrow G(\mathbb{C}^N) \cong G(\mathbb{C})^N$.

The structure of the present paper is as follows. In Section 2 we explain our setup and collect some structural information on the Lie algebras $\mathfrak{g}(\mathcal{A})$. Section 3 is completely independent of our representation theoretic framework. Its main result is Theorem 3.3, asserting that every multiplicative holomorphic map $\chi: \mathcal{A} \rightarrow \mathbb{C}$ on a unital commutative Banach algebra is a finite product of algebra homomorphisms. This observation is the key to our results on the characterization of the holomorphic line bundles $\mathcal{L}_{\chi} \rightarrow X(\mathcal{A})$ associated to holomorphic characters $\chi: P(\mathcal{A}) \rightarrow \mathbb{C}^{\times}$. Theorem 4.2 provides a characterization of those characters χ for which \mathcal{L}_{χ} has non-zero holomorphic sections, as those which are products of characters pulled back from dominant characters of $P(\mathbb{C})$ by homomorphisms $\varphi_{\eta}: P(\mathcal{A}) \rightarrow P(\mathbb{C})$ induced by algebra homomorphisms $\eta: \mathcal{A} \rightarrow \mathbb{C}$. This theorem is proved in Section 5 by showing first that, for the special case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, it follows from Theorem 3.3 and then deriving the general case by applying the \mathfrak{sl}_2 -case to \mathfrak{sl}_2 -subalgebras corresponding to simple roots of \mathfrak{g} . In Section 6 we apply all that to the special case where \mathcal{A} is a unital C^* -algebra. Finally, we show in Section 7 that, in general, the space $\mathcal{O}_{\chi}(G(\mathcal{A}))$ of holomorphic sections of \mathcal{L}_{χ} is not finite-dimensional, although the corresponding line bundle is a pullback of a finite dimensional one.

For $\mathcal{A} = C^k(\mathbb{S}^1)$, $k \in \mathbb{N}_0$, the groups $G(\mathcal{A})$ are variants of complex loop groups. It is well-known that, at least for $k = \infty$, these groups have interesting central extensions with a very rich representation theory (by unbounded operators) (cf. [PS86], [Ne01]), so that our results can also be understood as a contribution to the description of those representations of the centrally extended groups which are trivial on the center.

As a consequence of our main result, the space of holomorphic sections of \mathcal{L}_χ contains a finite dimensional $G(\mathcal{A})$ -invariant subspace whenever it is non-zero. In particular, this leads to a natural class of finite dimensional representations of groups of the type $G(\mathcal{A})$ that deserve the name *evaluation representations*. Finite dimensional representations of Lie algebras of the form $\mathfrak{g} \otimes \mathcal{A}$, \mathcal{A} a unital commutative algebra, are presently under active investigation from an algebraic point of view. In [Se09] one finds a survey on this theory for the case where \mathcal{A} is an algebra of Laurent polynomials. For the larger class of Lie algebras of the form $(\mathfrak{g} \otimes \mathcal{A})^\Gamma$, where \mathcal{A} is the algebra of regular functions on an affine variety and Γ a finite group acting on \mathfrak{g} and \mathcal{A} , the irreducible finite dimensional representations have recently been classified by Neher, Savage and Senesi ([NSS09]).

Also closely related to our setting is the notion of a Weyl module introduced in [CP01]. These are the maximal finite dimensional modules of algebras of the form $\mathfrak{g} \otimes \mathcal{A}$ generated by eigenvectors of $\mathfrak{b} \otimes \mathcal{A}$, where \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . The connection to our context is as follows. For any line bundle \mathcal{L}_χ as above, we can identify its space of holomorphic sections with a certain space $\mathcal{O}_\chi(G(\mathcal{A}))$ of holomorphic functions on $G(\mathcal{A})$, so that the evaluation $\text{ev}_1: \mathcal{O}_\chi(G(\mathcal{A})) \rightarrow \mathbb{C}$ is a morphism of $P(\mathcal{A})$ -modules. In particular, ev_1 can be viewed as a $P(\mathcal{A})$ -eigenvector in the dual $G(\mathcal{A})$ -module $\mathcal{O}_\chi(G(\mathcal{A}))^*$. If $\mathcal{O}_\chi(G(\mathcal{A}))$ is finite dimensional (which is in particular the case if \mathcal{A} is finite dimensional; cf. [MNS09, Cor. 3.9]), then $\mathcal{O}_\chi(G(\mathcal{A}))^*$ is a finite dimensional $G(\mathcal{A})$ -module generated by a $P(\mathcal{A})$ -eigenvector. If the parabolic $P(\mathcal{A})$ is minimal, these dual modules are Weyl modules. Conversely, the description of $\mathcal{O}_\chi(G(\mathcal{A}))$ as coinduced modules on the Lie algebra level in [MNS09, Sec. 2] implies that, whenever one can translate between the analytic and the algebraic setting, Weyl modules can be realized as duals of finite dimensional $G(\mathcal{A})$ -invariant spaces of holomorphic sections of some line bundle \mathcal{L}_χ . For recent results on the structure of Weyl modules we refer to [FoLi07], [FL04].

2 Preliminaries

Let \mathfrak{g} be a finite dimensional complex semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\Delta \subseteq \mathfrak{h}^*$ be the corresponding root system, so that we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

We write $\check{\alpha} \in \mathfrak{h}$ for the coroot associated to $\alpha \in \Delta$, i.e., the unique element $\check{\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\check{\alpha}) = 2$. Fix a positive system Δ^+ , and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ denote the corresponding simple roots.

In the following \mathcal{A} always denotes a complex unital commutative Banach algebra. Then $\mathfrak{g}(\mathcal{A}) := \mathfrak{g} \otimes \mathcal{A}$, equipped with Lie bracket defined by

$$[x_1 \otimes a_1, x_2 \otimes a_2] := [x_1, x_2] \otimes a_1 a_2$$

is a Banach–Lie algebra with respect to the natural tensor product topology, for which $\mathfrak{g}(\mathcal{A}) \cong \mathcal{A}^{\dim \mathfrak{g}}$ as a Banach space. We consider $\mathfrak{g}(\mathcal{A})$ as a Lie algebra over the algebra \mathcal{A} , hence sometimes write $x \otimes a \in \mathfrak{g} \otimes \mathcal{A}$ also as ax . From the \mathfrak{h} -weight space decomposition

$$\mathfrak{g}(\mathcal{A}) = (\mathfrak{h} \otimes \mathcal{A}) \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}_\alpha \otimes \mathcal{A}),$$

we derive that $\mathfrak{g}(\mathcal{A})$ is weakly Δ -graded in the sense of [MNS09, Def. 1.1] because it contains $\mathfrak{g}_\Delta := \mathfrak{g} \otimes \mathbf{1}$ and we have the \mathfrak{h} -weight decomposition from above.

To each subset $\Pi_\Sigma \subseteq \Pi$, we associate a parabolic system of roots, defined by

$$\Sigma := (-\Delta^+) \cup (\Delta \cap \text{span}_{\mathbb{Z}}(\Pi \setminus \Pi_\Sigma)). \quad (1)$$

If $x_\Sigma \in \mathfrak{h}$ is such that

$$\alpha_i(x_\Sigma) = \begin{cases} 0 & \text{for } \alpha_i \notin \Pi_\Sigma \\ -1 & \text{for } \alpha_i \in \Pi_\Sigma, \end{cases}$$

then $\Sigma = \{\alpha \in \Delta \mid \alpha(x_\Sigma) \geq 0\}$. Let $\mathfrak{p} := \mathfrak{p}(\mathbb{C}) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ be the parabolic subalgebra corresponding to Σ . Then $\mathfrak{p}(\mathcal{A}) := \mathfrak{p} \otimes \mathcal{A}$ is called a *parabolic subalgebra* of $\mathfrak{g}(\mathcal{A})$.

The Lie-algebra $\mathfrak{g}(\mathcal{A})$ integrates to a Banach–Lie group. In fact, if we choose some faithful representation $\mathfrak{g} \rightarrow \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{g}(\mathcal{A})$ is a closed subalgebra of the Banach–Lie algebra $\mathfrak{gl}_n(\mathcal{A})$ of $n \times n$ -matrices with entries in \mathcal{A} . This Lie algebra integrates to the Lie group $\text{GL}_n(\mathcal{A})$ of all invertible matrices with entries in \mathcal{A} . Hence $\mathfrak{g}(\mathcal{A})$ is a closed Lie subalgebra of a Lie algebra of a linear Lie group, and therefore integrates to a Lie group ([Mais62]). Let $G(\mathcal{A})$, resp., $G(\mathbb{C})$ be simply connected Banach–Lie groups with Lie algebras $\mathfrak{g}(\mathcal{A})$, resp., \mathfrak{g} , and define Lie subgroups $P(\mathcal{A})$, resp., $P(\mathbb{C})$ as the connected subgroups with Lie algebras $\mathfrak{p}(\mathcal{A})$, resp., $\mathfrak{p}(\mathbb{C}) = \mathfrak{p}$.

