

# Unitary highest weight modules of locally affine Lie algebras

Karl-Hermann Neeb

**ABSTRACT.** Locally affine Lie algebras are generalizations of affine Kac–Moody algebras with Cartan subalgebras of infinite rank whose root system is locally affine. In this note we study a class of representations of locally affine algebras generalizing integrable highest weight modules. In particular, we construct such an integrable representation for each integral weight not vanishing on the center and show that, over the complex numbers, we thus obtain unitary representations w.r.t. a unitary real form.

We also use Yoshii’s recent classification of locally affine root systems to derive a classification of so-called minimal locally affine Lie algebras and give realizations as twisted loop algebras.

## Introduction

It is an important feature of the integrable highest weight modules  $L(\lambda)$  of a Kac–Moody Lie algebra  $\mathfrak{g}(A)$ , that the set  $\mathcal{P}_\lambda$  of weights can be described as

$$(0.1) \quad \mathcal{P}_\lambda = \text{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}),$$

where  $\mathcal{Q} = \text{span}_{\mathbb{Z}} \Delta$  is the root lattice. This description implies in particular that  $\text{conv}(\mathcal{P}_\lambda) = \text{conv}(\mathcal{W}\lambda)$  and that each weight  $\lambda$  is an extreme point of this set. Over the field  $\mathbb{C}$  of complex numbers, the Lie algebra  $\mathfrak{g}(A)$  has a natural antilinear involution defining a so-called *unitary real form*  $\mathfrak{k}(A)$ , and if  $A$  is symmetrizable, each integrable highest weight module  $L(\lambda)$  is unitary in the sense that it carries a positive definite  $\mathfrak{k}(A)$ -invariant hermitian form ([Ka90]). If  $A$  is of affine (or finite) type, then one even has natural realizations of these unitary modules in spaces of holomorphic sections of holomorphic line bundles over certain coadjoint orbits with Kähler structures (cf. [PS86], [Ne01], where this is discussed for loop groups).

On the other hand, split locally finite Lie algebras  $(\mathfrak{g}, \mathfrak{h})$  (i.e.,  $\mathfrak{g}$  has a root decomposition with respect to  $\mathfrak{h}$ ) also have a natural class of representations fitting into this scheme. In this context the situation becomes more tricky because one cannot choose a positive system, resp., a set of simple roots, a priori, but for each integral weight  $\lambda$  there exists a positive system for which  $\lambda$  is dominant. As we have seen in [Ne98], this also leads to integrable highest weight modules  $L(\lambda)$  which are uniquely determined by their weight set  $\mathcal{P}_\lambda$ , given by (0.1). In the complex case, they are unitary with respect to a unitary real form of the Lie algebra  $\mathfrak{g}$ . In

---

2000 *Mathematics Subject Classification.* Primary 17B70, Secondary 17B10.

particular, these modules do not depend on the positive system with respect to which they are dominant, and their equivalence classes are parameterized by the set  $\mathcal{P}/\mathcal{W}$  of Weyl group orbits in the set  $\mathcal{P}$  of integral weights (which in finite dimensions is usually identified with the set of dominant integral weights with respect to some positive system  $\Delta^+$ ). Under suitable boundedness conditions on the weight, the realization of these modules in holomorphic line bundles is discussed in the context of Banach–Lie groups in [Ne04].

Comparing these two classes of representations which exhibit a rich geometric structure, we would like to understand if there are larger classes of Lie algebras with similar types of representations, or even if there is a common roof for the Kac–Moody cases and the locally finite situation. In this note we address this question for the class of locally affine Lie algebras (LALAs), a subclass of the recently introduced locally extend affine Lie algebras (LEALAs) ([MY06]), which in turn are infinite rank variants of extended affine Lie algebras ([AA-P97]).

From a geometric point of view, the choice of a positive system with respect to which an integral weight  $\lambda$  is dominant is a redundant feature of the theory. It is much more natural to work with the stabilizer  $\mathfrak{p}_\lambda$  of the highest weight ray in  $L(\lambda)$ . A key observation that makes our approach work is that for an affine Kac–Moody algebra, the property of an integral weight  $\lambda$  to be an extremal weight of an integrable highest weight module is equivalent to  $\lambda$  not vanishing on the center, a property that no longer refers to the choice of a positive system, so that it also makes sense for locally affine Lie algebras.

In the first section we introduce the framework of split quadratic Lie algebras  $(\mathfrak{g}, \mathfrak{h}, \kappa)$ , where  $\kappa$  is an invariant symmetric bilinear form and  $\mathfrak{h}$  is a splitting Cartan subalgebra. In this context one has a natural concept of an integrable root generalizing the notion of a real root of a Kac–Moody Lie algebra, resp., an anisotropic root of an EALA. In this framework we define LEALAs and LALAs following [MY06] and [YY08]. Since notions such as the Weyl group, coroots and integral weights make sense in general, it becomes an interesting issue to understand the interactions of the axiomatic framework and geometric, resp., representation theoretic features of these Lie algebras. The main new observation in Section 1 is that any integral weight  $\lambda$  defines a natural  $\mathbb{Q}$ -grading  $\mathfrak{g} = \bigoplus_{q \in \mathbb{Q}} \mathfrak{g}^q(\lambda)$  and that this becomes a  $\mathbb{Z}$ -grading if

$$(2\text{-Aff}) \quad \beta(\tilde{\alpha})\alpha(\tilde{\beta}) \leq 4 \text{ for } \alpha, \beta \in \Delta_i,$$

where  $\Delta_i$  denotes the set of integrable roots (Definition 1.1). This means that the subalgebras  $\mathfrak{g}(\alpha, \beta)$  generated by two integrable root- $\mathfrak{sl}_2$ -algebras are either finite dimensional or affine Kac–Moody.

In Section 2 we then turn to LEALAs. Since all these Lie algebras satisfy (2-Aff), each integral weight  $\lambda$  defines a  $\mathbb{Z}$ -grading of  $\mathfrak{g}$ . For affine Kac–Moody Lie algebras, the  $\mathbb{Z}$ -gradings corresponding to integrable highest weight modules  $L(\lambda, \Delta^+)$  have the property that all roots of the 0-component  $\mathfrak{g}^0(\lambda)$  of the grading are integrable. We therefore focus on integral weights with this property. The first main result of Section 2 is that the existence of an integral weight with this property implies that  $\mathfrak{g}$  is a locally affine Lie algebra (Theorem 2.8). This is a representation theoretic characterization of LALAs among LEALAs. From this point on we restrict our considerations to locally affine Lie algebras.

In Section 3 we turn to the structure of locally affine Lie algebras. After analyzing how they can be exhausted by affine Kac–Moody algebras in a controlled

fashion, we show that isomorphisms of locally affine root systems “extend” to isomorphisms of the cores of the corresponding Lie algebras and even to all of  $\mathfrak{g}$ , provided  $\mathfrak{g}$  is *minimal* (cf. Definition 3.6).<sup>1</sup> From that it also follows that for a minimal locally affine Lie algebra two Cartan subalgebras with isomorphic root systems are conjugate under  $\text{Aut}(\mathfrak{g})$  (cf. [Sa08] for related results) and that, over  $\mathbb{C}$ , minimal affine Lie algebras have a unitary real form.

Section 4 is completely devoted to representation theoretic issues. Here we construct for each integral weight  $\lambda$  of a locally extended affine Lie algebra  $\mathfrak{g}$  a simple module  $L(\lambda) := L(\lambda, \mathfrak{p}_\lambda)$  by induction of a generalized parabolic subalgebra  $\mathfrak{p}_\lambda = \sum_{q \geq 0} \mathfrak{g}^q(\lambda)$ . Our main representation theoretic result is that these modules  $L(\lambda)$  are integrable and that, over  $\mathbb{C}$ , they are unitary with respect to the unitary real form. Moreover,  $\mathcal{P}_\lambda$  is given by (0.1) and  $L(\lambda) \cong L(\mu)$  if and only if  $\mu$  is conjugate to  $\lambda$  under the Weyl group  $\mathcal{W}$ .

In a first appendix we recall Yoshii’s classification of the locally affine root systems of infinite rank, derive some more information that is needed in Section 3 and describe a realization of the corresponding minimal locally affine Lie algebras in terms of twisted loop algebras. In a second appendix we show that Yoshii’s seven types of infinite rank locally affine root systems lead to four isomorphy classes of minimal locally affine Lie algebras.

In the following  $\mathbb{K}$  denotes a field of characteristic zero, if not explicitly said otherwise.

**Acknowledgement:** We thank Y. Yoshii for illuminating discussions during the preparation of this manuscript and for making the manuscripts [MY08] and [YY08] available.

## 1. Split quadratic Lie algebras

DEFINITION 1.1. (a) We call an abelian subalgebra  $\mathfrak{h}$  of the Lie algebra  $\mathfrak{g}$  a *splitting Cartan subalgebra* if  $\mathfrak{h}$  is maximal abelian and  $\text{ad } \mathfrak{h}$  is simultaneously diagonalizable. Then the pair  $(\mathfrak{g}, \mathfrak{h})$ , resp.,  $\mathfrak{g}$ , is called a *split Lie algebra* and we have a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})[h, x] = \alpha(h)x\}$  and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

is the corresponding *root system*. Note that  $\mathfrak{g}_0 = \mathfrak{h}$  because  $\mathfrak{h}$  is maximal abelian.

(b) We call a root  $\alpha \in \Delta$  *integrable* if there exist  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  with  $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$  such that the operators  $\text{ad } x_{\pm\alpha}$  on  $\mathfrak{g}$  are locally nilpotent. Then  $\dim \mathfrak{g}_{\pm\alpha} = 1$  and we write  $\check{\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$  for the unique element satisfying  $\alpha(\check{\alpha}) = 2$ , called the corresponding *coroot*. We also write

$$\mathfrak{g}(\alpha) := \mathbb{K}\check{\alpha} + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2(\mathbb{K})$$

for the associated three dimensional subalgebra (cf. [Ne00b, Prop. I.6]). The set of integrable roots is denoted  $\Delta_i$ . For a subset  $S \subseteq \Delta_i$ , we write  $\mathfrak{g}(S)$  for the subalgebra generated by the  $\mathfrak{g}(\alpha)$ ,  $\alpha \in S$ . The subalgebra  $\mathfrak{g}_c := \mathfrak{g}(\Delta_i)$  is called the

<sup>1</sup>After this paper was written, we learned about the work in progress [MY08] of Morita and Yoshii, where they also show that the cores or locally affine Lie algebras with isomorphic root systems are isomorphic. However, their approach is somewhat different.

core of  $\mathfrak{g}$ . In the following, we shall frequently assume that  $\mathfrak{g}$  is *coral*, i.e.,  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$ . Then  $\mathfrak{g}_c$  is a perfect ideal of  $\mathfrak{g}$ , hence coincides with the commutator algebra  $[\mathfrak{g}, \mathfrak{g}]$ .

(c) A linear functional  $\lambda \in \mathfrak{h}^*$  is called an *integral weight* if  $\lambda(\check{\alpha}) \in \mathbb{Z}$  for each  $\alpha \in \Delta_i$ .

(d) The Weyl group  $\mathcal{W}$  of  $\mathfrak{g}$  is the subgroup of  $\mathrm{GL}(\mathfrak{h})$  generated by the reflections  $r_\alpha(h) := h - \alpha(h)\check{\alpha}$ . Then each  $r_\alpha$  extends to an automorphism of  $\mathfrak{g}$  preserving  $\mathfrak{h}$  and acting on  $\mathfrak{h}^*$  by the adjoint linear map  $r_\alpha(\beta) := \beta - \beta(\check{\alpha})\alpha$ . The action of the Weyl group preserves the root system  $\Delta$  and the subset  $\Delta_i$  of integrable roots (cf. [Ne00b, Def. I.8, Lemma I.11]).

(e) We call two integrable roots  $\alpha$  and  $\beta$  *connected* if there exists a chain  $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_n = \beta$  in  $\Delta_i$  with  $\alpha_{i+1}(\check{\alpha}_i) \neq 0$  for  $i = 0, \dots, n-1$ . Then we define  $\mathrm{dist}(\alpha, \beta)$  as the minimal length  $n$  of such a chain. Connectedness defines an equivalence relation on  $\Delta_i$  (cf. Proposition 1.3 below) and on each connected component of  $\Delta_i$  (the corresponding equivalence classes),  $\mathrm{dist}$  defines a metric.

DEFINITION 1.2. A *quadratic Lie algebra* is a pair  $(\mathfrak{g}, \kappa)$ , consisting of a Lie algebra  $\mathfrak{g}$  and a non-degenerate invariant symmetric bilinear form  $\kappa$  on  $\mathfrak{g}$ . If  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a split quadratic Lie algebra, then the root spaces satisfy  $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$  for  $\alpha + \beta \neq 0$ , and in particular,  $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$  is non-degenerate. We thus obtain an injective map  $\flat: \mathfrak{h} \rightarrow \mathfrak{h}^*, h \mapsto h^\flat, h^\flat(x) := \kappa(x, h)$ . For  $\alpha \in \mathfrak{h}^\flat := \flat(\mathfrak{h})$  we put  $\alpha^\sharp := \flat^{-1}(\alpha)$  and define a symmetric bilinear map on  $\mathfrak{h}^\flat$  by  $(\alpha, \beta) := \kappa(\alpha^\sharp, \beta^\sharp)$ . Note that (1.2) in Remark 1.4 below implies in particular that  $\Delta \subseteq \mathfrak{h}^\flat$ , so that  $(\alpha, \beta)$  is defined for  $\alpha, \beta \in \Delta$ .

A split quadratic Lie algebra  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is called a *locally extended affine Lie algebra* (LEALA) if the following two conditions are satisfied

(Irr)  $\Delta_i$  is connected.<sup>2</sup>

(Int) All non-isotropic roots are integrable, i.e.,  $\{\alpha \in \Delta: (\alpha, \alpha) \neq 0\} \subseteq \Delta_i$ .

The following proposition combines Cor. III.8 and Prop. III.9 in [Ne00b].

PROPOSITION 1.3. For  $\alpha, \beta \in \Delta_i$ , the following assertions hold:

(i)  $\alpha(\check{\beta}) \in \mathbb{Z}$ ,  $\alpha(\check{\beta})\beta(\check{\alpha}) \geq 0$ , and  $\alpha(\check{\beta}) = 0$  implies  $\beta(\check{\alpha}) = 0$ .

(ii) If  $\kappa$  is an invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ , then  $\kappa(\check{\alpha}, \check{\alpha}) \neq 0$ , and if  $\beta(\check{\alpha}) \neq 0$ , then

$$\kappa(\check{\beta}, \check{\beta})/\kappa(\check{\alpha}, \check{\alpha}) = \alpha(\check{\beta})/\beta(\check{\alpha}) \in \mathbb{Q}_+^\times := \{q \in \mathbb{Q}: q > 0\}.$$

REMARK 1.4. Let  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  be a split quadratic Lie algebra.

(a) For  $h \in \mathfrak{h}$  and  $x_\pm \in \mathfrak{g}_{\pm\alpha}$  we have

$$(1.1) \quad \alpha(h)\kappa(x_\alpha, x_{-\alpha}) = \kappa([h, x_\alpha], x_{-\alpha}) = \kappa(h, [x_\alpha, x_{-\alpha}]),$$

so that the non-degeneracy of  $\kappa$  on  $\mathfrak{h}$  leads to  $\alpha \in \mathfrak{h}^\flat$  and

$$(1.2) \quad [x_\alpha, x_{-\alpha}] = \kappa(x_\alpha, x_{-\alpha})\alpha^\sharp$$

(cf. [MY08]).<sup>3</sup> If  $\alpha$  is integrable and  $\check{\alpha} = [x_\alpha, x_{-\alpha}]$ , then (1.1) and  $\alpha(\check{\alpha}) = 2$  imply  $\kappa(\check{\alpha}, \check{\alpha}) = 2\kappa(x_\alpha, x_{-\alpha})$ , which leads for  $\beta \in \mathfrak{h}^\flat$  to

$$(1.3) \quad \alpha^\sharp = \frac{2\check{\alpha}}{\kappa(\check{\alpha}, \check{\alpha})}, \quad (\alpha, \alpha) = \frac{4}{\kappa(\check{\alpha}, \check{\alpha})} \quad \text{and} \quad (\beta, \alpha) = \frac{2\beta(\check{\alpha})}{\kappa(\check{\alpha}, \check{\alpha})}.$$

<sup>2</sup>This is equivalent to  $\Delta_i$  being *irreducible* in the sense that it cannot be decomposed into two proper mutually orthogonal subsets.

<sup>3</sup>This argument shows that  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is an admissible triple in the sense of [MY06].

(b) If, in addition,  $\Delta_i$  is connected, then Proposition 1.3(ii) implies that, after multiplication with a suitable element of  $\mathbb{K}^\times$ ,  $\kappa$  satisfies  $\kappa(\check{\alpha}, \check{\alpha}) \in \mathbb{Q}_+^\times$  for each  $\alpha \in \Delta_i$ , and thus  $(\alpha, \alpha) \in \mathbb{Q}_+^\times$ . In the following, we shall always assume this normalization whenever  $\Delta_i$  is connected.

(c) It is easy to see that for  $S \subseteq \Delta_i$  we have

$$(1.4) \quad \mathfrak{h} \cap \mathfrak{g}(S) = \text{span } \check{S} \quad \text{and in particular} \quad \mathfrak{h} \cap \mathfrak{g}_c = \text{span } \check{\Delta}_i$$

(cf. [Ne00b, Lemma I.10]). If  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$ , then we further have

$$(1.5) \quad \Delta \subseteq \text{span}_{\mathbb{Z}} \Delta_i \quad \text{and} \quad (\forall \beta \in \Delta)(\exists \alpha \in \Delta_i) \beta - \alpha \in \Delta.$$

In fact, for each  $\beta \in \Delta$ , the root space  $\mathfrak{g}_\beta$  is spanned by brackets of root vectors of integrable roots. Hence there exists an  $\alpha \in \Delta_i$  with  $\beta - \alpha \in \Delta$ . From (1.4) we derive that

$$(1.6) \quad \mathfrak{g}_c^\perp = \mathfrak{h} \cap (\check{\Delta}_i)^\perp = \Delta_i^\perp = \Delta^\perp = \mathfrak{z}(\mathfrak{g}).$$

LEMMA 1.5. *Let  $(\mathfrak{g}, \kappa)$  be a split quadratic Lie algebra with  $(\alpha, \alpha) \in \mathbb{Q}_+^\times$  for each integrable root  $\alpha$ . For  $\alpha, \beta \in \Delta_i$ , the form  $(\cdot, \cdot)$  is positive semidefinite on  $\text{span}_{\mathbb{Q}}\{\alpha, \beta\}$  if and only if*

$$(1.7) \quad \alpha(\check{\beta})\beta(\check{\alpha}) \in \{0, 1, 2, 3, 4\}.$$

PROOF. If  $\alpha$  and  $\beta$  are linearly dependent over  $\mathbb{Q}$ , then  $\beta \in \{\pm\alpha\}$  ([Ne00b, Prop. I.6]), which implies that  $\beta(\check{\alpha})\alpha(\check{\beta}) = 4$ .

Now we assume that  $\alpha$  and  $\beta$  are linearly independent. If  $\beta(\check{\alpha}) = \alpha(\check{\beta}) = 0$ , then  $(\alpha, \beta) = 0$ , so that  $(\cdot, \cdot)$  is positive definite on  $\text{span}_{\mathbb{Q}}\{\alpha, \beta\}$ . We may therefore assume that  $(\alpha, \beta) \neq 0$ . Then the assertion is equivalent to the positive semidefiniteness of the rational Gram matrix  $\begin{pmatrix} (\alpha, \alpha) & (\alpha, \beta) \\ (\alpha, \beta) & (\beta, \beta) \end{pmatrix}$ , which means that  $(\alpha, \beta)^2 \leq (\alpha, \alpha)(\beta, \beta)$ . In view of Remark 1.4(a), this is equivalent to  $\alpha(\check{\beta})\beta(\check{\alpha}) \leq 4$ . Since both factors on the left are integers of the same sign, their product is non-negative, and the assertion follows (Proposition 1.3(i)).  $\square$

LEMMA 1.6. *If  $\alpha, \beta \in \Delta_i$  are connected, then the orbit  $\mathcal{W}\beta$  contains an element  $\gamma$  with  $\gamma(\check{\alpha}) \neq 0$ .*

PROOF. If  $\text{dist}(\alpha, \beta) \leq 1$ , then  $\beta(\check{\alpha}) \neq 0$ , and there is nothing to show. If  $\text{dist}(\alpha, \beta) > 1$ , we show that there exists an element  $\gamma \in \mathcal{W}\beta$  with  $\text{dist}(\gamma, \alpha) < \text{dist}(\alpha, \beta)$ . Then the assertion follows by induction.

Let  $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_n = \beta$  be a minimal chain of integrable roots with  $\alpha_i(\check{\alpha}_{i+1}) \neq 0$  for  $i = 0, \dots, n-1$ . We consider the root

$$\gamma := r_{\alpha_{n-1}}(\beta) = \beta - \beta(\check{\alpha}_{n-1})\alpha_{n-1} \in \Delta_i$$

(cf. Definition 1.1(d)). Since  $n$  is minimal and  $> 1$ , we have  $\beta(\check{\alpha}_{n-2}) = 0$  and therefore  $\gamma(\check{\alpha}_{n-2}) = -\beta(\check{\alpha}_{n-1})\alpha_{n-1}(\check{\alpha}_{n-2}) \neq 0$ . This implies that  $\text{dist}(\alpha, \gamma) < n$ .  $\square$

PROPOSITION 1.7. *Suppose that the split quadratic Lie algebra  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  satisfies (2-Aff). If  $\alpha$  and  $\beta$  are connected with  $(\beta, \beta)/(\alpha, \alpha) > 1$ , then  $(\beta, \beta)/(\alpha, \alpha) \in \{2, 3, 4\}$ . If, in addition,  $\Delta_i$  is connected, then  $(\alpha, \alpha)$  has at most 3 values for  $\alpha \in \Delta_i$ .*

PROOF. In view of Remark 1.4(b), we may w.l.o.g. assume that  $(\alpha, \alpha) \in \mathbb{Q}_+^\times$ , so that the connectedness of  $\beta$  with  $\alpha$  implies  $(\beta, \beta) \in \mathbb{Q}_+^\times$ . Assume that  $(\beta, \beta) > (\alpha, \alpha)$ . In view of Lemma 1.6, there exists  $\gamma \in \mathcal{W}\beta$  with  $\gamma(\check{\alpha}) \neq 0$ . Since the form  $(\cdot, \cdot)$  on  $\text{span}_{\mathbb{Z}} \Delta$  is invariant under the action of the Weyl group (which is easily checked on the generators with Remark 1.4(a)), we have  $(\gamma, \gamma) = (\beta, \beta) > (\alpha, \alpha)$ . By combining Proposition 1.3(ii) with Remark 1.4, we find that

$$\frac{(\beta, \beta)}{(\alpha, \alpha)} = \frac{(\gamma, \gamma)}{(\alpha, \alpha)} = \frac{\kappa(\check{\alpha}, \check{\alpha})}{\kappa(\check{\gamma}, \check{\gamma})} = \frac{\gamma(\check{\alpha})}{\alpha(\check{\gamma})}.$$

Therefore  $|\alpha(\check{\gamma})| < |\gamma(\check{\alpha})|$ , so that the integrality of these numbers and (2-Aff) lead to  $|\alpha(\check{\gamma})| = 1$ . This proves that  $\frac{(\beta, \beta)}{(\alpha, \alpha)} = |\gamma(\check{\alpha})| \in \{2, 3, 4\}$ .

