Parabolic Subgroups of Real Direct Limit Lie Groups

Elizabeth Dan-Cohen*, Ivan Penkov† & Joseph A. Wolf‡

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Abstract

Let $G_{\mathbb{R}}$ be a classical real direct limit Lie group, and $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. The parabolic subalgebras of the complexification $\mathfrak{g}_{\mathbb{C}}$ were described by the first two authors. In the present paper we extend these results to $\mathfrak{g}_{\mathbb{R}}$. This also gives a description of the parabolic subgroups of $G_{\mathbb{R}}$. Furthermore, we give a geometric criterion for a parabolic subgroup $P_{\mathbb{C}}$ of $G_{\mathbb{C}}$ to intersect $G_{\mathbb{R}}$ in a parabolic subgroup. This criterion involves the $G_{\mathbb{R}}$–orbit structure of the flag ind–manifold $G_{\mathbb{C}}/P_{\mathbb{C}}$.

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1 Introduction and Basic Definitions

We start with the three classical simple locally finite countable–dimensional Lie algebras $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{n,\mathbb{C}}$, and their real forms $\mathfrak{g}_{\mathbb{R}}$. The Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are the classical direct limits, $\mathfrak{sl}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C}) = \varinjlim \mathfrak{so}(2n; \mathbb{C}) = \varinjlim \mathfrak{so}(2n+1; \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sp}(n; \mathbb{C})$, where the direct systems are given by the inclusions of the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$. See [1] or [2]. We often consider the locally reductive algebra $\mathfrak{gl}(\infty; \mathbb{C}) = \varinjlim \mathfrak{gl}(n; \mathbb{C})$ along with $\mathfrak{sl}(\infty; \mathbb{C})$.

The real forms of these classical simple locally finite countable–dimensional complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$ have been classified by A. Baranov in [1]. A slight reformulation of [1, Theorem 1.4] says that the following is a complete list of the real forms of $\mathfrak{g}_{\mathbb{C}}$.

<u>If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(\infty; \mathbb{C})$</u>, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

$\mathfrak{sl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sl}(n; \mathbb{R})$, the real special linear Lie algebra,

$\mathfrak{sl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{sl}(n; \mathbb{H})$, the quaternionic special linear Lie algebra, where $\mathfrak{sl}(n; \mathbb{H}) := \mathfrak{gl}(n; \mathbb{H}) \cap \mathfrak{sl}(2n; \mathbb{C})$,

$\mathfrak{su}(p, \infty) = \varinjlim \mathfrak{su}(p, n)$, the complex special unitary Lie algebra of finite real rank p,

$\mathfrak{su}(\infty, \infty) = \varinjlim \mathfrak{su}(p, q)$, the complex special unitary Lie algebra of infinite real rank.

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If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

$\mathfrak{so}(p,\infty) = \varinjlim \mathfrak{so}(p,n)$, the real orthogonal Lie algebra of finite real rank p,

$\mathfrak{so}(\infty,\infty) = \varinjlim \mathfrak{so}(p,q)$, the real orthogonal Lie algebra of infinite real rank,

$\mathfrak{so}^*(2\infty) = \varinjlim \mathfrak{so}^*(2n)$, with $\mathfrak{so}^*(2n) = \{\xi \in \mathfrak{sl}(n;\mathbb{H}) \mid \kappa_n(\xi x, y) + \kappa_n(x, \xi y) = 0 \ \forall x, y \in \mathbb{H}^n\}$, where $\kappa_n(x,y) := \sum_\ell x^\ell i \bar{y}^\ell = {}^t x i \bar{y}$. Equivalently, $\mathfrak{so}^*(2n) = \mathfrak{so}(2n;\mathbb{C}) \cap \mathfrak{u}(n,n)$ with $\mathfrak{so}(2n;\mathbb{C})$ defined by $(u,v) = \sum_1^n (u_{2j-1}v_{2j} + u_{2j}w_{2j-1})$ and $\mathfrak{u}(n,n)$ by $\langle u, v\rangle = \sum_1^n (u_{2j-1}\overline{v_{2j-1}} - u_{2j}\overline{v_{2j}})$.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

$\mathfrak{sp}(\infty;\mathbb{R}) = \varinjlim \mathfrak{sp}(n;\mathbb{R})$, the real symplectic Lie algebra,

$\mathfrak{sp}(p,\infty) = \varinjlim \mathfrak{sp}(p,n)$, the quaternionic unitary Lie algebra of finite real rank p,

$\mathfrak{sp}(\infty,\infty) = \varinjlim \mathfrak{sp}(p,q)$, the quaternionic unitary Lie algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following:

$\mathfrak{gl}(\infty;\mathbb{R}) = \varinjlim \mathfrak{gl}(n;\mathbb{R})$, the real general linear Lie algebra,

$\mathfrak{gl}(\infty;\mathbb{H}) = \varinjlim \mathfrak{gl}(n;\mathbb{H})$, the quaternionic general linear Lie algebra,

$\mathfrak{u}(p,\infty) = \varinjlim \mathfrak{u}(p,n)$, the complex unitary Lie algebra of finite real rank p,

$\mathfrak{u}(\infty,\infty) = \varinjlim \mathfrak{u}(p,q)$, the complex unitary Lie algebra of infinite real rank.

The *defining representations* of $\mathfrak{g}_{\mathbb{C}}$ are characterized as direct limits of minimal–dimensional nontrivial representations of simple subalgebras. It is well known that that $\mathfrak{sl}(\infty;\mathbb{C})$ and $\mathfrak{gl}(\infty;\mathbb{C})$ have two inequivalent defining representations V and W, whereas each of $\mathfrak{so}(\infty;\mathbb{C})$ and $\mathfrak{sp}(\infty;\mathbb{C})$ has only one (up to equivalence) V. In particular the restrictions to $\mathfrak{so}(\infty;\mathbb{C})$ or $\mathfrak{sp}(\infty;\mathbb{C})$ of the two defining representations of $\mathfrak{sl}(\infty;\mathbb{C})$ are equivalent. The real forms $\mathfrak{g}_{\mathbb{R}}$ listed above also have *defining representations*, as detailed below, which are particular restrictions of the defining representations of $\mathfrak{g}_{\mathbb{C}}$. We denote an element of $\mathbb{Z}_{\geqq 0} \cup \{\infty\}$ by $*$.

Suppose that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{sl}(\infty;\mathbb{R})$ or $\mathfrak{gl}(\infty;\mathbb{R})$. The defining representation spaces of $\mathfrak{g}_{\mathbb{R}}$ are the finitary (i.e. with finitely many nonzero entries) column vectors $V_{\mathbb{R}} = \mathbb{R}^\infty$ and the finitary row vectors $W_{\mathbb{R}} = \mathbb{R}^\infty$. The algebra of $\mathfrak{g}_{\mathbb{R}}$–endomorphisms of $V_{\mathbb{R}}$ or $W_{\mathbb{R}}$ is $\mathbb{R}$. The restriction of the pairing of V and W is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$–invariant $\mathbb{R}$–bilinear pairing of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$.

The defining representation space $V_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(*,\infty)$ consists of the finitary real column vectors. The algebra of $\mathfrak{g}_{\mathbb{R}}$–endomorphisms of $V_{\mathbb{R}}$ (the commuting algebra) is $\mathbb{R}$. The restriction of the symmetric form on V to $V_{\mathbb{R}}$ is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$–invariant symmetric $\mathbb{R}$–bilinear form.

The defining representation space $V_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(\infty;\mathbb{R})$ consists of the finitary real column vectors. The algebra of $\mathfrak{g}_{\mathbb{R}}$–endomorphisms of $V_{\mathbb{R}}$ is $\mathbb{R}$. The restriction of the antisymmetric form on V to $V_{\mathbb{R}}$ is a nondegenerate $\mathfrak{g}_{\mathbb{R}}$–invariant antisymmetric $\mathbb{R}$–bilinear form.

In both of these cases the defining representation of $\mathfrak{g}_{\mathbb{R}}$ is a real form of the defining representation of $\mathfrak{g}_{\mathbb{C}}$, i.e. $V = V_{\mathbb{R}} \otimes \mathbb{C}$.