Remark 2.1 (a) The Lie algebra \mathfrak{p} is a semidirect sum $\mathfrak{p} = \mathfrak{u} \ltimes \mathfrak{l}$, where

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha(x_\Sigma)=0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u} = \bigoplus_{\alpha(x_\Sigma)>0} \mathfrak{g}_\alpha.$$

Moreover, the subalgebra \mathfrak{l} is a semidirect sum $\mathfrak{l} = \mathfrak{c} \ltimes \mathfrak{s}$, where

$$\mathfrak{c} := \text{span } \check{\Pi}_\Sigma \quad \text{and} \quad \mathfrak{s} := [\mathfrak{l}, \mathfrak{l}] = \left(\bigoplus_{\alpha(x_\Sigma)=0} \mathbb{C}\check{\alpha} \right) \oplus \bigoplus_{\alpha(x_\Sigma)=0} \mathfrak{g}_\alpha.$$

Let $U := \exp \mathfrak{u}$ and $L := N_{P(\mathbb{C})}(\mathfrak{l}) \cap N_{P(\mathbb{C})}(\mathfrak{u})$. Then the multiplication map $L \times U \rightarrow P(\mathbb{C})$, $(l, u) \mapsto lu$, is a holomorphic isomorphism. In particular, L is connected because $P(\mathbb{C})$ is connected.

(b) We define the groups C and S as the integral subgroups of L with Lie algebras \mathfrak{c} and \mathfrak{s} , respectively. Let H_S be the integral subgroup of S with Lie algebra $\mathfrak{h}_S := \mathfrak{h} \cap \mathfrak{s}$. Then the integral subgroup H with Lie algebra \mathfrak{h} satisfies

$$H \cong \mathfrak{h} / \ker(\exp|_{\mathfrak{h}}) = \mathfrak{h} / 2\pi i \mathbb{Z}[\check{\Delta}] \cong (\mathbb{C}^\times)^r,$$

and this implies that the multiplication map $C \times H_S \rightarrow H$ is an isomorphism of abelian complex Lie groups. The product group $C \cdot S \subseteq G$, being the integral subgroup with Lie algebra $\mathfrak{c} \oplus \mathfrak{s}$, equals L . Hence the multiplication map $C \times S \rightarrow L$ is surjective. To see that it is also injective, we note that its kernel is discrete and normal, hence contained in the center $Z(C \times S) \subseteq C \times H_S$. The injectivity now follows since the restriction of the multiplication to $C \times H_S$ is injective.

Let $\mathfrak{n}(\mathcal{A}) := \sum_{\alpha \in \Delta \setminus \Sigma} \mathfrak{g}_\alpha \otimes \mathcal{A}$, and let $N(\mathcal{A}) := \exp(\mathfrak{n}(\mathcal{A}))$ be the corresponding integral subgroup of $G(\mathcal{A})$. Recall from [MNS09, Prop. 1.11] that the multiplication map

$$N(\mathcal{A}) \times P(\mathcal{A}) \rightarrow G(\mathcal{A}), \quad (n, p) \mapsto np \tag{2}$$

is biholomorphic onto an open subset. It follows that $P(\mathcal{A})$ is a complemented Lie subgroup, so that the quotient space $X(\mathcal{A}) := G(\mathcal{A})/P(\mathcal{A})$ is a complex Banach manifold and the projection map $\pi_{\mathcal{A}} : G(\mathcal{A}) \rightarrow X(\mathcal{A})$ is a holomorphic submersion defining a holomorphic $P(\mathcal{A})$ -principal bundle.

3 Multiplicative holomorphic functions

In this section we are concerned with holomorphic functions $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ on a commutative unital Banach algebra which are multiplicative, i.e., $\varphi(ab) = \varphi(a)\varphi(b)$ for $a, b \in \mathcal{A}$. Clearly, every algebra homomorphism has this property, and so does every finite product of algebra homomorphisms. The main result of this section (Theorem 3.3) asserts the converse, namely that any such φ is a finite product of algebra homomorphisms.

We start with a simple algebraic observation.

Lemma 3.1 *Let Γ be a finite group, and let $\sigma : \Gamma \rightarrow \text{Aut}(\mathcal{A})$ be a representation of Γ as automorphisms of a unital algebra \mathcal{A} over a field of characteristic zero. Let \mathcal{A}^Γ denote the subalgebra of Γ -invariants. Then every proper left ideal I in \mathcal{A}^Γ generates a proper left ideal in \mathcal{A} .*

Proof. Since the representation σ of Γ on \mathcal{A} is locally finite, we can decompose \mathcal{A} as a finite direct sum $\mathcal{A} = \bigoplus_{\tau \in \hat{\Gamma}} \mathcal{A}_\tau$, where $\hat{\Gamma}$ denotes the set of irreducible representations of Γ , and \mathcal{A}_τ is the τ -isotypic component, i.e., the sum of all irreducible subrepresentations of σ which are equivalent to τ . Observe that \mathcal{A}^Γ is the isotypic component of the trivial representation. For any

$a_0 \in \mathcal{A}^\Gamma$, the map $x \mapsto a_0x$ is Γ -equivariant, so that $\mathcal{A}^\Gamma \mathcal{A}_\tau \subseteq \mathcal{A}_\tau$ for any $\tau \in \widehat{\Gamma}$. Assume now that the left ideal $\mathcal{A}I$ generated by I is not proper, i.e., that $\mathbf{1} \in \mathcal{A}I = \bigoplus_\tau \mathcal{A}_\tau I$. Since $\mathbf{1} \in \mathcal{A}^\Gamma$, it follows that $\mathbf{1} \in \mathcal{A}^\Gamma I \subseteq I$, which contradicts that I is a proper ideal. Hence $\mathcal{A}I$ is a proper ideal. ■

Proposition 3.2 *If \mathcal{A} is a unital commutative complex Banach algebra and $\Gamma \subseteq \text{Aut}(\mathcal{A})$ a finite subgroup, then any algebra homomorphism $\varphi: \mathcal{A}^\Gamma \rightarrow \mathbb{C}$ extends to an algebra homomorphism $\tilde{\varphi}: \mathcal{A} \rightarrow \mathbb{C}$ and any such homomorphism is continuous.*

Proof. The kernel $I := \ker \varphi$ is a proper ideal in the subalgebra \mathcal{A}^Γ , so that the preceding lemma implies that I is contained in a proper ideal of \mathcal{A} . In particular, it is contained in a maximal ideal of \mathcal{A} , so that the Gelfand–Mazur Theorem implies the existence of a (continuous) homomorphism $\tilde{\varphi}: \mathcal{A} \rightarrow \mathbb{C}$ with $\ker \tilde{\varphi} \cap \mathcal{A}^\Gamma = \ker \varphi$ ([Ru91, Thm. 11.5]). Since $\mathcal{A}^\Gamma = I \oplus \mathbb{C}\mathbf{1}$, it follows that $\tilde{\varphi}$ extends φ . Its continuity follows from [Ru91, Thm. 11.10]. ■

Theorem 3.3 *Let \mathcal{A} be a unital commutative Banach algebra and $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ be a holomorphic character of the multiplicative semigroup (\mathcal{A}, \cdot) . Then there exist finitely many continuous algebra homomorphisms $\chi_1, \dots, \chi_n: \mathcal{A} \rightarrow \mathbb{C}$ such that*

$$\varphi = \chi_1 \cdots \chi_n. \quad (3)$$

Proof. We first claim that φ is a homogeneous polynomial of some degree n , i.e., there exists a symmetric n -linear map $\tilde{\varphi}: \mathcal{A}^n \rightarrow \mathbb{C}$ with $\varphi(a) = \tilde{\varphi}(a, \dots, a)$ for every $a \in \mathcal{A}$. Indeed, the map $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \varphi(z\mathbf{1})$, is holomorphic and multiplicative, hence of the form $z \mapsto z^n$ for some $n \in \mathbb{N}_0$. From this, we get the homogeneity condition

$$\varphi(za) = z^n \varphi(a) \quad \text{for } z \in \mathbb{C}, a \in \mathcal{A}. \quad (4)$$

On the other hand, we have a power series expansion

$$\varphi = \sum_{k=0}^{\infty} \varphi_k \quad (5)$$

at the origin, where $\varphi_k: \mathcal{A} \rightarrow \mathbb{C}$ is a homogeneous polynomial of degree k . Comparison of (4) and (5) yields $\varphi = \varphi_n$. We can thus write $\varphi(a) = \tilde{\varphi}(a, \dots, a)$ for a continuous n -linear map $\tilde{\varphi}: \mathcal{A}^n \rightarrow \mathbb{C}$.

Now let $\mathcal{A}^{\otimes n}$ denote the projective n -fold tensor product of \mathcal{A} , which is the completion of the algebraic tensor product with respect to the maximal cross norm. It has the universal property that continuous linear maps $\mathcal{A}^{\otimes n} \rightarrow X$ to a Banach space are in one-to-one correspondence with continuous n -linear maps $\mathcal{A}^n \rightarrow X$. From the universal property and the associativity of projective tensor products it easily follows that $\mathcal{A}^{\otimes n}$ carries a natural unital commutative Banach algebra structure, determined by

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_n) := a_1 b_1 \otimes \cdots \otimes a_n b_n.$$

The symmetric group S_n acts by automorphisms on $\mathcal{A}^{\otimes n}$ by the continuous linear extensions of the maps which permute the tensor factors, i.e.

$$\sigma(a_1 \otimes \cdots \otimes a_n) := a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}, \quad \sigma \in S_n.$$

The fixed point algebra $S^n(\mathcal{A}) := (\mathcal{A}^{\otimes n})^{S_n}$ also is a unital Banach algebra. It is topologically generated by tensors of the form

$$a_1 \vee \cdots \vee a_n := \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

and by polarization it is actually generated by the diagonal elements $a \vee \cdots \vee a = a \otimes \cdots \otimes a$.

