EXCEPTIONAL SEQUENCES OF INVERTIBLE SHEAVES ON RATIONAL SURFACES

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Abstract. In this article we consider exceptional sequences of invertible sheaves on smooth complete rational surfaces. We show that to every such sequence one can associate a smooth complete toric surface in a canonical way. We use this structural result to prove various theorems on exceptional and strongly exceptional sequences of invertible sheaves on rational surfaces. We construct full strongly exceptional sequences for a large class of rational surfaces. For the case of toric surfaces we give a complete classification of full strongly exceptional sequences of invertible sheaves.

Contents

1. Introduction

The study of derived categories of coherent sheaves on algebraic varieties has gained much attention since the mid-90's, with some of the main motivations coming from Kontsevich's homological mirror symmetry conjecture [Kon95] and, evolving from this, the use of derived categories for D-branes in superstring theory [Dou01]. The object one studies is the derived category $D^b(X)$ of coherent sheaves over a smooth algebraic variety X defined over some algebraically closed field $\mathbb{K}$. By definition, $D^b(X)$ is a categorial framework for the homological algebra of coherent sheaves on X. It turns out that $D^b(X)$ carries a very rich structure and encodes information which might not directly be visible from the geometry of X. For an overview we refer to the book [Huy06] and the survey article [Bri06]. However, despite of many interesting and deep results, the theory seems far from being developed enough to make $D^b(X)$ an easily accessible object in any sense. A particular open problem is the construction of suitable generating sets, for which the framework of exceptional sequences has been developed by the Seminaire Rudakov [Rud90]:

Definition: A coherent sheaf $\mathcal{E}$ on X is called *exceptional* if $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) = \mathbb{K}$ and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) = 0$ for every $i \neq 0$. A sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of exceptional sheaves is called an *exceptional* sequence if $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all k and for all $i > j$. If an exceptional sequence generates $D^b(X)$, then it is called *full*. A *strongly* exceptional sequence is an exceptional sequence such that $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $k > 0$ and all i, j.

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If a full exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ exists on X and $\langle \mathcal{E}_i \rangle$ denotes the minimal triangulated subcategory of $D^b(X)$ containing $\mathcal{E}_i$, then $\langle \mathcal{E}_1 \rangle, \ldots \langle \mathcal{E}_n \rangle$ forms a *semi-orthogonal decomposition* of $D^b(X)$, i.e. we have $\langle \mathcal{E}_j \rangle \subset \langle \mathcal{E}_i \rangle^\perp$ for all $i > j$. Such decompositions naturally arise in birational geometry (see [Orl93], [Kaw08]) and for Fourier-Mukai transforms (see [HvdB07]). Full strongly exceptional sequences provide an even stronger characterization of $D^b(X)$ in terms of representation theory of algebras [Hap88]. By theorems of Baer [Bae88] and Bondal [Bon90] for such a sequence there exists an equivalence of categories

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, \,.\,) : D^b(X) \longrightarrow D^b(\mathrm{End}(\mathcal{T}) - \mathrm{mod}),$$

where $\mathcal{T} := \bigoplus_{i=1}^n \mathcal{E}_i$, which is sometimes called a tilting sheaf. This way the algebra $\mathrm{End}(\mathcal{T})$, at least in the derived sense, represents a non-commutative coordinate system of X.

Strongly exceptional sequences have classically been known for the case of $\mathbb{P}^n$ (see [Beĭ78] and [DL85]). However, exceptional or strongly exceptional sequences must not exist in general, and their existence still is an open problem. For instance, on Calabi-Yau varieties it follows from Serre duality that there do not even exist exceptional sheaves. On the other hand, by now, exceptional sequences have been constructed in many interesting cases, including certain types of homogeneous spaces [Kap86], [Kap88], [Kuz05], [Sam07], del Pezzo surfaces and almost del Pezzo surfaces [Gor89], [KO95], [Kul97], [KN98], and some higher dimensional Fano varieties [Nog94], [Sam05].