Suppose that $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{su}(*,\infty)$ or $\mathfrak{u}(*,\infty)$. Then $\mathfrak{g}_{\mathbb{R}}$ has two defining representations, one on the space $V_{\mathbb{R}} = \mathbb{C}^{*,\infty}$ of finitary complex column vectors and the other on the space $W_{\mathbb{R}}$ of finitary complex row vectors. Thus the two defining representations of $\mathfrak{g}_\mathbb{C}$ remain irreducible as a representations of $\mathfrak{g}_\mathbb{R}$, the respective algebras of $\mathfrak{g}_\mathbb{R}$–endomorphisms of $V_\mathbb{R}$ and $W_\mathbb{R}$ are $\mathbb{C}$, and $V = V_\mathbb{R}$ and $W = W_\mathbb{R}$. The pairing of V and W defines a $\mathfrak{g}_\mathbb{R}$–invariant hermitian form of signature $(*, \infty)$ on $V_\mathbb{R}$.

Suppose that $\mathfrak{g}_\mathbb{R}$ is $\mathfrak{sl}(\infty; \mathbb{H})$ or $\mathfrak{gl}(\infty; \mathbb{H})$. The two defining representation spaces of $\mathfrak{g}_\mathbb{R}$ consist of the finitary column vectors $V_\mathbb{R} = \mathbb{H}^\infty$ and finitary row vectors $W_\mathbb{R} = \mathbb{H}^\infty$. The algebra of $\mathfrak{g}_\mathbb{R}$–endomorphisms of $V_\mathbb{R}$ or $W_\mathbb{R}$ is $\mathbb{H}$. The defining representations of $\mathfrak{g}_\mathbb{C}$ on V and W restrict to irreducible representations of $\mathfrak{g}_\mathbb{R}$, and $V_\mathbb{R} = \mathbb{H}^\infty = \mathbb{C}^\infty + \mathbb{C}^\infty j = \mathbb{C}^{2\infty} = V$. The pairing of V and W is a nondegenerate $\mathfrak{g}_\mathbb{R}$–invariant $\mathbb{R}$–bilinear pairing of $V_\mathbb{R}$ and $W_\mathbb{R}$.

The defining representation space $V_\mathbb{R} = \mathbb{H}^{*,\infty}$ of $\mathfrak{sp}(*, \infty)$ consists of the finitary quaternionic vectors. The algebra of $\mathfrak{sp}(*, \infty)$–endomorphisms of $V_\mathbb{R}$ is $\mathbb{H}$. The form on $V_\mathbb{R}$ is a nondegenerate $\mathfrak{sp}(*, \infty)$–invariant quaternionic–hermitian form of signature $(*, \infty)$. In this case $V_\mathbb{R} = \mathbb{H}^{*,\infty} = \mathbb{C}^{2*,2\infty} = V$.

The defining representation space $V_\mathbb{R} = \mathbb{H}^\infty$ of $\mathfrak{so}^*(2\infty)$ consists of the finitary quaternionic vectors. The algebra of $\mathfrak{so}^*(2\infty)$–endomorphisms of $V_\mathbb{R}$ is $\mathbb{H}$. The form on $V_\mathbb{R}$ is the nondegenerate $\mathfrak{so}^*(2\infty)$–invariant quaternionic–skew–hermitian form κ which is the limit of the forms κ_n. In this case again $V_\mathbb{R} = \mathbb{H}^\infty = \mathbb{C}^{2\infty} = V$.

The Lie ind–group (direct limit group) corresponding to $\mathfrak{gl}(\infty; \mathbb{C})$ is the general linear group $GL(\infty; \mathbb{C})$, which consists of all invertible linear transformations of V of the form $g = g' + \mathrm{Id}$ where $g' \in \mathfrak{gl}(\infty; \mathbb{C})$. The subgroup of $GL(\infty; \mathbb{C})$ corresponding to $\mathfrak{sl}(\infty; \mathbb{C})$ is the special linear group $SL(\infty; \mathbb{C})$, consisting of elements of determinant 1. The connected ind–subgroups of $GL(\infty; \mathbb{C})$ whose Lie algebras are $\mathfrak{so}(\infty; \mathbb{C})$ and $\mathfrak{sp}(\infty; \mathbb{C})$ are denoted by $SO_0(\infty; \mathbb{C})$ and $Sp(\infty; \mathbb{C})$.

In Section 2 we recall the structure of parabolic subalgebras of complex finitary Lie algebras from [4]. A *parabolic subalgebra* of a complex Lie algebra is by definition a subalgebra that contains a maximal locally solvable (that is, *Borel*) subalgebra. Parabolic subalgebras of complex finitary Lie algebras are classified in [4]. We recall the structural result that every parabolic subalgebra is a subalgebra (technically: defined by infinite trace conditions) of the stabilizer of a *taut couple* of generalized flags in the defining representations, and we strengthen this result by studying the non–uniqueness of the flags in the case of the orthogonal Lie algebra. As in the finite–dimensional case, we define a *parabolic subalgebra* of a real locally reductive Lie algebra $\mathfrak{g}_\mathbb{R}$ as a subalgebra $\mathfrak{p}_\mathbb{R}$ whose complexification $\mathfrak{p}_\mathbb{C}$ is parabolic in $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. It is a well–known fact that already in the finite–dimensional case a parabolic subalgebra of $\mathfrak{g}_\mathbb{R}$ does not neccesarily contain a subalgebra whose complexification is a Borel subalgebra of $\mathfrak{g}_\mathbb{C}$.

In Section 3 we prove our main result. It extends the classification in [4] to the real case. The key difference from the complex case is that one must take into account the additional structure of a defining representation space of $\mathfrak{g}_\mathbb{R}$ as a module over its algebra of $\mathfrak{g}_\mathbb{R}$–endomorphisms.

In Section 4 we give a geometric criterion for a parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$ to be the complexification of a parabolic subalgebra of $\mathfrak{g}_\mathbb{R}$. The criterion is based on an observation of one of us from the 1960's, concerning the structure of closed real group orbits on finite–dimensional complex flag manifolds. We recall that result, appropriately reformulated, and indicate its extension to flag ind–manifolds.

2 Complex Parabolic Subalgebras

2A Generalized Flags

Let V and W be countable–dimensional right vector spaces over a real division algebra $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, together with a nondegenerate bilinear pairing $\langle\cdot,\cdot\rangle : V \times W \to \mathbb{D}$. Then V and W are endowed with the Mackey topology, and the closure of a subspace $F \subset V$ is $F^{\perp\perp}$, where $\perp$ refers to the pairing $\langle\cdot,\cdot\rangle$. A set of $\mathbb{D}$–subspaces of V (or W) is called a *chain* in V (or W) if it is totally ordered by inclusion. A $\mathbb{D}$*–generalized flag* is a chain in V (or W) such that each subspace has an immediate predecessor or an immediate successor in the inclusion ordering, and every nonzero vector of V (or W) is caught between an immediate predecessor–successor pair [5].

Definition 2.1. [4] *A $\mathbb{D}$–generalized flag $\mathcal{F}$ in V (or W) is said to be* semiclosed *if for every immediate predecessor–successor pair $F' \subset F''$ the closure of F' is either F' or F''.* $\diamondsuit$

If $\mathcal{C}$ is a chain in V (or W), then we denote by $\mathcal{C}^\perp$ the chain in W (or V) consisting of the perpendicular complements of the subspaces of $\mathcal{C}$.