With the universal property of the Banach space $S^n(\mathcal{A})$, we find a continuous linear map $\psi: S^n(\mathcal{A}) \rightarrow \mathbb{C}$ with $\tilde{\varphi}(a, \dots, a) = \psi(a \otimes \cdots \otimes a)$ for $a \in \mathcal{A}$. For the diagonal generators of $S^n(\mathcal{A})$ we now have

$$\begin{aligned} \psi((a \otimes \cdots \otimes a)(b \otimes \cdots \otimes b)) &= \psi(ab \otimes \cdots \otimes ab) = \varphi(ab) = \varphi(a)\varphi(b) \\ &= \psi(a \otimes \cdots \otimes a)\psi(b \otimes \cdots \otimes b). \end{aligned}$$

From the linearity of ψ and its multiplicativity on a set of topological linear generators of the Banach algebra $S^n(\mathcal{A})$, it now follows that ψ is an algebra homomorphism. By Proposition 3.2, we can extend ψ to an algebra homomorphism $\chi: \mathcal{A}^{\otimes n} \rightarrow \mathbb{C}$. Then

$$\begin{aligned} \varphi(a) &= \chi(a \otimes \cdots \otimes a) \\ &= \chi(a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})\chi(\mathbf{1} \otimes a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) \cdots \chi(\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes a) \\ &= \chi_1(a) \cdots \chi_n(a), \end{aligned}$$

where $\chi_i: \mathcal{A} \rightarrow \mathbb{C}$ is the character

$$\chi_i(a) := \chi(\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})$$

with a occurring as the i th factor. Since every character of \mathcal{A} is automatically continuous ([Ru91, Thm. 11.10]), this proves the theorem. \blacksquare

Remark 3.4 The preceding theorem basically asserts that for the commutative Banach algebra $S^n(\mathcal{A})$, the natural map

$$(\mathcal{A}^{\otimes n})^{\wedge} \cong \widehat{\mathcal{A}}^n \rightarrow S^n(\mathcal{A})^{\wedge}$$

is surjective, and it is not hard to see that this leads to a topological isomorphism

$$S^n(\mathcal{A})^{\wedge} \cong \widehat{\mathcal{A}}^n / S_n,$$

where the symmetric group S_n acts on $\widehat{\mathcal{A}}^n$ by permutations.

It is an interesting problem to find a non-commutative analog of this result (for C^* -algebras) (cf. [Ar87] for some results pointing in this direction).

Remark 3.5 There is an interesting algebraic version of the preceding theorem which may also be of interest in other contexts. Let \mathcal{A} be a finitely generated unital commutative algebra over an algebraically closed field \mathbb{K} of characteristic zero and $\varphi: \mathcal{A} \rightarrow \mathbb{K}$ a non-zero multiplicative polynomial map. Then there exists an $n \in \mathbb{N}_0$ with $\varphi(z\mathbf{1}) = z^n$ for all $z \in \mathbb{K}$ and algebra homomorphisms $\chi_1, \dots, \chi_n: \mathcal{A} \rightarrow \mathbb{K}$ with $\varphi = \prod_{i=1}^n \chi_i$.

To verify this claim, we first consider the polynomial $\mathbb{K} \rightarrow \mathbb{K}, z \mapsto \varphi(z\mathbf{1})$. Since it is multiplicative and non-zero, it maps \mathbb{K}^\times into \mathbb{K}^\times , so that it has no zero in \mathbb{K}^\times . This implies that $\varphi(z\mathbf{1}) = z^n$ for some $n \in \mathbb{N}_0$ and all $z \in \mathbb{K}$. We conclude that φ is a homogeneous polynomial of degree n .

Following the same line of argument as above, we have to extend a homomorphism $\psi: S^n(\mathcal{A}) \rightarrow \mathbb{K}$ to an algebra homomorphism $\mathcal{A}^{\otimes n} \rightarrow \mathbb{K}$. In view of Lemma 3.1, this reduces to the problem to show that every maximal ideal J of $\mathcal{A}^{\otimes n}$ has the property that $\mathcal{A}^{\otimes n}/J \cong \mathbb{K}$. Since \mathcal{A} is assumed to be finitely generated, the same holds for the quotient field of $\mathcal{A}^{\otimes n}$, so that it is a quotient of some polynomial ring $\mathbb{K}[x_1, \dots, x_N]$. Therefore the assertion follows from Hilbert's Nullstellensatz.

4 Homogeneous line bundles

Let $\chi: P(\mathcal{A}) \rightarrow \mathbb{C}^\times$ be a holomorphic character. We define the associated holomorphic homogeneous line bundle

$$\mathcal{L}_\chi := (G(\mathcal{A}) \times \mathbb{C})/P(\mathcal{A}) := G(\mathcal{A}) \times_\chi \mathbb{C}$$

and write its elements as $[g, v]$, which are the orbits for the $P(\mathcal{A})$ -action on $G(\mathcal{A}) \times \mathbb{C}$ by $p.(g, v) := (gp^{-1}, \chi(p)v)$. We identify the space of holomorphic sections of \mathcal{L}_χ with

$$\begin{aligned} \mathcal{O}_\chi(G(\mathcal{A})) \\ := \{f \in \mathcal{O}(G(\mathcal{A})) : (\forall g \in G(\mathcal{A}))(\forall p \in P(\mathcal{A})) f(gp) = \chi(p)^{-1}f(g)\} \end{aligned}$$

by assigning to $f \in \mathcal{O}_\chi(G(\mathcal{A}))$ the section defined by $s_f(gP(\mathcal{A})) := [g, f(g)]$ (cf. [MNS09]).

Let $\widehat{\mathcal{A}}$ denote the space of all unital continuous algebra homomorphisms $\mathcal{A} \rightarrow \mathbb{C}$. We recall that this is a compact space with respect to the weak-* topology on the topological dual space \mathcal{A}' , and that the Gelfand transform

$$\mathcal{G}: \mathcal{A} \rightarrow C(\widehat{\mathcal{A}}), \quad a \mapsto \widehat{a}, \quad \widehat{a}(\eta) := \eta(a)$$

is a homomorphism of Banach algebras (cf. [Ru91, Ch. 11]).

For any $\eta \in \widehat{\mathcal{A}}$, we obtain a homomorphism of Lie algebras

$$\varphi_\eta: \mathfrak{g}(\mathcal{A}) \rightarrow \mathfrak{g}, \quad x \otimes a \mapsto \eta(a)x,$$

and, since $G(\mathcal{A})$ is 1-connected, it integrates to a holomorphic homomorphism of complex Banach–Lie groups

$$\varphi_\eta^G: G(\mathcal{A}) \rightarrow G(\mathbb{C}).$$

From $\varphi_\eta(\mathfrak{p}(\mathcal{A})) \subseteq \mathfrak{p}(\mathbb{C})$, we derive $\varphi_\eta^G(P(\mathcal{A})) \subseteq P(\mathbb{C})$ because $P(\mathcal{A})$ is connected by definition. Since the quotient map $\pi_{\mathcal{A}}: G(\mathcal{A}) \rightarrow X(\mathcal{A})$ is a holomorphic submersion, φ_η^G thus induces a holomorphic map

$$\varphi_\eta^X : X(\mathcal{A}) = G(\mathcal{A})/P(\mathcal{A}) \rightarrow X(\mathbb{C}) = G(\mathbb{C})/P(\mathbb{C})$$

such that the diagram

$$\begin{array}{ccc} G(\mathcal{A}) & \xrightarrow{\varphi_\eta^G} & G(\mathbb{C}) \\ \downarrow \pi_{\mathcal{A}} & & \downarrow \pi_{\mathbb{C}} \\ X(\mathcal{A}) & \xrightarrow{\varphi_\eta^X} & X(\mathbb{C}) \end{array}$$

commutes.

Remark 4.1 If $\xi : P(\mathbb{C}) \rightarrow \mathbb{C}$ is a holomorphic character, then by the isomorphism $P(\mathbb{C}) \cong U \rtimes L \cong U \rtimes (S \rtimes C)$ (cf. Remark 2.1), and the fact that S is connected, semisimple, and $\mathfrak{u} \subseteq [\mathfrak{p}(\mathbb{C}), \mathfrak{p}(\mathbb{C})]$, it follows that ξ is uniquely determined by its restriction to the subgroup C . Hence the group $\widehat{P}(\mathbb{C})$ of holomorphic characters $P(\mathbb{C}) \rightarrow \mathbb{C}^\times$ is generated by the characters of the form ξ_α , where $L(\xi_\alpha)|_{\mathfrak{h}} = \omega_\alpha \in \mathfrak{h}^*$, $\alpha \in \Pi_\Sigma$, is the fundamental weight with $\omega_\alpha(\check{\beta}) = \delta_{\alpha, \beta}$ for $\beta \in \Pi$. For each $\xi \in \widehat{P}(\mathbb{C})$ we thus obtain

$$\xi = \prod_{\alpha \in \Pi_\Sigma} \xi_\alpha^{L(\xi)(\check{\alpha})}$$

and ξ is dominant if and only if $L(\xi)(\check{\alpha}) \geq 0$ holds for each $\alpha \in \Pi_\Sigma$, which in turn implies $L(\xi)(\check{\alpha}) \geq 0$ for each $\alpha \in \Delta^+$.

According to the classical Borel–Weil Theorem, in our convention (1) for the positive system, ξ is dominant if and only if the holomorphic line bundle \mathcal{L}_ξ over $X(\mathbb{C})$ has non-zero holomorphic sections.

Assume now that χ is the pullback of a holomorphic character ξ of $P(\mathbb{C})$ with respect to $\varphi_\eta^G|_{P(\mathcal{A})} : P(\mathcal{A}) \rightarrow P(\mathbb{C})$, i.e., it is of the form $\chi = \xi \circ \varphi_\eta^G|_{P(\mathcal{A})}$. Then the corresponding line bundle \mathcal{L}_χ is the pullback of the line bundle \mathcal{L}_ξ over $X(\mathbb{C})$ with respect to φ_η^X . If ξ is dominant, we can produce holomorphic sections of \mathcal{L}_χ by pulling back holomorphic sections to \mathcal{L}_ξ . This proves one half of the following theorem.