In this paper we consider exceptional sequences on smooth complete rational surfaces which consist of invertible sheaves. This special setting is motivated by a conjecture of King [Kin97], which states that on every smooth complete toric variety there exists a strongly exceptional sequence of invertible sheaves. Invertible sheaves on toric varieties can be described in very explicit combinatorial terms and a number of examples were well-known when the conjecture was stated. Also of interest here is the fact that toric varieties can nicely be represented as moduli spaces of certain quiver representations and their universal sheaf is a good candidate for a (partial) tilting sheaf. Examples of strongly exceptional sequences have been given from this point of view in [Kin97] and [AH99] (see also [Bro06], [CS06], [BP08]). Other constructions have been given in [CM04], [CM05], and for toric stacks in [BH08]. Typically, general constructions are only available for very special situations such as iterated projective bundles, or small Picard number. It is known that strongly exceptional sequences of invertible sheaves exist on the toric 3-Fanos, and computer experiments indicate that this is also true for 4-Fanos. However, general existence theorems are only available for exceptional sequences which are not strongly exceptional. So it has been shown in [Hil04] that exceptional sequences of invertible sheaves exist on smooth toric surfaces. The existence of exceptional sequences which do not necessarily consist of invertible sheaves has been shown for general smooth projective toric stacks by Kawamata [Kaw06]. Despite a lot of positive evidence, the existence of strongly exceptional sequences still is an open problem for toric varieties. In [HP06] an example was given of a toric surface which does not admit a strongly exceptional sequence of invertible sheaves, the second Hirzebruch surface iteratively blown up three times. This counterexample at that time seemed somewhat mysterious, in particular because, having Picard number 5, it is surprisingly small. For general rational surfaces there is no bound for the Picard number. This can be shown by well-known examples, such as simultaneous blow-ups of $\mathbb{P}^2$ in several points, by which any Picard number can be realized (see Theorem 5.9). In the toric case, explicit positive examples with higher Picard numbers were known to the authors, including further blow-ups of the counterexample (see example 8.4). So the question is, what is the obstruction for the existence of a (strongly) exceptional sequence of invertible sheaves on a toric or more general rational surface? It turns out that toric surfaces are at the heart of the problem, even for the case of general rational surfaces. The most important structural insight of this paper is the following remarkable observation:

Theorem (3.5)**:** *Let X be a smooth complete rational surface, let $\mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_n)$ be a full exceptional sequence of invertible sheaves on X, and set $E_{n+1} := E_1 - K_X$. Then to this sequence there is associated in a canonical way a smooth complete toric surface with torus invariant prime divisors $D_1, \ldots, D_n$ such that $D_i^2 + 2 = \chi\big(\mathcal{O}_X(E_{i+1} - E_i)\big)$ for all $1 \leq i \leq n$.*

Of course, this theorem deserves a more detailed explanation which will be given below. For the convenience of the reader we want first to present the most important consequences derived from this. Our first main result shows the existence of exceptional sequences in general:

Theorem (5.6)**:** *On every smooth complete rational surface there exists a full exceptional sequence of invertible sheaves.*

We point out that for rational surfaces this theorem is not a big surprise and can also be derived from results of Orlov [Orl93]. However, as noted above, an analogous theorem does not hold if we require the sequences to be strongly exceptional. A necessary condition for the existence of a full strongly exceptional sequence seems to be that the surface is not too far away from a minimal model. By the Enriques classification, every smooth complete rational surface is a blow-up of the projective plane or some Hirzebruch surface. In fact, we can prove that such sequences exist on a surface which comes from blowing up a Hirzebruch surface once or twice, possibly in several points in every step.

Theorem (5.9)**:** *Any smooth complete rational surface which can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step) has a full strongly exceptional sequence of invertible sheaves.*

In the toric case, we can show that the converse is also true:

Theorem (8.2)**:** *Let $\mathbb{P}^2 \neq X$ be a smooth complete toric surface. Then there exists a full strongly exceptional sequence of invertible sheaves on X if and only if X can be obtained from a Hirzebruch surface in at most two steps by blowing up torus fixed points.*

Note that the blow-up of $\mathbb{P}^2$ at any point is isomorphic to the first Hirzebruch surface. So there is no loss of generality if only blow-ups of Hirzebruch surfaces are considered. In particular, Theorem 8.2 implies that the Picard number of a toric surface on which a full strongly exceptional sequence of invertible sheaves exists is at most 14. On the other hand, the example given in [HP06] is a minimal example which does not satisfy the condition of the theorem.