We fix an identification of V and W with the defining representations of $\mathfrak{gl}(\infty;\mathbb{D})$ as follows. To identify V and W with the defining representations of $\mathfrak{gl}(\infty;\mathbb{D})$, it suffices to find bases in V and W dual with respect to the pairing $\langle\cdot,\cdot\rangle$. If $\mathbb{D} \neq \mathbb{H}$, the existence of dual bases in V and W with respect to any nondegenerate $\mathbb{D}$–bilinear pairing is a result of Mackey [9, p. 171]. Now suppose that $\mathbb{D} = \mathbb{H}$. Then there exist $\mathbb{C}$–subspaces $V_\mathbb{C} \subset V$ and $W_\mathbb{C} \subset W$ such that $V = V_\mathbb{C} \oplus V_\mathbb{C} j$ and $W = W_\mathbb{C} \oplus W_\mathbb{C} j$. The restriction of $\langle\cdot,\cdot\rangle$ to $V_\mathbb{C} \times W_\mathbb{C}$ is a nondegenerate $\mathbb{C}$–bilinear pairing. The result of Mackey therefore implies the existence of dual bases in $V_\mathbb{C}$ and $W_\mathbb{C}$, which are also dual bases of V and W over $\mathbb{H}$. In all cases we identify the right multiplication of vectors in V by elements of $\mathbb{D}$ with the action of the algebra of $\mathfrak{g}_\mathbb{R}$–endomorphisms of $V_\mathbb{R}$.

Definition 2.2. [4] *Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathbb{D}$–semiclosed generalized flags in V and W, respectively. We say $\mathcal{F}$ and $\mathcal{G}$ form a* taut couple *if $\mathcal{F}^\perp$ is stable under the $\mathfrak{gl}(\infty;\mathbb{D})$–stabilizer of $\mathcal{G}$ and $\mathcal{G}^\perp$ is stable under the $\mathfrak{gl}(\infty;\mathbb{D})$–stabilizer of $\mathcal{F}$. If we have a fixed isomorphism $f : V \to W$ then we say that $\mathcal{F}$ is self–taut if $\mathcal{F}$ and $f(\mathcal{F})$ form a taut couple.* $\diamondsuit$

If one has a fixed isomorphism between V and W, then there is an induced bilinear form on V. A semiclosed generalized flag $\mathcal{F}$ in V is self–taut if and only if $\mathcal{F}^\perp$ is stable under the $\mathfrak{gl}(\infty;\mathbb{D})$–stabilizer of $\mathcal{F}$, where $\mathcal{F}^\perp$ is taken with respect to the form on V.

Remark 2.3. Fix a nondegenerate bilinear form on V. If V is finite dimensional, a self–taut generalized flag in V consists of a finite number of isotropic subspaces together with their perpendicular complements. In this case, the stabilizer of a self–taut generalized flag equals the stabilizer of its isotropic subspaces. If V is infinite dimensional, the non–closed non–isotropic subspaces in a self–taut generalized flag in V influence its stabilizer, but it is still true that every subspace is either isotropic or coisotropic. Indeed, let $\mathcal{F}$ be a self–taut generalized flag, and let $F \in \mathcal{F}$. By [4, Proposition 3.2], $F^\perp$ is a union of elements of $\mathcal{F}$ if it is a nontrivial proper subspace of V. Hence $\mathcal{F} \cup \{F^\perp\}$ is a chain that contains both F and $F^\perp$. Thus either $F \subset F^\perp$ or $F^\perp \subset F$, so F is either isotropic or coisotropic. $\diamondsuit$

We will need the following lemma when we pass to consideration of real parabolic subalgebras.

Lemma 2.4. *Suppose that $\mathbb{D} = \mathbb{H}$. Fix $\mathbb{H}$–generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in W. Then $\mathcal{F}$ and $\mathcal{G}$ form a taut couple if and only if they are form taut couple as $\mathbb{C}$–generalized flags.*

Proof. It is immediate from the definition that $\mathcal{F}$ and $\mathcal{G}$ are semiclosed $\mathbb{C}$–generalized flags if and only if they are semiclosed $\mathbb{H}$–generalized flags. The proof of [4, Proposition 3.2] holds in the quaternionic case as well. Thus if $\mathcal{F}$ and $\mathcal{G}$ form a taut couple as either $\mathbb{C}$–generalized flags or $\mathbb{H}$–generalized flags, then as long as $F^{\perp}$ is a nontrivial proper subspace of W, it is a union of elements of $\mathcal{G}$ for any $F \in \mathcal{F}$. Thus $F^{\perp}$ is stable under both the $\mathfrak{gl}(\infty; \mathbb{C})$–stabilizer and the $\mathfrak{gl}(\infty; \mathbb{H})$–stabilizer of $\mathcal{G}$ for any $F \in \mathcal{F}$. Similarly, if $G \in \mathcal{G}$ then $G^{\perp}$ is stable under both the $\mathfrak{gl}(\infty; \mathbb{C})$–stabilizer and the $\mathfrak{gl}(\infty; \mathbb{H})$–stabilizer of $\mathcal{F}$. □

2B Trace Conditions

Let $\mathfrak{g}$ be a locally finite Lie algebra over a field of characteristic zero. A subalgebra of $\mathfrak{g}$ is *locally solvable* (resp. *locally nilpotent*) if every finite subset of $\mathfrak{g}$ is contained in a solvable (resp. nilpotent) subalgebra. The sum of all locally solvable ideals is again a locally solvable ideal, the *locally solvable radical* of $\mathfrak{g}$. If $\mathfrak{r}$ is the locally solvable radical of $\mathfrak{g}$ then $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$ is a locally nilpotent ideal in $\mathfrak{g}$. Indeed, note that $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] = \bigcup_n (\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]) \cap \mathfrak{g}_n$ for any exhaustion $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ by finite–dimensional subalgebras $\mathfrak{g}_n$, and furthermore $(\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]) \cap \mathfrak{g}_n$ is nilpotent for all n by standard finite–dimensional Lie theory.

Let $\mathfrak{g}$ be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{D})$, that is, a subalgebra containing the Jordan components of its elements), and let $\mathfrak{r}$ be its locally solvable radical. The *linear nilradical* $\mathfrak{m}$ of $\mathfrak{g}$ is defined to be the set of all nilpotent elements in $\mathfrak{r}$.

Lemma 2.5. *Let $\mathfrak{g}$ be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{D})$. Then its linear nilradical $\mathfrak{m}$ is a locally nilpotent ideal. If $\mathbb{D} = \mathbb{R}$, then the complexification $\mathfrak{m}_{\mathbb{C}}$ is the linear nilradical of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. If $\xi, \eta \in \mathfrak{m}$ they are both contained in the solvable radical of a finite–dimensional subalgebra of $\mathfrak{g}$, so $\xi + \eta$ and $[\xi, \eta]$ are nilpotent. Thus, by Engel's Theorem, $\mathfrak{m}$ is a locally nilpotent subalgebra of $\mathfrak{g}$. Although it is only stated for complex Lie algebras, [4, Proposition 2.1] shows that $\mathfrak{m} \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$, so $[\mathfrak{m}, \mathfrak{g}] \subset [\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$, and thus $\mathfrak{m}$ is an ideal in $\mathfrak{g}$. This proves the first statement. For the second let $\mathfrak{r}$ be the locally solvable radical of $\mathfrak{g}$ and note that $\mathfrak{r}_{\mathbb{C}}$ is the locally solvable radical of $\mathfrak{g}_{\mathbb{C}}$, so the assertion follows from finite–dimensional theory. □

Definition 2.6. *Let $\mathfrak{g}$ be a splittable subalgebra of $\mathfrak{gl}(\infty; \mathbb{F})$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and and let $\mathfrak{m}$ be its linear nilradical. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is* defined by trace conditions on $\mathfrak{g}$ *if $\mathfrak{m} \subset \mathfrak{p}$ and*

$$[\mathfrak{g}, \mathfrak{g}]/\mathfrak{m} \subset \mathfrak{p}/\mathfrak{m} \subset \mathfrak{g}/\mathfrak{m},$$

in other words if there is a family Tr of Lie algebra homomorphisms $f : \mathfrak{g} \to \mathbb{F}$ with joint kernel equal to $\mathfrak{p}$. Further, $\mathfrak{p}$ is defined by infinite trace conditions *if every $f \in \mathrm{Tr}$ annihilates every finite–dimensional simple ideal in $[\mathfrak{g}, \mathfrak{g}]/\mathfrak{m}$.* ♢

We write $\mathrm{Tr}^{\mathfrak{p}}$ for the maximal family Tr of Definition 2.6. On the group level we have corresponding *determinant conditions* and *infinite determinant conditions*. Note that infinite trace conditions and infinite determinant conditions do not occur when $\mathfrak{g}$ and G are finite dimensional.