Theorem 4.2 *Let $G(\mathcal{A})$ be a 1-connected Banach–Lie group with Lie algebra $\mathfrak{g}(\mathcal{A})$, $P(\mathcal{A})$ a connected parabolic subgroup of $G(\mathcal{A})$, and $\chi : P(\mathcal{A}) \rightarrow \mathbb{C}^\times$ be a holomorphic character. Then the line bundle \mathcal{L}_χ over $X(\mathcal{A}) = G(\mathcal{A})/P(\mathcal{A})$ has nonzero global holomorphic sections if and only if there exist $\eta_1, \dots, \eta_m \in \widehat{A}$ and fundamental holomorphic characters ξ_1, \dots, ξ_m of $P(\mathbb{C})$ such that*

$$\chi = \prod_{j=1}^m (\varphi_{\eta_j}^G)^* \xi_j. \quad (6)$$

This implies in particular that \mathcal{L}_χ is the tensor product of line bundles of the form $(\varphi_{\eta_j}^X)^ \mathcal{L}_{\xi_j}$, where $L(\xi_j)$ is a fundamental weight of \mathfrak{g} .*

Remark 4.3 (a) If the group $G(\mathcal{A})$ is connected, but not simply connected, then the preceding theorem applies to the universal covering group $q: \widetilde{G}(\mathcal{A}) \rightarrow G(\mathcal{A})$. If $\widetilde{P}(\mathcal{A})$, resp., $P(\mathcal{A})$, denote the connected subgroups of $\widetilde{G}(\mathcal{A})$, resp., $G(\mathcal{A})$ with Lie algebra $\mathfrak{p}(\mathcal{A})$, then we derive that $\mathcal{O}_\chi(G(\mathcal{A})) \neq \{0\}$ implies that the character $\widetilde{\chi} := q^*\chi: \widetilde{P}(\mathcal{A}) \rightarrow \mathbb{C}^\times$ is a product

$$\widetilde{\chi} = \prod_{j=1}^m (\varphi_{\eta_j}^G)^* \xi_j. \quad (7)$$

Since, in general, an algebra homomorphism $\eta_j: \mathcal{A} \rightarrow \mathbb{C}$ does not lead to a group homomorphism $G(\mathcal{A}) \rightarrow G(\mathbb{C})$, we have to face the difficulty to express the information directly with respect to the group $G(\mathcal{A})$.

(b) However, if $G(\mathcal{A})$ is functorially attached to \mathcal{A} , such as the groups $\mathrm{SL}_n(\mathcal{A})_0$, $\mathrm{Sp}_{2n}(\mathcal{A})_0$ or $\mathrm{SO}_n(\mathcal{A})_0$, every algebra homomorphism $\eta: \mathcal{A} \rightarrow \mathbb{C}$ induces a morphism of Banach–Lie groups $\varphi_\eta^G: G(\mathcal{A}) \rightarrow G(\mathbb{C})$, regardless of whether $G(\mathcal{A})$ is simply connected or not. Then (7) implies that

$$q^*\chi = \widetilde{\chi} = q^* \prod_{j=1}^m (\varphi_{\eta_j}^G)^* \xi_j,$$

which immediately leads to

$$\chi = \prod_{j=1}^m (\varphi_{\eta_j}^G)^* \xi_j \quad (8)$$

because $q: \widetilde{P}(\mathcal{A}) \rightarrow P(\mathcal{A})$ is surjective.

Remark 4.4 The preceding theorem implies in particular that if \mathcal{L}_χ has non-zero holomorphic sections, then it is a tensor product of pullbacks of finite dimensional line bundles over $X(\mathbb{C})$. Accordingly, the products of the pullbacks of the finite dimensional spaces of holomorphic sections of these line bundles over $X(\mathbb{C})$ form a finite dimensional non-zero G -invariant subspace of $\mathcal{O}_\chi(G(\mathcal{A}))$. The results in [MNS09] imply that this subspace is contained in every closed $G(\mathcal{A})$ -invariant subspace. However, the $G(\mathcal{A})$ -module structure on the Banach space $\mathcal{O}_\chi(G(\mathcal{A}))$ is far from being semisimple. As we shall see in Section 7 below, the space $\mathcal{O}_\chi(G(\mathcal{A}))$ can be infinite dimensional, although it contains a finite dimensional minimal non-zero subspace.

5 Proof of Theorem 4.2

5.1 The \mathfrak{sl}_2 -case

In this section we consider the special case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g}(\mathcal{A}) = \mathfrak{sl}_2(\mathcal{A})$, where $G(\mathcal{A}) := \widetilde{\mathrm{SL}}_2(\mathcal{A})_0$ is the simply connected covering group of the identity component $\mathrm{SL}_2(\mathcal{A})_0$ of $\mathrm{SL}_2(\mathcal{A})$. We consider the parabolic subalgebra \mathfrak{p} of upper triangular matrices in $\mathfrak{sl}_2(\mathbb{C})$, and put

$$\mathfrak{h} = \mathbb{C}\check{\alpha}, \quad \check{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta^+ = \{-\alpha\}, \quad \text{and} \quad \mathfrak{u} = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Definition 5.1 For $z \in \mathcal{A}^\times$, we define $\tilde{h}(z) \in G(\mathcal{A})$ by

$$\tilde{h}(z) := \exp \begin{pmatrix} 0 & 0 \\ z^{-1} - 1 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 \\ z - 1 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & -z^{-1} \\ 0 & 0 \end{pmatrix},$$

and observe that $\tilde{h}(\mathbf{1}) = \mathbf{1}$. If $q: G(\mathcal{A}) \rightarrow \mathrm{SL}_2(\mathcal{A})_0$ is the universal covering map with $L(q) = \mathrm{id}_{\mathrm{sl}_2(\mathcal{A})}$, then

$$h(z) := q(\tilde{h}(z)) = \begin{pmatrix} 1 & 0 \\ z^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

Lemma 5.2 For each $a \in \mathcal{A}$, we have $\tilde{h}(\exp_{\mathcal{A}} a) = \exp \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, where

$$\exp_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}^\times, \quad x \mapsto e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the exponential function of the Banach-Lie group \mathcal{A}^\times .

Proof. From $q(\tilde{h}(\exp_{\mathcal{A}} a)) = h(\exp_{\mathcal{A}} a) = \exp_{\mathrm{SL}_2(\mathcal{A})} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, we derive that $\tilde{h} \circ \exp_{\mathcal{A}}: \mathcal{A} \rightarrow G(\mathcal{A})$ is the unique continuous lift of the map

$$\mathcal{A} \rightarrow \mathrm{SL}_2(\mathcal{A}), \quad a \mapsto h(\exp_{\mathcal{A}}(a)) = \exp_{\mathrm{SL}_2(\mathcal{A})}(a\check{\alpha})$$

satisfying $\tilde{h}(\exp(0)) = \mathbf{1}$. Since $a \mapsto \exp(a\check{\alpha})$ is another lift with this property, the uniqueness of lifts implies the assertion. \blacksquare

Proposition 5.3 Let $\chi: P(\mathcal{A}) \rightarrow \mathbb{C}^\times$ be a holomorphic character and observe that it defines a holomorphic character

$$\chi_{\mathcal{A}}: (\mathcal{A}, +) \rightarrow \mathbb{C}^\times, \quad a \mapsto \chi(\exp(a\check{\alpha}))^{-1}. \quad (9)$$

If the line bundle \mathcal{L}_χ over $G(\mathcal{A})/P(\mathcal{A})$ admits nonzero holomorphic sections, then $\chi_{\mathcal{A}}$ vanishes on the kernel $\ker(\exp_{\mathcal{A}})$ of the exponential function

$$\exp_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{A}_0^\times,$$

and induces a holomorphic character $\bar{\chi}_{\mathcal{A}}: \mathcal{A}_0^\times \rightarrow \mathbb{C}^\times$ which extends to a holomorphic character $(\mathcal{A}, \cdot) \rightarrow \mathbb{C}$ of the multiplicative semigroup (\mathcal{A}, \cdot) .

Proof. We identify the space of holomorphic sections of \mathcal{L}_χ with the space $\mathcal{O}_\chi(G(\mathcal{A}))$ of holomorphic functions $f: G(\mathcal{A}) \rightarrow \mathbb{C}$ which are equivariant for $P(\mathcal{A})$ in the sense that $f(gp) = \chi(p)^{-1}f(g)$ holds for $g \in G(\mathcal{A})$ and $p \in P(\mathcal{A})$.