Now we assume, in addition, that  $\Delta_i$  is connected. We may w.l.o.g. assume that the minimal value of  $(\alpha, \alpha)$  for  $\alpha \in \Delta_i$  is 1. Then the argument above implies that  $(\beta, \beta) \in \{1, 2, 3, 4\}$  for  $\beta \in \Delta_i$ . If there exists a  $\beta$  with  $(\beta, \beta) = 3$ , then  $(\gamma, \gamma) \in \{2, 4\}$  is ruled out for  $\gamma \in \Delta_i$  by the preceding argument. Therefore the only possibility for the set of square lengths is  $\{1, 3\}$  or a subset of  $\{1, 2, 4\}$ .  $\square$

The preceding proposition has an interesting consequence.

THEOREM 1.8. *Let  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  be a split quadratic Lie algebra with  $\kappa(\check{\alpha}, \check{\alpha}) \in \mathbb{Q}$  for  $\alpha \in \Delta_i$  and  $\Delta \subseteq \text{span}_{\mathbb{Z}} \Delta_i$ . Then each integral weight  $\lambda \in \mathfrak{h}^*$  defines a  $\mathbb{Q}$ -grading of  $\mathfrak{g}$  by*

$$\mathfrak{g}^q := \mathfrak{g}^q(\lambda) := \bigoplus_{\lambda(\alpha^\sharp)=q} \mathfrak{g}_\alpha \quad \text{for } q \in \mathbb{Q}.$$

*If, in addition, (2-Aff) holds and  $\Delta_i$  is connected, then the support of this grading lies in a cyclic subgroup of  $\mathbb{Q}$ , so that we actually obtain a  $\mathbb{Z}$ -grading.*

PROOF. Clearly, the map  $\mathfrak{h}^\flat \rightarrow \mathbb{K}, \alpha \mapsto \alpha^\sharp(h)$ , is additive, and for  $\alpha \in \Delta_i$  we have  $\lambda(\alpha^\sharp) = \frac{2\lambda(\check{\alpha})}{\kappa(\check{\alpha}, \check{\alpha})} \in \mathbb{Q}$ . Now the first assertion follows from  $\Delta \subseteq \text{span}_{\mathbb{Z}}(\Delta_i)$ .

Let us now assume that (2-Aff) is satisfied and  $\Delta_i$  is connected. We have to show that the set of rational numbers of the form

$$\lambda(\alpha^\sharp) = \frac{2\lambda(\check{\alpha})}{\kappa(\check{\alpha}, \check{\alpha})} = \frac{1}{2}(\alpha, \alpha)\lambda(\check{\alpha}) \in \frac{1}{2}(\alpha, \alpha)\mathbb{Z}$$

is contained in a cyclic subgroup of  $(\mathbb{Q}, +)$ . Since each finitely generated subgroup of  $\mathbb{Q}$  is cyclic, this follows from Proposition 1.7.  $\square$

In the following, we are mainly interested in those integral weights for which all roots  $\alpha$  with  $\lambda(\alpha^\sharp) = 0$  are integrable. The existence of such weights for LEALAs will be investigated in Section 2 below.

The following proposition completely describes the meaning of (2-Aff) for Kac–Moody algebras.<sup>4</sup>

PROPOSITION 1.9. *A Kac–Moody algebra  $\mathfrak{g}(A)$  satisfies (2-Aff) if and only if it either is finite-dimensional or affine.*

PROOF. If  $\mathfrak{g}(A)$  is affine or finite-dimensional, then the canonical bilinear form on  $\text{span } \Delta$  is positive semidefinite, so that (2-Aff) follows from Lemma 1.5.

<sup>4</sup>We thank Pierre-Emmanuel Caprace for suggesting a different proof of this result, based on the geometry of the corresponding Coxeter group.

Suppose, conversely, that  $\mathfrak{g}(A)$  is infinite dimensional and satisfies (2-Aff). Let  $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta_i$  be a system of simple roots and define the height of a root by  $\text{ht}(\sum_i n_i \alpha_i) := \sum_i n_i$ . Let  $\delta = \sum_{i=1}^r n_i \alpha_i \in \Delta^+$  be a non-integrable root of minimal height (cf. [Ka90]). Then there exists a simple root  $\alpha \in \Pi$  with  $\beta := \delta - \alpha \in \Delta$ , and since the height of  $\delta$  is minimal,  $\beta$  is integrable. Next we use [Ne00b, Prop. III.14] to see that the subalgebra  $\mathfrak{g}(\alpha, \beta)$  generated by  $\mathfrak{g}(\alpha)$  and  $\mathfrak{g}(\beta)$  is the commutator algebra of an affine Kac–Moody algebra of rank 2. In particular, for each  $n \in \mathbb{N}$ ,  $\alpha + n\delta \in \Delta_i$  ([Ne00b, Thm. III.5]). Since (2-Aff) implies in particular that  $|\gamma(\tilde{\eta})| \leq 4$  for  $\gamma, \eta \in \Delta_i$ , it follows that  $\delta(\tilde{\eta}) = 0$  for each integrable root  $\eta$ .

If some coefficient  $n_i$  of  $\delta$  vanishes, then the connectedness of  $\Pi$  implies the existence of some  $j$  with  $n_j > 0$  and  $\alpha_j(\tilde{\alpha}_i) < 0$ . Since  $\alpha_i(\tilde{\alpha}_k) \leq 0$  holds for each  $i \neq k$ , we thus arrive at the contradiction  $\delta(\tilde{\alpha}_i) < 0$ . Therefore the vector  $\mathbf{n} = (n_1, \dots, n_r)^\top \in \mathbb{N}^r$  has no zero entry and satisfies  $\mathbf{A}\mathbf{n} = 0$ , so that Vinberg’s classification of generalized Cartan matrices ([MP95, Prop. 3.6.5]) implies that  $A$  is of affine type.  $\square$

We record the following proposition because we shall use it later on to obtain a classification of minimal locally affine Lie algebras.

**DEFINITION 1.10.** A subset  $\Pi \subseteq \Delta$  is called a *simple system* if  $\alpha - \beta \notin \Delta$  for  $\alpha, \beta \in \Pi$ .

**PROPOSITION 1.11.** *Let  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  be a split quadratic Lie algebra and  $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta_i$  be a linearly independent simple system. Then the following assertions hold:*

- (i)  $\tilde{\Pi}$  is linearly independent in  $\mathfrak{h}$ .
- (ii) The matrix  $A_\Pi := (\alpha_i(\tilde{\alpha}_j))_{i,j \in \Pi}$  is a symmetrizable generalized Cartan matrix.
- (iii) Let  $k := \text{rank}(A_\Pi)$  and  $n := 2r - k$ . Put  $h_i := \tilde{\alpha}_i$  for  $i = 1, \dots, r$  and choose elements  $h_{r+1}, \dots, h_n$  such that  $h_1, \dots, h_n$  are linearly independent and their span  $\mathfrak{h}_\Pi$  separates the points in  $\text{span } \Pi$ . Then

$$\mathfrak{h}_\Pi + \mathfrak{g}(\Pi) \cong \mathfrak{g}(A_\Pi),$$

the Kac–Moody Lie algebra associated to  $A_\Pi$ .

- (iv) If, in addition,  $A_\Pi$  is of affine type, then  $n = r + 1$  and we may choose any  $h_n \in \mathfrak{h}$  on which the isotropic roots of  $\mathfrak{g}(A_\Pi)$  do not vanish.

**PROOF.** (i) In view of Remark 1.4(a),  $\tilde{\alpha}_i$  is a linear multiple of  $\alpha_i^\sharp$ . Hence (i) follows from the injectivity of the linear map  $\sharp$  on  $\text{span } \Delta$ .

(ii) That  $A_\Pi$  is a generalized Cartan matrix follows from Proposition 1.3(i) and the symmetrizability follows from the symmetry of the matrix with entries  $(\alpha_i, \alpha_j)$  and Remark 1.4(a).

(iii) The choice of the  $h_i$  implies that  $(\mathfrak{h}_\Pi, \Pi, \tilde{\Pi})$  is a realization of the generalized Cartan matrix  $A_\Pi$ . Let  $\mathfrak{g}(A_\Pi)$  be the corresponding Kac–Moody algebra. Since  $A_\Pi$  is symmetrizable, the Gabber–Kac Theorem [GK81, Thm. 2] implies that it is defined by the generators  $e_1, \dots, e_r; f_1, \dots, f_r; h_1, \dots, h_n$  and the Serre relations.

For  $\alpha \in \Pi$  pick  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  with  $[x_\alpha, x_{-\alpha}] = \tilde{\alpha}$ . Then [Ne00b, Prop. II.11] implies the existence of a unique homomorphism  $\varphi: \mathfrak{g}(A_\Pi) \rightarrow \mathfrak{g}$  which is the identity on the  $h_i$  and maps  $e_i$  to  $x_{\alpha_i}$ . Then  $\varphi(\mathfrak{g}(A_\Pi)) = \mathfrak{h}_\Pi + \mathfrak{g}(\Pi)$  and it remains to see

that  $\varphi$  is injective. In view of [Ne00b, Lemma VII.5], its kernel is central in  $\mathfrak{g}(A_\Pi)$ , but since  $\varphi|_{\mathfrak{h}_\Pi}$  is injective, it is injective.

(iv) If  $A_\Pi$  is of affine type, then  $\text{rank } A_\Pi = r - 1$  implies  $n = r + 1$ . If  $\delta$  is an isotropic root of  $\mathfrak{g}(A_\Pi)$ , then  $\mathbb{K}\delta = \check{\Pi}^\perp \cap \text{span}(\Pi)$ , so that we obtain with any element  $h_n \in \mathfrak{h}$  with  $\delta(h_n) \neq 0$  an  $n$ -dimensional space  $\mathfrak{h}_\Pi$  separating the points of  $\text{span } \Pi$ .  $\square$

LEMMA 1.12. *Each group homomorphism*

$$\chi: \mathcal{Q} = \text{span}_{\mathbb{Z}} \Delta \rightarrow \mathbb{K}^\times$$

defines an automorphism  $\varphi_\chi \in \text{Aut}(\mathfrak{g})$  by  $\varphi_\chi(x) := \chi(\alpha)x$  for  $x \in \mathfrak{g}_\alpha$ .

If, conversely, there exists a rationally linearly independent subset  $\mathcal{B} \subseteq \Delta_i$  with  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}(\mathcal{B})$ , then each automorphism  $\varphi \in \text{Aut}(\mathfrak{g})$  fixing  $\mathfrak{h}$  pointwise is of the form  $\varphi_\chi$  as above.

PROOF. The first assertion is trivial. For the second, let  $\varphi \in \text{Aut}(\mathfrak{g})$  fix  $\mathfrak{h}$  pointwise. Then  $\varphi$  preserves all root spaces, so that there exists for each  $\alpha \in \Delta_i$  a number  $\lambda_\alpha \in \mathbb{K}^\times$  with  $\varphi(x_\alpha) = \lambda_\alpha x_\alpha$  for  $x_\alpha \in \mathfrak{g}_\alpha$ . Since  $\mathcal{B}$  is linearly independent over  $\mathbb{Q}$ , it generates a free subgroup of  $\mathcal{Q}$  and there exists a group homomorphism  $\chi: \mathcal{Q} \rightarrow \mathbb{K}^\times$  with  $\chi(\alpha) = \lambda_\alpha$  for each  $\alpha \in \mathcal{B}$ . Now  $\varphi_\chi^{-1} \circ \varphi \in \text{Aut}(\mathfrak{g})$  fixes  $\mathfrak{h}$  pointwise and likewise all subalgebras  $\mathfrak{g}(\alpha)$ ,  $\alpha \in \mathcal{B}$ . Therefore it also fixes  $\mathfrak{h} + \mathfrak{g}(\mathcal{B}) = \mathfrak{g}$  pointwise, i.e.,  $\varphi = \varphi_\chi$ .  $\square$

PROBLEM 1.13. Let  $\mathfrak{g}$  be a coral split Lie algebra for which  $\Delta_i$  is connected. Does (2-Aff) imply the existence of an invariant symmetric bilinear form  $\kappa$  on the commutator algebra  $[\mathfrak{g}, \mathfrak{g}]$ ? If such a form exists and extends to a non-degenerate form on  $\mathfrak{g}$ , then  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  would be a split quadratic Lie algebra.

A necessary condition for that is that the matrix  $(\alpha(\check{\beta}))_{\alpha, \beta \in \Delta_i}$  is symmetrizable. In view of [KN01, Prop. 2.3], this is the case if for any 3-element set  $\{\alpha_1, \alpha_2, \alpha_3\}$  of integrable roots, the matrix  $(\alpha_i(\check{\alpha}_j))_{i, j=1, \dots, 3}$  is symmetrizable.

## 2. Locally extended affine Lie algebras

Throughout this section,  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  denotes an LEALA for which  $\kappa$  is normalized such that  $(\alpha, \alpha) \in \mathbb{Q}_+^\times$  for each integrable root. The goal of this section is to see that the existence of an integral weight  $\lambda$  for which all roots in

$$\Delta^\lambda := \{\alpha \in \Delta: \lambda(\alpha^\sharp) = 0\}$$

are integrable implies that  $\mathfrak{g}$  is locally affine (cf. Section 3). If

$$\mathfrak{g}^0(\lambda) = \mathfrak{h} + \sum_{\lambda(\alpha^\sharp)=0} \mathfrak{g}_\alpha$$

is generated by  $\mathfrak{h}$  and its core, then this is also equivalent to the local finiteness of  $\mathfrak{g}^0(\lambda)$  (cf. Proposition 2.10). Conversely, for each locally affine algebra such weights exist, as we shall derive from Yoshi's description of the locally affine root systems.

The following result is of central importance for the structure theory of LEALAs ([MY06, Thm. 3.10]):

THEOREM 2.1 (Morita–Yoshii). *The form  $(\cdot, \cdot)$  on  $V := \text{span}_{\mathbb{Q}} \Delta$  is positive semidefinite.*

In view of Lemma 1.5, this implies



**COROLLARY 2.2.** *Each LEALA satisfies (2-Aff). For  $\alpha, \beta \in \Delta_i$  with  $(\beta, \beta) \geq (\alpha, \alpha)$ , we have  $(\beta, \beta)/(\alpha, \alpha) \in \{1, 2, 3, 4\}$ .*

**PROBLEM 2.3.** Let  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  be coral split quadratic with  $\Delta_i$  connected. Does (2-Aff) already imply that it is an LEALA, i.e., that all non-isotropic roots are integrable? This would be a nice characterization of LEALAs in terms of rank-2 subalgebras.

To proceed, we have to take a closer look at root systems.

**DEFINITION 2.4.** (cf. [YY08]) Let  $V$  be a rational vector space with a positive semidefinite bilinear form  $(\cdot, \cdot)$  and  $\mathcal{R} \subseteq V$  a subset. The triple  $(V, \mathcal{R}, (\cdot, \cdot))$  is called a *locally extended affine root system* or *LEARS* for short, if the following conditions are satisfied:<sup>5</sup>

- (A1)  $(\alpha, \alpha) \neq 0$  for each  $\alpha \in \mathcal{R}$  and  $\text{span } \mathcal{R} = V$ .
- (A2)  $\langle \beta, \alpha \rangle := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$  for  $\alpha, \beta \in \mathcal{R}$ .
- (A3)  $r_\alpha(\beta) := \beta - \langle \beta, \alpha \rangle \alpha \in \mathcal{R}$  for  $\alpha, \beta \in \mathcal{R}$ .
- (A4) If  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$  with  $(\mathcal{R}_1, \mathcal{R}_2) \subseteq \{0\}$ , then either  $\mathcal{R}_1$  or  $\mathcal{R}_2$  is empty ( $\mathcal{R}$  is irreducible).

A LEARS is said to be *reduced* if, in addition,

- (R)  $2\alpha \notin \mathcal{R}$  for each  $\alpha \in \mathcal{R}$ .

The root system  $(V, \mathcal{R})$  is called *locally affine* (LARS), if, in addition, the following condition is satisfied:

- (A5) The subspace  $V^0 := \{v \in V : (v, V) = \{0\}\}$  intersects  $\text{span}_{\mathbb{Z}} \mathcal{R}$  in a non-trivial cyclic group.

In view of (A1), (A5) implies that  $\dim V^0 = 1$ . If, in addition,  $V$  is finite-dimensional, then  $(V, \mathcal{R})$  is called *affine*.<sup>6</sup>

The quotient space  $\overline{V} := V/V^\perp$  inherits a positive definite form for which the image  $\overline{\mathcal{R}}$  still satisfies (A1)-(A4) and is a locally finite root system in the sense of [LN04], which is not necessarily reduced (cf. [MY06]).

The importance of these root systems is due to the following observation, which, in view of Definition 1.1 and Remark 1.4, is an immediate consequence of the Morita–Yoshii Theorem 2.1.

**PROPOSITION 2.5.** *If  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a LEALA, then  $(\text{span}_{\mathbb{Q}} \Delta_i, \Delta_i, (\cdot, \cdot))$  is a reduced LEARS.*

In the following we put  $V := \text{span}_{\mathbb{Q}} \Delta_i$ . Generalizing the finite-dimensional case in [ABGP97, Ch. 2], the structure of LEARS is described in detail in [YY08]. In the following we shall only need very specific information, which we now recall. In the locally finite root system  $\overline{\Delta} := \{\overline{\alpha} : \alpha \in \Delta_i\} \subseteq \overline{V}$ , we write

$$\overline{\Delta}_{\text{red}} := \{\overline{\alpha} : \overline{\alpha} \notin 2\overline{\Delta}\}$$

<sup>5</sup>In [AA-P97] extended affine root systems are defined in such a way that they may also contain isotropic roots. To use [YY08], we follow Yoshii's approach.

<sup>6</sup>Although it is not obvious, this concept of an affine root system is consistent with Macdonald's ([Mac72]). He assumes, instead of (A5), the properness of the action of the corresponding affine Weyl group. After tensoring with  $\mathbb{R}$ , condition (A5) is equivalent to the discreteness of the root system and the discreteness of the root system implies in his context the local finiteness of the associated system of affine hyperplanes, which in turn is equivalent to the properness of the action of the Weyl group ([HoG04, Cor. 3.5.9]).

for the corresponding reduced root system. Let  $V' \subseteq V$  a complementary subspace which is a *reflectable section*, i.e.,  $\Delta_{\text{red}} := V' \cap \Delta$  contains an inverse image of each element of  $\overline{\Delta}_{\text{red}}$ . We thus obtain a locally finite subsystem of  $\Delta_{\text{red}} \subseteq \Delta$  spanning a hyperplane of  $V$ , on which the bilinear form is positive definite. Since  $\Delta$  is reduced, we cannot hope for  $V' \cap \Delta$  to map surjectively onto all of  $\overline{\Delta}$  if  $\overline{\Delta}$  is not reduced. Therefore reflectable sections are optimal in the sense they intersect  $\Delta$  in a maximal subset (cf. [YY08, Lemma 4]).

In the following we identify  $\overline{V}$  with  $V'$  and write  $\overline{\alpha} \in V'$  for the projection of  $\alpha$  onto  $V'$  along  $V^0$ . The Weyl group  $\overline{W}$  of  $\overline{\Delta}$  has at most 3 orbits, determined by the square length ([LN04, Prop. 4.4, Cor. 5.6]).<sup>7</sup> Accordingly, we write  $\overline{\Delta}_{\text{sh}}, \overline{\Delta}_{\text{lg}}, \overline{\Delta}_{\text{ex}}$  for the set of short roots (with minimal length), extralong roots (twice the length of a short root) and long roots (all others). There are no extralong roots if and only if the root system  $\overline{\Delta}$  is reduced. Correspondingly, we obtain a disjoint decomposition

$$\Delta_i = \Delta_{\text{sh}} \dot{\cup} \Delta_{\text{lg}} \dot{\cup} \Delta_{\text{ex}}.$$

We now write  $\Delta_i = \bigcup_{\overline{\alpha} \in \overline{\Delta}} (\overline{\alpha} + S_{\overline{\alpha}})$ , where  $S_{\overline{\alpha}} := \{\beta \in V^0 : \overline{\alpha} + \beta \in \Delta\}$  is a subset only depending on the  $\overline{W}$ -orbit of  $\overline{\alpha} \in \overline{\Delta}$  ([YY08]). For  $\alpha$  short, we put  $S := S_{\overline{\alpha}}$ , for  $\alpha$  long, we put  $L := S_{\overline{\alpha}}$ , and for  $\alpha$  extralong, we put  $E := S_{\overline{\alpha}}$ , so that

$$\Delta_{\text{sh}} = \overline{\Delta}_{\text{sh}} + S, \quad \Delta_{\text{lg}} = \overline{\Delta}_{\text{lg}} + L \quad \text{and} \quad \Delta_{\text{ex}} = \overline{\Delta}_{\text{ex}} + E.$$

Finally, we write  $\Delta^0 := \Delta \cap V^0$  for the set of isotropic roots.

LEMMA 2.6. *Let  $\mathfrak{g}$  be a coral LEALA and  $G := \langle S, L, E \rangle \subseteq V^0$  denote the subgroup generated by  $S, L$  and  $E$ . Then*

$$\Delta \subseteq (\overline{\Delta} \cup \{0\}) \oplus G \subseteq V' \oplus V^0$$

and there exists an  $m \in \mathbb{N}$  with

$$mG + \Delta_i \subseteq \Delta_i \quad \text{and} \quad mG \subseteq \Delta^0 \cup \{0\}.$$

PROOF. First we show that  $\Delta^0 \subseteq G$ . In view of (1.5) in Remark 1.4(c), for each  $\delta \in \Delta^0$ , there exists an  $\alpha \in \Delta_i$  with  $\delta + \alpha \in \Delta$ . Then  $(\delta + \alpha, \delta + \alpha) = (\alpha, \alpha) > 0$  implies that  $\delta + \alpha$  is integrable. Now  $\alpha, \alpha + \delta \in \overline{\alpha} + S_{\overline{\alpha}}$  shows that  $\delta = (\alpha + \delta) - \alpha \in G$ . This proves the first assertion.