Another important aspect of exceptional sequences is their relation to helix theory as developed in [Rud90].

Definition: An infinite sequence of sheaves $\ldots, \mathcal{E}_i, \mathcal{E}_{i+1}, \ldots$ is called a *cyclic (strongly) exceptional sequence* if there exists an n such that $\mathcal{E}_{i+n} \cong \mathcal{E}_i \otimes \mathcal{O}(-K_X)$ for every $i \in \mathbb{Z}$ and if every winding (i.e. every subinterval $\mathcal{E}_{i+1}, \ldots, \mathcal{E}_{i+n}$) forms a (strongly) exceptional sequence. A cyclic exceptional sequence is *full* if every winding is a full exceptional sequence.

Our notion of cyclic strongly exceptional sequences is very close to the geometric helices of [BP94], but we want to point out that these notions do not coincide, as we do not require that our cyclic exceptional sequences are generated by mutations. In fact, if we consider a winding $\mathcal{E}_{i+1}, \ldots, \mathcal{E}_{i+n}$ as the foundation of a helix, then the n-th right mutation of $\mathcal{E}_i$ coincides with $\mathcal{E}_{i+n}$ up to a shift in the derived category. By results of [Bon90] a foundation of a helix generates the derived category precisely if any foundation does. Hence a cyclic exceptional sequence is full if and only if it has any winding which is a full exceptional sequence. By a result of Bondal and Polishchuk, the maximal periodicity of a geometric helix on a surface is 3, which implies that $\mathbb{P}^2$ is the only rational surface which admits a full geometric helix. Our weaker notion admits a bigger class of surfaces, but still imposes very strong conditions:

Theorem (5.13)**:** *Let X be a smooth complete rational surface on which a full cyclic strongly exceptional sequence of invertible sheaves exists. Then* $\operatorname{rk}\operatorname{Pic}(X) \leq 7$.

So not even every del Pezzo surface admits such a sequence. However:

Theorem (5.14)**:** *Let X be a del Pezzo surface with* $\operatorname{rk}\operatorname{Pic}(X) \leq 7$*, then there exists a full cyclic strongly exceptional sequence of invertible sheaves on X.*

The condition that $-K_X$ is ample can be weakened in general. In the toric case we obtain a complete characterization for toric surfaces admitting cyclic strongly exceptional sequences:

Theorem (8.5 & 8.6)**:** *Let X be a smooth complete toric surface, then there exists a full cyclic strongly exceptional sequence of invertible sheaves on X if and only if $-K_X$ is nef.*

Note that cyclic strongly exceptional sequences have been considered before, most notably in physics literature (see [HHV06], [Asp08], [BP06], [HK06]), but usually under different names. Theorems 8.5 and

8.6 have been conjectured in this context. The particular interest here comes from the fact that the total space $\pi : \omega_X \to X$ of the canonical bundle $\mathcal{O}_X(K_X)$ is a local Calabi-Yau manifold. It follows from results of Bridgeland [Bri05] that a full strongly exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ on X can be extended to a cyclic strongly exceptional sequence iff the pullbacks $\pi^*\mathcal{E}_1, \ldots, \pi^*\mathcal{E}_n$ form a sequence on ω_X which is almost exceptional in the sense that the $\pi^*\mathcal{E}_i$ generate $D^b(\omega_X)$ and $\operatorname{Ext}^k(\pi^*\mathcal{E}_i, \pi^*\mathcal{E}_j) = 0$ for every i, j and all $k > 0$ (however, due to the fact that ω_X is not complete, we cannot expect that any Hom-groups among the $\pi^*\mathcal{E}_i$ vanish). Another interesting observation is that for the toric singularities which arise from contracting the zero section in ω_X, the endomorphism algebras of $\bigoplus_{i=1}^n \pi^*\mathcal{E}_i$ give examples for non-commutative resolutions in the sense of van den Bergh [vdB04a], [vdB04b].