If this space is nonzero, then [MNS09, Thm. 3.7] implies the existence of an $\mathfrak{n}(\mathcal{A})$ -invariant function $f \in \mathcal{O}_\chi(G(\mathcal{A}))$ with $f(\mathbf{1}) = 1$. This implies that f is $N(\mathcal{A})$ -left invariant and hence in particular $f(N(\mathcal{A})) = \{1\}$. Next we note that χ vanishes on $U(\mathcal{A})$ since $\mathfrak{u}(\mathcal{A}) \subseteq [\mathfrak{p}(\mathcal{A}), \mathfrak{p}(\mathcal{A})]$, so that f is also $U(\mathcal{A})$ -right invariant. Therefore

$$f(\tilde{h}(z)) = f\left(\exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & 0 \\ z-1 & 0 \end{pmatrix}\right),$$

and the right hand side defines a holomorphic function on \mathcal{A} . On the hand, Lemma 5.2 implies that, for $a \in \mathcal{A}$,

$$f(\tilde{h}(\exp_{\mathcal{A}} a)) = f(\exp(a\check{\alpha})) = \chi(\exp(a\check{\alpha}))^{-1} = \chi_{\mathcal{A}}(a).$$

First, this proves that $\ker \exp_{\mathcal{A}} \subseteq \ker \chi_{\mathcal{A}}$, so that $\chi_{\mathcal{A}}$ factors through a holomorphic character $\bar{\chi}_{\mathcal{A}}: \mathcal{A}_0^\times = \exp_{\mathcal{A}}(\mathcal{A}) \rightarrow \mathbb{C}^\times$ with $\bar{\chi}_{\mathcal{A}} \circ \exp_{\mathcal{A}} = \chi_{\mathcal{A}}$. For $z \in \mathcal{A}_0^\times$, we now have $\bar{\chi}_{\mathcal{A}}(z) = f(\tilde{h}(z))$, and we have just seen that this function extends to a holomorphic function on all of \mathcal{A} . Since this function is multiplicative on all pairs in the open subset \mathcal{A}_0^\times , it follows by analytic continuation that it is multiplicative. \blacksquare

We can now prove Theorem 4.2 for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Proof. (of Theorem 4.2 for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$) If \mathcal{L}_χ has nonzero holomorphic sections, then the preceding proposition implies the existence of a multiplicative holomorphic function $\bar{\chi}_{\mathcal{A}}: \mathcal{A} \rightarrow \mathbb{C}$ with

$$\chi(\exp(a\check{\alpha}))^{-1} = \bar{\chi}_{\mathcal{A}}(\exp_{\mathcal{A}} a) \quad \text{for } a \in \mathcal{A}.$$

Theorem 3.3 now implies that $\bar{\chi}_{\mathcal{A}} = \eta_1 \cdots \eta_n$ for algebra homomorphisms $\eta_j: \mathcal{A} \rightarrow \mathbb{C}$, and this implies that

$$\chi(\exp(a\check{\alpha}))^{-1} = \prod_{j=1}^n \eta_j(\exp_{\mathcal{A}} a) \quad \text{for } a \in \mathcal{A}.$$

With the dominant fundamental character (with respect to $\Delta^+ = \{-\alpha\}$)

$$\xi: P(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C} \right\} \rightarrow \mathbb{C}^\times, \quad \xi\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) := a^{-1},$$

we now obtain

$$\begin{aligned} \chi(\exp(a\check{\alpha})) &= \prod_{j=1}^n \eta_j(\exp_{\mathcal{A}} a)^{-1} = \prod_{j=1}^n e^{-\eta_j(a)} = \prod_{j=1}^n \xi(\exp_{\mathrm{PSL}_2(\mathbb{C})} \eta_j(a)\check{\alpha}) \\ &= \prod_{j=1}^n ((\varphi_{\eta_j}^G)^* \xi)(\exp(a\check{\alpha})). \end{aligned}$$

As $P(\mathcal{A}) = \exp(\mathcal{A}\check{\alpha})U(\mathcal{A})$ and χ vanishes on $U(\mathcal{A})$, this implies that

$$\chi = \prod_{j=1}^n (\varphi_{\eta_j}^G)^* \xi. \quad (10)$$

The character ξ satisfies $L(\xi)(-\check{\alpha}) = 1$, hence it is dominant and fundamental. The corresponding line bundle \mathcal{L}_ξ then admits nonzero holomorphic sections by the classical Borel–Weil Theorem, and this implies that the line bundle $\mathcal{L}_\chi \cong \otimes_{j=1}^n (\varphi_{\eta_j}^G)^* \mathcal{L}_\xi$ has non-zero holomorphic sections. ■

Remark 5.4 The line bundle \mathcal{L}_ξ is the bundle of hyperplane sections over the Riemann sphere. Its space of holomorphic sections is the two-dimensional fundamental representation of $\mathrm{SL}_2(\mathbb{C})$ on the dual space of \mathbb{C}^2 .

5.2 The general case

Proof. (of Theorem 4.2) We have already seen that (6) is sufficient for the existence of non-zero holomorphic sections. We now prove that it is also necessary.

For any simple root $\alpha \in \Pi$, consider the $\mathfrak{sl}_2(\mathbb{C})$ -subalgebra

$$\mathfrak{g}^\alpha := \mathbb{C}\check{\alpha} + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} \subseteq \mathfrak{g} \quad \text{and} \quad \mathfrak{p}^\alpha := \mathbb{C}\check{\alpha} + \mathfrak{g}_\alpha.$$

For the corresponding 1-connected Banach–Lie groups, we then have morphisms

$$\gamma_\alpha^G: G^\alpha(\mathcal{A}) \rightarrow G(\mathcal{A})$$

integrating the inclusion maps $\mathfrak{g}^\alpha(\mathcal{A}) \hookrightarrow \mathfrak{g}(\mathcal{A})$. Clearly, $\gamma_\alpha^G(P^\alpha(\mathcal{A})) \subseteq P(\mathcal{A})$ holds for the corresponding connected parabolic subgroups $P^\alpha(\mathcal{A}) \subseteq G^\alpha(\mathcal{A})$.

If $\mathcal{O}_\chi(G(\mathcal{A})) \neq \{0\}$, then, in view of the left invariance of this space, we also have that

$$\{0\} \neq \gamma_\alpha^* \mathcal{O}_\chi(G(\mathcal{A})) \subseteq \mathcal{O}_{\chi^\alpha}(G^\alpha(\mathcal{A})), \quad \text{where} \quad \chi^\alpha := \chi \circ \gamma_\alpha^G|_{P^\alpha(\mathcal{A})}.$$

From the \mathfrak{sl}_2 -case we thus obtain algebra homomorphisms $\eta_1^\alpha, \dots, \eta_{n_\alpha}^\alpha \in \widehat{\mathcal{A}}$ with

$$\chi^\alpha = \prod_{j=1}^{n_\alpha} (\varphi_{\eta_j^\alpha})^* \xi,$$

where $\varphi_{\eta_j^\alpha}: G^\alpha(\mathcal{A}) \rightarrow \mathrm{SL}_2(\mathbb{C})$ is the corresponding evaluation homomorphism.

If $\iota_\alpha: G^\alpha(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \rightarrow G(\mathbb{C})$ is the homomorphism integrating the inclusion $\mathfrak{g}^\alpha \rightarrow \mathfrak{g}$, then the corresponding fundamental weight $\xi_\alpha \in \widehat{P}(\mathbb{C})$ satisfies $\iota_\alpha^* \xi_\alpha = \xi$ because $L(\xi)(\check{\alpha}) = -1$. With the evaluation homomorphisms $\varphi_{\eta_j^\alpha}^G = \iota_\alpha \circ \varphi_{\eta_j^\alpha}: G^\alpha(\mathcal{A}) \rightarrow G(\mathbb{C})$, we thus obtain

$$\chi^\alpha = \prod_{j=1}^{n_\alpha} (\varphi_{\eta_j^\alpha}^G)^* \xi_\alpha.$$

Since

$$\mathfrak{p}(\mathcal{A}) = [\mathfrak{p}(\mathcal{A}), \mathfrak{p}(\mathcal{A})] \oplus \bigoplus_{\alpha \in \Pi_{\Sigma}} \mathcal{A}\check{\alpha}$$

(cf. Remark 2.1), the restrictions to the subgroups $P^{\alpha}(\mathcal{A})$ of $P(\mathcal{A})$, $\alpha \in \Pi_{\Sigma}$, determine the character χ via

$$\chi\left(\prod_{\alpha \in \Pi_{\Sigma}} \exp(a_{\alpha}\check{\alpha})\right) = \prod_{\alpha \in \Pi_{\Sigma}} \chi^{\alpha}(\exp(a_{\alpha}\check{\alpha})) = \prod_{\alpha \in \Pi_{\Sigma}} \prod_{j=1}^{n_{\alpha}} \chi_j(\exp a_{\alpha})^{-1}$$

(cf. Remark 4.1). We conclude that χ is a product of pullbacks of dominant fundamental characters ξ_{α} by certain evaluation homomorphisms φ_{χ}^G , and this completes the proof. \blacksquare

6 The case of C^* -algebras

Theorem 3.3 provides a complete classification of all homogeneous holomorphic line bundles over the spaces $X(\mathcal{A})$ with non-zero holomorphic sections. The main reason for these bundles playing a role in the classical context $\mathcal{A} = \mathbb{C}$ is that the Borel–Weil Theorem asserts that the corresponding spaces of holomorphic sections are always irreducible and that every irreducible finite dimensional holomorphic representation of $G(\mathbb{C})$ can be realized in this way if $P(\mathbb{C})$ is a Borel subgroup, i.e., $\Pi_{\Sigma} = \Pi$.

For general commutative Banach algebras, one cannot expect such a sharp picture, as the examples discussed in [NS09] show. Here the main source of the lacking semisimplicity of the representations lies in the algebra, the simplest examples arising for the two-dimensional algebra $\mathcal{A} = \mathbb{C}[\varepsilon]$ of dual numbers, where $\varepsilon^2 = 0$. As we know from the Gelfand theory of commutative Banach algebras, commutative C^* -algebras are the prototype of commutative Banach algebras, and any semisimple commutative Banach algebra \mathcal{A} embeds continuously into the C^* -algebra $C(\widehat{\mathcal{A}})$ by the Gelfand transform.