For the second, we write  $k := (\beta, \beta)/(\alpha, \alpha) \in \{2, 3\}$  for  $\beta \in \Delta_{\text{lg}}$  and  $\alpha \in \Delta_{\text{sh}}$  (cf. Proposition 1.7). Then

$$S + 2G \subseteq S, \quad L + kS \subseteq L \quad \text{and} \quad E + 4S \subseteq E$$

(cf. [YY08], [AA-P97]) imply the existence of an  $m \in \mathbb{N}$  with  $mG + S_{\overline{\alpha}} \subseteq S_{\overline{\alpha}}$  for each  $\alpha \in \Delta_i$  and hence that  $mG + \Delta_i \subseteq \Delta_i$ .

If  $\alpha \in \Delta_i$  and  $\delta \in G \setminus \{0\}$  satisfy  $\beta := \alpha + \delta \in \Delta$ , then

$$\beta(\tilde{\alpha}) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2(\alpha, \alpha)}{(\alpha, \alpha)} = 2 > 0$$

leads to  $\delta = \beta - \alpha \in \Delta$  (cf. [Ne00b, Prop. I.7]). In particular, we see that  $mG \subseteq \Delta^0 \cup \{0\}$ .  $\square$

LEMMA 2.7. *If  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$ , then  $\alpha \in V$  is contained in  $V^0$  if and only if  $\alpha^\# \in \mathfrak{z}(\mathfrak{g})$ .*

<sup>7</sup>The most convenient normalization of the scalar product is that  $(\alpha, \alpha) = 2$  for all long roots.

PROOF. In fact,  $\alpha \in V^0$  is equivalent to  $\alpha \perp \Delta_i$ , which in turn is equivalent to  $\Delta_i(\alpha^\sharp) = \{0\}$ , and hence to  $\alpha^\sharp \in \mathfrak{z}(\mathfrak{g})$  (Remark 1.4(c)).  $\square$

THEOREM 2.8. *Let  $\mathfrak{g}$  be a coral locally extended affine Lie algebra for which there exists an integral weight  $\lambda \in \mathfrak{h}^*$  with  $\Delta^\lambda = \{\alpha \in \Delta : \lambda(\alpha^\sharp) = 0\} \subseteq \Delta_i$ . Then  $(V, \Delta_i, (\cdot, \cdot))$  is a locally affine or a locally finite root system.*

PROOF. Since  $V = \text{span}_{\mathbb{Q}} \Delta_i$ , Lemma 2.6 implies that  $G$  spans  $V^0$ . Therefore it suffices to show that  $G$  is cyclic. In fact, Lemma 2.6 yields  $\text{span}_{\mathbb{Z}} \Delta \subseteq V' \oplus G$ , so that the group  $(\text{span}_{\mathbb{Z}} \Delta) \cap V^0 \subseteq G$  is also cyclic if  $G$  has this property.

For  $\delta \in mG$ ,  $\alpha \in \Delta_i$  and  $\beta := \alpha + \delta$  we have  $\beta^\sharp = \alpha^\sharp + \delta^\sharp$ , and since  $\delta^\sharp$  is central (Lemma 2.7),  $\beta(\beta^\sharp) = \beta(\alpha^\sharp) = (\beta, \alpha) = (\alpha, \alpha)$ , which leads to

$$\check{\beta} = \frac{2}{(\alpha, \alpha)}(\alpha^\sharp + \delta^\sharp) = \check{\alpha} + \frac{2}{(\alpha, \alpha)}\delta^\sharp.$$

In view of Corollary 2.2, the relation  $\frac{2}{(\alpha, \alpha)}\lambda(\delta^\sharp) \in \mathbb{Z}$  for each integrable root  $\alpha$  implies that  $\lambda(G^\sharp)$  is contained in a cyclic subgroup of  $\mathbb{Q}$ . The condition  $\Delta^\lambda \subseteq \Delta_i$  is equivalent to  $\ker \lambda \cap (\Delta^0)^\sharp = \emptyset$ , so that Lemma 2.6 implies that  $\lambda|_{mG^\sharp} : mG^\sharp \rightarrow \mathbb{Q}$  is injective with cyclic image. Therefore  $G$  is a cyclic group.

If  $r := \text{rank}((\text{span}_{\mathbb{Z}} \Delta) \cap V^0)$ , then the preceding argument implies that the root system  $\Delta$  is either locally finite (for  $r = 0$ ) or locally affine (for  $r = 1$ ).  $\square$

For the classification of locally finite and locally affine root systems, we refer to the appendix (Theorem 5.1).

DEFINITION 2.9. In the following we call an integral weight  $\lambda \in \mathfrak{h}^*$  of *transversal type* if  $\Delta^\lambda \subseteq \Delta_i$ , because this is equivalent to  $\ker \lambda \cap (V^0)^\sharp = \emptyset$ , i.e.,  $\ker \lambda$  is a hyperplane of  $\mathfrak{h}$  transversal to  $(V^0)^\sharp$ .

The following proposition sheds some extra light on the condition  $\Delta^\lambda \subseteq \Delta_i$  in terms of the structure of the split quadratic Lie algebra  $(\mathfrak{g}^0(\lambda), \mathfrak{h}, \kappa)$ .

PROPOSITION 2.10. *For a coral quadratic split Lie algebra  $(\mathfrak{g}, \mathfrak{h}, \kappa)$ , the following are equivalent:*

- (i)  $\mathfrak{g}$  is locally finite.
- (ii)  $\Delta = \Delta_i$ .

PROOF. From [Ne00b, Thm. VI.3] we know that (ii) implies (i). If  $\mathfrak{g}$  is locally finite, then the Levi decomposition of locally finite split Lie algebras ([St99, Thm. III.16]) shows that  $\mathfrak{g}_c = (\text{span } \check{\Delta}_i) + \sum_{\alpha \in \Delta_i} \mathfrak{g}_\alpha$ , so that all roots of  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$  are integrable.  $\square$

REMARK 2.11. (a) The four dimensional split oscillator algebra  $\mathfrak{osc}$  is a  $\mathbb{K}$ -Lie algebra with basis  $h, c, p, q$ , where  $c$  is central,  $[p, q] = c$  and  $[h, p] = p, [h, q] = -q$ . Then  $\mathfrak{h} = \mathbb{K}c + \mathbb{K}h$  is a splitting Cartan subalgebra, and for  $\alpha(c) = 0, \alpha(h) = 1$  we have  $\Delta = \{\pm\alpha\}$  and  $\Delta_i = \emptyset$ . In particular,  $\mathfrak{osc}$  is not coral, so that the coralality is necessary for the implication (i)  $\Rightarrow$  (ii) in Proposition 2.10.

(b) If  $\mathfrak{g}$  is a LEALA and  $\bar{\lambda}$  an integral weight of  $\bar{\Delta}$ , then  $\alpha^\sharp \mapsto \bar{\lambda}(\bar{\alpha}^\sharp)$  defines a linear functional on  $V^\sharp \subseteq \mathfrak{h}$  which we extend to a linear functional  $\lambda$  on all of  $\mathfrak{h}$ . For each integrable root  $\alpha$  we then have

$$\lambda(\check{\alpha}) = \frac{2}{(\alpha, \alpha)}\lambda(\alpha^\sharp) = \frac{2}{(\bar{\alpha}, \bar{\alpha})}\bar{\lambda}(\bar{\alpha}^\sharp) = \bar{\lambda}(\check{\bar{\alpha}}) \in \mathbb{Z}.$$

Therefore  $\lambda$  is integral, but  $\lambda$  vanishes on the center, so that  $\Delta^\lambda$  contains non-integrable roots.

### 3. Locally affine Lie algebras

In the preceding section we have seen that the existence of an integral weight  $\lambda$  for which all roots in  $\mathfrak{g}^0(\lambda)$  are integrable implies for a LEALA that its root system is locally affine or locally finite. This leads to a natural concept of a locally affine Lie algebra, and this section is dedicated to a discussion of the structure of these Lie algebras.

The first main results in this section describes how locally affine Lie algebras can be described as direct limits of affine Kac–Moody Lie algebras. Based on this information, we then show that isomorphisms of locally affine root systems “extend” to isomorphisms of the corresponding minimal locally affine Lie algebras. From that it also follows that for a minimal locally affine Lie algebra, any two Cartan subalgebras with isomorphic root systems are conjugate under an automorphism.

**DEFINITION 3.1.** We call a LEALA  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  satisfying  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$  a *coral locally affine Lie algebra* if  $\Delta_i$  is a locally affine root system (in its rational span) and  $\Delta \neq \Delta_i$ .

The following lemma helps to translate between the rational vector space generated by the integrable roots and its  $\mathbb{K}$ -span.

**LEMMA 3.2.** *If  $\mathfrak{g}$  is a coral locally affine Lie algebra and  $V := \text{span}_{\mathbb{Q}} \Delta_i$ , then the canonical map  $V \otimes_{\mathbb{Q}} \mathbb{K} \rightarrow \mathfrak{h}^*$ ,  $(v, \lambda) \mapsto \lambda v$  is a linear isomorphism onto  $\text{span}_{\mathbb{K}} \Delta$ .*

**PROOF.** We have to show that if  $\alpha_0, \dots, \alpha_n \in V$  are linearly independent over  $\mathbb{Q}$ , then they are also linearly independent over  $\mathbb{K}$ . We may assume that  $\alpha_0$  is contained in the one-dimensional space  $V^0$ . Since  $(\cdot, \cdot)$  is non-degenerate modulo  $V^0$ , there exist  $\alpha_1^*, \dots, \alpha_n^* \in V$  with  $(\alpha_i, \alpha_j^*) = \delta_{ij}$  for  $i, j = 1, \dots, n$ .

Suppose that  $\sum_{i=0}^n t_i \alpha_i = 0$  for  $t_i \in \mathbb{K}$ . Then  $0 = (\sum_{i=1}^n t_i \alpha_i, \alpha_j^*) = t_j$  for  $j = 1, \dots, n$ , and hence  $t_0 \alpha_0 = 0$ . As  $\alpha_0$  is non-zero, it also follows that  $t_0 = 0$ . This proves the lemma.  $\square$

**PROPOSITION 3.3.** *For a coral locally affine Lie algebra  $\mathfrak{g}$ , a generator  $\delta$  of the group  $V^0 \cap \text{span}_{\mathbb{Z}} \Delta_i$  and an element  $h_0 \in \mathfrak{h}$  with  $\delta(h_0) \neq 0$ , the following assertions hold:*

- (i)  $\dim(\mathfrak{z}(\mathfrak{g}_c)) = 1$ .
- (ii)  $\Delta_i$  is the directed union of all finite connected subsets  $F$  with  $\delta \in \text{span}_{\mathbb{Z}} F$ . For each such  $F$ , the following assertions hold for  $V_F := \text{span}_{\mathbb{Q}} F$ :
  - (a) If  $\Delta_i^F := V_F \cap \Delta_i$ , then  $(\Delta_i^F, V_F)$  is an affine root system.
  - (b)  $\Delta_i^F$  contains a linearly independent simple system  $\Pi_F$ , i.e.,  $\alpha - \beta \notin \Delta$  for  $\alpha, \beta \in \Pi_F$ .
  - (c) The subalgebra  $\mathfrak{g}(\Pi_F) + \mathbb{K}h_0$  is isomorphic to the affine Kac–Moody algebra  $\mathfrak{g}(A_{\Pi_F})$  and its root system is  $\Delta^F = \Delta_i^F \cup (\mathbb{Z} \setminus \{0\})\delta$ .
- (iii)  $\delta \in \Delta$  and  $\Delta^0 = \mathbb{Z}\delta \setminus \{0\}$  is the set of isotropic roots.
- (iv)  $\Delta_i$  contains a linearly independent subset  $\mathcal{B}$  with  $\Delta \subseteq \text{span}_{\mathbb{Z}} \mathcal{B}$ . We call such a set an integral base of  $\Delta$ .

**PROOF.** (i) First we recall from Remark 1.4(c) that  $\Delta^\# \subseteq \mathfrak{h} \cap \mathfrak{g}_c = \text{span}_{\mathbb{K}} \check{\Delta}_i$ , so that  $\#: \text{span}_{\mathbb{K}} \Delta \rightarrow \mathfrak{h} \cap \mathfrak{g}_c$  is a linear isomorphism. Next we observe that, for

$\alpha \in \text{span}_{\mathbb{K}} \Delta$ , the relation  $\alpha \in V^0$  is equivalent to  $\alpha^{\sharp} \in \mathfrak{z}(\mathfrak{g}_c)$  (Lemma 2.7). Now (i) follows from  $\dim V^0 = 1$ .

(ii), (iii) If  $M \subseteq \Delta_i$  is a finite subset, then the connectedness of  $\Delta_i$  implies the existence of a finite connected subset  $\widetilde{M} \subseteq \Delta_i$  containing  $M$ . Since  $\delta \in \text{span}_{\mathbb{Z}} \Delta_i$ , it follows that  $\Delta_i$  is the directed union of all finite connected subsets  $F$  with  $\delta \in \text{span}_{\mathbb{Z}} F$ .

Clearly,  $(V_F, \Delta_i^F)$  satisfies (A1)-(A3). Since  $V_F$  is spanned by a connected set of roots,  $\Delta_i^F$  is irreducible, so that it is an affine root system. Moreover,  $\Delta_i^F$  is discrete in  $V_F \otimes_{\mathbb{Q}} \mathbb{R}$  because its image in  $\overline{V}$  is finite (cf. [AA-P97, Lemma 2.8]) and the fibers of the map  $\Delta \rightarrow \overline{\Delta}$ , i.e., the sets  $S_{\overline{\alpha}}$  are contained in a cyclic group. That each affine root system contains a linearly independent simple system follows from the discussion in ([ABGP97, Sect 2]).

Next we use Proposition 1.11(iv) to see that  $\mathfrak{g}(\Pi_F) + \mathbb{K}h_0$  is isomorphic to the affine Kac-Moody algebra  $\mathfrak{g}(A_{\Pi_F})$ . From [Ka90, §5.5] we know that  $\mathbb{Z}\delta \setminus \{0\} \subseteq \Delta^F \subseteq \Delta$ , and since  $V^0 \cap \text{span}_{\mathbb{Z}} \Delta = \mathbb{Z}\delta$ , it follows that  $\Delta^0 = \mathbb{Z}\delta \setminus \{0\}$ . This completes the proof of (ii), and (iii) also follows.

(iv) First we recall from [St99, Thm. VI.6] that the locally finite root system  $\overline{\Delta}$  has an integral base  $\overline{\mathcal{B}}$  (see also [LN04, Cor. 6.5]). Let  $\mathcal{B}_1 \subseteq \Delta$  be a subset mapping bijectively onto  $\overline{\mathcal{B}}$  and observe that this implies that  $(\cdot, \cdot)$  is positive definite on  $\text{span}_{\mathbb{Q}} \mathcal{B}_1$ .

For  $V_F$  as above, we may w.l.o.g. assume that  $\Delta^F$  contains an element  $\alpha_1 \in \mathcal{B}_1$ . This element is part of a simple system of the affine root system  $\Delta^F$ , so that  $\alpha_0 := \delta - \alpha_1 \in \Delta_i^F$ . Then we put  $\mathcal{B} := \mathcal{B}_1 \cup \{\alpha_0\}$  (cf. Proposition 3.3). Since  $\overline{\mathcal{B}}$  is linearly independent and  $\delta \in V^0$ , the subset  $\mathcal{B} \subseteq V$  is also linearly independent.

To see that  $\Delta \subseteq \text{span}_{\mathbb{Z}} \mathcal{B} = \text{span}_{\mathbb{Z}} \mathcal{B}_1 + \mathbb{Z}\delta$ , let  $\beta \in \Delta$ . Since  $\overline{\mathcal{B}}$  is an integral base of  $\overline{\Delta}$ , there exist  $\alpha_1, \dots, \alpha_N \in \mathcal{B}_1$  such and  $n_i \in \mathbb{Z}$  such that

$$\beta - \sum_{j=1}^N n_j \alpha_j \in V^0 \cap \text{span}_{\mathbb{Z}} \Delta = \mathbb{Z}\delta.$$

This proves that  $\beta \in \text{span}_{\mathbb{Z}} \mathcal{B}$ . □

**PROPOSITION 3.4.** *If  $\mathfrak{g}$  is a coral locally affine Lie algebra, then for an integral weight  $\lambda \in \mathfrak{h}^*$ , the condition  $\Delta^\lambda \subseteq \Delta_i$  is equivalent to  $\lambda|_{\mathfrak{z}(\mathfrak{g}_c)} \neq 0$ . Such weights exist.*

**PROOF.** Since  $\Delta \setminus \Delta_i = \mathbb{Z}\delta \setminus \{0\}$  for an isotropic root  $\delta$  (Proposition 3.3),  $\Delta^\lambda \subseteq \Delta_i$  is equivalent to  $\lambda(\delta^{\sharp}) = 0$ . As  $\delta^{\sharp}$  generates  $\mathfrak{z}(\mathfrak{g}_c)$  (Lemma 2.7), this in turn is equivalent to  $\lambda|_{\mathfrak{z}(\mathfrak{g}_c)} \neq 0$ .

That weights of transversal type actually exist can be derived from the description of the affine root system  $\Delta$  in terms of the locally finite subsystem  $\Delta_{\text{red}}$ . Since the assertion is trivial for the finite-dimensional case, we may assume that  $\dim V = \infty$ . From Yoshii's classification in Theorem 5.1 below, it follows that

$$\Delta \subseteq (\Delta_{\text{red}} + \mathbb{Z}\delta) \cup (2(\Delta_{\text{red}})_{\text{sh}} + (2\mathbb{Z} + 1)\delta).$$

From this information one can easily calculate the possible coroots. For a root of the form  $n\alpha + m\delta$ ,  $\alpha \in \Delta_{\text{red}}$ ,  $n \in \{1, 2\}$ , the corresponding coroot is determined by  $(n\alpha + m\delta)^{\sim} \in \mathbb{Q}(n\alpha^{\sharp} + m\delta^{\sharp})$  and  $(n\alpha + m\delta)((n\alpha + m\delta)^{\sim}) = 2$ , which leads to

$$(3.1) \quad (n\alpha + m\delta)^{\sim} = \frac{2}{n^2(\alpha, \alpha)}(n\alpha^{\sharp} + m\delta^{\sharp}), \quad n \in \{1, 2\}, m \in \mathbb{Z}.$$

Therefore a linear functional  $\lambda \in \mathfrak{h}^*$  vanishing on  $\check{\Delta}_{\text{red}}$  is integral if

$$\lambda(\delta^\sharp) \in 2(\alpha, \alpha)\mathbb{Z}$$

holds for each integrable root  $\alpha$ . In this case we have  $\Delta^\lambda = \Delta_{\text{red}}$ . Since at most three square lengths occur (Proposition 1.7), this proves the existence of integral weights  $\lambda$  of transversal type.  $\square$

REMARK 3.5. (i) If  $\Delta = \Delta_i$ , then  $\mathfrak{g}$  is locally finite ([Ne00b, Thm. VI.3]). If, in addition,  $\mathfrak{g}$  is perfect and  $\Delta_i$  connected, then  $\mathfrak{g}$  carries the structure of a locally extended affine Lie algebra for which  $(\cdot, \cdot)$  is positive definite on  $\text{span}_{\mathbb{Q}} \Delta$  ([LN04, Thm. 4.2]). In this case  $\Delta^\lambda \subseteq \Delta_i$  trivially holds for any integral weight  $\lambda$ .

(ii) Suppose that  $\mathfrak{g}$  is affine and that  $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta$  is a generating linearly independent simple system. Let  $\lambda \in \mathfrak{h}^*$  be a dominant integral weight not vanishing on all coroots, i.e.,

$$\Pi_\lambda := \{\alpha \in \Pi: \lambda(\check{\alpha}) = 0\} \neq \Pi.$$

We claim that  $\lambda$  does not vanish on the center. Using the notation of [Ka90], we write a generator of the center as  $K = \sum_{j=1}^r \check{a}_j \check{\alpha}_j$ , where all coefficients  $\check{a}_j$  are positive. Then

$$\lambda(K) = \sum_{\alpha_i \notin \Pi_\lambda} \check{a}_i \lambda(\check{\alpha}_i) > 0.$$

**Minimal locally affine Lie algebras.** The following notion of minimality distinguishes a class of locally affine Lie algebras which, as we shall see, are uniquely determined by their root systems.

DEFINITION 3.6. We call a locally affine Lie algebra  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  *minimal* if  $\mathfrak{g}_c$  is a hyperplane in  $\mathfrak{g}$  and there exists an element  $d \in \mathfrak{h}$  for which  $\{\alpha \in \Delta_i: \alpha(d) = 0\}$  is a reflectable section. Then  $\delta(d) \neq 0$  and we may normalize  $d$  by  $\delta(d) = 1$  (cf. Proposition 3.3).

To analyze how minimal locally affine Lie algebras can be reconstructed from their core, we need the concept of a double extension of a quadratic Lie algebra (cf. [MR85]).

DEFINITION 3.7. Let  $(\bar{\mathfrak{g}}, \bar{\kappa})$  be a quadratic Lie algebra and  $D \in \text{der}(\bar{\mathfrak{g}}, \bar{\kappa})$  be a derivation which is skew-symmetric with respect to  $\bar{\kappa}$ . Then  $\omega_D(x, y) := \bar{\kappa}(Dx, y)$  defines a 2-cocycle on  $\bar{\mathfrak{g}}$  and  $D$  extends to a derivation  $\tilde{D}(z, x) := (0, Dx)$  of the corresponding central extension  $\mathbb{K} \oplus_{\omega_D} \bar{\mathfrak{g}}$ . The Lie algebra

$$\mathfrak{g} = (\mathbb{K} \oplus_{\omega_D} \bar{\mathfrak{g}}) \rtimes_{\tilde{D}} \mathbb{K}$$

with the Lie bracket

$$[(z, x, t), (z', x', t')] = (\omega_D(x, x'), [x, x'] + tDx' - t'Dx, 0)$$

is called the corresponding *double extension*. It carries a non-degenerate invariant symmetric bilinear form

$$\kappa((z, x, t), (z', x', t)) = \bar{\kappa}(x, x') + zt' + z't,$$

so that  $(\mathfrak{g}, \kappa)$  also is a quadratic Lie algebra.

REMARK 3.8. (a) Each affine Kac–Moody algebra is a minimal locally affine Lie algebra.