Now we give some more technical explanations concerning Theorem 3.5 and its consequences. The key idea is astoundingly simple. Let X be a smooth complete rational surface and $E_1, \ldots, E_n$ Cartier divisors on X such that $\mathcal{O}_X(E_1), \ldots, \mathcal{O}_X(E_n)$ form an exceptional sequence of invertible sheaves. For these sheaves, there are natural isomorphisms $\operatorname{Ext}^k_{\mathcal{O}_X}\big(\mathcal{O}_X(E_i), \mathcal{O}_X(E_j)\big) \cong H^k\big(X, \mathcal{O}_X(E_j - E_i)\big)$ and therefore it is convenient to bring this exceptional sequence into a normal form by passing to differences. We set $A_i := E_{i+1} - E_i$ for $1 \leq i < n$ and $A_n := -K_X - \sum_{i=1}^{n-1} A_i$, where K_X denotes the canonical divisor. The reason for adding A_n will become clear below. The fact that the E_i form an exceptional sequence then implies $H^k\big(X, \mathcal{O}_X(-\sum_{i \in I} A_i)\big) = 0$ for every interval $I \subset [1, \ldots, n-1]$ and every $k > 0$. It is an easy consequence of the Riemann-Roch theorem that moreover the A_i have the following properties:

(i) $A_i.A_{i+1} = 1$ for $1 \leq i < n$ and $A_1.A_n = 1$;
(ii) $A_i.A_j = 0$ for $i \neq j$, $\{i, j\} \neq \{1, n\}$, and $\{i, j\} \neq \{k, k+1\}$ for any $1 \leq k < n$;
(iii) $\sum_{i=1}^n A_i = -K_X$.

Definition: We call a set of divisors on X which satisfy the conditions (i), (ii), (iii) above a *toric system*.

With respect to a toric system we consider the short exact sequence

$$0 \longrightarrow \operatorname{Pic}(X) \xrightarrow{A} \mathbb{Z}^n \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

where A maps a divisor class D to the tuple $(A_1.D, \ldots, A_n.D)$. The images $l_1, \ldots, l_n$ of the standard basis of $\mathbb{Z}^n$ in $\mathbb{Z}^2$ are the *Gale duals* of $\mathcal{A} = A_1, \ldots, A_n$. It is now an exercise in linear algebra (see Proposition 2.7) to show that the l_i generate the fan of a smooth complete toric surface which we denote $Y(\mathcal{A})$. This means, by passing from $E_1, \ldots, E_n$ via its toric system to the vectors $l_1, \ldots, l_n$, we have a canonical way of associating a toric surface to a strongly exceptional sequence of invertible sheaves on any rational surface. This correspondence is even stronger; as Gale duality is indeed a duality, we can as well consider the A_i as Gale duals of the l_i. But by a standard fact of toric geometry, the Gale duals of the l_i can be interpreted as the classes of the torus invariant prime divisors $D_1, \ldots, D_n$ on $Y(\mathcal{A})$. Hence, we can identify $\operatorname{Pic}(X)$ and $\operatorname{Pic}\big(Y(\mathcal{A})\big)$ and the respective intersection products in a natural way, such that $A_i^2 = D_i^2$ for all i. In particular, note that the set of invariant irreducible divisors forms a toric system for any smooth complete toric surface.