In this section we therefore study the special case where \mathcal{A} is a commutative unital C^* -algebra. Let $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ be an involutive antilinear automorphism satisfying $\sigma(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{-\alpha}$ for each root α and observe that this implies that $\sigma(\mathfrak{h}) = \mathfrak{h}$ and

$$\sigma(\check{\alpha}) = -\check{\alpha} \quad \text{for } \alpha \in \Delta.$$

We combine σ with the involution $a \mapsto a^*$ of \mathcal{A} to an antilinear map

$$*: \mathfrak{g}(\mathcal{A}) \rightarrow \mathfrak{g}(\mathcal{A}), \quad x \otimes a \mapsto -\sigma(x) \otimes a^*,$$

satisfying

$$(x^*)^* = x \quad \text{and} \quad [x, y]^* = [y^*, x^*] \quad \text{for } x, y \in \mathfrak{g}(\mathcal{A}).$$

Thus $(\mathfrak{g}(\mathcal{A}), *)$ is an *involutive Banach–Lie algebra*. In view of the Gelfand isomorphism, we have $\mathcal{A} \cong C(\widehat{\mathcal{A}})$ and, accordingly, $\mathfrak{g}(\mathcal{A}) \cong C(\widehat{\mathcal{A}}, \mathfrak{g})$ with $f^*(\chi) =$

$-\sigma(f(\chi))$. Since $G(\mathcal{A})$ was assumed to be simply connected, there exists an antiholomorphic involution $g \mapsto g^*$ on $G(\mathcal{A})$ satisfying

$$(gh)^* = h^*g^* \quad \text{and} \quad (\exp x)^* = \exp(x^*) \quad \text{for} \quad g, h \in G(\mathcal{A}), x \in \mathfrak{g}(\mathcal{A}).$$

A *holomorphic involutive representation* of $(G(\mathcal{A}), *)$ is a pair (π, \mathcal{H}) consisting of a complex Hilbert space \mathcal{H} and a holomorphic homomorphism $\pi: G(\mathcal{A}) \rightarrow \mathrm{GL}(\mathcal{H})$ which is compatible with the involutions in the sense that

$$\pi(g^*) = \pi(g)^* \quad \text{for} \quad g \in G(\mathcal{A}).$$

Such a representation is said to be *irreducible* if \mathcal{H} contains no non-trivial $G(\mathcal{A})$ -invariant closed subspace. We write $\mathfrak{d}\pi: \mathfrak{g}(\mathcal{A}) \rightarrow B(\mathcal{H})$ for the derived representation of $\mathfrak{g}(\mathcal{A})$ by bounded operators on \mathcal{H} .

Assume that (π, \mathcal{H}) is an irreducible involutive holomorphic representation of $G(\mathcal{A})$ and $P(\mathcal{A})$ is a connected parabolic subgroup. Then [MNS09, Thm. 5.1] implies that $E := \mathcal{H}/\overline{\mathfrak{u}(\mathcal{A})}\mathcal{H}$ carries an irreducible holomorphic representation ρ of $P(\mathcal{A})$ with $U(\mathcal{A}) \subseteq \ker \rho$, and the quotient map $\beta: \mathcal{H} \rightarrow E$ leads to an inclusion of holomorphic $G(\mathcal{A})$ -representations

$$\beta_G: \mathcal{H} \rightarrow \mathcal{O}_\rho(G(\mathcal{A}), E), \quad \beta_G(v)(g) := \beta(\pi(g)^{-1}v),$$

where

$$\begin{aligned} & \mathcal{O}_\rho(G(\mathcal{A}), E) \\ & := \{f \in \mathcal{O}(G(\mathcal{A}), E) : (\forall g \in G(\mathcal{A}))(\forall p \in P(\mathcal{A})) f(gp) = \rho(p)^{-1}f(g)\} \end{aligned}$$

corresponds to the space of holomorphic sections of the associated holomorphic vector bundle $G(\mathcal{A}) \times_\rho E$ over $X(\mathcal{A})$. The closure of the image of β_G is the unique minimal closed $G(\mathcal{A})$ -invariant subspace of $\mathcal{O}_\rho(G(\mathcal{A}), E)$.

From the construction of the involution $*$ on $\mathfrak{g}(\mathcal{A})$, we immediately derive that $\mathfrak{u}(\mathcal{A})^* = \mathfrak{n}(\mathcal{A})$ and $\mathfrak{l}(\mathcal{A})^* = \mathfrak{l}(\mathcal{A})$. In particular, the subgroup $L(\mathcal{A}) \subseteq G(\mathcal{A})$ is $*$ -invariant, hence also carries the structure of a complex involutive Banach-Lie group. Since the subspace $\mathfrak{u}(\mathcal{A})\mathcal{H}$ of \mathcal{H} is $L(\mathcal{A})$ -invariant and $L(\mathcal{A})^* = L(\mathcal{A})$, the orthogonal complement $E \cong (\mathfrak{u}(\mathcal{A})\mathcal{H})^\perp$ is also $L(\mathcal{A})$ -invariant, so that the representation (ρ, E) of $L(\mathcal{A})$ actually is involutive.

If, in addition, $P(\mathcal{A})$ is minimal parabolic, then $\mathfrak{l} = \mathfrak{h}$ shows that $\mathfrak{l}(\mathcal{A})$ is abelian, so that Schur's Lemma implies that $\dim E = 1$, and thus $\rho: P(\mathcal{A}) \rightarrow \mathrm{GL}(E) \cong \mathbb{C}^\times$ is a holomorphic character and the results developed above apply. In particular, Remark 4.4 implies that the minimal $G(\mathcal{A})$ -invariant subspace of $\mathcal{O}_\rho(G(\mathcal{A}), E)$ is finite-dimensional, so that $\dim \mathcal{H} < \infty$. Further, the representation of $G(\mathcal{A})$ on the minimal submodule factors through some evaluation homomorphism

$$(\varphi_{\eta_1}^G, \dots, \varphi_{\eta_m}^G): G(\mathcal{A}) \rightarrow G(\mathbb{C}^m) \cong G(\mathbb{C})^m,$$

where $\rho = \prod_{j=1}^m (\varphi_{\eta_j}^G)^* \xi_j$. To see how this information can be made compatible with the involution, we note that $\ker \pi \trianglelefteq \mathfrak{g}(\mathcal{A})$ is a $*$ -invariant ideal. Hence it

is in particular \mathfrak{g} -invariant, and since $\mathfrak{g}(\mathcal{A}) \cong \mathfrak{g} \otimes \mathcal{A}$ is an isotypical semisimple \mathfrak{g} -module, the closed \mathfrak{g} -submodule $\ker \pi$ is of the form $\ker \pi = \mathfrak{g} \otimes I$, where $I \subseteq \mathcal{A}$ is a closed $*$ -invariant subspace. As $\ker \pi$ is an ideal, the relation

$$[x \otimes a, \ker \pi] = [x, \mathfrak{g}] \otimes aI$$

implies that $I \trianglelefteq \mathcal{A}$ is an ideal. As $\mathfrak{d}\pi(\mathfrak{g}(\mathcal{A}))$ is finite dimensional, \mathcal{A}/I is a finite dimensional C^* -algebra, so that the Gelfand Representation Theorem implies that $\mathcal{A}/I \cong \mathbb{C}^N$ as C^* -algebras. We conclude that

$$\pi(\mathfrak{g}(\mathcal{A})) \cong \mathfrak{g} \otimes \mathbb{C}^N \cong \mathfrak{g}^N,$$

as involutive Lie algebras. Since $(\mathfrak{g}, -\sigma)$ has an involutive Hilbert representation, the fixed point algebra \mathfrak{g}^σ is compact, so that σ actually is a Cartan involution and \mathfrak{g}^σ is a compact real form.

We collect the result of the preceding discussion in the following theorem.

Theorem 6.1 *Let $(\mathcal{A}, *)$ be a commutative C^* -algebra, $\sigma \in \text{Aut}(\mathfrak{g})$ be an involutive automorphism with $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for each $\alpha \in \Delta$ and define an involution on $\mathfrak{g}(\mathcal{A})$ by $(x \otimes a)^* := -\sigma(x) \otimes a^*$. Let $(G(\mathcal{A}), *)$ be the 1-connected involutive Banach–Lie group corresponding to $(\mathfrak{g}(\mathcal{A}), *)$. Then every irreducible involutive representation (π, \mathcal{H}) of $G(\mathcal{A})$ is finite dimensional and factors through an involutive surjective multi-evaluation homomorphism*

$$\varphi_\eta^G: G(\mathcal{A}) \rightarrow G(\mathbb{C})^N, \quad g \mapsto (\varphi_{\eta_1}^G(g), \dots, \varphi_{\eta_N}^G(g)).$$

Remark 6.2 From the preceding theorem we can now easily derive a description of all irreducible involutive representations of $G(\mathcal{A})$. Since every finite dimensional involutive representation is a direct sum of irreducible ones, this implies a classification of all finite dimensional ones.

As every irreducible involutive representation factors through an involutive representation of some group $G(\mathbb{C})^N$, the classification problem reduces to a description of all irreducible holomorphic involutive representations of this group. In view of Weyl’s Unitary Trick, this is equivalent to the classification of irreducible unitary representations of the maximal compact subgroup

$$\{g = (g_1, \dots, g_N) \in G(\mathbb{C})^N : (\forall j) \sigma_G(g_j) = g_j\} \cong K^N,$$

where σ_G is the antiholomorphic involution of $G(\mathbb{C})$ with $L(\sigma_G) = \sigma$ and $K := G(\mathbb{C})^\sigma$. As all representations of this product group are tensor products of irreducible representations of the factor groups K , their classification follows from the Cartan–Weyl classification in terms of highest weight modules of $\mathfrak{g}(\mathbb{C})$.