(b) If  $\mathfrak{g}_c$  is the core of a locally affine Lie algebra, then  $\mathfrak{g}_c$  is graded by the root group  $\mathcal{Q} = \text{span}_{\mathbb{Z}} \Delta$ . Let  $\lambda \in \mathfrak{h}^*$  not vanish on the center and pick  $c \in \mathfrak{z}(\mathfrak{g}_c)$  with  $\lambda(c) = 1$ . We extend  $\lambda$  to a linear functional, also called  $\lambda$ , on  $\mathfrak{g}$ , vanishing on all root spaces. Then  $\ker \lambda \subseteq \mathfrak{g}_c$  is a subspace mapped bijectively onto the centerless core  $\mathfrak{g}_{cc} := \mathfrak{g}_c / \mathfrak{z}(\mathfrak{g}_c)$ . For  $x, x' \in \ker \lambda$  and  $a, a' \in \mathbb{K}$  we have

$$[x + ac, x' + a'c] = [x, x'] = ([x, x'] - \lambda([x, x'])c) + \lambda([x, x'])c,$$

so that  $\mathfrak{g}_c$  is the central extension  $\mathbb{K} \oplus_{\omega} \mathfrak{g}_{cc}$  defined by the cocycle  $\omega(\bar{x}, \bar{x}') = \lambda([x, x'])$ , where  $\bar{x} := x + \mathfrak{z}(\mathfrak{g}_c)$ .

Next we observe that  $\lambda$  defines a diagonal derivation  $\tilde{D}_{\lambda} \in \text{der}(\mathfrak{g}_c)$  by

$$\tilde{D}_{\lambda} x = \lambda(\alpha^{\sharp}) x \quad \text{for } x \in \mathfrak{g}_{\alpha}.$$

This derivation also induces a derivation  $D$  on the centerless core  $\mathfrak{g}_{cc} = \mathfrak{g}_c / \mathfrak{z}(\mathfrak{g}_c)$  preserving the induced non-degenerate symmetric bilinear form  $\bar{\kappa}$ .

(c) Let  $\widehat{\mathfrak{g}} = (\mathbb{K} \oplus_{\omega} \mathfrak{g}_{cc}) \rtimes_D \mathbb{K}$  be the corresponding double extension (Definition 3.7). With  $\mathfrak{h}_c := \mathfrak{h} \cap \mathfrak{g}_c$  we now obtain a splitting Cartan subalgebra  $\widehat{\mathfrak{h}} := \mathbb{R} \oplus \mathfrak{h}_c \oplus \mathbb{R}$  of  $\widehat{\mathfrak{g}}$ , for which the corresponding root decomposition coincides with the  $\mathcal{Q}$ -grading. In particular, we obtain a realization of the locally affine root system  $\Delta$  in  $\mathfrak{h}^*$ .

To obtain a minimal locally affine Lie algebra  $\widehat{\mathfrak{g}}$  with this procedure, we have to assume, in addition, that  $\Delta^{\lambda} = \{\alpha \in \Delta : \lambda(\check{\alpha}) = 0\} = \Delta_{\text{red}}$  holds for a reflectable section. Since such functionals exist by Proposition 3.4, we derive the existence of a minimal realization for  $\Delta$ . We refer to the appendix for minimal realizations of the infinite rank affine root systems by twisted loop algebras.

(d) Let us assume, in addition, that there exists an element  $d \in \mathfrak{h}$  with  $\lambda = d^{\flat}$ . Then  $\lambda(\alpha^{\sharp}) = \kappa(d, \alpha^{\sharp}) = \alpha(d)$  implies that  $\tilde{D} = \text{ad } d$  and

$$\omega_D(\bar{x}, \bar{x}') = \kappa([d, x], x') = \kappa(d, [x, x']) = \lambda([x, x']).$$

We also note that for  $x, x' \in \ker \lambda \cap \mathfrak{g}_c$  and  $a, a', b, b' \in \mathbb{K}$  we have

$$\kappa(ac + x + bd, a'c + x' + b'd) = \kappa(x, x') + ba' + ab',$$

which shows that with  $\mathfrak{h}_c := \mathfrak{h} \cap \mathfrak{g}_c$  we obtain an isomorphism

$$(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}}, \widehat{\kappa}) \cong (\mathfrak{g}_c + \mathbb{K}d, \mathfrak{h}_c + \mathbb{K}d, \kappa)$$

of split quadratic Lie algebras. We thus find a minimal locally affine subalgebra of  $\mathfrak{g}$ .

### The Extension Theorem.

DEFINITION 3.9. Two locally extended affine root systems  $(V_1, \mathcal{R}_1)$  and  $(V_2, \mathcal{R}_2)$  are said to be isomorphic if there exists a linear isomorphism  $\psi: V_1 \rightarrow V_2$  with  $\psi(\mathcal{R}_1) = \mathcal{R}_2$ .

Since the quadratic form is part of the concept of a LEARS, it should also be taken into account for the concept of an isomorphism, but the following lemma shows that this is redundant ([YY08, Lemma 9]):

LEMMA 3.10. *If  $\varphi: (V_1, \mathcal{R}_1) \rightarrow (V_2, \mathcal{R}_2)$  is an isomorphism of locally extended affine root systems, then*

$$(3.2) \quad \langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle \quad \text{for } \alpha, \beta \in \mathcal{R}_1,$$

*$\varphi$  preserves the quadratic form up to a factor, and  $\varphi \circ r_\alpha \circ \varphi^{-1} = r_{\varphi(\alpha)}$  for  $\alpha \in \mathcal{R}_1$ .*

Together with Proposition 3.3, the following lemma is the key ingredient in our Extension Theorem. It provides the required local information.

LEMMA 3.11. *Let  $(\mathfrak{g}_1, \mathfrak{h}_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2)$  be affine Kac–Moody Lie algebras and  $\psi: \Delta_1 \rightarrow \Delta_2$  an isomorphism of affine root systems. Further, let  $\mathcal{B} \subseteq \Delta_{1,i}$  be an integral base and pick  $0 \neq x_\alpha \in \mathfrak{g}_{1,\alpha}$  and  $0 \neq y_\alpha \in \mathfrak{g}_{2,\psi(\alpha)}$  for  $\alpha \in \mathcal{B}$ . Then there exists a unique isomorphism of Lie algebras*

$$\varphi: (\mathfrak{g}_1)_c \rightarrow (\mathfrak{g}_2)_c \quad \text{with} \quad \varphi(\mathfrak{g}_{1,\alpha}) = \mathfrak{g}_{2,\psi(\alpha)} \quad \text{for } \alpha \in \Delta_1$$

*and  $\varphi(x_\alpha) = y_\alpha$  for  $\alpha \in \mathcal{B}$ .*

PROOF. Let  $\Pi_1 \subseteq \Delta_1$  be a linearly independent generating simple system and  $\Pi_2 := \psi(\Pi_1)$ . In view of (3.2),

$$\psi(\alpha)(\psi(\beta)) = \langle \psi(\alpha), \psi(\beta) \rangle = \langle \alpha, \beta \rangle = \alpha(\tilde{\beta}) \quad \text{for } \alpha, \beta \in \Pi_1,$$

so that  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  correspond to the same generalized Cartan matrix, hence are isomorphic (cf. [Ka90, Ch. 1]).

Let  $\gamma: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  be an isomorphism with  $\gamma(\mathfrak{h}_1) = \mathfrak{h}_2$ , inducing the isomorphism  $\varphi: \Delta_1 \rightarrow \Delta_2$ . Since the non-isotropic root spaces  $\mathfrak{g}_{2,\psi(\alpha)}$  are 1-dimensional, there exist scalars  $\lambda_\alpha \in \mathbb{K}^\times$  with

$$\gamma(x_\alpha) = \lambda_\alpha y_\alpha \quad \text{for } \alpha \in \mathcal{B}.$$

Since  $\mathcal{B} \subseteq \Delta_1$  is linearly independent, there exists a group homomorphism

$$\chi: \text{span}_{\mathbb{Z}} \mathcal{B} \rightarrow \mathbb{K}^\times \quad \text{with} \quad \chi(\alpha) = \lambda_\alpha \quad \text{for } \alpha \in \mathcal{B}.$$

Then  $\varphi_\chi(x) := \chi(\alpha)x$  for  $x \in \mathfrak{g}_{1,\alpha}$  defines an automorphism of Lie algebras (Lemma 1.12) and

$$\varphi := \gamma \circ \varphi_\chi^{-1}: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$$

maps each  $x_\alpha$ ,  $\alpha \in \mathcal{B}$ , to the corresponding element  $y_\alpha \in \mathfrak{g}_2$ . This proves the existence of  $\varphi$ .

For the uniqueness, we assume that  $\tilde{\varphi}: (\mathfrak{g}_1)_c \rightarrow (\mathfrak{g}_2)_c$  is another isomorphism with the same properties. Then  $\Phi := \tilde{\varphi}^{-1} \circ \varphi: (\mathfrak{g}_1)_c \rightarrow (\mathfrak{g}_1)_c$  is an isomorphism preserving each root space and fixing each  $x_\alpha$ ,  $\alpha \in \mathcal{B}$ . We have to show that this implies that  $\Phi = \text{id}_{(\mathfrak{g}_1)_c}$ .

On each 3-dimensional subalgebra  $\mathfrak{g}(\beta)$ ,  $\beta \in \Delta_{1,i}$ ,  $\Phi$  induces an automorphism preserving the root decomposition. This implies that  $\Phi(\tilde{\beta}) = \tilde{\beta}$  and that  $\Phi(x_\beta) = \mu_\beta x_\beta$  for some  $\mu_\beta \in \mathbb{K}^\times$ . Let  $\Pi_1 = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta_1$  be a generating simple system and  $\mu_j := \mu_{\alpha_j}$ . Let  $\nu: \text{span}_{\mathbb{Z}} \Pi_1 \rightarrow \mathbb{K}^\times$  be the unique group homomorphism mapping  $\alpha_j$  to  $\mu_j$ . Then  $\varphi_\nu \in \text{Aut}(\mathfrak{g}_1)$  is the unique automorphism fixing  $\mathfrak{h}_1$  pointwise and multiplying each  $x_{\alpha_j}$  with  $\mu_j$ . We conclude that  $\Phi = \varphi_\nu$ , which implies that  $\nu(\alpha) = 1$  for each  $\alpha \in \mathcal{B}$ . Now  $\Delta \subseteq \text{span}_{\mathbb{Z}} \mathcal{B}$  finally leads to  $\nu = 1$ , so that  $\Phi = \varphi_\nu = \text{id}_{(\mathfrak{g}_1)_c}$ .  $\square$



**THEOREM 3.12 (Extension Theorem).** *Let  $(\mathfrak{g}_1, \mathfrak{h}_1, \kappa_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2, \kappa_2)$  be locally affine Lie algebras. If  $\psi: (V_1, \Delta_1) \rightarrow (V_2, \Delta_2)$  is an isomorphism of locally affine root systems, then there exists an isomorphism of Lie algebras*

$$\varphi: (\mathfrak{g}_1)_c \rightarrow (\mathfrak{g}_2)_c \quad \text{with} \quad \varphi(\mathfrak{g}_{1,\alpha}) = \mathfrak{g}_{2,\psi(\alpha)} \quad \text{for} \quad \alpha \in \Delta_1.$$

*If  $\mathcal{B} \subseteq \Delta_{1,i}$  is an integral base  $0 \neq x_\alpha \in \mathfrak{g}_{1,\alpha}$ ,  $0 \neq y_\alpha \in \mathfrak{g}_{2,\psi(\alpha)}$  for  $\alpha \in \mathcal{B}$ , then there exists a unique such  $\varphi$  with*

$$(3.3) \quad \varphi(x_\alpha) = y_\alpha \quad \text{for} \quad \alpha \in \mathcal{B}.$$

**PROOF.** Let  $\mathcal{B} \subseteq \Delta_{1,i}$  be an integral base (Proposition 3.3(iv)). For each  $\alpha \in \mathcal{B}$ , we pick non-zero elements  $x_\alpha \in \mathfrak{g}_{1,\alpha}$  and  $y_\alpha \in \mathfrak{g}_{2,\psi(\alpha)}$ .

Let  $F \subseteq \mathcal{B}$  be a connected finite subset with  $\delta \in \text{span}_{\mathbb{Z}} F$  and  $\Delta^F := \Delta \cap \text{span } F$ , so that the subalgebra  $\mathfrak{g}_1(\Delta_i^F) \subseteq \mathfrak{g}_1$  is the core of an affine Kac–Moody algebra and  $F \subseteq \Delta^F$  is an integral base (Proposition 3.3). With Lemma 3.11 we now obtain a unique isomorphism

$$\varphi_F: (\mathfrak{g}_{1,F})_c \rightarrow (\mathfrak{g}_{2,\psi(F)})_c \quad \text{with} \quad \varphi_F(x_\alpha) = y_\alpha \quad \text{for} \quad \alpha \in F.$$

For any larger finite subset  $E \supseteq F$  with the same properties, we likewise obtain a unique isomorphism

$$\varphi_E: (\mathfrak{g}_{1,E})_c \rightarrow (\mathfrak{g}_{2,\psi(E)})_c \quad \text{with} \quad \varphi_E(x_\alpha) = y_\alpha \quad \text{for} \quad \alpha \in E,$$

and the uniqueness of  $\varphi_F$  implies that  $\varphi_E|_{\mathfrak{g}_F} = \varphi_F$ . We conclude that the isomorphisms  $\varphi_F$  combine to a unique isomorphism

$$\varphi: \mathfrak{g}_{1,c} = \bigcup_F (\mathfrak{g}_{1,F})_c \rightarrow \mathfrak{g}_{2,c} = \bigcup_F (\mathfrak{g}_{2,F})_c \quad \text{with} \quad \varphi(x_\alpha) = y_\alpha \quad \text{for} \quad \alpha \in \mathcal{B}.$$

□

**COROLLARY 3.13.** *If  $(\mathfrak{g}_1, \mathfrak{h}_1, \kappa_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2, \kappa_2)$  are locally affine Lie algebras with isomorphic root systems, then their cores are isomorphic.*

**REMARK 3.14.** In general there is no unique extension of  $\varphi$  to all of  $\mathfrak{g}_1$ . If  $h_1 \in \mathfrak{h}_1$ , then any such extension mapping  $\mathfrak{h}_1$  into  $\mathfrak{h}_2$  would have to map  $h$  to an element  $h_2$  satisfying  $\psi(\alpha)(h_2) = \alpha(h_1)$  for each  $\alpha \in \Delta_1$ . This determines  $h_2$  uniquely up to a central element. On the other hand, every linear map  $\mathfrak{g}_1/[\mathfrak{g}_1, \mathfrak{g}_1] \rightarrow \mathfrak{z}(\mathfrak{g}_2)$  is a homomorphism of Lie algebras that can be added to any homomorphism  $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ .

**THEOREM 3.15 (Uniqueness Theorem).** *If  $(\mathfrak{g}_1, \mathfrak{h}_1, \kappa_1)$  and  $(\mathfrak{g}_2, \mathfrak{h}_2, \kappa_2)$  are minimal locally affine Lie algebras with isometrically isomorphic root systems, then there exists an isomorphism  $\varphi: (\mathfrak{g}_1, \mathfrak{h}_1, \kappa_1) \rightarrow (\mathfrak{g}_2, \mathfrak{h}_2, \kappa_2)$  of quadratic split Lie algebras.*

**PROOF.** Since both Lie algebras  $\mathfrak{g}_j$  are minimal locally affine, there exist  $d_j \in \mathfrak{h}_j$ ,  $j = 1, 2$ , such that

$$\Delta_{j,\text{red}} := \{\alpha \in \Delta_j : \alpha(d_j) = 0\}$$

define reflectable sections and  $\delta_j(d_j) = 1$  holds for the respective basic isotropic roots  $\delta_j$ . In view of Theorem 5.2 below, there exists an isomorphism  $\psi: \Delta_{1,i} \rightarrow \Delta_{2,i}$  of root systems mapping  $\Delta_{1,\text{red}}$  to  $\Delta_{2,\text{red}}$ , and we may further assume that  $\psi(\delta_1) = \delta_2$  (which can be achieved by replacing  $\psi$  by  $-\psi$  if necessary). Then  $\psi(\alpha)(d_2) = \alpha(d_1)$  holds for  $\alpha \in \Delta_{1,\text{red}}$  and also for  $\alpha = \delta_1$ , hence for each  $\alpha \in \Delta_1$ .

Now we apply Theorem 3.12 to obtain an isomorphism  $\varphi: \mathfrak{g}_{1,c} \rightarrow \mathfrak{g}_{2,c}$  with  $\varphi(\mathfrak{g}_{1,\alpha}) = \mathfrak{g}_{2,\psi(\alpha)}$  for  $\alpha \in \Delta_{1,i}$ . For  $x \in \mathfrak{g}_{1,\alpha}$  we have

$$\varphi([d_1, x]) = \alpha(d_1)\varphi(x) = \psi(\alpha)(d_2)\varphi(x) = [d_2, \varphi(x)],$$

which implies that  $\varphi \circ \text{ad } d_1 = \text{ad } d_2 \circ \varphi$ .

Next we claim that  $\varphi$  is isometric. To this end, we consider the symmetric bilinear form  $\kappa := \varphi^* \kappa_2 - \kappa_1$  on  $\mathfrak{g}_{1,c}$ . To see that  $\kappa$  vanishes, we note that its radical is an ideal of  $\mathfrak{g}_{1,c}$ . If we can show that it contains  $\check{\Delta}_{i,1}$ , then it also contains all subalgebras  $\mathfrak{g}(\alpha)$ ,  $\alpha \in \Delta_i$ , and therefore all of  $\mathfrak{g}_{1,c}$ . It therefore remains to show that for  $\alpha, \beta \in \Delta_{1,i}$  we have

$$(3.4) \quad \kappa_2(\varphi(\check{\alpha}), \varphi(\check{\beta})) = \kappa_1(\check{\alpha}, \check{\beta}).$$

From  $\varphi(\mathfrak{g}_{1,\alpha}) = \mathfrak{g}_{2,\psi(\alpha)}$  we derive that  $\varphi(\check{\alpha}) = \psi(\alpha)$ , so that

$$\kappa_1(\check{\alpha}, \check{\beta}) = \frac{4(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} = \frac{4(\psi(\alpha), \psi(\beta))}{(\psi(\alpha), \psi(\alpha))(\psi(\beta), \psi(\beta))} = \kappa_2(\varphi(\check{\alpha}), \varphi(\check{\beta}))$$

implies (3.4). This proves that  $\varphi: \mathfrak{g}_{1,c} \rightarrow \mathfrak{g}_{2,c}$  is isometric.

In particular, we derive that the induced isomorphism  $\bar{\varphi}: \mathfrak{g}_{1,cc} \rightarrow \mathfrak{g}_{2,cc}$  of the centerless cores is isometric and intertwines the derivations  $D_j$  induced by  $\text{ad } d_j$  on  $\mathfrak{g}_{j,cc}$ . Therefore  $\bar{\varphi}$  extends to an isomorphism

$$\widehat{\varphi}: \widehat{\mathfrak{g}}_1 \rightarrow \widehat{\mathfrak{g}}_2, \quad (z, x, t) \mapsto (z, \bar{\varphi}(x), t)$$

of the corresponding double extensions (Definition 3.7). Finally, Remark 3.8(d) implies that  $\widehat{\mathfrak{g}}_j \cong \mathfrak{g}_j$ , and the assertion follows.  $\square$

**COROLLARY 3.16.** *If  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a minimal locally affine Lie algebra and  $\mathfrak{h}'$  is another splitting Cartan subalgebra for which the corresponding root system  $\Delta'$  is isomorphic to  $\Delta$ , then there exists an automorphism  $\varphi$  of  $\mathfrak{g}$  with  $\varphi(\mathfrak{h}) = \mathfrak{h}'$ .*

**Unitary real forms.** In this subsection, we consider complex Lie algebras and suitable real forms which are compatible with all the relevant structure. The main point is the existence of unitary real forms of minimal locally affine Lie algebras.

def:unitaryrealform

**DEFINITION 3.17.** An *involution* of a complex quadratic split Lie algebra  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is an involutive antilinear antiautomorphism  $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}, x \mapsto x^*$  satisfying

- (I1)  $\alpha(x) \in \mathbb{R}$  for  $x = x^* \in \mathfrak{h}$ .
- (I2)  $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$  for  $\alpha \in \Delta$ .
- (I3)  $\kappa(\sigma(x), \sigma(y)) = \overline{\kappa(x, y)}$  for  $x, y \in \mathfrak{g}$ .

Then we call  $(\mathfrak{g}, \mathfrak{h}, \kappa, \sigma)$  an *involution quadratic split Lie algebra* and write

$$\mathfrak{k} := \mathfrak{k}(\sigma) := \{x \in \mathfrak{g} : x^* = -x\}$$

for the corresponding real form of  $\mathfrak{g}$ . We call  $\sigma$  and the corresponding real form  $\mathfrak{k}$  *unitary* if the hermitian form

$$\kappa_\sigma(x, y) := \kappa(x, \sigma(y))$$

is positive semidefinite on  $\mathfrak{g}_c$ .

**LEMMA 3.18.** *Let  $(\mathfrak{g}, \mathfrak{h}, \kappa, \sigma)$  be an involutive affine Kac–Moody algebra with a unitary real form. For  $\alpha \in \Delta_i$ , pick  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  with  $[x_\alpha, x_{-\alpha}] = \check{\alpha}$ . Then*

$$x_\alpha^* = \lambda_\alpha x_{-\alpha} \quad \text{for some } \lambda_\alpha > 0.$$

PROOF. It is easy to verify this by computation because the positivity of  $\lambda_\alpha$  is equivalent to  $\mathfrak{k} \cap \mathfrak{g}(\alpha) \cong \mathfrak{su}_2(\mathbb{C})$ , the compact real form of  $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{g}(\alpha)$ .  $\square$

PROPOSITION 3.19. *If  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a complex minimal locally affine Lie algebra, then  $\mathfrak{g}$  has a unitary real form.*

PROOF. Let  $\bar{\mathfrak{g}}$  denote the same Lie algebra  $\mathfrak{g}$ , endowed with the complex conjugate scalar multiplication  $z \bullet x := \bar{z}x$ . Then  $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}, \bar{\kappa})$  also is a complex minimal locally affine Lie algebra with  $\bar{\mathfrak{g}}_\alpha = \mathfrak{g}_{-\bar{\alpha}}$ . Note that  $\bar{\alpha}: \bar{\mathfrak{h}} \rightarrow \mathbb{C}$  is complex linear because  $\bar{\mathfrak{h}}$  carries the opposite complex structure.