Implicitly, toric systems have already shown up in the classical analysis of del Pezzo surfaces. In modern form, this seems first to be written in the first edition of [Man86] (see also [Dem80]). Consider X a t-fold blow-up of $\mathbb{P}^2$, i.e. $X = X_t \xrightarrow{b_t} X_{t-1} \xrightarrow{b_{t-1}} \cdots \xrightarrow{b_2} X_1 \xrightarrow{b_1} \mathbb{P}^2$. Then we get a nice basis $H, R_1, \ldots, R_t$ of $\operatorname{Pic}(X)$, where H is the pull-back of the class of a line on $\mathbb{P}^2$, and R_i is the pull-back of the exceptional divisor of the blow-up b_i. This basis diagonalizes the intersection product of $\operatorname{Pic}(X)$, i.e. $H^2 = 1$, $R_i^2 = -1$ and $H.R_i = 0$ for all i, and $R_i.R_j = 0$ for all $i \neq j$. For simplicity, let us assume that $t > 5$. Then we construct a graph as follows. For the vertices, we set $A_0 := H - R_1 - R_2 - R_3$ and $A_i := R_i - R_{i+1}$ for $i = 1, \ldots, t-1$ and we draw an edge between A_i and A_j whenever $A_i.A_j \neq 0$. This way we obtain a graph of type $\mathbb{E}_t$ which is indefinite for $t > 8$. For $t \leq 8$ it is shown in [Man86] that the set of divisors $\{D \in \operatorname{Pic}(X) \mid \chi(-D) = -K_X.D = 0\}$ forms a root system which is generated by the A_i. In case of $t = 6$ this root system represents the symmetries of the famous 27 lines on the cubic surface. The system of divisors $A_0, \ldots, A_{t-1}$ is almost a toric system. We can turn it into a proper toric system by removing A_0 and adding $A_t := R_t$, $A_{t+1} := H - \sum_{i=1}^t R_i$, $A_{t+2} := H$, and $A_{t+3} := H - R_1$. This toric system always represents an exceptional sequence which is of the form $\mathcal{O}_X, \mathcal{O}_X(R_1), \ldots, \mathcal{O}_X(R_t), \mathcal{O}_X(H), \mathcal{O}_X(2H)$. In case that the b_i commute, this sequence is even strongly exceptional. Note that there always are ambiguities concerning the enumeration of the A_i; we always can try to change it cyclically or even choose the reverse enumeration.

This sequence gives an example of an exceptional sequence which is an augmentation of the standard sequence on $\mathbb{P}^2$. On $\mathbb{P}^2$ there exists a unique toric system, which is of the form H, H, H. After blowing up once, we can augment this toric system by inserting R_1 in any place and subtracting R_i in the two neighbouring positions, i.e., up to symmetries, we obtain a toric system $H - R_1, R_1, H - R_1, H$ on X_1. Continuing with this, we essentially get two possibilities on X_2, namely

$$H - R_1 - R_2, R_2, R_1 - R_2, H - R_1, H$$
$$H - R_1, R_1, H - R_1 - R_2, R_2, H - R_2.$$

It is easy to see that all of these examples lead to strongly exceptional sequences for almost all enumerations which keep the cyclic order. The only exception being the first one in the case where b_2 is a blow-up of an infinitesimal point. Here, we necessarily have to choose the enumeration of the A_i such that $A_n = R_1 - R_2$.

Similarly, on any Hirzebruch surface $\mathbb{F}_a$ there exist, in fact infinitely many, toric systems of the form $P, sP + Q, P, -(a + s)P + Q$ with $s \geq -1$, which correspond to strongly exceptional sequences. Here, P and Q are the two generators of the nef cone in $\operatorname{Pic}(\mathbb{F}_a)$, where P is the class of a fiber of the $\mathbb{P}^1$-fibration $\mathbb{F}_a \to \mathbb{P}^1$ and Q is the generator with $Q^2 = a$. We can extend these toric systems along blow-ups in an analogous fashion. We call toric systems obtained this way *standard augmentations* (see Definition 5.4). It turns out that Theorem 8.2 is a consequence of the following characterization of strongly exceptional sequences arising from standard augmentations.