2C Complex Parabolic Subalgebras

Recall that a *parabolic subalgebra* of a complex Lie algebra is by definition a subalgebra that contains a Borel subalgebra, i.e. a maximal locally solvable subalgebra.

Theorem 2.7. [4] *Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{gl}(\infty, \mathbb{C})$ or $\mathfrak{sl}(\infty, \mathbb{C})$, and let V and W be its defining representation spaces. A subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. subgroup of $G_{\mathbb{C}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer (resp. $G_{\mathbb{C}}$–stabilizer) of a (necessarily unique) taut couple of $\mathbb{C}$–generalized flags $\mathcal{F}$ in V and $\mathcal{G}$ in W.*

Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{so}(\infty, \mathbb{C})$ or $\mathfrak{sp}(\infty, \mathbb{C})$. and let V be its defining representation space. A subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. subgroup of $G_{\mathbb{C}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer (resp. $G_{\mathbb{C}}$–stabilizer) of a self–taut $\mathbb{C}$–generalized flag $\mathcal{F}$ in V. In the $\mathfrak{sp}(\infty, \mathbb{C})$ case the flag $\mathcal{F}$ is necessarily unique.

In contrast to the finite dimensional case, the normalizer of a parabolic subalgebra can be larger than the parabolic algebra. For example, Theorem 2.7 implies that $\mathfrak{sl}(\infty, \mathbb{C})$ is parabolic in $\mathfrak{gl}(\infty; \mathbb{C})$, since it is the elements of the stabilizer of the trivial generalized flags $\{0, V\}$ and $\{0, W\}$ whose usual trace is 0. To understand the origins of this example, one should consider the explicit construction in [6] of a locally nilpotent Borel subalgebra of $\mathfrak{gl}(\infty; \mathbb{C})$. The normalizer of a parabolic subalgebra equals the stabilizer of the corresponding generalized flags [4], which is in general larger than the parabolic subalgebra because of the infinite determinant conditions. The self–normalizing parabolics are thus those for which $\mathrm{Tr}^{\mathfrak{p}} = 0$. This is in contrast to the finite–dimensional setting, where there are no infinite trace conditions, and all parabolic subalgebras are self–normalizing.

In [4] the uniqueness issue was discussed for $\mathfrak{gl}(\infty, \mathbb{C})$, $\mathfrak{sl}(\infty, \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C})$, but not for $\mathfrak{so}(\infty, \mathbb{C})$. In the orthogonal setting one can have three different self–taut generalized flags with the same stabilizer (see [3] and [7], where the non–uniqueness is discussed in special cases.)

Theorem 2.8. *Let $\mathfrak{p}$ be a parabolic subalgebra given by infinite trace conditions on the $\mathfrak{so}(\infty; \mathbb{C})$–stabilizer of a self–taut generalized flag $\mathcal{F}$ in V. Then there are two possibilities:*

1. *$\mathcal{F}$ is uniquely determined by $\mathfrak{p}$;*
2. *there are exactly three self-taut generalized flags with the same stabilizer as $\mathcal{F}$.*

The latter case occurs precisely when there exists an isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^{\perp}/L = 2$. The three flags with the same stabilizer are then

- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\}$
- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_1$
- $\{F \in \mathcal{F} \mid F \subset L \text{ or } L^{\perp} \subset F\} \cup M_2$

where M_1 and M_2 are the two maximal isotropic subspaces containing L.

Proof. The main part of the proof is to show that $\mathfrak{p}$ determines all the subspaces in $\mathcal{F}$, except a maximal isotropic subspace under the assumption that $\mathcal{F}$ has a closed isotropic subspaces L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$.

Let A denote the set of immediate predecessor–successor pairs of $\mathcal{F}$ such that both subspaces in the pair are isotropic. Let F'_α denote the predecessor and F''_α the successor of each pair $\alpha \in A$. Let M denote the union of all the isotropic subspaces in $\mathcal{F}$, i.e. $M = \bigcup_{\alpha\in A} F''_\alpha$. If $M \neq M^{\perp}$, then M has an immediate successor W in $\mathcal{F}$. Note that W is not isotropic, by the definition of M. Furthermore, one has $W^{\perp} = M$ since $\mathcal{F}$ is a self–taut generalized flag. If $M = M^{\perp}$, let us take $W = 0$.

Let C denote the set of all $\gamma \in A$ such that F'_γ is closed. For each $\gamma \in C$, it is seen in [4] that the coisotropic subspace $(F''_\gamma)^{\perp}$ has an immediate successor in $\mathcal{F}$. For each $\gamma \in C$, let G''_γ denote the immediate successor of $(F''_\gamma)^{\perp}$ in $\mathcal{F}$. It is also shown in [4] that $(G''_\gamma)^{\perp} = F'_\gamma$.

Since $\mathcal{F}$ is a self–taut generalized flag, $\mathcal{F}$ is uniquely determined by the set of subspaces

$$\{F''_\alpha \mid \alpha \in A\} \cup \{G''_\gamma \mid \gamma \in C \text{ such that } G''_\gamma \text{ is not closed}\} \cup \{W\}.$$

We use separate arguments for these three kinds of subspaces to show that they are determined by $\mathfrak{p}$, except for a maximal isotropic subspace and W under the assumption that $\mathcal{F}$ has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$. We must also show that we can determine from $\mathfrak{p}$ whether or not $\mathcal{F}$ has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^{\perp}/L = 2$.

Let $\widetilde{\mathfrak{p}}$ denote the normalizer in $\mathfrak{so}(\infty;\mathbb{C})$ of $\mathfrak{p}$. We use the classical identification $\mathfrak{so}(\infty;\mathbb{C}) \cong \Lambda^2(V)$ where $u \wedge v$ corresponds to the linear transformation $x \mapsto \langle x, v\rangle u - \langle x, u\rangle v$. With this identification, following [4] one has

$$\widetilde{\mathfrak{p}} = \sum_{\alpha\in A\setminus C} F''_\alpha \wedge (F'_\alpha)^{\perp} + \sum_{\gamma\in C} F''_\gamma \wedge G''_\gamma + \Lambda^2(W).$$

Let $\alpha \in A$, and let $x \in F''_\alpha \setminus F'_\alpha$. Then one may compute

$$\begin{aligned}
\widetilde{\mathfrak{p}} \cdot x &= \Big(\sum_{\alpha\in A\setminus C} F''_\alpha \wedge (F'_\alpha)^{\perp} + \sum_{\gamma\in C} F''_\gamma \wedge G''_\gamma + \Lambda^2(W)\Big) \cdot x \\
&= \Big(\sum_{\alpha\in A\setminus C} F''_\alpha \otimes (F'_\alpha)^{\perp} + \sum_{\gamma\in C} F''_\gamma \otimes G''_\gamma\Big) \cdot x \\
&= \Big(\bigcup_{x\notin (F'_\alpha)^{\perp\perp}} F''_\alpha\Big) \cup \Big(\bigcup_{x\notin (G''_\gamma)^{\perp}} F''_\gamma\Big).
\end{aligned}$$

As a result

$$\widetilde{\mathfrak{p}} \cdot x = \begin{cases} F'_\alpha & \text{if } \alpha \in A \setminus C \\ F''_\alpha & \text{if } \alpha \in C. \end{cases}$$

So far we have shown the following. If $x \in \widetilde{\mathfrak{p}} \cdot x$, then $F''_\alpha = \widetilde{\mathfrak{p}} \cdot x$. If $x \notin \widetilde{\mathfrak{p}} \cdot x$, then $F''_\alpha = (\widetilde{\mathfrak{p}} \cdot x)^{\perp\perp}$. Furthermore, if $x \notin M$, then $\widetilde{\mathfrak{p}} \cdot x$ is not isotropic, unless there exists a closed isotropic subspace

$L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^\perp/L = 2$, and x is an element of M_1 or M_2. We now consider the union of the subspaces $\widetilde{\mathfrak{p}} \cdot x$, where the union is taken over $x \in V$ for which $\widetilde{\mathfrak{p}} \cdot x$ is isotropic. If there does not exist L as described, then these subspaces will be the nested isotropic subspaces computed above, and indeed their union is M. If L exists, then these subspaces will exhaust L, and furthermore M_1 and M_2 will both appear in the union. Hence the union of the isotropic subspaces of the form $\widetilde{\mathfrak{p}} \cdot x$ for $x \in V$ when L exists is $L^\perp$. As a result, if the union of all the isotropic subspaces of the form $\widetilde{\mathfrak{p}} \cdot x$ for $x \in V$ is itself isotropic, then we conclude that no such L exists and we have constructed the subspace M. If that union is not isotropic, then we conclude that there exists a closed isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}} L^\perp/L = 2$, and the union is $L^\perp$. In the latter case, L is recoverable from $\mathfrak{p}$, as it equals $L^{\perp\perp}$. We have now shown that we can determine whether $\mathcal{F}$ has a closed isotropic subspace L with $\dim_{\mathbb{C}} L^\perp/L = 2$, that F''_α is determined by $\mathfrak{p}$ for all $\alpha \in A$ in the latter case, and that F''_α is determined by $\mathfrak{p}$ for all $\alpha \in A$ such that $F''_\alpha \subset L$ in the former case.