Now  $\psi: \Delta(\mathfrak{g}, \mathfrak{h}) \rightarrow \Delta(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}), \alpha \mapsto -\bar{\alpha}$  is an isometric isomorphism of locally affine root systems. With Theorem 3.15 we obtain an isometric isomorphism

$$\tilde{\sigma}: (\mathfrak{g}, \kappa) \rightarrow (\bar{\mathfrak{g}}, \bar{\kappa}) \quad \text{with} \quad \tilde{\sigma}(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\bar{\alpha}} \quad \text{and} \quad \tilde{\sigma}(d) = -d.$$

We next define an antilinear map  $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}, x \mapsto -\tilde{\sigma}(x)$  and obtain an involutive antiautomorphism of  $\mathfrak{g}_c$  satisfying  $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$  for each  $\alpha \in \Delta$ .

Let  $\mathcal{B} \subseteq \Delta_i$  be an integral base and pick  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}, \alpha \in \mathcal{B}$ , in such a way that  $[x_\alpha, x_{-\alpha}] = \check{\alpha}$ . In view of Theorem 3.12 (cf. also Lemma 1.12), we may choose  $\sigma$  in such a way that  $\sigma(x_\alpha) = x_{-\alpha}$ . Then  $\sigma^2(x_\alpha) = x_\alpha$  for each  $\alpha \in \mathcal{B}$ , so that the uniqueness assertion in Theorem 3.12 implies that  $\sigma^2 = \text{id}$  on  $\mathfrak{g}_c$ , so that  $\sigma(d) = d$  leads to  $\sigma^2 = \text{id}_{\mathfrak{g}}$ , i.e.,  $\sigma$  defines a real form  $\mathfrak{k} := \{x \in \mathfrak{g}: \sigma(x) = x\}$ .

To see that  $\mathfrak{k}$  is unitary, we first show that whenever  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  satisfy  $[x_\alpha, x_{-\alpha}] = \check{\alpha}$ , then  $\sigma(x_\alpha) = \lambda_\alpha x_{-\alpha}$  for some real  $\lambda_\alpha > 0$  (Lemma 3.18). Let  $V_F$  be as in Proposition 3.3(iv) and  $\mathfrak{g}_F \subseteq \mathfrak{g}$  be a corresponding affine Kac–Moody subalgebra. Then  $\sigma(\mathfrak{g}_{F,c}) = \mathfrak{g}_{F,c}$ , and  $\sigma$  induces an involution on the core  $\mathfrak{g}_{F,c}$  of  $\mathfrak{g}_F$ . We know from [Ka90, §2.7 and Thm. 11.7] that there exists a unitary involution  $\sigma_c$  on  $\mathfrak{g}_F$ . We thus obtain a complex linear automorphism

$$\varphi := \sigma \circ \sigma_c: \mathfrak{g}_{F,c} \rightarrow \mathfrak{g}_{F,c}$$

preserving all root spaces and satisfying  $\varphi(x_\alpha) = \mu_\alpha x_\alpha$  with  $\mu_\alpha > 0$  for each  $\alpha \in F \subseteq \mathcal{B}$  and  $x_{\pm\alpha}$  as above. Since  $F$  is an integral base of  $\Delta_F$ , we have  $\varphi|_{\mathfrak{g}_F} = \varphi_\chi$  for a homomorphism  $\chi: \text{span}_{\mathbb{Z}} \Delta_F \rightarrow \mathbb{C}^\times$  with  $\chi(\alpha) = \mu_\alpha$  for  $\alpha \in \mathcal{B}$  (cf. Lemma 1.12). Then  $\text{im}(\chi) \subseteq \mathbb{R}_+^\times$ , so that  $\chi(\alpha) > 0$  for each root.

For the corresponding hermitian form, we therefore have on  $\mathfrak{g}_\alpha, \alpha \in \Delta_F$ :

$$\kappa_\sigma(x, x) = \kappa(x, \sigma(x)) = \chi(\alpha) \kappa(x, \sigma_c(x)) \geq 0$$

and on  $\text{span } \check{F} \subseteq \mathfrak{h}$  we have

$$\kappa_\sigma(x, x) = \kappa(x, \sigma(x)) = \kappa(x, \sigma_c(x)) \geq 0.$$

Therefore  $\sigma$  is unitary.  $\square$

#### 4. Highest weight modules

In this section we construct for each integral weight  $\lambda$  of a coral locally affine Lie algebra  $\mathfrak{g}$  a simple module  $L(\lambda, \mathfrak{p}_\lambda)$  and show that it is integrable if  $\lambda$  does not vanish on  $\mathfrak{z}(\mathfrak{g}_c)$ . For complex Lie algebras, we further show that for  $\lambda = \lambda^*$ , the module  $L(\lambda, \mathfrak{p}_\lambda)$  is unitary with respect to any unitary real form  $\mathfrak{k}$  of  $\mathfrak{g}$ .

DEFINITION 4.1. Let  $(\mathfrak{g}, \mathfrak{h})$  be a split Lie algebra. A triple  $(\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-)$  is said to define a *split triangular decomposition* if there exist subsets  $\Sigma^0, \Sigma^\pm \subseteq \Delta$  such that

$$(T1) \quad \Delta = \Sigma^+ \dot{\cup} \Sigma^0 \dot{\cup} \Sigma^- \quad \text{is a partition of } \Delta.$$

$$(T2) \quad \mathfrak{g}^\pm = \sum_{\alpha \in \Sigma^\pm} \mathfrak{g}_\alpha \text{ and } \mathfrak{g}^0 = \mathfrak{h} + \sum_{\alpha \in \Sigma^0} \mathfrak{g}_\alpha.$$

$$(T3) \quad [\mathfrak{g}^0, \mathfrak{g}^\pm] \subseteq \mathfrak{g}^\pm.$$

$$(T4) \quad \sum_{i=1}^n \alpha_i \neq 0 \text{ for } \alpha_i \in \Sigma^- \text{ and } n > 0.$$

Then  $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+$  is a direct sum of vector spaces and  $\mathfrak{g}^\pm \rtimes \mathfrak{g}^0$  are subalgebras (sometimes called *generalized parabolics*).

DEFINITION 4.2. Let  $(\mathfrak{g}^+, \mathfrak{g}^0, \mathfrak{g}^-)$  be a split triangular decomposition,  $\mathfrak{p} := \mathfrak{g}^0 + \mathfrak{g}^+$  and  $\lambda \in \mathfrak{h}^*$ . We extend  $\lambda$  to a linear functional  $\lambda: \mathfrak{p} \rightarrow \mathbb{K}$  vanishing on all root spaces. We assume that  $\lambda([\mathfrak{p}, \mathfrak{p}]) = \lambda([\mathfrak{g}^0, \mathfrak{g}^0]) = \{0\}$ , so that  $\lambda$  defines a one-dimensional  $\mathfrak{p}_\lambda$ -module  $\mathbb{K}_\lambda$ . We write

$$M(\lambda, \mathfrak{p}) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{K}_\lambda$$

for the corresponding *generalized Verma module* (cf. [JK85]). This is a  $\mathfrak{g}$ -module generated by a 1-dimensional  $\mathfrak{p}$ -module  $[1 \otimes \mathbb{K}_\lambda]$  isomorphic to  $\mathbb{K}_\lambda$ , and each other  $\mathfrak{g}$ -module with this property is a quotient of  $M(\lambda, \mathfrak{p})$ .

The Poincaré–Birkhoff–Witt Theorem implies that the multiplication map  $\mathcal{U}(\mathfrak{g}^-) \otimes \mathcal{U}(\mathfrak{p}) \rightarrow \mathcal{U}(\mathfrak{g})$  is a linear isomorphism, so that

$$M(\lambda, \mathfrak{p}) \cong \mathcal{U}(\mathfrak{g}_-) \otimes_{\mathbb{K}} \mathbb{K}_\lambda$$

as  $\mathfrak{h}$ -modules. In particular,  $M(\lambda, \mathfrak{p})$  has an  $\mathfrak{h}$ -weight decomposition, all weights are contained in the set  $\lambda + \text{span}_{\mathbb{N}_0} \Sigma^-$ , and (T4) implies that the multiplicity of the weight  $\lambda$  is 1. Therefore  $M(\lambda, \mathfrak{p})$  contains a unique maximal submodule, namely the sum of all submodules not containing  $v_\lambda$ . This is a submodule whose set of weights does not contain  $\lambda$ , therefore it is proper (cf. [Ne00a, Prop. IX.1.12] and [JK85]). We write  $L(\lambda, \mathfrak{p})$  for the corresponding unique simple quotient and call it the  *$\mathfrak{p}$ -highest weight module* defined by  $\lambda$ .

**Highest weight modules for integral weights.** Let  $\mathfrak{g}$  be a coral locally extended affine Lie algebra and  $\lambda \in \mathfrak{h}^*$  an integral weight. We have seen in Theorem 1.8 that  $\mathfrak{g}^q(\lambda) := \bigoplus_{\lambda(\alpha^\#)=q} \mathfrak{g}_\alpha$  defines a grading of  $\mathfrak{g}$  by a cyclic subgroup of  $\mathbb{Q}$ . We claim that the three sets

$$\Sigma^\pm := \{\alpha \in \Delta: \pm \lambda(\alpha^\#) > 0\} \quad \text{and} \quad \Sigma^0(\lambda) := \{\alpha \in \Delta: \lambda(\alpha^\#) = 0\},$$

resp., the corresponding subalgebras

$$\mathfrak{g}^0(\lambda) := \sum_{\lambda(\alpha^\#)=0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}^\pm(\lambda) := \sum_{\pm q > 0} \mathfrak{g}_q(\lambda)$$

define a split triangular decomposition. Clearly (T1/2) hold by definition, (T3) follows from  $(\Sigma^0(\lambda) + \Sigma^\pm(\lambda)) \cap \Delta \subseteq \Sigma^\pm(\lambda)$ , and (T4) is an immediate consequence of the definition of  $\Sigma^-(\lambda)$ .

The linear functional  $\lambda \in \mathfrak{h}^*$  extends in a natural way to a linear functional

$$\lambda: \mathfrak{p}_\lambda := \mathfrak{g}^+(\lambda) \rtimes \mathfrak{g}^0(\lambda) = \sum_{\lambda(\alpha^\#) \geq 0} \mathfrak{g}_\alpha \rightarrow \mathbb{K}$$

vanishing on all root spaces. In view of the following lemma, we thus obtain a homomorphism of Lie algebras and we obtain from Definition 4.2 a simple  $\mathfrak{g}$ -module  $L(\lambda) := L(\lambda, \mathfrak{p}_\lambda)$ .

LEMMA 4.3.  $\lambda: \mathfrak{p}_\lambda \rightarrow \mathbb{K}$  is a homomorphism of Lie algebras, i.e., it vanishes on the commutator algebra.

PROOF. In view of  $\mathfrak{h} \subseteq \mathfrak{p}_\lambda$ , the commutator algebra of  $\mathfrak{p}_\lambda$  is adapted to the root decomposition. Therefore it suffices to observe that  $\lambda$  vanishes on

$$\mathfrak{h} \cap [\mathfrak{p}_\lambda, \mathfrak{p}_\lambda] = \sum_{\lambda(\alpha^\#)=0} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \sum_{\lambda(\alpha^\#)=0} \mathbb{K}\alpha^\#$$

(Remark 1.4(a)).  $\square$

DEFINITION 4.4. (a) A  $\mathfrak{g}$ -module  $M$  is said to be *integrable* if it has a weight decomposition with respect to  $\mathfrak{h}$  and, for each integrable root  $\alpha \in \Delta_i$ , it is a locally finite  $\mathfrak{g}(\alpha)$ -module (which is equivalent to  $\mathfrak{g}_{\pm\alpha}$  acting nilpotently on  $M$ ) (cf. [Ka90, Prop. 3.6]).

(b) A subset  $\Delta^+ \subseteq \Delta$  is called a *positive system* if

(PS1)  $\Delta = \Delta^+ \cup -\Delta^+$ .

(PS2)  $\sum_{i=1}^n \alpha_i \neq 0$  for  $\alpha_i \in \Delta^+$  and  $n > 0$ .

Note that (PS2) implies in particular that  $\Delta^+ \cap -\Delta^+ = \emptyset$  and if  $\alpha, \beta \in \Delta^+$  and  $\alpha + \beta$  is a root, then it is positive (cf. [Ne98, Lemma 1.2]). To each positive system  $\Delta^+$  corresponds a split triangular decomposition with  $\Sigma^\pm = \Delta^\pm$  and  $\Sigma^0 = \emptyset$ . The corresponding generalized parabolic subalgebra is

$$\mathfrak{p} := \mathfrak{p}(\Delta^+) := \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Since each  $\lambda \in \mathfrak{h}^*$  extends to a homomorphism  $\mathfrak{p} \rightarrow \mathbb{K}$ , we obtain a simple  $\mathfrak{g}$ -module  $L(\lambda, \Delta^+) := L(\lambda, \mathfrak{p}(\Delta^+))$  by the construction in Definition 4.2. It is the unique simple  $\mathfrak{g}$ -module of highest weight  $\lambda$ , i.e., generated by a  $\mathfrak{p}(\Delta^+)$ -weight vector of weight  $\lambda$ . We call these eigenvectors *primitive*.

REMARK 4.5. Let  $M$  be an integrable  $\mathfrak{g}$ -module and  $\pi: \mathfrak{g} \rightarrow \mathfrak{gl}(M)$  the corresponding representation.

(a) The set  $\mathcal{P}_M \subseteq \mathfrak{h}^*$  of  $\mathfrak{h}$ -weights of  $M$  consists of integral weights because the eigenvalues of  $\tilde{\alpha}$  on  $M$  are integral.

(b) By the definition of the integrable roots,  $\mathfrak{g}$  is an integrable  $\mathfrak{g}$ -module.

(c) If  $\alpha$  is integrable and  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ , then the operators  $\pi(x_{\pm\alpha})$  are locally nilpotent, so that  $r_\alpha^M := e^{\pi(x_\alpha)} e^{-\pi(x_{-\alpha})} e^{\pi(x_\alpha)} \in \text{GL}(M)$  is defined and satisfies

$$\pi(r_\alpha^M x) = r_\alpha^M \pi(x) (r_\alpha^M)^{-1} \quad \text{for } x \in \mathfrak{g}$$

(cf. [MP95, Prop. 6.1.3]). Since  $r_\alpha^M|_{\mathfrak{h}} = r_\alpha$ , it follows immediately that there exists for each  $w \in \mathcal{W}$  an element  $w^M \in \text{GL}(M)$  and an automorphism  $w^\mathfrak{g} \in \text{Aut}(\mathfrak{g})$  with  $w^\mathfrak{g}|_{\mathfrak{h}} = w$  and

$$\pi(w^\mathfrak{g} x) = w^M \pi(x) (w^M)^{-1} \quad \text{for } x \in \mathfrak{g}.$$

In particular, the representations  $\pi$  and its  $w^\mathfrak{g}$ -twist  $\pi \circ w^\mathfrak{g}$  are equivalent. This observation also implies that the weight set  $\mathcal{P}_M$  is  $\mathcal{W}$ -invariant.

PROPOSITION 4.6. *If  $\lambda \in \mathfrak{h}^*$  is an integral weight for which  $L(\lambda, \mathfrak{p}_\lambda)$  is integrable, then*

$$L(\lambda, \mathfrak{p}_\lambda) \cong L(w\lambda, \mathfrak{p}_{w\lambda}) \quad \text{for each } w \in \mathcal{W}.$$

PROOF. The main point is that the element  $w$  of the Weyl group is induced by an automorphism  $\varphi \in \text{Aut}(\mathfrak{g}, \mathfrak{h}, \kappa)$  with  $(\varphi^{-1})^* \mu = w\mu$  for  $\mu \in \mathfrak{h}^*$  (Remark 4.5(c)). Then

$$\varphi(\mathfrak{p}_\lambda) = \mathfrak{h} + \sum_{\lambda(\alpha^\#) \geq 0} \mathfrak{g}_{(\varphi^{-1})^* \alpha} = \mathfrak{h} + \sum_{\lambda(\alpha^\#) \geq 0} \mathfrak{g}_{w\alpha} = \mathfrak{h} + \sum_{(w\lambda)(\alpha^\#) \geq 0} \mathfrak{g}_\alpha = \mathfrak{p}_{w\lambda}.$$

On  $L(\lambda, \mathfrak{p}_\lambda)$  we now define the  $(\varphi^{-1})$ -twisted  $\mathfrak{g}$ -module structure by  $x \bullet v := \varphi^{-1}(x)v$ , for which  $[1 \otimes 1]$  becomes a  $\mathfrak{p}_{w\lambda}$ -eigenvector of weight  $w\lambda$ , which leads to an isomorphism to  $L(w\lambda, \mathfrak{p}_{w\lambda})$ . Now the assertion follows from Remark 4.5(c), asserting that the  $\varphi^{-1}$ -twist of  $L(\lambda, \mathfrak{p}_\lambda)$  is isomorphic to  $L(\lambda, \mathfrak{p}_\lambda)$ .  $\square$

**PROPOSITION 4.7.** *If  $\Delta^+$  is a positive system,  $L(\lambda, \Delta^+)$  is integrable and  $\mathfrak{g}^0(\lambda) = \mathfrak{h} + \mathfrak{g}(\Delta_i^\lambda)$  holds for  $\Delta_i^\lambda := \Delta_i \cap \Delta^\lambda$  (which is trivially satisfied if  $\Delta^\lambda \subseteq \Delta_i$ ), then*

$$L(\lambda, \Delta^+) \cong L(\lambda, \mathfrak{p}_\lambda).$$

*In particular,  $L(\lambda, \Delta^+)$  does not depend on the positive system  $\Delta^+$ .*

**PROOF.** Let  $v_\lambda \in L(\lambda, \Delta^+)$  be a generating primitive element. Then, for each  $\alpha \in \Delta_i$ ,  $v_\lambda$  generates an integrable  $\mathfrak{g}(\alpha)$ -module of highest weight  $\lambda(\check{\alpha})$ , for which  $\pm\lambda(\check{\alpha})$  are the maximal and minimal eigenvalues of  $\check{\alpha}$ . If  $\lambda(\check{\alpha}) = 0$ , this implies that  $\mathfrak{g}(\alpha)v_\lambda = \{0\}$ . Now our assumption implies that  $v_\lambda$  is a  $\mathfrak{p}_\lambda$ -eigenvector. Therefore  $L(\lambda, \Delta^+)$  is a simple  $\mathfrak{g}$ -module generated by a  $\mathfrak{p}_\lambda$ -weight vector of weight  $\lambda$ , so that the universal property of  $L(\lambda, \mathfrak{p}_\lambda)$  implies that it is isomorphic to  $L(\lambda, \Delta^+)$ .  $\square$

The main feature of the construction in Definition 4.2 is that it provides a construction of simple ‘‘highest weight’’ modules without referring to a positive system. Proposition 4.7 now tells us that in all classical cases, it produces the same result. This provides a new perspective on highest weight modules which is more natural for infinite rank algebras, because they have no distinguished  $\mathcal{W}$ -conjugacy class of positive systems (cf. [Ne98]).

**REMARK 4.8.**<sup>8</sup> Let  $(\mathfrak{g}, \mathfrak{h})$  be a split Lie algebra and assume that all roots are integrable, i.e.,  $\Delta = \Delta_i$ . In view of [Ne00b, Thm. VI.3],  $\mathfrak{g}$  is locally finite and its commutator algebra is a direct sum of simple split Lie algebras ([St99, Thm. III.11]).

Let  $\lambda \in \mathfrak{h}^*$  be an integral weight. Then there exists a positive system  $\Delta^+ \subseteq \Delta$  for which  $\lambda$  is dominant integral ([Ne98, Lemma I.18]), i.e.,  $\lambda(\check{\alpha}) \in \mathbb{N}_0$  for  $\alpha \in \Delta^+$ .

For finite-dimensional reductive split Lie algebras it is well-known that the simple highest weight module  $L(\lambda, \Delta^+)$  is finite-dimensional if and only if  $\lambda$  is dominant integral and that each integrable highest weight module is simple. Now  $\mathfrak{g} = \varinjlim \mathfrak{g}_j$  is a directed union of finite-dimensional reductive Lie algebras for which  $\mathfrak{h}_j := \mathfrak{h} \cap \mathfrak{g}_j$  is a splitting Cartan subalgebra of  $\mathfrak{g}_j$  and whose corresponding root systems are  $\Delta_j := \{\alpha \in \Delta : \mathfrak{g}_\alpha \subseteq \mathfrak{g}_j\}$ . Then  $\Delta_j^+ := \Delta_j \cap \Delta^+$  is a positive system, and  $\lambda_j := \lambda|_{\mathfrak{h}_j}$  is dominant integral for  $\mathfrak{g}_j$ . For  $\mathfrak{g}_j \subseteq \mathfrak{g}_k$ , the submodule  $\mathcal{U}(\mathfrak{g}_j)v_{\lambda_k} \subseteq L(\lambda_k, \Delta_k^+)$  generated by a primitive element  $v_{\lambda_k}$  is an integrable highest weight module, hence isomorphic to  $L(\lambda_j, \Delta_j^+)$ . We conclude that we may form a direct limit module  $\varinjlim L(\lambda_j, \Delta_j^+)$  which is a highest weight module of  $\mathfrak{g}$  of highest weight  $\lambda$ . As a direct limit of simple  $\mathfrak{g}_j$ -modules, it is simple, hence isomorphic to  $L(\lambda, \Delta^+)$ . This implies in particular that

$$L(\lambda, \Delta^+) \cong \varinjlim L(\lambda_j, \Delta_j^+)$$

<sup>8</sup>Many results stated in this remark have been obtained in [Ne98, Sect. I] and [Ne04, Sect. 3] in the context of unitary highest weight modules of complex involutive Lie algebras. Since we shall need it later, we now explain how one can argue in the algebraic context over a general field of characteristic zero.

is integrable and that its set of weights coincides with

$$(4.1) \quad \mathcal{P}_\lambda := \text{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}),$$

where  $\mathcal{Q} = \text{span}_{\mathbb{Z}} \Delta \subseteq \mathfrak{h}^*$  is the root group (cf. [Ne98, Thm. I.11]). From the integrability of  $L(\lambda, \Delta^+)$  and Proposition 4.7 we now derive that

$$L(\lambda) = L(\lambda, \mathfrak{p}_\lambda) \cong L(\lambda, \Delta^+)$$

does not depend on the choice of  $\Delta^+$  and Proposition 4.6, together with (4.1), further shows that  $L(\lambda) \cong L(\mu)$  if and only if  $\mu \in \mathcal{W}\lambda$  (cf. [Ne98, Thm. I.20]).

The observation summarized in the preceding remark was our original motivation to explore the approach to highest weight modules of locally affine Lie algebras developed in the present paper.

The following proposition explains the  $\Delta^+$ -independent picture for highest weight modules of affine Kac–Moody algebras.