Remark 6.3 (a) The preceding discussion applies in particular to the universal covering $G(\mathcal{A}) = \widetilde{\text{SL}}_n(\mathcal{A})_0$ of the group $\text{SL}_n(\mathcal{A})_0$ for a commutative unital C^* -algebra, if we take $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then the factorization of the representation through some $G(\mathbb{C})^N = \text{SL}_n(\mathbb{C})^N$ even implies that all irreducible involutive

representations of $G(\mathcal{A})$ factor through $\mathrm{SL}_n(\mathcal{A})_0$ because the evaluation homomorphisms to $\mathrm{SL}_n(\mathbb{C})^N$ have this property.

(b) The techniques developed above apply to groups of the form $G(\mathcal{A})$, where $\mathfrak{g}(\mathcal{A}) = \mathfrak{g} \otimes \mathcal{A}$ and \mathfrak{g} is semisimple. If we want to extend the results to reductive Lie algebras \mathfrak{g} , we observe that, in this case,

$$\mathfrak{g}(\mathcal{A}) = (\mathfrak{z}(\mathfrak{g}) \otimes \mathcal{A}) \oplus \mathfrak{g}'(\mathcal{A}),$$

where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the commutator algebra and $\mathfrak{z}(\mathfrak{g}) \otimes \mathcal{A} = \mathfrak{z}(\mathfrak{g}(\mathcal{A}))$ is central.

In view of Schur's Lemma, all irreducible involutive holomorphic representations (π, \mathcal{H}) of

$$G(\mathcal{A}) \cong Z(G(\mathcal{A}))_0 \times G'(\mathcal{A})$$

have the property that $\pi(Z(G(\mathcal{A}))) \subseteq \mathbb{C}^\times \mathbf{1}$, so that $\pi|_{G'(\mathcal{A})}$ is also irreducible. Therefore the classification in the reductive case splits into the classification in the semisimple case and the description of the holomorphic characters of

$$Z(G(\mathcal{A}))_0 \cong \mathfrak{z}(\mathfrak{g}) \otimes \mathcal{A} = \mathfrak{z}(\mathfrak{g}(\mathcal{A})),$$

which can be identified with the elements in the dual space $\mathfrak{z}(\mathfrak{g})^* \otimes \mathcal{A}'$, which are invariant under the canonical extension of $*$.

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ we have $\mathfrak{z}(\mathfrak{g}) = \mathbb{C}$, so that the holomorphic characters of $Z(G(\mathcal{A}))_0$ correspond to arbitrary hermitian functionals $\alpha: \mathcal{A} \rightarrow \mathbb{C}$. In particular, these functionals do not have to factor through evaluation maps. A typical example is the Riemann integral $I(f) = \int_0^1 f(x) dx$ on the commutative C^* -algebra $\mathcal{A} = C([0, 1])$.

7 An infinite dimensional space of sections

We have seen in Theorem 4.2 that, whenever the space $\mathcal{O}_\chi(G(\mathcal{A}))$ is nonzero for a holomorphic characters $\chi: P(\mathcal{A}) \rightarrow \mathbb{C}^\times$, then the corresponding line bundle $\mathcal{L}_\chi \rightarrow X(\mathcal{A})$ is a product of pullbacks of line bundles $\mathcal{L}_\xi \rightarrow X(\mathbb{C})$. Since the corresponding space $\mathcal{O}_\xi(G(\mathbb{C}))$ of holomorphic sections is finite dimensional, we obtain a $G(\mathcal{A})$ -invariant non-zero finite dimensional subspace of $\mathcal{O}_\chi(G(\mathcal{A}))$. In this section we describe an example where the space $\mathcal{O}_\chi(G(\mathcal{A}))$ is infinite dimensional.

Throughout this section, we fix a commutative unital Banach algebra \mathcal{A} and a unital algebra homomorphism $\eta: \mathcal{A} \rightarrow \mathbb{C}$. Let $\mathcal{I} := \ker \eta$. As we shall see below, an important ingredient in our construction is the space $T_\eta(\mathcal{A}) := \mathcal{I}/\overline{\mathcal{I}^2}$, and we shall assume that this space is infinite dimensional.

Examples 7.1 (a) The simplest examples where $T_\eta(\mathcal{A})$ is infinite dimensional arises as follows. For a Banach space E , we consider the unital Banach algebra

$$\mathcal{A} = E \oplus \mathbb{C} \quad \text{with} \quad (v, \lambda)(w, \mu) := (\lambda w + \mu v, \lambda \mu),$$

and the homomorphism $\eta: \mathcal{A} \rightarrow \mathbb{C}, \eta(v, \lambda) := \lambda$. Then $\mathcal{I} = E$ and $\mathcal{I}^2 = \{0\}$, so that $T_\eta(\mathcal{A}) \cong \mathcal{I}$ is infinite dimensional if E has this property.

(b) An example which reminds more of an algebra of functions can be constructed as follows. Let E be a Banach space and \mathcal{A} be the algebra of all analytic functions $f = \sum_{n=0}^{\infty} f_n$ (f_n homogeneous of degree n) with

$$\|f\| := \sum_{n=0}^{\infty} \|f_n\| < \infty,$$

where $\|f_n\| := \sup_{\|v\| \leq 1} \|f_n(v)\|$, so that \mathcal{A} can be considered as an algebra of functions on the closed unit ball of E . It is easy to verify that \mathcal{A} is a Banach algebra with respect to pointwise multiplication. Further $\eta(f) := f(0) = f_0$ defines a continuous homomorphism to \mathbb{C} and

$$\mathcal{I} = \ker \eta = \{f \in \mathcal{A} : f_0 = 0\}$$

implies that

$$\mathcal{I}^2 \subseteq \{f \in \mathcal{A} : f_0 = f_1 = 0\}.$$

In particular, we obtain an injection $E' \hookrightarrow T_\eta(\mathcal{A})$, so that this space is infinite dimensional if E is.

(c) Another class of examples can be produced by considering for a unital commutative Banach algebra \mathcal{B} the Banach algebra

$$\mathcal{A} := \left\{ f = \sum_{n=0}^{\infty} t^n f_n \in \mathcal{B}[[t]] : \sum_{n=0}^{\infty} \|f_n\| < \infty, f_0 \in \mathbb{C}\mathbf{1} \right\}$$

of formal \mathcal{B} -valued power series converging absolutely for $|t| \leq 1$. Then $\eta(f) := f(0) \in \mathbb{C}$ defines a homomorphism $\eta: \mathcal{A} \rightarrow \mathbb{C}$, for which

$$\mathcal{I} = \left\{ f \in \mathcal{A} : f = \sum_{n=1}^{\infty} t^n f_n \right\}, \quad \mathcal{I}^2 = \left\{ f \in \mathcal{A} : f = \sum_{n=2}^{\infty} t^n f_n \right\},$$

so that $T_\eta(\mathcal{A}) \cong \mathcal{B}$, as a Banach space.

Definition 7.2 Let E be a Banach space. A continuous linear functional $\delta: \mathcal{A} \rightarrow E$ is called an η -derivation if

$$\delta(ab) = \eta(a)\delta(b) + \eta(b)\delta(a) \quad \text{for } a, b \in \mathcal{A}. \quad (11)$$

We write $\text{Der}_\eta(\mathcal{A}, E)$ for the space of all η -derivations on \mathcal{A} .

Remark 7.3 Clearly, for $\delta \in \text{Der}_\eta(\mathcal{A}, E)$, the relation $\delta(\mathbf{1}) = 2\delta(\mathbf{1})$ leads to $\delta(\mathbf{1}) = 0$, so that δ is determined by its values on the hyperplane ideal \mathcal{I} and (11) further implies that $\delta(\mathcal{I}^2) = \{0\}$. Using the relation

$$ab - \eta(ab)\mathbf{1} \in \eta(a)(b - \eta(b)\mathbf{1}) + \eta(b)(a - \eta(a)\mathbf{1}) + \mathcal{I}^2,$$

it is easy to see that, conversely, every continuous linear map $\alpha \in \mathcal{I} \rightarrow E$ vanishing on \mathcal{I}^2 defines an η -derivation via

$$\delta(a) := \alpha(a - \eta(a)\mathbf{1}).$$

This leads to an isomorphism of Banach spaces

$$\text{Der}_\eta(\mathcal{A}, E) \cong B(T_\eta(\mathcal{A}), E) \cong \{\alpha \in B(\mathcal{I}, E) : \mathcal{I}^2 \subseteq \ker \alpha\}.$$

In this sense the map

$$\delta_u : \mathcal{A} \rightarrow T_\eta(\mathcal{A}), \quad a \mapsto [a - \eta(a)\mathbf{1}] := (a - \eta(a)\mathbf{1}) + \overline{\mathcal{I}^2}$$

is the universal η -derivation; every other η -derivation factors uniquely through δ_u .