We now turn our attention to a non–closed subspace G''_γ for $\gamma \in C$. Since G''_γ is not closed, the codimension of F''_γ in G''_γ is infinite. Thus if there exists $L \in \mathcal{F}$ as above, then $F''_\gamma \subset L$. So we have already shown that F''_γ, and indeed F'_γ as well, are recoverable from $\mathfrak{p}$ whether or not there exists $L \in \mathcal{F}$. Let $x \in (F'_\gamma)^\perp \setminus (F''_\gamma)^\perp$. Then there exists $v \in F''_\gamma$ such that $\langle v, x\rangle \neq 0$, and one has

$$(v \wedge G''_\gamma) \cdot x = \{(v \wedge y) \cdot x \mid y \in G''_\gamma\} = \{\langle x, y\rangle v - \langle x, v\rangle y \mid y \in G''_\gamma\}.$$

Since $v \wedge G''_\gamma \subseteq \widetilde{\mathfrak{p}}$ and $v \in F''_\gamma$, we see that $G''_\gamma = (v \wedge G''_\gamma) \cdot x + F''_\gamma \subset \widetilde{\mathfrak{p}} \cdot x + F''_\gamma \subset G''_\gamma$. Hence $G''_\gamma = \widetilde{\mathfrak{p}} \cdot x + F''_\gamma$, and we conclude that G''_γ is recoverable from $\mathfrak{p}$.

Finally, we must show that $\mathfrak{p}$ determines W under the assumption that no subspace $L \in \mathcal{F}$ as above exists. We have already shown that M is recoverable from $\mathfrak{p}$ under this assumption. If $M = M^\perp$, then $W = 0$. We claim that $W = \widetilde{\mathfrak{p}} \cdot x + M$ for any $x \in M^\perp \setminus M$ when $M \neq M^\perp$. Indeed, let X be any vector space complement of M in W. Since $x \notin M$ and $W^\perp = M$, one has $\langle x, X\rangle \neq 0$. Furthermore, the restriction of the symmetric bilinear form on V to X is symmetric and nondegenerate. Then $\Lambda^2(X) \cdot x = X$ because $\dim_{\mathbb{C}} X \geqq 3$. Since $\Lambda^2(X) \subset \widetilde{\mathfrak{p}}$, we conclude that $\widetilde{\mathfrak{p}} \cdot x + M = W$. Thus W can be recovered from $\mathfrak{p}$.

If $\mathcal{F}$ is a self–taut generalized flag without any isotropic subspace $L \in \mathcal{F}$ such that $\dim_{\mathbb{C}} L^\perp/L = 2$, then we have now shown that $\mathcal{F}$ is uniquely determined by $\mathfrak{p}$. Finally, suppose that there does exist an isotropic subspace $L \in \mathcal{F}$ such that $\dim_{\mathbb{C}} L^\perp/L = 2$. Then we have shown that every subspace of $\mathcal{F}$ which does not lie strictly between L and $L^\perp$ is determined by $\mathfrak{p}$. There are exactly two maximal isotropic subspaces M_1 and M_2 containing L, and both M_1 and M_2 are stable under the $\mathfrak{so}(\infty;\mathbb{C})$–stabilizer of L. Hence the three self-taut generalized flags listed in the statement are precisely the self–taut generalized flags whose stabilizers equal the stabilizer of $\mathcal{F}$. □

3 Real Parabolic Subalgebras

Recall that a *parabolic subalgebra* of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is a subalgebra whose complexification is a parabolic subalgebra of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{g}_{\mathbb{C}}$ be one of $\mathfrak{gl}(\infty,\mathbb{C})$, $\mathfrak{sl}(\infty,\mathbb{C})$, $\mathfrak{so}(\infty,\mathbb{C})$, and $\mathfrak{sp}(\infty,\mathbb{C})$, and let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{R}}$ be the corresponding connected real subgroup of $G_{\mathbb{C}}$. When $\mathfrak{g}_{\mathbb{R}}$ has two inequivalent defining representations, we denote them by $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$, and when $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation, we denote it by $V_{\mathbb{R}}$. Let $\mathbb{D}$ denote the algebra of $\mathfrak{g}_{\mathbb{R}}$–endomorphisms of $V_{\mathbb{R}}$.

Theorem 3.1. *Suppose that $\mathfrak{g}_{\mathbb{R}}$ has two inequivalent defining representations. A subalgebra of $\mathfrak{g}_{\mathbb{R}}$ (resp. subgroup of $G_{\mathbb{R}}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{R}}$–stabilizer (resp. $G_{\mathbb{R}}$–stabilizer) of a taut couple of $\mathbb{D}$–generalized flags $\mathcal{F}$ in $V_{\mathbb{R}}$ and $\mathcal{G}$ in $W_{\mathbb{R}}$.*

Suppose that $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation. A subalgebra of $\mathfrak{g}_{\mathbb{R}}$ (resp. subgroup) of $G_{\mathbb{R}}$ is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_{\mathbb{R}}$–stabilizer (resp. $G_{\mathbb{R}}$–stabilizer) of a self–taut $\mathbb{D}$–generalized flag $\mathcal{F}$ in $V_{\mathbb{R}}$.

Proof. We will prove the statements for the Lie algebras in question. The statements on the level of Lie ind–groups follow immediately, since infinite determinant conditions on a Lie ind–group are equivalent to infinite trace conditions on its Lie algebra.

Suppose that $\mathfrak{p}_{\mathbb{R}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. By definition, the complexification $\mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is defined by infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer of a taut couple of generalized flags in V and W or on a self–taut generalized flag in V. As $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ is stable under complex conjugation it is the complexification of the real subspace $(\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}} := \{t \in \mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}} \mid \tau(t) = t\}$ where τ comes from complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}_{\mathbb{R}}$. We will use this to show case by case that $\mathfrak{p}_{\mathbb{R}}$ is defined by trace conditions on the $\mathfrak{g}_{\mathbb{R}}$–stabilizer of the appropriate generalized flag(s).

The first cases we treat are those where the defining representation space $V_{\mathbb{R}}$ is the fixed point set of a complex conjugation $\tau : V \to V$. The real forms fitting this description are $\mathfrak{sl}(\infty;\mathbb{R})$, $\mathfrak{so}(\infty,\infty)$, $\mathfrak{so}(p,\infty)$, $\mathfrak{sp}(\infty;\mathbb{R})$, and $\mathfrak{gl}(\infty;\mathbb{R})$. Consider the $\mathfrak{sl}(\infty;\mathbb{R})$ case, and note that the proof also holds in the $\mathfrak{gl}(\infty;\mathbb{R})$ case. Let $\mathcal{F}$ and $\mathcal{G}$ be the taut couple of generalized flags in V and W given in Theorem 2.7, and note that $W_{\mathbb{R}}$ is the fixed points of complex conjugation $\tau : W \to W$. Evidently $\tau(\mathfrak{p}_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}}$, so $\tau(\mathcal{F}) = \mathcal{F}$ and $\tau(\mathcal{G}) = \mathcal{G}$ by the uniqueness claim of Theorem 2.7. Since the generalized flags $\mathcal{F}$ and $\mathcal{G}$ are τ–stable, every subspace in them is τ–stable. (Explicitly, for any $F \in \mathcal{F}$, we have $\tau(F) \in \mathcal{F}$, so either $\tau(F) \subset F$ or $F \subset \tau(F)$. Since $\tau^2 = \mathrm{Id}$, we have $F = \tau(F)$ for any $F \in \mathcal{F}$.) Hence every subspace in $\mathcal{F}$ and $\mathcal{G}$ has a real form, obtained as the intersection with $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$, respectively. The generalized flags $\mathcal{F}_{\mathbb{R}} := \{F \cap V_{\mathbb{R}} \mid F \in \mathcal{F}\}$ and $\mathcal{G}_{\mathbb{R}} := \{G \cap W_{\mathbb{R}} \mid G \in \mathcal{G}\}$ form a taut couple as $\mathbb{R}$–generalized flags in $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. Now $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on the $\mathfrak{sl}(\infty;\mathbb{R})$–stabilizer of the taut couple $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ of generalized flags in $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$.