Theorem (5.11)**:** *Let $\mathbb{P}^2 \neq X$ be a smooth complete rational surface which admits a full strongly exceptional sequence whose associated toric system is a standard augmentation. Then X can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step).*

Standard augmentations provide a straightforward procedure which allows to produce strongly exceptional sequences of invertible sheaves on a large class of rational surfaces. It is natural to ask whether it is actually possible to get all such sequences this way. The answer so far is: probably yes. Indeed, Theorem 8.2 is a corollary of Theorem 5.11 and the following result:

Theorem (8.1)**:** *Let X be a smooth complete toric surface, then every full strongly exceptional sequence of invertible sheaves comes from a toric system which is a standard augmentation.*

Conjecturally, this Theorem should generalize to general rational surfaces. However, our result is based on a rather detailed analysis of cohomology vanishing on toric surfaces which we cannot easily extend to the general case. Moreover, a standard augmentation does not necessarily look like a standard augmentation at the first glance. In the phrase "comes from" in above theorem is hidden a normalization process which must be performed and, as such, is almost obvious (see the end of section 5 for details), but whose necessity significantly increases the difficulty of the classification. It turns out that in the toric case all "difficult" strongly exceptional sequences are related to cyclic exceptional sequences. These in turn are easier to understand, but in no case it is a priori clear whether a given strongly exceptional sequence is cyclic. We hope to obtain a more geometric understanding for this in future work.

Overview. In section 2, after surveying some standard facts on the geometry of smooth complete toric surfaces, we introduce toric systems and explain their relation to toric surfaces. In section 3 we derive some elementary properties from cohomology vanishing and show that to every exceptional sequence on a smooth complete rational surface there is associated a toric system. Section 4 contains some general results for cohomology vanishing on rational surfaces. Based on this, we prove in section 5 our results for exceptional sequences on general rational surfaces, except for Theorem 5.11, which is proved in section 6. Sections 7 to 10 are entirely devoted to the case of toric surfaces. In section 7 we give a detailed description of cohomology vanishing of divisors on smooth complete toric surfaces. Section 8 contains the main results on strongly exceptional sequences on toric surfaces. In sections 9 and 10 we give a proof of Theorem 8.1.

Notation and general conventions. For some positive integer l, we denote $[l] := \{1, \ldots, l\}$. If we use the letter n (or $n - 1$, $n + 1$, $n + k$, etc.), we will usually assume that the elements of $[n]$ are in cyclic order in the sense that we consider $[n]$ as a system of representatives of $\mathbb{Z}/n\mathbb{Z}$. In particular, for some $i \in [n]$ and some $j \in \mathbb{Z}$, we identify $i + j$ with the corresponding class in $[n]$. If we use some different letter, say t, then we will usually consider the standard total order on the set $[t]$. Depending on context, we may also consider other partial orders on the set $[t]$. An *interval* $I \subsetneq [n]$ is a subset $I = \{i, i+1, \ldots, i+k\}$, where $i \in [n]$, $i+k \leq n$ and $0 \leq k < n-1$. A *cyclic* interval $I \subsetneq [n]$ is either an interval or the union $I = I_1 \cup I_2$ of two intervals such that $1 \in I_1$ and $n \in I_2$. For any $\mathbb{Z}$-module K, we will denote $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$. For some divisor D on a variety X, we will usually omit the subscript X for the corresponding invertible sheaf $\mathcal{O}_X(D)$ if there is no ambiguity for X. We denote $h^i(D) := h^i\big(\mathcal{O}(D)\big) := \dim H^i\big(X, \mathcal{O}_X(D)\big)$. We will frequently make use of the fact that for any Cartier divisor D on an algebraic surface X and any blow-up $b : X' \to X$ there are isomorphisms $H^i\big(X', b^*\mathcal{O}_X(D)\big) \cong H^i\big(X, \mathcal{O}_X(D)\big)$ for every $i \in \mathbb{Z}$.

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2. The birational geometry of toric surfaces

For general reference on toric varieties, we refer to [Oda88] and [Ful93]. The specifics for toric surfaces are taken from [MO78] and [Oda88]. For Gale transformation, we refer to [GKZ94] and [OP91]. Let X be a smooth complete toric surface defined over some algebraically closed field $\mathbb{K}$. That is, there exists a two-dimensional torus $T \cong (\mathbb{K}^*)^2$ acting on X such that T itself is embedded as maximal open and dense orbit in X on which the action restricts to the group multiplication of T. It is clear that every such X is rational.