**PROPOSITION 4.9.** *Let  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  be an affine Kac–Moody algebra,  $\Pi \subseteq \Delta$  a fundamental system of simple roots and  $\lambda \in \mathfrak{h}^*$ . Then the following assertions hold:*

- (i) *If  $\lambda(\mathfrak{z}(\mathfrak{g})) \neq \{0\}$ , then  $\mathcal{W}\lambda$  contains a unique dominant weight  $\tilde{\lambda}$ , i.e.,  $\tilde{\lambda}(\check{\alpha}) \geq 0$  for  $\alpha \in \Pi$ .*
- (ii) *If  $\lambda(\check{\Delta}_i) \neq \{0\}$ , then the following are equivalent:*
  - (a)  *$(\mathcal{W}\lambda)(\check{\alpha})$  is bounded for each  $\alpha \in \Delta_i$ .*
  - (b)  *$(\mathcal{W}\lambda)(\check{\alpha})$  is bounded for some  $\alpha \in \Delta_i$ .*
  - (c)  *$\lambda(\mathfrak{z}(\mathfrak{g})) = \{0\}$ .*
- (iii) *If  $\lambda$  is dominant integral, then  $L(\lambda, \Delta^+) \cong L(\lambda, \mathfrak{p}_\lambda)$ .*

**PROOF.** (i) In [Ka90], a generator of the one-dimensional center is denoted  $K$ . Our assumption implies that  $\lambda(K) \neq 0$ , so that the assertion follows from [Ka90, Prop. 6.6], combined with [MP95, Thm. 16].

(ii) (a)  $\Rightarrow$  (b) is trivial.

(b)  $\Rightarrow$  (c): Pick  $m \in \mathbb{N}$  such that  $\alpha_k := \alpha + km\delta \in \Delta_i$  holds for each  $k \in \mathbb{N}$  (cf. Lemma 2.6) and use (3.1) to see that

$$\check{\alpha}_k = \check{\alpha} + \frac{2km}{(\alpha, \alpha)} \delta^\sharp.$$

Then

$$(r_{\alpha_k} \lambda)(\check{\alpha}) = \left( \lambda - \left( \lambda(\check{\alpha}) + \frac{2km}{(\alpha, \alpha)} \lambda(\delta^\sharp) \right) \alpha_k \right) (\check{\alpha}) = -\lambda(\check{\alpha}) - \frac{4km}{(\alpha, \alpha)} \lambda(\delta^\sharp).$$

If the set of these numbers is bounded (for  $k \in \mathbb{N}$ ), then  $\lambda(\delta^\sharp) = 0$ , and this is (c) (cf. Lemma 2.7).

(c)  $\Rightarrow$  (a): If  $\lambda|_{\mathfrak{z}(\mathfrak{g})} = \{0\}$ , then  $\lambda$  factors through a linear functional

$$(\text{span } \check{\Delta}_i) / \mathfrak{z}(\mathfrak{g}) \cong \text{span } \overline{\check{\Delta}}_i.$$

Since the root system  $\overline{\check{\Delta}}_i$  is finite and  $\mathcal{W}$  acts on it as a finite group, (a) follows.

(iii) First we recall from [Ka90, Lemma 10.1] or [MP95, Prop. 6.1.6] that  $L(\lambda, \Delta^+)$  is integrable if and only if  $\lambda$  is dominant integral. Next we observe that  $\Delta^+ \subseteq \text{span}_{\mathbb{N}_0} \Pi$  implies that  $\Sigma^0(\lambda) = \Delta \cap \text{span}_{\mathbb{Z}}(\Pi_\lambda)$ , so that  $\mathfrak{g}^0(\lambda) = \mathfrak{h} + \mathfrak{g}(\Pi_\lambda)$  (for  $\Pi_\lambda := \Pi \cap \Delta^\lambda$ ) follows from [MP95, Prop. 4.1.14]. Therefore the assumptions of Proposition 4.7 are satisfied, and (ii) follows.  $\square$

**Highest weight modules of locally affine Lie algebras.** We now come to our main results on highest weight modules.

**THEOREM 4.10.** *Let  $\mathfrak{g}$  be a locally affine Lie algebra and  $\lambda \in \mathfrak{h}^*$  an integral weight with  $\lambda(\mathfrak{z}(\mathfrak{g}_c)) \neq 0$ , so that  $\Delta^\lambda \subseteq \Delta_i$ . Then the following assertions hold:*

- (a)  $L(\lambda)$  is an integrable  $\mathfrak{g}$ -module.
- (b) Its set of weights is  $\mathcal{P}_\lambda = \text{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q})$ , where  $\mathcal{Q} = \text{span}_{\mathbb{Z}} \Delta$ .
- (c)  $L(\mu) \cong L(\lambda)$  if and only if  $\mu \in \mathcal{W}\lambda$ .

**PROOF.** (a) We write  $\mathfrak{g}$  as a directed union of subalgebras  $\mathfrak{g}_F = \mathfrak{h} + \mathfrak{g}(\Delta_i^F)$  as in Proposition 3.3, so that  $\mathfrak{g}_F$  is a direct sum of  $(\Delta^F)^\perp \subseteq \mathfrak{h}$  and the affine Kac–Moody algebra  $\mathfrak{g}(A_{\Pi_F})$ . Further,  $\mathfrak{p}_\lambda$  is a directed union of the subalgebras  $\mathfrak{p}_\lambda^F := \mathfrak{p}_\lambda \cap \mathfrak{g}_F$ . In the following,  $L(\lambda, \mathfrak{p}_\lambda^F)$  is always understood as a  $\mathfrak{g}_F$ -module. We now choose an element  $w_F \in \mathcal{W}$  such that  $\lambda_F := w_F \lambda$  is dominant integral with respect to  $\Pi_F$  (Proposition 4.9) and observe that, as  $\mathfrak{g}_F$ -modules, we have

$$L(\lambda, \mathfrak{p}_\lambda^F) \cong L(\lambda_F, \mathfrak{p}_{\lambda_F}^F) \cong L(\lambda_F, (\Delta^F)^+)$$

(Propositions 4.6 and 4.9(iii)). This implies that  $L(\lambda_F, \mathfrak{p}_{\lambda_F}^F)$  is an integrable  $\mathfrak{g}_F$ -module ([Ka90, Lemma 10.1] or [MP95, Prop. 6.1.6]).

From our construction, it follows that for  $F_1 \subseteq F_2$  we have a natural embedding of the simple integrable  $\mathfrak{g}_{F_1}$ -modules

$$L(\lambda_{F_1}, \mathfrak{p}_{\lambda_{F_1}}^{F_1}) \subseteq L(\lambda_{F_2}, \mathfrak{p}_{\lambda_{F_2}}^{F_2})$$

because  $U(\mathfrak{g}_{F_1})v_{\lambda_{F_2}} \subseteq L(\lambda_{F_2}, \mathfrak{p}_{\lambda_{F_2}}^{F_2})$  is an integrable highest weight module, hence simple ([Ka90, Cor. 10.4]), and therefore isomorphic to  $L(\lambda_{F_1}, \mathfrak{p}_{\lambda_{F_1}}^{F_1})$ . We conclude that, for each  $\mathfrak{g}_F$ ,  $L(\lambda, \mathfrak{p}_\lambda)$  is a direct limit of integrable  $\mathfrak{g}_F$ -modules, hence integrable. Since  $\mathfrak{g}_F$  was arbitrary, the assertion follows from the corresponding results in the affine case ([MP95, Prop. 6.2.7]).

(b) follows immediately from the direct limit description of  $L(\mu)$  under (a).

(c) In view of (a),  $L(w\lambda) \cong L(\lambda)$  for each  $w \in \mathcal{W}$  follows from Proposition 4.6). For the converse, we use (b) to see that in the rational affine space  $\lambda + \text{span}_{\mathbb{Q}} \mathcal{Q}$ , we have

$$\text{Ext}(\text{conv } \mathcal{P}_\lambda) = \text{Ext}(\text{conv}(\mathcal{W}\lambda)) \subseteq \mathcal{W}\lambda.$$

On the other hand,

$$\mathcal{P}_\lambda \subseteq \lambda + \text{span}_{\mathbb{N}_0} \Sigma^-(\lambda)$$

implies that  $\lambda \in \text{Ext}(\text{conv}(\mathcal{P}_\lambda))$ , so that the  $\mathcal{W}$ -invariance of  $\mathcal{P}_\lambda$  implies  $\mathcal{W}\lambda = \text{Ext}(\text{conv}(\mathcal{P}_\lambda))$ , and this implies (c).  $\square$

**THEOREM 4.11.** *Let  $\mathfrak{g}$  be a locally affine complex Lie algebra and  $\sigma$  a unitary involution preserving  $\mathfrak{h}$  (Definition 3.17). For  $\mu \in \mathfrak{h}^*$  put  $\mu^*(h) := \overline{\mu(\sigma(h))}$ . Let  $\lambda = \lambda^* \in \mathfrak{h}^*$  be an integral weight not vanishing on the center. Then  $L(\lambda, \mathfrak{p}_\lambda)$  carries a positive definite hermitian form invariant under the unitary real form  $\mathfrak{k}$  of  $\mathfrak{g}$ .*

**PROOF.** For any affine Kac–Moody Lie algebra  $\mathfrak{g}_F$ , we know from [Ka90, Thm. 11.7] that for each dominant integral weight  $\lambda_F = \lambda_F^*$ , the corresponding integrable weight module  $L(\lambda_F, \mathfrak{p}_{\lambda_F})$  has a  $\mathfrak{k}_F$ -invariant positive definite hermitian form, which is unique if normalized by  $(v_{\lambda_F}, v_{\lambda_F}) = 1$  on the highest weight vector  $v_{\lambda_F}$ .



Using this uniqueness and the description of  $L(\lambda, \mathfrak{p}_\lambda)$  as a direct limit of the  $\mathfrak{g}_F$ -modules  $L(\lambda_F, \mathfrak{p}_{\lambda_F})$  (Theorem 4.10), it follows that  $L(\lambda, \mathfrak{p}_\lambda)$  carries a  $\mathfrak{k}$ -invariant positive definite hermitian form.  $\square$

### 5. Appendix 1. Yoshii's classification

In this appendix we describe Yoshii's classification of locally affine root systems of infinite rank and show that two reflectable sections are conjugate under the automorphism group. We have already seen in Section 3 that this can be used to show that minimal locally affine Lie algebras are determined by their root system. Below we describe for each of the seven types of root systems a corresponding minimal locally affine Lie algebra which is a twisted loop algebra.

We first recall that each irreducible locally finite root system of infinite rank is isomorphic to one of the following (cf. [LN04, §8]). Here we realize the root systems in the free vector space  $\mathbb{Q}^{(J)}$  with basis  $\varepsilon_j$ ,  $j \in J$  and the canonical symmetric bilinear form defined by  $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ :

$$\begin{aligned} A_J &:= \{\varepsilon_j - \varepsilon_k : j, k \in J, j \neq k\}, \\ B_J &:= \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\}, \quad (B_J)_{\text{sh}} = \{\pm\varepsilon_j : j \in J\} \\ C_J &:= \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\}, \quad (C_J)_{\text{lg}} = \{\pm 2\varepsilon_j : j \in J\} \\ D_J &:= \{\pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\} = (B_J)_{\text{lg}} = (C_J)_{\text{sh}}. \\ BC_J &:= \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\}, \quad (BC_J)_{\text{ex}} = \{\pm 2\varepsilon_j : j \in J\}. \end{aligned}$$

For a root system  $(V, \mathcal{R}, (\cdot, \cdot))$ , we put  $\mathcal{R}^{(1)} := \mathcal{R} \times \mathbb{Z} \subseteq V \times \mathbb{Q}$ , where the scalar product on  $V \times \mathbb{Q}$  is defined by  $((\alpha, t), (\alpha', t')) := (\alpha, \alpha')$ . Now we can state Yoshii's classification ([YY08, Cor. 13]):

**THEOREM 5.1.** *Each irreducible reduced locally affine root system  $(V, \mathcal{R})$  of infinite rank is isomorphic to one of the following:  $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}$ , or*

$$\begin{aligned} B_J^{(2)} &:= ((B_J)_{\text{sh}} \times \mathbb{Z}) \cup ((B_J)_{\text{lg}} \times 2\mathbb{Z}) = (B_J \times 2\mathbb{Z}) \cup ((B_J)_{\text{sh}} \times (2\mathbb{Z} + 1)), \\ C_J^{(2)} &:= ((C_J)_{\text{sh}} \times \mathbb{Z}) \cup ((C_J)_{\text{lg}} \times 2\mathbb{Z}) = (C_J \times 2\mathbb{Z}) \cup (D_J \times (2\mathbb{Z} + 1)) \\ (BC)_J^{(2)} &:= \left( ((BC_J)_{\text{sh}} \cup (BC_J)_{\text{lg}}) \times \mathbb{Z} \right) \cup ((BC_J)_{\text{ex}} \times (2\mathbb{Z} + 1)) \\ &= (B_J \times 2\mathbb{Z}) \cup (BC_J \times (2\mathbb{Z} + 1)). \end{aligned}$$

Let  $(V, \mathcal{R})$  be a locally affine root system and recall that a subspace  $V' \subseteq V$  is called a reflectable section if  $V' \cap \Delta$  maps bijectively onto  $\overline{\Delta}_{\text{red}}$ . From the classification one easily derives the existence of a reflectable section. The following theorem proves their uniqueness up to conjugacy by automorphisms:

**THEOREM 5.2.** *If  $(V, \mathcal{R})$  is a locally affine root system of infinite rank and  $V', V'' \subseteq V$  two reflectable sections, then there exists an isometric automorphism  $\varphi \in \text{Aut}(V, \mathcal{R})$  with  $\varphi(V') = V''$ , inducing the identity on  $\overline{V}$ .*

**PROOF.** We think of a reflectable section as being realized by a linear section  $\sigma: \overline{V} \rightarrow V$  of the quotient map  $V \rightarrow \overline{V}$ . Any other section  $\sigma': \overline{V} \rightarrow V$  is of the form  $\sigma' = \sigma + \gamma \cdot \delta$ , where  $\gamma: \overline{V} \rightarrow \mathbb{Q}$  is a linear functional with  $\sigma'(\overline{\Delta}_{\text{red}}) \subseteq \Delta$ .

We fix a reflectable section  $V'$ , the corresponding map  $\sigma$  and the corresponding reduced root system  $\Delta_{\text{red}} = \sigma(\overline{\Delta}_{\text{red}})$ . Accordingly, we identify  $V$  with  $\overline{V} \times \mathbb{Q}$  with

$\delta = (0, 1)$ , so that  $\Delta \subseteq \overline{\Delta} \times \mathbb{Z}$ , as in the classification. We now have to determine all other reflectable sections of  $\Delta$ .

In all cases, a necessary condition on  $\gamma$  is  $\gamma(\overline{\Delta}_{\text{red}}) \subseteq \mathbb{Z}$ . For the untwisted types  $X_J^{(1)}$ , this is also sufficient. For  $B_J^{(2)}$  and  $C_J^{(2)}$  we find the conditions  $\gamma(\alpha) \in \mathbb{Z}$  for  $\alpha$  short and  $\gamma(\alpha) \in 2\mathbb{Z}$  for  $\alpha$  long. For  $BC_J^{(2)}$  we need  $\gamma(\alpha) \in \mathbb{Z}$  for  $\alpha$  short or long.

In all these cases, it is easily verified that  $\varphi(\alpha) := \alpha + \gamma(\overline{\alpha})\delta$  defines an automorphism of  $\Delta$  mapping  $V'$  onto  $\sigma'(\overline{V})$ . Finally  $\delta \in V^0$  implies that  $\varphi$  is isometric.  $\square$

**REMARK 5.3.** We describe for each locally affine root system of infinite rank the set of all integral weights  $\lambda$  with  $\Delta_{\text{red}} \cong \Delta^\lambda = \{\alpha \in \Delta : \lambda(\alpha^\sharp) = 0\}$ , i.e., for which  $\Delta^\lambda$  is a reflectable setion. We use some information from the proof of Proposition 3.4, where we have shown that such weights exist.

We write each root  $\beta \in \Delta$  as  $\beta = n\alpha + m\delta$  with  $n = 1$  (if  $\beta$  is short or long) or  $n = 2$  (if  $\beta$  is extralong and  $\alpha$  is short). We also normalize the scalar product on roots in such a way that long roots have square length 2. Then short roots have square length 1 (if they occur) and extralong roots have square length 4. In the proof of Proposition 3.4 we have seen in (3.1) that

$$(n\alpha + m\delta)^\vee = \frac{2}{n^2(\alpha, \alpha)}(n\alpha^\sharp + m\delta^\sharp) = \frac{1}{n}\check{\alpha} + \frac{2m}{n^2(\alpha, \alpha)}\delta^\sharp.$$

For the untwisted cases  $\mathcal{R}^{(1)}$ , we have  $\Delta = \Delta_{\text{red}} \oplus \mathbb{Z}\delta$ , and we find the condition  $\lambda(\delta^\sharp) \in \mathbb{Z}$  by considering long roots  $\alpha$ .

For  $B_J^{(2)}$  and  $C_J^{(2)}$  and a short root  $\beta$ ,  $m \in \mathbb{Z}$  is arbitrary, which leads to  $2\lambda(\delta^\sharp) \in \mathbb{Z}$ , and for a long root we have  $m \in 2\mathbb{Z}$ , which leads to the same condition  $\lambda(\delta^\sharp) \in \frac{1}{2}\mathbb{Z}$ .

For  $BC_J^{(2)}$  we find for extralong roots the condition  $\lambda(\delta^\sharp) \in 2\mathbb{Z}$ , which is also sufficient for short and long roots.

**Realization of minimal locally affine Lie algebras.** In this subsection we combine Yoshii's classification of locally affine root systems (Theorem 5.1) with the Uniqueness Theorem 3.15 to realize all infinite rank minimal locally affine Lie algebras as twisted loop algebras.

We start with a description of doubly extended loop algebras.

**EXAMPLE 5.4.** (cf. [MY06, §5]) Let  $(\overline{\mathfrak{g}}, \overline{\mathfrak{h}}, \kappa_{\overline{\mathfrak{g}}})$  be a split quadratic Lie algebra whose root system  $\overline{\Delta} = \overline{\Delta}_i$  is locally finite and connected, resp., irreducible. Further, let  $\Gamma \subseteq \mathbb{Q}$  be a subgroup containing 1, and  $\mathbb{K}[\Gamma]$  be the algebra whose generators we write as formal exponentials  $t^q$ ,  $q \in \Gamma$ .

(a) We form the Lie algebra  $\mathcal{L}^\Gamma(\overline{\mathfrak{g}}) := \mathbb{K}[\Gamma] \otimes \overline{\mathfrak{g}}$ , which is a generalization of a loop algebra (which we obtain for  $\Gamma = \mathbb{Z}$ , for which we simply write  $\mathcal{L}(\overline{\mathfrak{g}})$ ). It is a  $\Gamma$ -graded Lie algebra with grading spaces  $\mathcal{L}^\Gamma(\overline{\mathfrak{g}})_q = t^q \otimes \overline{\mathfrak{g}}$  and

$$\overline{\kappa}(t^q \otimes x, t^s \otimes y) := \delta_{q, -s} \kappa_{\overline{\mathfrak{g}}}(x, y)$$

is a non-degenerate invariant symmetric bilinear form on  $\mathcal{L}^\Gamma(\overline{\mathfrak{g}})$ . Further,  $D(t^q \otimes x) := qt^q \otimes x$  defines a  $\overline{\kappa}$ -skew symmetric derivation on  $\mathcal{L}^\Gamma(\overline{\mathfrak{g}})$ , so that we may form the associated double extension

$$\mathfrak{g} := \widehat{\mathcal{L}}^\Gamma(\overline{\mathfrak{g}}) := (\mathbb{K} \oplus_{\omega_D} \mathcal{L}^\Gamma(\overline{\mathfrak{g}})) \rtimes_{\overline{D}} \mathbb{K},$$

where  $\omega_D(x, y) = \bar{\kappa}(Dx, y)$  is a 2-cocycle and  $\tilde{D}(z, x) := (0, Dx)$  is the canonical extension of  $D$  to the central extension  $\mathbb{K} \oplus_{\omega_D} \mathcal{L}^\Gamma(\bar{\mathfrak{g}})$  (cf. Definition 3.7). Now

$$\kappa((z, x, t), (z', x', t')) := zt' + z't + \bar{\kappa}(x, x')$$

is an invariant symmetric non-degenerate bilinear form on  $\mathfrak{g}$  and  $\mathfrak{h} := \mathbb{K} \oplus \bar{\mathfrak{h}} \oplus \mathbb{K}$  is a splitting Cartan subalgebra, so that  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a split quadratic Lie algebra. The element  $c := (1, 0, 0)$  is central and the eigenvalue of  $d := (0, 0, 1)$  on  $t^q \otimes \bar{\mathfrak{g}}$  is  $q$ .

It is now easy to verify that the root system of  $(\mathfrak{g}, \mathfrak{h})$  can be identified with the set

$$\bar{\Delta} \times \Gamma \subseteq \{0\} \times \bar{\mathfrak{h}}^* \times \mathbb{K}, \quad \text{where} \quad (\alpha, q)(z, h, t) := (0, \alpha, q)(z, h, t) = \alpha(h) + tq,$$

and that the set of integrable roots is  $\Delta_i = \bar{\Delta} \times \Gamma$ .

For root vectors  $x_{(\alpha, q)} = t^q \otimes x_\alpha \in \mathfrak{g}_{(\alpha, q)}$  with  $[x_\alpha, x_{-\alpha}] = \check{\alpha}$ , we have

$$[t^q \otimes x_\alpha, t^{-q} \otimes x_{-\alpha}] = (q\kappa_{\bar{\mathfrak{g}}}(x_\alpha, x_{-\alpha}), \check{\alpha}) = \left( \frac{2q}{(\alpha, \alpha)}, \check{\alpha} \right)$$

(cf. Remark 1.4). Since  $(\alpha, q)$  takes the value 2 on this element, it follows that

$$(\alpha, q)^\check{=} = \left( \frac{2q}{(\alpha, \alpha)}, \check{\alpha} \right) \quad \text{and} \quad \kappa((\alpha, q)^\check{=}, (\beta, r)^\check{=}) = \kappa_{\bar{\mathfrak{g}}}(\check{\alpha}, \check{\beta}).$$

From that we easily derive for the scalar product of the roots

$$((\alpha, q), (\beta, r)) = (\alpha, \beta),$$

which implies that  $(\mathfrak{g}, \mathfrak{h}, \kappa)$  is a LEALA.