If $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{so}(*,\infty)$ or $\mathfrak{sp}(\infty;\mathbb{R})$, Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is defined by infinite trace conditions on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer a self–taut generalized flag $\mathcal{F}$ in V. The arguments of the $\mathfrak{sl}(\infty;\mathbb{R})$ case show that $\mathcal{F}$ is τ–stable, provided that $\tau(\mathfrak{p}_{\mathbb{C}}) = \mathfrak{p}_{\mathbb{C}}$ forces $\tau(\mathcal{F}) = \mathcal{F}$. That is ensured by the uniqueness claim in Theorem 2.7 for the symplectic case, and by Theorem 2.8 in the orthogonal cases where uniqueness holds. Uniqueness fails precisely when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(\infty,\infty)$ and there exists an isotropic subspace $L \in \mathcal{F}$ with $\dim_{\mathbb{C}}(L^{\perp}/L) = 2$. We may assume that $\mathcal{F}$ is the first of the three generalized flags listed in the statement of Theorem 2.8. Then $\tau(\mathcal{F})$ is one of the three generalized flags listed in the statement of Theorem 2.8, and since $\mathcal{F}$ is contained in any of those three, the subspaces of $\mathcal{F}$ are all τ–stable. Finally, the generalized flag $\mathcal{F}_{\mathbb{R}} := \{F \cap V_{\mathbb{R}} \mid F \in \mathcal{F}\}$ in $V_{\mathbb{R}}$ is self–taut, and $\mathfrak{p}_{\mathbb{R}}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}})_{\mathbb{R}}$ on its $\mathfrak{g}_{\mathbb{R}}$–stabilizer.

Second, suppose that $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(*,\infty)$. Note that the arguments for $\mathfrak{su}(*,\infty)$ apply without change to $\mathfrak{u}(*,\infty)$. By Theorem 2.7, $\mathfrak{p}_\mathbb{C}$ is given by infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_\mathbb{C}}$ on the $\mathfrak{gl}(\infty;\mathbb{C})$–stabilizer of a taut couple $\mathcal{F}$ and $\mathcal{G}$ of generalized flags in V and W. There exists an isomorphism of $\mathfrak{g}_\mathbb{R}$–modules $f : V \to W$. Both $\mathcal{G}$ and $f(\mathcal{F})$ are stabilized by $\mathfrak{p}_\mathbb{R}$, hence also by $\mathfrak{p}_\mathbb{C}$, so the uniqueness claim of Theorem 2.7 tells us that $\mathcal{G} = f(\mathcal{F})$. Thus $\mathcal{F}$ is self–taut. We conclude that $\mathfrak{p}_\mathbb{R}$ is given by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_\mathbb{C}})_\mathbb{R}$ on the stabilizer of the self–taut generalized flag $\mathcal{F}$.

The third case we consider is that of $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}(\infty;\mathbb{H})$. Note that the $\mathfrak{gl}(\infty;\mathbb{H})$ case is proved in the same manner. Then $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(2\infty;\mathbb{C})$, where we have the identifications $V = \mathbb{C}^{2\infty} = \mathbb{C}^\infty + \mathbb{C}^\infty j = \mathbb{H}^\infty = V_\mathbb{R}$ and $W = W_\mathbb{R}$. The quaternionic scalar multiplication $v \mapsto vj$ is a complex conjugate–linear transformation J of $\mathbb{C}^{2\infty}$ of square $-\mathrm{Id}$, and the complex conjugation τ of $\mathfrak{g}_\mathbb{C}$ over $\mathfrak{g}_\mathbb{R}$ is given by $\xi \mapsto J\xi J^{-1} = J^{-1}\xi J$. Let $\mathcal{F}$ and $\mathcal{G}$ be the unique taut couple given by Theorem 2.7. Since $\mathfrak{p}_\mathbb{C} = \tau(\mathfrak{p}_\mathbb{C})$, we have $\mathcal{F} = J(\mathcal{F})$ and $\mathcal{G} = J(\mathcal{G})$. Since $J^2 = -\mathrm{Id}$, every subspace of $\mathcal{F}$ and $\mathcal{G}$ is preserved by J. In other words $\mathcal{F}$ and $\mathcal{G}$ consist of $\mathbb{H}$-subspaces of $V_\mathbb{R}$ and $W_\mathbb{R}$. The fact that $\mathcal{F}$ and $\mathcal{G}$ form a taut couple of $\mathbb{C}$–generalized flags in V and W implies via Lemma 2.4 that they form a taut couple of $\mathbb{H}$–generalized flags in $V_\mathbb{R}$ and $W_\mathbb{R}$. Hence $\mathfrak{p}_\mathbb{R}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_\mathbb{C}})_\mathbb{R}$ on the stabilizer of the taut couple $\mathcal{F}$, $\mathcal{G}$.

The fourth case we consider is that of $\mathfrak{sp}(*,\infty)$. Then $V_\mathbb{R}$ has an invariant quaternion–hermitian form of signature $(*,\infty)$ and a complex conjugate–linear transformation J of square $-\mathrm{Id}$ as described above. Let $\mathcal{F}$ be the unique self–taut generalized flag in V given by Theorem 2.7. By the uniqueness of $\mathcal{F}$, we have $\mathcal{F} = J(\mathcal{F})$, so as before $\mathcal{F}$ consists of $\mathbb{H}$–subspaces of $V_\mathbb{R}$. Lemma 2.4 implies that $\mathcal{F}$ is self–taut when considered as an $\mathbb{H}$–generalized flag in $V_\mathbb{R}$. Hence $\mathfrak{p}_\mathbb{R}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_\mathbb{C}})_\mathbb{R}$ on the stabilizer of $\mathcal{F}$.

The fifth and final case and is that of $\mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(2\infty)$. Any subspace of V which is stable under the $\mathbb{C}$-conjugate linear map J which corresponds to $x \mapsto xj$ is an $\mathbb{H}$–subspace of $V_\mathbb{R}$. Let $\mathcal{F}$ be a self–taut generalized flag in V as given by Theorem 2.7. Since $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(\infty;\mathbb{C})$, Theorem 2.8 says that either $\mathcal{F}$ is unique or there are exactly three possibilities for $\mathcal{F}$. When $\mathcal{F}$ is unique, we must have $\mathcal{F} = J(\mathcal{F})$, so $\mathcal{F}$ is an $\mathbb{H}$–generalized flag. When $\mathcal{F}$ is not unique, we may assume that $\mathcal{F}$ is the first of the three generalized flags listed in the statement of Theorem 2.8, the one with an immediate predecessor–successor pair $L \subset L^\perp$ where L is closed and $\dim_\mathbb{C}(L^\perp/L) = 2$. Then $J(\mathcal{F})$ has the same property so $J(\mathcal{F}) = \mathcal{F}$. In all cases Lemma 2.4 implies that $\mathcal{F}$ is self–taut when considered as an $\mathbb{H}$–generalized flag. Hence $\mathfrak{p}_\mathbb{R}$ is defined by the infinite trace conditions $(\mathrm{Tr}^{\mathfrak{p}_\mathbb{C}})_\mathbb{R}$ on the $\mathfrak{so}^*(2\infty)$–stabilizer of the self–taut $\mathbb{H}$–generalized flag $\mathcal{F}$.