For the construction of examples where $\mathcal{O}_\chi(G(\mathcal{A}))$ is infinite dimensional, we shall focus on the case where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and χ is the square of a pullback character, i.e., $m = 2$ in the notation of Theorem 4.2, resp., (10):

$$\chi = (\varphi_\eta^G)^* \xi^2.$$

In this case the representation of $\text{SL}_2(\mathbb{C})$ in the space $\mathcal{O}_{\xi^2}(\text{SL}_2(\mathbb{C}))$ is equivalent to the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$, so that the pullback by φ_η^G leads to an injection

$$\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathcal{O}_\chi(G(\mathcal{A})),$$

where $G(\mathcal{A})$, resp., the quotient group $\text{SL}_2(\mathcal{A})$ acts on this space by

$$g.x := \text{Ad}(\varphi_\eta^G(g))x.$$

Accordingly, we write $\mathfrak{sl}_2(\mathbb{C})_\eta$ for the $G(\mathcal{A})$ -module $\mathfrak{sl}_2(\mathbb{C})$, endowed with this action. Using [MNS09, Prop. 2.13], we see that the quotient module $\mathcal{O}_\chi(G(\mathcal{A}))/\mathfrak{sl}_2(\mathbb{C})_\eta$ is trivial because the only eigenvalue of $\tilde{\alpha}$ on this space is zero. We take this as a motivation to study $\mathfrak{g}(\mathcal{A})$ -modules V which are extensions of a trivial module W by $\mathfrak{sl}_2(\mathbb{C})_\eta$:

$$\mathbf{0} \rightarrow \mathfrak{sl}_2(\mathbb{C})_\eta \hookrightarrow V \twoheadrightarrow W \rightarrow \mathbf{0}.$$

As $\mathfrak{sl}_2(\mathbb{C})$ is finite dimensional, the Hahn–Banach Theorem implies that any such module can be written as a direct sum $V = \mathfrak{sl}_2(\mathbb{C})_\eta \oplus W$ of Banach spaces, and the action is given by

$$(x \otimes a).(y, w) = (\eta(a)[x, y] + f(w)(x \otimes a), 0),$$

where

$$f : W \rightarrow Z^1(\mathfrak{sl}_2(\mathcal{A}), \mathfrak{sl}_2(\mathbb{C})_\eta)$$

is a continuous linear map with values in the Banach space of 1-cocycles on $\mathfrak{sl}_2(\mathcal{A})$ with values in $\mathfrak{sl}_2(\mathbb{C})$. We therefore have to analyze the space $Z^1(\mathfrak{sl}_2(\mathcal{A}), \mathfrak{sl}_2(\mathbb{C})_\eta)$. Averaging over the compact group corresponding to the subalgebra $\mathfrak{su}_2(\mathbb{C}) \subseteq \mathfrak{sl}_2(\mathbb{C}) \subseteq \mathfrak{sl}_2(\mathcal{A})$, it follows that every cocycle $\beta \in Z^1(\mathfrak{sl}_2(\mathcal{A}), \mathfrak{sl}_2(\mathbb{C})_\eta)$ is cohomologous to an $\mathfrak{su}_2(\mathbb{C})$ -equivariant one, and by complex linearity, it is even $\mathfrak{sl}_2(\mathbb{C})$ -equivariant. Since $\mathfrak{sl}_2(\mathbb{C})$ is a simple $\mathfrak{sl}_2(\mathbb{C})$ -module, the description of $\mathfrak{sl}_2(\mathcal{A})$ as $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{A}$ exhibits \mathcal{A} as a multiplicity

space, and we conclude that any $\mathfrak{sl}_2(\mathbb{C})$ -equivariant map $\mathfrak{sl}_2(\mathcal{A}) \rightarrow \mathfrak{sl}_2(\mathbb{C})$ is of the form

$$\beta(x \otimes a) = d(a)x$$

for a continuous linear map $d: \mathcal{A} \rightarrow \mathbb{C}$. Now

$$\beta([x \otimes a, y \otimes b]) = \beta([x, y] \otimes ab) = d(ab)[x, y]$$

and

$$(x \otimes a).\beta(y \otimes b) = (x \otimes a)d(b)y = \eta(a)d(b)[x, y]$$

imply that β is a 1-cocycle if and only if d is an η -derivation. This shows that

$$Z^1(\mathfrak{sl}_2(\mathcal{A}), \mathfrak{sl}_2(\mathbb{C})_\eta)^{\mathfrak{sl}_2(\mathbb{C})} \cong \text{Der}_\eta(\mathcal{A}, \mathbb{C}).$$

We therefore consider the Banach space

$$V := \mathfrak{sl}_2(\mathbb{C})_\eta \oplus T_\eta(\mathcal{A})' \cong \mathfrak{sl}_2(\mathbb{C})_\eta \oplus \text{Der}_\eta(\mathcal{A}, \mathbb{C})$$

and observe that

$$(x \otimes a).(y, \alpha) := (\eta(a)[x, y] + \alpha(a)x, 0)$$

defines a continuous action of $\mathfrak{sl}_2(\mathcal{A})$ on V because $f(\alpha)(x \otimes a) := \alpha(a)x$ defines a continuous linear map

$$f: \text{Der}_\eta(\mathcal{A}, \mathbb{C}) \rightarrow Z^1(\mathfrak{sl}_2(\mathcal{A}), \mathfrak{sl}_2(\mathbb{C})).$$

Since the group $G(\mathcal{A}) = \widetilde{\text{SL}}_2(\mathcal{A})_0$ is simply connected, this $\mathfrak{sl}_2(\mathcal{A})$ -module structure integrates to a holomorphic representation (π, V) of $G(\mathcal{A})$. It remains to show that V injects into the space $\mathcal{O}_\chi(G(\mathcal{A}))$ for $\chi = (\varphi_\eta^G)^*(\xi^2)$.

In the following we recall the basis

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{C})$ with the relations

$$[e, f] = h, \quad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f.$$

We write (h^*, e^*, f^*) for the corresponding dual basis of $\mathfrak{sl}_2(\mathbb{C})^*$ and recall the fundamental character

$$\xi: P(\mathbb{C}) \rightarrow \mathbb{C}^\times, \quad \xi\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) := a^{-1}.$$

Lemma 7.4 *For $\chi = (\varphi_\eta^G)^*(\xi^2)$, the continuous linear functional $\mu \in V'$, defined by*

$$\mu(x, \alpha) := f^*(x)$$

defines an embedding

$$\mu_G: V \rightarrow \mathcal{O}_\chi(G), \quad \mu_G(v)(g) := \mu(g^{-1}.v)$$

of holomorphic Banach $G(\mathcal{A})$ -modules.

Proof. In view of [MNS09, Thm. A.6], we have to show that μ is $P(\mathcal{A})$ equivariant if the action on \mathbb{C} is defined by χ , and that μ is $G(\mathcal{A})$ -cyclic, i.e., that $\mu(G(\mathcal{A}).v) = \{0\}$ implies $v = 0$.

First we verify the equivariance. For $x \otimes a \in \mathfrak{p}(\mathcal{A})$ with $x = \gamma h + \beta e$, we have

$$\begin{aligned}\mu((x \otimes a).(y, \alpha)) &= \mu(\eta(a)[x, y] + \alpha(a)x, 0) = \eta(a)f^*([x, y]) = \gamma\eta(a)f^*([h, y]) \\ &= -2\gamma\eta(a)f^*(y) = -2\gamma\eta(a)\mu(y, \alpha) = 2\mathbf{L}(\xi)(x)\eta(a)\mu(y, \alpha) \\ &= \mathbf{L}(\chi)(x \otimes a)\mu(y, \alpha),\end{aligned}$$

so that the connectedness of $P(\mathcal{A})$ implies the equivariance of μ .

Next we show that μ is cyclic. Let $v = (y, \alpha) \in V$ be such that $\mu(G(\mathcal{A}).v) = \{0\}$. By taking first order derivatives, we obtain $\mu(\mathfrak{sl}_2(\mathcal{A}).v) = \{0\}$, which implies in particular that, for each $a \in \mathcal{A}$, we have

$$\begin{aligned}0 &= \mu(f \otimes a.(y, \alpha)) = \mu(\eta(a)[f, y] + \alpha(a)f, 0) \\ &= \eta(a)f^*([f, y]) + \alpha(a)f^*(f) = \eta(a)f^*([f, y]) + \alpha(a).\end{aligned}$$

For $a = \mathbf{1}$ this leads to $f^*([f, y]) = 0$, so that $y \in \text{span}\{f, e\}$. For $a \in \mathcal{I}$ we further obtain $\alpha = 0$. Now $0 = \mu(y, \alpha) = f^*(y)$ implies that $y = \lambda e$ for some $\lambda \in \mathbb{C}$. We also have

$$\begin{aligned}0 &= \mu((f \otimes a)^2.(y, 0)) = \eta(a)\mu(\eta(a)[f, [f, y]], 0) = \eta(a)^2 f^*([f, [f, y]]) \\ &= \lambda\eta(a)^2 f^*([f, -h]) = (-2)\lambda\eta(a)^2,\end{aligned}$$

which leads to $\lambda = 0$, and hence to $(y, \alpha) = 0$. This proves that μ is cyclic. \blacksquare

The preceding lemma implies in particular that $\mathcal{O}_\chi(G(\mathcal{A}))$ is infinite dimensional if $T_\eta(\mathcal{A})$ has this property. This completes the proof of:

Proposition 7.5 *Let \mathcal{A} be a unital commutative Banach algebra and $\chi \in \widehat{\mathcal{A}}$ with kernel \mathcal{I} such that $T_\chi(\mathcal{A}) = \mathcal{I}/\overline{\mathcal{I}^2}$ is infinite dimensional. Let $G := \widetilde{\text{SL}}_2(\mathcal{A})_0$ and $\varphi_G^\eta: G \rightarrow \text{SL}_2(\mathbb{C})$ denote the group homomorphism induced by η . Then, for the holomorphic character*

$$\xi\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) := a^{-1}$$

of $P(\mathbb{C})$, and the pullback of its square $\chi := (\varphi_\eta^G)^(\xi^2): P(\mathcal{A}) \rightarrow \mathbb{C}^\times$, the Banach space $\mathcal{O}_\chi(G)$ is infinite dimensional.*

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