(b) Writing a linear function on  $\mathfrak{h}$  as a triple  $\lambda = (z, \lambda_0, t) \in \mathbb{K} \times \bar{\mathfrak{h}}^* \times \mathbb{K}$ , we conclude that  $\lambda$  is integral if and only if

$$\frac{2qz}{(\alpha, \alpha)} + \lambda_0(\check{\alpha}) \in \mathbb{Z}$$

holds for each  $q \in \Gamma$  and  $\alpha \in \bar{\Delta}$ . This means that  $\lambda_0 \in \bar{\mathfrak{h}}^*$  is an integral weight of  $\bar{\mathfrak{g}}$  and  $z \in \frac{(\alpha, \alpha)}{2q}\mathbb{Z}$  for each  $q \in \Gamma$ . The latter condition has a non-zero solution  $z$  if and only if the subgroup  $\Gamma$  is cyclic. We conclude that there are integral weights  $\lambda$  not vanishing on the central element  $(1, 0, 0)$  if and only if  $\Gamma \cong \mathbb{Z}$ , which corresponds to the classical case of loop algebras (cf. Theorem 2.8).

REMARK 5.5. If  $\Gamma$  is not cyclic, then the group  $\Gamma$  is a directed union of cyclic infinite groups  $\mathbb{Z}q_j$ , so that the Lie algebra  $\mathfrak{g} = \mathcal{L}^\Gamma(\bar{\mathfrak{g}})$  is a direct limit of doubly extended loop algebras isomorphic to  $\hat{\mathcal{L}}(\bar{\mathfrak{g}})$ . If  $\bar{\mathfrak{g}}$  is finite-dimensional, this exhibits  $\hat{\mathcal{L}}^\Gamma(\bar{\mathfrak{g}})$  as a direct limit of affine Kac–Moody algebras, but it is not locally affine in the sense of Definition 3.1 (cf. [YY08]).

If  $X_J \in \{A_J, B_J, C_J, D_J\}$  is one of the irreducible locally affine Lie algebras, then we have the following corresponding locally affine simple Lie algebras.

For a set  $J$  and a field  $\mathbb{K}$ , we write  $\mathfrak{gl}_J(\mathbb{K})$  for the set of all  $(J \times J)$ -matrices with finitely many non-zero entries, i.e., the finitely supported functions on  $J \times J$ . Then the set  $\mathfrak{h} = \text{span}_{\mathbb{K}}\{E_{jj} : j \in J\}$  of diagonal matrices is a splitting Cartan subalgebra with the root system  $A_J$ , where  $\varepsilon_j(E_{kk}) := \delta_{jk}$ . Its commutator algebra is the simple Lie algebra  $\mathfrak{sl}_J(\mathbb{K})$ .

Next, let  $2J := J \dot{\cup} (-J)$  be a disjoint union, where  $-J$  denotes a copy of the set  $J$  whose elements are denoted by  $-j$ ,  $j \in J$ . We define  $S \in \mathbb{K}^{2J \times 2J}$  (the set of all  $J \times J$ -matrices) by

$$S_{\pm} := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \pm \mathbf{1} & \mathbf{0} \end{pmatrix} = \sum_{j \in J} (E_{j, -j} \pm E_{-j, j}).$$

Then

$$\mathfrak{sp}_{2J}(\mathbb{K}) := \{x \in \mathfrak{gl}_{2J}(\mathbb{K}) : x^{\top} S_{-} + S_{-} x = 0\}$$

and

$$\mathfrak{o}_{2J}(\mathbb{K}) := \{x \in \mathfrak{gl}_{2J}(\mathbb{K}) : x^{\top} S_{+} + S_{+} x = 0\}$$

are split Lie algebras with respect to the space

$$\mathfrak{h} = \text{span}\{E_{jj} - E_{-j, -j} : j \in J\}$$

of diagonal matrices. If we define  $\varepsilon_j(E_{kk} - E_{-k, -k}) := \delta_{jk}$ , then the corresponding root systems are  $C_J$  for  $\mathfrak{sp}_{2J}(\mathbb{K})$  and  $D_J$  for  $\mathfrak{o}_{2J}(\mathbb{K})$ .

To realize the root system  $B_J$ , we put  $2J+1 := 2J \dot{\cup} \{0\}$  (disjoint union) and

$$S := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} = E_{00} + \sum_{j \in J} (E_{j, -j} + E_{-j, j}).$$

Then

$$\mathfrak{o}_{2J+1}(\mathbb{K}) := \{x \in \mathfrak{gl}_{2J+1}(\mathbb{K}) : x^{\top} S + Sx = 0\}$$

is a split Lie algebra with respect to  $\mathfrak{h} = \text{span}\{E_{jj} - E_{-j, -j} : j \in J\}$  and the root system  $B_J$ . Since the quadratic spaces  $(\mathbb{K}^{2J}, S_{+})$  and  $(\mathbb{K}^{2J+1}, S)$  are isomorphic, the Lie algebras  $\mathfrak{o}_{2J+1}(\mathbb{K})$  and  $\mathfrak{o}_{2J}(\mathbb{K})$  are isomorphic, although they have two non-isomorphic root decompositions with respect to non-conjugate Cartan subalgebras (cf. [NS01, Lemma I.4]).

On all these Lie algebras, there is a natural non-degenerate invariant symmetric bilinear form, given by  $\kappa(x, y) := \text{tr}(xy)$ .

To obtain realizations of minimal locally affine Lie algebras, we now turn to twisted loop algebras. Let  $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}, \kappa_{\bar{\mathfrak{g}}})$  be one of the four types of simple locally finite split quadratic Lie algebras with root system of type  $X_J$ . Further, let  $\sigma \in \text{Aut}(\bar{\mathfrak{g}})$  be an involutive automorphism fixing  $\bar{\mathfrak{h}}$ . Then  $\sigma$  induces an automorphism of the root system which is isometric because of the positive definiteness of the form and the fact that every homomorphism  $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{R}_{+}^{\times}$  is trivial. This implies that  $\bar{\kappa}$  is  $\sigma$ -invariant.

Let  $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{+} \oplus \bar{\mathfrak{g}}_{-}$  be the  $\sigma$ -eigenspace decomposition of  $\bar{\mathfrak{g}}$ , and put  $\bar{\mathfrak{h}}_{\pm} := \bar{\mathfrak{h}} \cap \bar{\mathfrak{g}}_{\pm}$ . We assume that  $\bar{\mathfrak{h}}_{+}$  is maximal abelian in  $\bar{\mathfrak{g}}_{+}$ , hence a splitting Cartan subalgebra and write  $\bar{\Delta}_{\pm} \subseteq \bar{\mathfrak{h}}_{+}^{*}$  for the set of non-zero weights of  $\bar{\mathfrak{g}}_{+}$ , resp., the set of  $\bar{\mathfrak{h}}_{+}$ -weights in  $\bar{\mathfrak{g}}_{-}$ .

Define  $\tilde{\sigma} \in \text{Aut}(\mathcal{L}(\bar{\mathfrak{g}}))$  by  $\tilde{\sigma}(t^q \otimes x) := (-1)^q t^q \otimes \sigma(x)$  and consider the corresponding twisted loop algebra

$$\mathcal{L}(\bar{\mathfrak{g}}, \sigma) := \{\xi \in \mathcal{L}(\bar{\mathfrak{g}}) : \tilde{\sigma}(\xi) = \xi\} = (\mathbb{K}[t^{\pm 2}] \otimes \bar{\mathfrak{g}}_{+}) \oplus (t\mathbb{K}[t^{\pm 2}] \otimes \bar{\mathfrak{g}}_{-}).$$

This Lie algebra is invariant under the canonical derivation  $D$  of the loop algebra, so that we also obtain a corresponding double extension  $\mathfrak{g} := \widehat{\mathcal{L}}(\bar{\mathfrak{g}}, \sigma) \subseteq \widehat{\mathcal{L}}(\bar{\mathfrak{g}})$ , which is the set of fixed points for the involution on  $\widehat{\mathcal{L}}(\bar{\mathfrak{g}}, \sigma)$ , defined by  $\widehat{\sigma}(z, \xi, t) := (z, \tilde{\sigma}(\xi), t)$  (which makes sense because  $\tilde{\sigma}$  leaves  $\bar{\kappa}$  invariant).

The subalgebra  $\mathfrak{h} := \mathbb{K} \oplus \bar{\mathfrak{h}}_+ \oplus \mathbb{K}$  is a splitting Cartan subalgebra of  $\mathfrak{g}$  and the restriction of the quadratic invariant form of  $\widehat{\mathcal{L}}(\bar{\mathfrak{g}})$  is non-degenerate on  $\mathfrak{g}$ . Its root system is given by

$$\Delta_i = (\bar{\Delta}_+ \times 2\mathbb{Z}) \dot{\cup} (\bar{\Delta}_- \times (2\mathbb{Z} + 1)).$$

In the proof of the classification theorem, we need the following elementary geometric lemma.

LEMMA 5.6. *Let  $(V, \beta)$  be a quadratic space,  $v \in V$  be non-isotropic and*

$$g(x) := x - \frac{2\beta(v, x)}{\beta(v, v)}v$$

*be the orthogonal reflection in the hyperplane  $v^\perp$ . Then  $\text{Ad}(g)X := gXg^{-1}$  is an involutive automorphism of  $\mathfrak{o}(V, \beta)$ , and for the corresponding eigenspaces  $\mathfrak{o}(V, \beta)_{\pm 1}$ , we have*

$$\mathfrak{o}(V, \beta)_1 \cong \mathfrak{o}(v^\perp, \beta),$$

*and the map*

$$\varphi: v^\perp \rightarrow \mathfrak{o}(V, \beta)_{-1}, \quad \varphi(x) := \beta_{v, x} - \beta_{x, v}, \quad \beta_{v, w}(u) := \beta(v, u)w,$$

*is a linear isomorphism.*

PROOF. Let  $V_\pm := V_\pm(g)$  denote the eigenspaces of  $g$ , so that  $V_+ = v^\perp$  and  $V_- = \mathbb{K}v$ . Then  $\mathfrak{o}(V, \beta)_1$  consists of all skew-symmetric linear maps commuting with  $g$ , i.e., preserving both  $g$ -eigenspaces. In view of  $\mathfrak{o}_1(\mathbb{K}) = \{0\}$ , this leads to the given description of  $\mathfrak{o}(V, \beta)_1$ .

On the other hand, the relation  $\text{Ad}(g)X = -X$  is equivalent to  $XV_\pm \subseteq V_\mp$ . In view of  $\beta_{v, w}^\top = \beta_{w, v}$ , the image of  $\varphi$  lies in  $\mathfrak{o}(V, \beta)$  and it clearly maps  $v^\perp$  into  $\mathbb{K}v$  and  $v$  into  $v^\perp$ . Conversely, let  $\gamma \in \mathfrak{o}(V, \beta)_{-1}$ . Then  $\gamma(v) \in v^\perp$  and we put  $x := \beta(v, v)^{-1}\gamma(v)$ . We claim that  $\gamma = \varphi(x)$ . Clearly,

$$\gamma(v) = \beta(v, v)x = \beta_{v, x}(v) = \varphi(x)(v).$$

For  $y \in v^\perp$  we have  $\gamma(y) \in \mathbb{K}v$  and

$$\beta(v, \gamma(y)) = -\beta(\gamma(v), y) = -\beta(v, v)\beta(x, y) = -\beta(v, \beta_{x, v}(y)) = \beta(v, \varphi(x)(y)),$$

which implies that  $\gamma(y) = \varphi(x)(y)$ , and hence that  $\gamma = \varphi(x)$ .  $\square$

THEOREM 5.7. *For the irreducible reduced locally affine root systems of infinite rank, the corresponding minimal locally affine Lie algebras can be constructed as follows:*

- (i) *For the root systems of type  $X_J^{(1)}$  and a simple split Lie algebra  $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$  with root system  $X_J$ , the doubly extended loop algebra  $\widehat{\mathcal{L}}(\bar{\mathfrak{g}})$  is minimal locally affine with the root system  $X_J^{(1)} = X_J \times \mathbb{Z}$ .*
- (ii) *For the root systems of type  $X_J^{(2)}$ , the doubly extended twisted loop algebra  $\widehat{\mathcal{L}}(\bar{\mathfrak{g}}, \sigma)$  is minimal locally affine with the root system  $X_J^{(2)}$ , where  $B_J^{(2)}: \bar{\mathfrak{g}} := \mathfrak{o}_{2J'}(\mathbb{K})$  with  $J' := J \cup \{j_0\}$ ,  $j_0 \notin J$  and  $\sigma = \text{Ad}(g)$  for the orthogonal reflection in the hyperplane  $(e_{j_0} - e_{-j_0})^\perp \subseteq \mathbb{K}^{2J'}$ .*<sup>9</sup>

<sup>9</sup>Note that our description of the Lie algebra of type  $B_J^{(2)}$  is more explicit than the one in [YY08].

$$\begin{aligned}
C_J^{(2)}: \bar{\mathfrak{g}} &= \mathfrak{sl}_{2J}(\mathbb{K}) \text{ and } \sigma(x) = -Sx^\top S^{-1}, \text{ where } S = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \\
&= \sum_{j \in J} (E_{j,-j} - E_{-j,j}). \\
BC_J^{(2)}: \bar{\mathfrak{g}} &= \mathfrak{sl}_{2J+1}(\mathbb{K}) \text{ and } \sigma(x) = -Sx^\top S^{-1}, \text{ where} \\
S &:= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} = E_{j_0, j_0} + \sum_{j \in J} (E_{j,-j} + E_{-j,j}).
\end{aligned}$$

PROOF. (i) follows immediately from Example 5.4.

(ii)  $B_J^{(2)}$ : In  $\bar{\mathfrak{g}}$  we consider the canonical Cartan subalgebra

$$\bar{\mathfrak{h}} = \text{span}\{E_{jj} - E_{-j,-j} : j \in J'\}.$$

Then  $v := e_{j_0} - e_{-j_0} \in \mathbb{K}^{2J'}$  is a non-isotropic vector defining an orthogonal reflection  $g \in O_{2J'}(\mathbb{K})$  in  $v^\perp$ , and we obtain an involution of  $\mathfrak{o}_{2J'}(\mathbb{K})$  by  $\sigma(x) := \text{Ad}(g)x = gxg^{-1}$ . It is easy to verify that  $\text{Ad}(g)$  preserves  $\bar{\mathfrak{h}}$  with

$$\bar{\mathfrak{h}}_+ = \text{span}\{E_{jj} - E_{-j,-j} : j \in J\} \quad \text{and} \quad \bar{\mathfrak{h}}_- = \mathbb{K}(E_{j_0, j_0} - E_{-j_0, -j_0}).$$

From Lemma 5.6 we now derive that

$$\bar{\mathfrak{g}}_+ \cong \mathfrak{o}_{2J+1}(\mathbb{K}) \quad \text{and} \quad \bar{\Delta}_+ = B_J.$$

This lemma also shows that  $\bar{\mathfrak{g}}_- \cong v^\perp \cong \mathbb{K}^{2J} \oplus \mathbb{K}(e_{j_0} - e_{-j_0})$ , so that the set of non-zero weights of  $\bar{\mathfrak{h}}_+$  is  $\bar{\Delta}_- = \{\pm \varepsilon_j : j \in J\} = (B_J)_{\text{sh}}$ , and this leads to

$$\Delta = (B_J \times 2\mathbb{Z}) \cup ((B_J)_{\text{sh}} \times (2\mathbb{Z} + 1)).$$

$C_J^{(2)}$ : We have  $\mathfrak{sl}_{2J}(\mathbb{K})_+ = \mathfrak{sp}_{2J}(\mathbb{K})$  with the Cartan subalgebra

$$\mathfrak{h}_+ = \text{span}\{E_{jj} - E_{-j,-j} : j \in J\}.$$

The condition  $\sigma(x) = -x$  is equivalent to  $(Sx)^\top = -Sx$ , which for  $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is equivalent to  $a^\top = d, b^\top = -b$  and  $c^\top = -c$ . From that it is easy to see that  $\bar{\Delta}_-$  is the root system  $D_J$ , so that

$$\Delta = (C_J \times 2\mathbb{Z}) \cup (D_J \times (2\mathbb{Z} + 1)) = C_J^{(2)}.$$

The corresponding minimal locally affine Lie algebra is the doubly extended twisted loop algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma)$ .

$BC_J^{(2)}$ : In this case  $\mathfrak{sl}_{2J+1}(\mathbb{K})_+ = \mathfrak{o}_{2J+1}(\mathbb{K})$  and

$$\mathfrak{h}_+ = \text{span}\{E_{jj} - E_{-j,-j} : j \in J\}$$

is a splitting Cartan subalgebra of  $\mathfrak{o}_{2J+1}(\mathbb{K})$  for which the root system is  $B_J$ .

The condition  $\sigma(x) = -x$  is equivalent to  $(Sx)^\top = Sx$ . Evaluating this condition by writing  $x$  as a  $(3 \times 3)$ -block matrix according to the decomposition  $2J+1 = J \cup \{0\} \cup -J$ , we see that  $\bar{\Delta}_-$  is the root system  $BC_J$ , so that

$$\Delta = (B_J \times 2\mathbb{Z}) \cup (BC_J \times (2\mathbb{Z} + 1)) = BC_J^{(2)}.$$

The corresponding minimal locally affine Lie algebras is the doubly extended twisted loop algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J+1}(\mathbb{K}), \sigma)$ .  $\square$

REMARK 5.8. For  $C_J^{(2)}$  we also describe an alternative realization, which is a geometric variant of Kac' approach via diagram automorphisms which is more implicit (cf. [Ka90]).

On  $\bar{\mathfrak{g}} := \mathfrak{sl}_{2J}(\mathbb{K})$  we consider the involutive automorphism defined by  $\sigma(x) = -Sx^\top S^{-1}$ , where

$$S = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \sum_{j \in J} (E_{j,-j} + E_{-j,j}).$$

Then  $\mathfrak{sl}_{2J}(\mathbb{K})_+ = \mathfrak{o}_{2J}(\mathbb{K})$  with the Cartan subalgebra

$$\mathfrak{h}_+ = \text{span}\{E_{jj} - E_{-j,-j} : j \in J\}.$$

The condition  $\sigma(x) = -x$  is equivalent to  $(Sx)^\top = Sx$ , which for  $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is equivalent to  $a^\top = d, b^\top = b$  and  $c^\top = c$ . From that it is easy to see that  $\bar{\Delta}_-$  is the root system  $C_J$ , so that

$$\Delta = (D_J \times 2\mathbb{Z}) \cup (C_J \times (2\mathbb{Z} + 1)) = (D_J \times \mathbb{Z}) \cup ((C_J)_{\text{lg}} \times (2\mathbb{Z} + 1)).$$

Then  $\bar{\Delta} = C_J$ , but  $D_J \times \{0\}$  does not correspond to a reflectable section. To obtain a reflectable section, we consider instead the hyperplane

$$V' := \text{span}\{(2\varepsilon_j, 1) : j \in J\},$$

which leads to  $\Delta_{\text{red}} = C_J$  and

$$\Delta \cong (C_J \times 2\mathbb{Z}) \cup (D_J \times (2\mathbb{Z} + 1)) = C_J^{(2)}.$$

The corresponding minimal locally affine Lie algebra is the doubly extended twisted loop algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma)$ .

## 6. Appendix 2. Isomorphisms of twisted loop algebras

Let  $\bar{\mathfrak{g}}$  be a  $\mathbb{K}$ -Lie algebra and  $\sigma \in \text{Aut}(\bar{\mathfrak{g}})$  an automorphism with  $\sigma^m = \text{id}_{\bar{\mathfrak{g}}}$ . Suppose that  $\mathbb{K}$  contains a primitive  $m$ -th root of unity  $\zeta \in \mathbb{K}^\times$ , i.e.,  $\text{ord}(\zeta) = m$ . We define  $\tilde{\sigma} \in \text{Aut}(\mathcal{L}(\bar{\mathfrak{g}}))$  by  $\tilde{\sigma}(t^q \otimes x) := \zeta^q t^q \otimes \sigma(x)$  and consider the corresponding twisted loop algebra

$$\mathcal{L}(\bar{\mathfrak{g}}, \sigma) := \{\xi \in \mathcal{L}(\bar{\mathfrak{g}}) : \tilde{\sigma}(\xi) = \xi\}.$$

LEMMA 6.1. *Let  $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}})$  be a locally finite split simple Lie algebra and  $\mathcal{L}(\bar{\mathfrak{g}}) = \mathbb{K}[t^\pm] \otimes \bar{\mathfrak{g}}$  be the corresponding loop algebra. Then the following assertions hold:*

- (i) *Each ideal of  $\mathcal{L}(\bar{\mathfrak{g}})$  is of the form  $I \otimes \bar{\mathfrak{g}}$  for an ideal  $I \trianglelefteq R$ .*
- (ii) *If  $\mathbb{K}$  is algebraically closed, then the maximal ideals of  $\mathcal{L}(\bar{\mathfrak{g}})$  are the kernels of the evaluation maps  $\text{ev}_z : \mathcal{L}(\bar{\mathfrak{g}}) \rightarrow \bar{\mathfrak{g}}$ ,  $z \in \mathbb{K}^\times$ , sending  $r \otimes x$  to  $r(z)x$ .*

PROOF. (i) First we note that  $\bar{\mathfrak{g}}$  is a central simple  $\bar{\mathfrak{g}}$ -module, so that  $\mathcal{L}(\bar{\mathfrak{g}})$  is an isotypic semisimple  $\bar{\mathfrak{g}}$ -module of type  $\bar{\mathfrak{g}}$ . Let  $R := \mathbb{K}[t^\pm]$  be the ring of Laurent polynomials. Then  $\mathcal{L}(\bar{\mathfrak{g}}) = R \otimes_{\mathbb{K}} \bar{\mathfrak{g}}$  and we may identify  $R$  with the multiplicity space  $\text{Hom}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{g}}, \mathcal{L}(\bar{\mathfrak{g}}))$  by assigning to  $r \in R$  the embedding  $x \mapsto r \otimes x$ . In fact, let  $\psi \in \text{Hom}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{g}}, \mathcal{L}(\bar{\mathfrak{g}}))$  and  $0 \neq x \in \bar{\mathfrak{g}}$ . We write  $\psi(x) = \sum_i r_i \otimes y_i$  with linearly independent elements  $r_i \in R$  and  $y_i \in \bar{\mathfrak{g}}$  and observe that this implies that

$$\psi(\bar{\mathfrak{g}}) = \psi(\mathcal{U}(\bar{\mathfrak{g}})x) \subseteq \mathcal{U}(\bar{\mathfrak{g}}) \left( \sum_i r_i \otimes \bar{\mathfrak{g}} \right) \subseteq \sum_i r_i \otimes \bar{\mathfrak{g}},$$

where we use the canonical action of the enveloping algebra  $\mathcal{U}(\bar{\mathfrak{g}})$  on  $\mathcal{L}(\bar{\mathfrak{g}})$ . We derive the existence of  $\varphi_i = \lambda_i \text{id} \in \text{End}_{\bar{\mathfrak{g}}}(\bar{\mathfrak{g}}) \cong \mathbb{K}$  with  $\varphi(z) = \sum_i r_i \otimes \varphi_i(z)$  for each  $z \in \bar{\mathfrak{g}}$ , and this leads to  $\psi(z) = (\sum_i \lambda_i r_i) \otimes z$  for each  $z \in \bar{\mathfrak{g}}$ .