Conversely, suppose that $\mathfrak{p}_\mathbb{R}$ is defined by infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_\mathbb{R}}$ on the $\mathfrak{g}_\mathbb{R}$–stabilizer of a taut couple $\mathcal{F}_\mathbb{R}$, $\mathcal{G}_\mathbb{R}$ or a self–taut generalized flag $\mathcal{F}_\mathbb{R}$, as appropriate. Either $V = V_\mathbb{R} \otimes \mathbb{C}$ or $V = V_\mathbb{R}$.

Suppose first that $V = V_\mathbb{R} \otimes \mathbb{C}$. Let $\mathcal{F} := \{F \otimes \mathbb{C} \mid F \in \mathcal{F}_\mathbb{R}\}$. If $\mathfrak{g}_\mathbb{C}$ has only one defining representation V, then $\mathcal{F}$ is a self–taut generalized flag in V, and $\mathfrak{p}_\mathbb{C}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_\mathbb{R}} \otimes \mathbb{C}$ on the $\mathfrak{g}_\mathbb{C}$–stabilizer of $\mathcal{F}$. Now suppose that $\mathfrak{g}_\mathbb{C}$ has two inequivalent defining representations. If $\mathfrak{g}_\mathbb{R}$ also has two inequivalent defining representations, let $\mathcal{G} := \{G \otimes \mathbb{C} \mid G \in \mathcal{G}_\mathbb{R}\}$. If $\mathfrak{g}_\mathbb{R}$ has only one defining representation, then let $\mathcal{G}$ be the image of $\mathcal{F}$ under the $\mathfrak{g}_\mathbb{R}$–module isomorphism $V \to W$. Then $\mathcal{F}$, $\mathcal{G}$ are a taut couple, and $\mathfrak{p}_\mathbb{C}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_\mathbb{R}} \otimes \mathbb{C}$ on the $\mathfrak{g}_\mathbb{C}$–stabilizer of $\mathcal{F}$, $\mathcal{G}$.

Suppose that $V = V_{\mathbb{R}}$. Then $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ have the same number of defining representations. If $\mathfrak{g}_{\mathbb{R}}$ has two defining representations, then Lemma 2.4 implies that $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{G}_{\mathbb{R}}$ are a taut couple when considered as $\mathbb{C}$–generalized flags. Then $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer of $\mathcal{F}_{\mathbb{R}}$, $\mathcal{G}_{\mathbb{R}}$. If $\mathfrak{g}_{\mathbb{R}}$ has only one defining representation, then Lemma 2.4 implies that $\mathcal{F}_{\mathbb{R}}$ is a self–taut generalized flag when considered as a $\mathbb{C}$–generalized flag. Thus $\mathfrak{p}_{\mathbb{C}}$ is defined by the infinite trace conditions $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{R}}} \otimes \mathbb{C}$ on the $\mathfrak{g}_{\mathbb{C}}$–stabilizer of $\mathcal{F}_{\mathbb{R}}$.

In each case, Theorem 2.7 implies that $\mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, so by definition $\mathfrak{p}_{\mathbb{R}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. □

Theorem 3.2. *Let $\mathfrak{p}_{\mathbb{R}}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$. If $\mathfrak{g}_{\mathbb{R}} \not\cong \mathfrak{so}(\infty,\infty)$, then there is a unique taut couple or self–taut generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ by* Theorem 3.1*. The real analogue of* Theorem 2.8 *holds for $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(\infty,\infty)$.*

Proof. If there is a unique taut couple or self–taut generalized flag associated to $\mathfrak{p}_{\mathbb{C}}$, then the uniqueness of the taut couple or self–taut generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ is immediate from the proof of Theorem 3.1. If $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}(\infty,\infty)$, then each of the $\mathbb{C}$–generalized flags of Theorem 2.8 has a real form, hence the real analogue of Theorem 2.8 holds in this case. Now suppose that $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{so}^*(2\infty)$ and the self–taut generalized flag $\mathcal{F}$ associated to $\mathfrak{p}_{\mathbb{C}}$ has a closed isotropic subspace L with $\dim_{\mathbb{C}}(L^{\perp}/L) = 2$. The proof of Theorem 3.1 shows that L and $L^{\perp}$ are $\mathbb{H}$–subspaces, and the quaternionic codimension of L in $L^{\perp}$ is 1. Hence the $\mathbb{H}$–generalized flag associated to $\mathfrak{p}_{\mathbb{R}}$ has no subspaces strictly between L and $L^{\perp}$, which forces it to be unique. □

Remark 3.3. Theorem 3.1 simplifies sharply in the $\mathfrak{su}(p,\infty)$, $\mathfrak{so}(p,\infty)$, $\mathfrak{sp}(p,\infty)$, and $\mathfrak{u}(p,\infty)$ cases when $p \in \mathbb{Z}_{\geqq 0}$. Because p is the maximal dimension of an isotropic subspace of $V_{\mathbb{R}}$ (and thus the maximal codimension of a closed coisotropic subspace), a self–taut generalized flag must be finite. No infinite trace conditions arise. The stabilizer of such a self–taut generalized flag coincides with the joint stabilizer of its isotropic subspaces and at most one non–closed coisotropic subspace. (The perpendicular complement of the single non–closed coisotropic subspace, when it occurs, is the largest isotropic subspace.) ♢

Remark 3.4. The special case where the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (or $\mathfrak{g}_{\mathbb{R}}$) is a direct limit of parabolics of the $\mathfrak{g}_{n,\mathbb{C}}$ (or the $\mathfrak{g}_{n,\mathbb{R}}$) has been studied in a number of contexts such as [8] and [10], and in particular in connection with direct limits of principal series representations [12]. Any direct limit of parabolic subalgebras is a parabolic subalgebra in the general sense of this paper. ♢

4 A Geometric Interpretation

Our geometric interpretation is modeled on a criterion from the finite–dimensional case. Let $G_{\mathbb{C}}$ be a finite–dimensional classical Lie ind–group, and $G_{\mathbb{R}}$ a real form of $G_{\mathbb{C}}$. Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup, and let $Z := G_{\mathbb{C}}/P$ be the corresponding flag manifold. Then $G_{\mathbb{R}}$ acts on Z as a subgroup of $G_{\mathbb{C}}$. One knows [11, Theorem 3.6] that there is a unique closed $G_{\mathbb{R}}$–orbit F on Z, and that $\dim_{\mathbb{R}} F \geqq \dim_{\mathbb{C}} Z$, with equality precisely when F is a real form of Z. Thus real and complex dimensions satisfy $\dim_{\mathbb{R}} F = \dim_{\mathbb{C}} Z$ if and only if F is a totally real submanifold of Z. This is the motivation for our geometric interpretation, for F is a totally real submanifold of Z if and only if $G_{\mathbb{R}}$ has a parabolic subgroup whose complexification is $G_{\mathbb{C}}$–conjugate to P. Then that real parabolic subgroup is the $G_{\mathbb{R}}$–stabilizer of a point of the closed orbit F. Here note that if any $G_{\mathbb{R}}$–orbit in Z is totally real then it has real dimension $\leqq \dim_{\mathbb{C}} Z$, so it must be the closed orbit.

Let now $G_{\mathbb{C}}$ be one of the Lie ind–groups $GL(\infty;\mathbb{C})$, $SL(\infty;\mathbb{C})$, $SO_0(\infty;\mathbb{C})$ and $Sp(\infty;\mathbb{C})$. Fix an exhaustion of $G_{\mathbb{C}}$ by classical connected finite–dimensional subgroups $G_{n,\mathbb{C}}$, and let $G_{n,\mathbb{R}}$ be nested real forms of $G_{n,\mathbb{C}}$. Then $G_{\mathbb{R}} := \varinjlim G_{n,\mathbb{R}}$ is a real form of $G_{\mathbb{C}}$. Let $P_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$. As described in Section 2C, $P_{\mathbb{C}}$ is defined by infinite determinant conditions on the stabilizer $\widetilde{P_{\mathbb{C}}}$ of a taut couple or a self–taut generalized flag. Here $\widetilde{P_{\mathbb{C}}}$ is the normalizer of P_C in G_C. We use the usual notation for the Lie algebras of all these Lie ind–groups.