We conclude that each simple  $\bar{\mathfrak{g}}$ -submodule of  $\mathcal{L}(\bar{\mathfrak{g}})$  is of the form  $r \otimes \bar{\mathfrak{g}}$ , and since each submodule is semisimple, hence a sum of simple submodules, it is of the form  $M \otimes \bar{\mathfrak{g}}$  for a unique subspace  $M \subseteq R$ .

Assume, in addition, that  $M \otimes \bar{\mathfrak{g}}$  is an ideal of  $\mathcal{L}(\bar{\mathfrak{g}})$ . Then

$$M \otimes \bar{\mathfrak{g}} \supseteq [t \otimes \bar{\mathfrak{g}}, M \otimes \bar{\mathfrak{g}}] = tM \otimes [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = tM \otimes \bar{\mathfrak{g}}$$

implies that  $tM \subseteq M$ , and we likewise obtain  $t^{-1}M = M$ , showing that  $M \trianglelefteq R$  is an ideal.

(ii) If  $\mathbb{K}$  is algebraically closed, then the maximal ideals of  $\mathbb{K}[t^{\pm}]$  are the kernels of the point evaluations  $\text{ev}_z$ ,  $z \in \mathbb{K}^{\times}$ , so that the assertion follows from (i).  $\square$

**PROPOSITION 6.2.** *Let  $(\bar{\mathfrak{g}}_j, \bar{\mathfrak{h}}_j)$ ,  $j = 1, 2$ , be locally finite split simple Lie algebras,  $m \in \mathbb{N}$ , and  $\sigma_j \in \text{Aut}(\bar{\mathfrak{g}}_j)$  be automorphisms with  $\sigma_j^m = \text{id}_{\bar{\mathfrak{g}}_j}$ . Then*

$$\mathcal{L}(\bar{\mathfrak{g}}_1, \sigma_1) \cong \mathcal{L}(\bar{\mathfrak{g}}_2, \sigma_2) \quad \Rightarrow \quad \bar{\mathfrak{g}}_1 \cong \bar{\mathfrak{g}}_2.$$

**PROOF.** Let  $\bar{\mathbb{K}}$  denote the algebraic closure of  $\mathbb{K}$ . If we can prove the assertion for the Lie algebras  $\bar{\mathbb{K}} \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_j$ , then we arrive at an isomorphism  $\bar{\mathbb{K}} \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_1 \cong \bar{\mathbb{K}} \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_2$ , so that the classification of locally finite split simple Lie algebras implies that  $\bar{\mathfrak{g}}_1 \cong \bar{\mathfrak{g}}_2$  because the isomorphism class is determined by the type of the corresponding root systems (cf. [NS01, Thm. VI.7]). We may therefore assume that  $\mathbb{K}$  is algebraically closed.

Let  $S := \mathbb{K}[t^{\pm}]$  and let  $R := \mathbb{K}[t^{\pm m}]$  be the subring generated by  $t^{\pm m}$ . According to [ABP04, Lemma 4.3],  $\mathcal{L}(\bar{\mathfrak{g}}_j, \sigma_j)$  is central over  $R$  and an  $S/R$ -form of  $R \otimes_{\mathbb{K}} \bar{\mathfrak{g}}$ , i.e.,

$$S \otimes_R \mathcal{L}(\bar{\mathfrak{g}}_j, \sigma_j) \cong S \otimes \bar{\mathfrak{g}}_j.$$

Since  $\mathbb{K}$  is algebraically closed, each element of  $\mathbb{K}$  has  $m$ -th roots, so that [ABP04, Thm. 4.6] shows that

$$(6.1) \quad \mathcal{L}(\bar{\mathfrak{g}}_1, \sigma_1) \cong_{\mathbb{K}} \mathcal{L}(\bar{\mathfrak{g}}_2, \sigma_2) \quad \Rightarrow \quad \mathcal{L}(\bar{\mathfrak{g}}_1, \sigma_1) \cong_R \mathcal{L}(\bar{\mathfrak{g}}_2, \sigma_2).$$

This in turn leads to

$$S \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_1 \cong_S S \otimes_R \mathcal{L}(\bar{\mathfrak{g}}_1, \sigma_1) \cong_S S \otimes_R \mathcal{L}(\bar{\mathfrak{g}}_2, \sigma_2) \cong_S S \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_2.$$

Finally Lemma 6.1 shows that all quotients of  $S \otimes_{\mathbb{K}} \bar{\mathfrak{g}}_j$  by maximal ideals are isomorphic to  $\bar{\mathfrak{g}}_j$ , so that we obtain  $\bar{\mathfrak{g}}_1 \cong_{\mathbb{K}} \bar{\mathfrak{g}}_2$ .  $\square$

**THEOREM 6.3.** *We have isomorphism between the minimal locally affine Lie algebras corresponding to the following pairs of root systems:*

$$(B_J^{(1)}, D_J^{(1)}), \quad (C_J^{(2)}, BC_J^{(2)}) \quad \text{and} \quad (B_J^{(1)}, B_J^{(2)}).$$

**PROOF.** (a) From the isomorphism of the Lie algebras  $\mathfrak{o}_{2J}(\mathbb{K}) \cong \mathfrak{o}_{2J+1}(\mathbb{K})$  ([NS01, Lemma I.4]) and the fact that any isomorphism is (up to a factor) isometric with respect to the invariant quadratic form, it follows that the corresponding doubly extended loop algebras are also isomorphic. Therefore the non-isomorphic root systems  $B_J^{(1)}$  and  $D_J^{(1)}$  correspond to isomorphic minimal locally affine Lie algebras.



(b) In Remark 5.8, we have also seen how to realize the root system  $C_J^{(2)}$  by a twisted loop algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma)$ , where  $\mathfrak{sl}_{2J}(\mathbb{K})_+ \cong \mathfrak{o}_{2J}(\mathbb{K})$ . From the isomorphism  $(\mathbb{K}^{(2J)}, \beta_1) \cong (\mathbb{K}^{(2J+1)}, \beta_2)$  of quadratic spaces, with

$$\beta_j(x, y) = x^\top S_j y, \quad S_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

we obtain an isomorphism  $(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma_1) \cong (\mathfrak{sl}_{2J+1}(\mathbb{K}), \sigma_2)$  of Lie algebras with involution. Combining Theorem 5.7 with Remark 5.8, it now follows that the minimal locally affine Lie algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J+1}(\mathbb{K}), \sigma_2)$  of type  $BC_J^{(2)}$  is isomorphic to the minimal locally affine Lie algebra  $\widehat{\mathcal{L}}(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma_1)$  of type  $C_J^{(2)}$ .

(c) We realize  $B_J^{(2)}$  as in Theorem 5.7 via the quadratic space  $(V = \mathbb{K}^{(2J')}, \beta')$ , where

$$\beta'(x, y) = \sum_{j \in J'} x_j y_{-j} + x_{-j} y_j$$

and  $\sigma = \text{Ad}(g)$ , where  $g$  is the orthogonal reflection in  $v := e_{j_0} - e_{-j_0}$ .

Next we choose an orthogonal decomposition  $V = \mathbb{K}v \oplus V_1 \oplus V_2$ , where  $V_k = (\mathbb{K}^{2J_k}, \beta_k)$ ,  $k = 1, 2$ , and  $\beta_k(x, y) = \sum_{j \in J_k} x_j y_{-j} + x_{-j} y_j$  is the canonical form on  $\mathbb{K}^{2J_k}$ . Accordingly, we obtain a decomposition

$$V = \mathbb{K}v \oplus (\mathbb{K}^{(J_1)} \oplus \mathbb{K}^{(-J_1)}) \oplus (\mathbb{K}^{(J_2)} \oplus \mathbb{K}^{(-J_2)}),$$

so that we may represent linear maps on  $V$  accordingly by  $(5 \times 5)$ -block matrices. We thus obtain a group homomorphism

$$\alpha: \mathbb{K}^\times \rightarrow \text{O}(V, \beta), \quad \alpha(t) := \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & t^{-1}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Thinking of elements  $\xi$  of  $\mathcal{L}(\mathfrak{o}_{2J'}(\mathbb{K}), \sigma)$  as maps  $\mathbb{K}^\times \rightarrow \mathfrak{o}_{2J'}(\mathbb{K})$  satisfying

$$\xi(-t) = \text{Ad}(g)(\xi(t)) \quad \text{for } t \in \mathbb{K}^\times,$$

it is now easy to see that

$$\xi \mapsto \xi', \quad \xi'(t) := \text{Ad}(\alpha(t))(\xi(t))$$

defines an isomorphism of Lie algebras

$$\mathcal{L}(\mathfrak{o}_{2J'}(\mathbb{K}), \sigma) \rightarrow \mathcal{L}(\mathfrak{o}_{2J'}(\mathbb{K}), \sigma'),$$

where  $\sigma' = \text{Ad}(\alpha(-1)) \text{Ad}(g) = \text{Ad}(\alpha(-1)g) = \text{Ad}(g')$ , where

$$g' := \alpha(1)g = \begin{pmatrix} -1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

is the orthogonal reflection in the subspace  $V_2 \subseteq V$ . Next we observe that the triple  $(V, \beta, g')$  of a quadratic space with an orthogonal reflection is isomorphic to the triple  $(V_1 \oplus V_2, \beta_1 \oplus \beta_2, \alpha(-1))$ , so that

$$\mathcal{L}(\mathfrak{o}_{2J'}(\mathbb{K}), \sigma') \cong \mathcal{L}(\mathfrak{o}_{2(J_1+J_2)}(\mathbb{K}), \alpha(1)).$$

Reversing the argument above, we further derive

$$\mathcal{L}(\mathfrak{o}_{2(J_1+J_2)}(\mathbb{K}), \alpha(1)) \cong \mathcal{L}(\mathfrak{o}_{2(J_1+J_2)}(\mathbb{K}), \text{id}) \cong \mathcal{L}(\mathfrak{o}_{2J}(\mathbb{K})).$$

□

For the proof of the following theorem, we recall some facts on automorphisms of  $\mathfrak{sl}_J(\mathbb{K})$  from [St01]:

REMARK 6.4. A matrix  $A \in \mathbb{K}^{J \times J}$  defines a linear endomorphism of the free vector space  $\mathbb{K}^{(J)}$  if and only if each column has only finitely many non-zero entries. We write  $\text{GL}_J(\mathbb{K})_f \subseteq \text{GL}(\mathbb{K}^{(J)})$  for the subgroup of those linear automorphisms  $\varphi(x) = Ax$ ,  $A \in \mathbb{K}^{J \times J}$ , for which the adjoint map, which is represented by the transposed matrix  $A^\top$ , preserves the subspace  $\mathbb{K}^{(J)}$  of  $(\mathbb{K}^{(J)})^* \cong \mathbb{K}^J$ .

It is shown in [St01] that an automorphism of  $\mathfrak{sl}_J(\mathbb{K})$  either is of the form  $\varphi_A(x) = AxA^{-1}$  for some  $A \in \text{GL}_J(\mathbb{K})_f$ , or of the form  $\tilde{\varphi}_A(x) := -Ax^\top A^{-1}$  for some  $A \in \text{GL}_J(\mathbb{K})_f$ .

Both types of automorphisms can easily be distinguished by their action on the invariant polynomial  $p_3(x) := \text{tr}(x^3)$  of degree 3, which is non-zero for  $|J| > 2$ . In fact,  $p_3$  is invariant under automorphisms of the form  $\varphi_A$  and  $p_3 \circ \tilde{\varphi}_A = -p_3$ .

THEOREM 6.5. *The minimal locally affine Lie algebras corresponding to the root systems  $A_J^{(1)}$  and  $C_J^{(2)}$  are not isomorphic.*

PROOF. In view of Theorem 5.7, it suffices to show that the Lie algebras  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))$  and  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma)$  are not isomorphic if  $\sigma$  is an involutive automorphism of the form  $\sigma(x) = \tilde{\varphi}_S(x) = -Sx^\top S^{-1}$ , where  $S$  is any matrix defining an involutive automorphism of  $\mathfrak{sl}_J(\mathbb{K})$ . After base field extension to the algebraic closure  $\overline{\mathbb{K}}$  of  $\mathbb{K}$ , we may w.l.o.g. assume that  $\mathbb{K}$  is algebraically closed.

We argue by contradiction. If  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K})) \cong \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \text{id}) \cong \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma)$ , then [ABP04, Thm. IV.6] implies that these Lie algebras are isomorphic over the ring  $R := \mathbb{K}[t^{\pm 2}] \subseteq S := \mathbb{K}[t^{\pm 1}]$ . We also recall from [ABP04, Lemma IV.3] that

$$S \otimes_R \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma) \cong \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma) \oplus t \cdot \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma) = S \otimes_{\mathbb{K}} \mathfrak{sl}_J(\mathbb{K}) = \mathcal{L}(\mathfrak{sl}_J(\mathbb{K})).$$

We therefore obtain an  $S$ -automorphism  $\varphi \in \text{Aut}(\mathcal{L}(\mathfrak{sl}_J(\mathbb{K})))$ , mapping the  $R$ -subalgebra  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma)$  to  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \text{id}) = R \otimes_{\mathbb{K}} \mathfrak{sl}_J(\mathbb{K})$ .

From Lemma 6.1, we know that the maximal ideals of  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))$  all have the form  $S_z \otimes \mathfrak{sl}_J(\mathbb{K}) = S_z \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))$ , with  $S_z := \{f \in S : f(z) = 0\}$  for some  $z \in \mathbb{K}^\times$ . Since  $\varphi$  is  $S$ -linear, it therefore preserves all maximal ideals of  $\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))$ , hence induces for each  $z \in \mathbb{K}^\times$  an automorphism  $\varphi_z \in \text{Aut}(\overline{\mathfrak{g}})$  via  $\varphi_z(x) = \varphi(1 \otimes x)(z)$ , which in turn implies

$$\varphi(f)(z) = \varphi_z(f(z)) \quad \text{for } z \in \mathbb{K}^\times, f \in \mathcal{L}(\mathfrak{sl}_J(\mathbb{K})).$$

Let  $\tilde{\sigma}(f)(z) := \sigma(f(-z))$ , so that

$$\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma) = \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))_+ \quad \text{and} \quad t\mathcal{L}(\mathfrak{sl}_J(\mathbb{K}), \sigma) = \mathcal{L}(\mathfrak{sl}_J(\mathbb{K}))_-$$

is the eigenspace decomposition of  $\tilde{\sigma}$ . Likewise

$$\mathcal{L}(\mathfrak{sl}_J(\mathbb{K})) = (R \otimes \mathfrak{sl}_J(\mathbb{K})) \oplus (tR \otimes \mathfrak{sl}_J(\mathbb{K}))$$

is the eigenspace decomposition of the involution defined by  $\tilde{\text{id}}(f)(z) := f(-z)$ . Since  $\varphi$  maps the  $\tilde{\sigma}$ -eigenspaces to the corresponding  $\tilde{\text{id}}$ -eigenspaces, we obtain

$$\varphi \circ \tilde{\sigma} = \tilde{\text{id}} \circ \varphi,$$

which leads to

$$\varphi_z \circ \sigma = \varphi_{-z} \quad \text{for } z \in \mathbb{K}^\times,$$

and hence to the factorization

$$\sigma = \varphi_{-1} \circ \varphi_1^{-1}.$$

Pick  $x \in \mathfrak{sl}_J(\mathbb{K})$  with  $p_3(x) := \text{tr}(x^3) \neq 0$ . Then the function  $\mathbb{K}^\times \rightarrow \mathbb{K}$ ,  $z \mapsto p_3(\varphi_z(x))$  is a Laurent polynomial, and we know from Remark 6.4 that its only possible values are  $\pm p_3(x)$ , so that it is constant. This in turn implies that all automorphisms  $\varphi_z$  either fix  $p_3$  or reverse its sign. In particular,  $p_3$  is invariant under  $\sigma$ , but this contradicts  $\sigma = \tilde{\varphi}_S$  (cf. Remark 6.4).  $\square$

Combining the results from the preceding two theorems with Proposition 6.2 and the classification for the locally finite case in [NS01], we finally obtain the following classification of minimal locally affine Lie algebras:

**THEOREM 6.6** (Classification Theorem). *For each infinite set  $J$ , there are four isomorphism classes of minimal locally affine Lie algebras with  $|\Delta| = |J|$ . They are represented by the split Lie algebras with the root systems  $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}$  and  $C_J^{(2)}$ , resp., the loop algebras  $\mathcal{L}(\bar{\mathfrak{g}})$  with  $\mathfrak{g}$  of type  $A_J, B_J$  or  $C_J$ , and the twisted loop algebra  $\mathcal{L}(\mathfrak{sl}_{2J}(\mathbb{K}), \sigma)$  with  $\sigma(x) = -Sx^\top S^{-1}$  and  $S = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ .*

**REMARK 6.7.** In [Sa08], Salmasian deals with the closely related conjugacy problem for maximal abelian splitting subalgebras of loop algebras of the form  $\mathfrak{L}(\mathfrak{k})$ , where  $\mathfrak{k}$  is coral locally finite. It turns out that the Lie algebras corresponding to the root systems  $A_J$  and  $C_J$  yield only one conjugacy class, but for type  $B_J$  and  $D_J$ , it is only shown that the number of conjugacy classes is  $\leq 5$ . Since  $B_J$  and  $D_J$  correspond to isomorphic Lie algebras, it is  $\geq 2$ .

## References

- [AA-P97] Allison, B., Azam, S., Berman, S., Gao, Y., and A. Pianzola, “Extended affine Lie algebras and their root systems,” *Memoirs of the Amer. Math. Soc.* **603**, 1997
- [ABGP97] Allison, B. N., S. Berman, Y. Gao, and A. Pianzola, *A characterization of affine Kac-Moody Lie algebras*, *Comm. Math. Phys.* **185:3** (1997), 671–688
- [ABP04] Allison, B. N., S. Berman, and A. Pianzola, *Covering Algebras II: Isomorphism of loop algebras*, *J. reine angew. Math.* **571** (2004), 39–71
- [GK81] Gabber, O., and V. G. Kac., *On defining relations of certain infinite dimensional Lie algebras*, *Bull of the Amer. Math. Soc.* **5** (1981), 185–189
- [HoG04] Hofmann, G., “The Geometry of Reflection Groups,” Thesis, TU Darmstadt, Shaker Verlag, 2004
- [JK85] Jakobsen, H. P., and V. Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras*, in “Non-linear equations in classical and quantum field theory,” N. Sanchez ed., Springer Verlag, Berlin, Heidelberg, New York, *Lecture Notes in Physics* **226** (1985), 1–20
- [Ka90] Kac, V., “Infinite-dimensional Lie Algebras,” Cambridge University Press, 3<sup>rd</sup> printing, 1990
- [KN01] Kürner, B., and K.-H. Neeb, *Invariant symmetric bilinear forms for reflection groups*, *J. geom.* **71** (2001), 99–127
- [LN04] Loos, O., and E. Neher, “Locally finite root systems,” *Memoirs of the Amer. Math. Soc.*, Vol. 171, **811**, 2004
- [Mac72] Macdonald, I. G., *Affine root systems and Dedekind’s  $\eta$ -Function*, *Invent. Math.* **15** (1972), 91–143
- [MR85] A. Medina and P. Revoy, *Algèbres de Lie et produit scalaire invariant*, *Ann. scient. Éc. Norm. Sup. 4<sup>e</sup> série* **18**(1985), 533–561

- [MP95] Moody, R., and A. Pianzola, “Lie algebras with triangular decompositions”, Canad. Math. Soc. Series of Monographs and advanced texts, Wiley Interscience, 1995
- [MY06] Morita, Y., and Y. Yoshii, *Locally extended affine Lie algebras*, J. Algebra **301** (2006), 59–81
- [MY08] —, *Locally loop algebras and locally affine Lie algebras*, in preparation
- [Ne98] Neeb, K.-H., *Holomorphic highest weight representations of infinite dimensional complex classical groups*, J. Reine Angew. Math. **497** (1998), 171–222
- [Ne00a] —, “Holomorphy and Convexity in Lie Theory,” Expositions in Mathematics **28**, de Gruyter Verlag, Berlin, 2000
- [Ne00b] —, *Integrable roots in split graded Lie algebras*, J. Algebra **225** (2000), 534–580
- [Ne01] —, *Borel–Weil theory for loop groups*, in “Infinite Dimensional Kähler Manifolds”, Eds. A. Huckleberry, T. Wurzbacher, DMV-Seminar **31**, Birkhäuser Verlag, 2001; 179–229
- [Ne04] —, *Infinite-dimensional Lie groups and their representations*, in “Lie Theory: Lie Algebras and Representations,” Progress in Math. **228**, Ed. J. P. Anker, B. Ørsted, Birkhäuser Verlag, 2004; 213–328
- [NS01] Neeb, K.-H., and N. Stumme, *The classification of locally finite split simple Lie algebras*, J. Algebra **553** (2001), 25–53
- [Neh08] Neher, E., *Extended affine Lie algebras and other generalizations of affine Lie algebras—a survey*, in “Developments and trends in infinite dimensional Lie theory”, Eds. K.-H. Neeb and A. Pianzola, Progress in Math., Birkhäuser Verlag, to appear
- [PS86] Pressley, A., and G. Segal, “Loop Groups,” Oxford University Press, Oxford, 1986
- [Sa08] Salmasian, H., *Conjugacy of maximal toral subalgebras of direct limits of loop algebras*, Preprint 2008
- [St99] Stumme, N., *The structure of locally finite split Lie algebras*, Journal of Algebra **220** (1999), 664–693
- [St01] —, *Automorphisms and conjugacy of compact real forms of the classical infinite dimensional matrix Lie algebras*, Forum Math. **13:6** (2001), 817–851
- [YY08] Y. Yoshii, *Locally extended affine root systems*, in this volume: “Quantum affine algebras, extended affine Lie algebras and applications”, Eds. Y. Gao et. al., Contemp. Math., to appear

FACHBEREICH MATHEMATIK, TU DARMSTADT, SCHLOSSGARTENSTRASSE 7, 64289-DARMSTADT, GERMANY

*E-mail address:* neeb@mathematik.tu-darmstadt.de