Lemma 4.1. *Consider the homogeneous space $Z = G_{\mathbb{C}}/\widetilde{P_{\mathbb{C}}}$. Write z_0 for the identity coset $1\cdot\widetilde{P_{\mathbb{C}}}$ in Z and define $Z_n = G_{n,\mathbb{C}}(z_0)$. Then each Z_n is a (finite–dimensional) complex homogeneous space and Z is the complex ind–manifold $\varinjlim Z_n$ (direct limit in the category of complex manifolds and holomorphic maps.)*

Proof. $\widetilde{P_{\mathbb{C}}}$ is a complex subgroup of $G_{\mathbb{C}}$, and $\widetilde{P_{\mathbb{C}}} = \varinjlim(G_{n,\mathbb{C}} \cap \widetilde{P_{\mathbb{C}}})$. Each finite–dimensional orbit Z_n is a complex manifold because $G_{n,\mathbb{C}} \cap \widetilde{P_{\mathbb{C}}}$ is a complex subgroup of $G_{n,\mathbb{C}}$, and the inclusions $Z_n \hookrightarrow Z_{n+1}$ are holomorphic embeddings. As in [10] now $Z = \varinjlim Z_n$ is a strict direct limit in the category of complex manifolds and holomorphic maps. In other words a function f on an open subset $U \subset Z$ is holomorphic if and only if each of the $f|_{U\cap Z_n} : U \cap Z_n \to \mathbb{C}$ is holomorphic. Note that separately holomorphic functions on open subsets $U \subset Z$ are jointly holomorphic because each $f|_{U\cap Z_n}$ is jointly holomorphic (and thus continuous) by Hartogs' Theorem. □

Lemma 4.2. *Let $Y = G_{\mathbb{R}}(z_0)$ and $Y_n = G_{n,\mathbb{R}}(z_0)$. Then Y is a totally real submanifold of Z if and only if each Y_n is a totally real submanifold of Z_n.*

Proof. Let J denote the complex structure operator for Z, linear transformation of square $-\mathrm{Id}$ on the complexified tangent space $T := T_{z_0,\mathbb{C}}(Z)$ of Z at z_0. Then J preserves each of the $T_n := T_{z_0,\mathbb{C}}(Z_n)$. Now Y is totally real if and only if the real tangent space $T_{\mathbb{R}} := T_{z_0}(Z)$ satisfies $J(T_{\mathbb{R}}) \cap T_{\mathbb{R}} = 0$, and Y_n is totally real if and only if the real tangent space $T_{n,\mathbb{R}} := T_{z_0}(Z_n)$ satisfies $J(T_{n,\mathbb{R}}) \cap T_{n,\mathbb{R}} = 0$. Since $T_{\mathbb{R}} = \varinjlim T_{n,\mathbb{R}}$ the assertion follows. □

Lemma 4.3. *$G_{n,\mathbb{R}} \cap \widetilde{P_{\mathbb{C}}}$ is a real form of $G_{n,\mathbb{C}} \cap \widetilde{P_{\mathbb{C}}}$ if and only if Y_n is totally real in Z_n.*

Proof. Denote $H_{n,\mathbb{C}} = G_{n,\mathbb{C}} \cap \widetilde{P_{\mathbb{C}}}$ and $H_{n,\mathbb{R}} = G_{n,\mathbb{R}} \cap \widetilde{P_{\mathbb{C}}}$. Suppose first that Y_n is totally real in Z_n. Then $\dim_{\mathbb{R}} G_{n,\mathbb{R}} - \dim_{\mathbb{R}} H_{n,\mathbb{R}} = \dim_{\mathbb{R}} Y_n \leqq \dim_{\mathbb{C}} Z_n = \dim_{\mathbb{C}} G_{n,\mathbb{C}} - \dim_{\mathbb{C}} H_{n,\mathbb{C}}$, so $\dim_{\mathbb{R}} H_{n,\mathbb{R}} \geqq \dim_{\mathbb{C}} H_{n,\mathbb{C}}$, forcing $\dim_{\mathbb{R}} H_{n,\mathbb{R}} = \dim_{\mathbb{C}} H_{n,\mathbb{C}}$. Now $H_{n,\mathbb{R}}$ is a real form of $H_{n,\mathbb{C}}$.

Conversely suppose that $H_{n,\mathbb{R}}$ is a real form of $H_{n,\mathbb{C}}$. Then the real tangent space to Y_n at z_0 is represented by any vector space complement $\mathfrak{m}_{n,\mathbb{R}}$ to $\mathfrak{h}_{n,\mathbb{R}}$ in $\mathfrak{g}_{n,\mathbb{R}}$, while the real tangent space to Z_n at z_0 is represented by the vector space complement $\mathfrak{m}_{n,\mathbb{R}} \otimes \mathbb{C}$ to $\mathfrak{h}_{n,\mathbb{C}}$ in $\mathfrak{g}_{n,\mathbb{C}}$, so Y_n is totally real in Z_n. □

Putting all this together, we have our geometric characterization of parabolic subgroups of the classical real Lie ind–groups.

Theorem 4.4. *Fix a parabolic subgroup $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ and consider the flag ind–manifold $Z = G_{\mathbb{C}}/\widetilde{P_{\mathbb{C}}}$. Then $P_{\mathbb{C}} \cap G_{\mathbb{R}}$ is a parabolic subgroup of $G_{\mathbb{R}}$ if and only if the following two conditions hold:*

(i) *the orbit $G_{\mathbb{R}}(z_0)$ of the base point $z_0 = \widetilde{P_{\mathbb{C}}}$ is a totally real submanifold of Z;*

(ii) *the set of all infinite trace conditions on $\widetilde{\mathfrak{p}_{\mathbb{C}}}$ satisfied by $\mathfrak{p}_{\mathbb{C}}$ is stable under the complex conjugation τ of $\mathfrak{g}_{\mathbb{C}}$ over $\mathfrak{g}_{\mathbb{R}}$.*

Proof. Lemmas 4.2 and 4.3 show that the orbit $G_{\mathbb{R}}(z_0)$ is a totally real submanifold of Z if and only if $G_{\mathbb{R}} \cap \widetilde{P_{\mathbb{C}}}$ is parabolic in $G_{\mathbb{R}}$.

If $G_{\mathbb{R}} \cap P_{\mathbb{C}}$ is parabolic in $G_{\mathbb{R}}$ then $G_{\mathbb{R}} \cap \widetilde{P_{\mathbb{C}}}$ is parabolic because it contains $G_{\mathbb{R}} \cap P_{\mathbb{C}}$, and the corresponding real set of infinite trace conditions complexifies to the set of infinite trace conditions by which $\mathfrak{p}_{\mathbb{C}}$ is defined from $\widetilde{\mathfrak{p}_{\mathbb{C}}}$. Thus (i) and (ii) follow.

Conversely assume (i) and (ii). From (i), $G_{\mathbb{R}} \cap \widetilde{P_{\mathbb{C}}}$ is a parabolic subgroup of $G_{\mathbb{R}}$, and from (ii), $\{x \in \mathfrak{g}_{\mathbb{R}} \cap \widetilde{\mathfrak{p}_{\mathbb{C}}} \mid x \text{ satisfies } \mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}}\} \otimes \mathbb{C} = \{x \in \widetilde{\mathfrak{p}_{\mathbb{C}}} \mid x \text{ satisfies } \mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}}\}$, where $\mathrm{Tr}^{\mathfrak{p}_{\mathbb{C}}}$ denotes the set of infinite trace conditions described in Definition 2.6. □

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E.D.-C.:
Department of Mathematics
Rice University
6100 S. Main St.
Houston TX 77005-1892
edc@rice.edu

I.P.:
School of Engineering and Science
Jacobs University Bremen
Campus Ring 1
28759 Bremen, Germany
i.penkov@jacobs-university.de

J.A.W.:
Department of Mathematics
University of California
Berkeley, CA 94720–3840
jawolf@math.berkeley.edu