We denote $M = \operatorname{Hom}(T, \mathbb{K}^*) \cong \mathbb{Z}^2$ and $N = \operatorname{Hom}(\mathbb{K}^*, T) \cong \mathbb{Z}^2$ the character and cocharacter groups of T, respectively. The toric surface X is completely determined by a collection of elements $l_1, \ldots, l_n \in N$ with the following properties. We assume that the l_i are circularly ordered and indexed by elements in $[n]$. Then for every $i \in [n]$ the pair l_i, l_{i+1} forms a positively oriented basis of N. Moreover, for every such pair there exists no other l_k such that $l_k = \alpha_i l_i + \alpha_{i+1} l_{i+1}$ for some nonnegative integers α_i, α_{i+1}. Every pair l_i, l_{i+1} generates a two-dimensional rational polyhedral cone in the vector space $N_{\mathbb{Q}}$, and the collection of faces of all these cones is the fan Δ associated to X. There is a one-to-one correspondence of 1-dimensional T-orbits in X and the rays in Δ, i.e. the one-dimensional cones, which have the l_i as primitive vectors. The corresponding orbit closures we denote by D_i. Every D_i is isomorphic to $\mathbb{P}^1$, and for every i, the divisors D_i and D_{i+1} intersect transversely in the torus fixed point associated to the cone generated by l_i and l_{i+1}, thus $D_i.D_{i+1} = 1$. This way, the D_i form a cycle of rational curves in X of arithmetic genus 1. Moreover, for every $i \in [n]$ there exists the unique relation

$$l_{i-1} + a_i l_i + l_{i+1} = 0,$$

where $a_i = D_i^2 \in \mathbb{Z}$ is the self-intersection number of D_i.

Clearly, if just the integers a_i are known, we can reconstruct the l_i from the a_i up to an automorphism of N. However, an arbitrary sequence of a_i's does not necessarily lead to a well-defined smooth toric surface. An admissible sequence $a_1, \ldots, a_n$ is determined by the minimal model program for toric surfaces. Whenever $a_i = -1$ for some i, we can equivariantly blow down the corresponding D_i and obtain another smooth toric surface X' on which T acts. This surface is specified by a sequence $a'_1, \ldots, a'_{i-1}, a'_{i+1}, \ldots a'_n$ (where, up to a cyclic change of enumeration, we can assume that $1 < i < n$) such that $a'_{i-1} = a_{i-1} + 1$, $a'_{i+1} = a_{i+1} + 1$, and $a'_k = a_k$ for $k \neq i-1, i, i+1$. Conversely, an equivariant blow-up at some point $D_i \cap D_{i+1}$ is described by changing $a_1, \ldots, a_i, a_{i+1}, \ldots a_n$ to $a_1, \ldots, a_{i-1}, a_i - 1, -1, a_{i+1} - 1, a_{i+2}, \ldots, a_n$. This way, we arrive at the same class of minimal models as in the case of general rational surfaces:

Theorem 2.1: *Every toric surface can be obtained by a finite sequence of equivariant blow-ups of $\mathbb{P}^2$ or some Hirzebruch surface $\mathbb{F}_a$.*

In particular, the sequences of self-intersection numbers associated to $\mathbb{P}^2$ and the $\mathbb{F}_a$ are $1, 1, 1$ for $\mathbb{P}^2$ and $0, a, 0, -a$ for $\mathbb{F}_a$. Every other admissible sequence $a_1, \ldots, a_n$ can be obtained by successive augmentation of one of these sequences by the aforementioned process. In particular, this implies

Proposition 2.2: *Let X be a smooth complete toric surface determined by self-intersection numbers $a_1, \ldots, a_n$. Then $\sum_{i=1}^n a_i = 12 - 3n$.*

There is also a local version of above theorem:

Proposition 2.3 ([MO78])**:** *Let $i < k$ such that l_i, l_k form a positively oriented basis. Then there exists a sequence of blow-downs from X to a smooth complete toric surface X' whose associated primitive vectors are $l_1, \dots, l_i, l_k, \dots, l_n$.*

The Picard group of X is generated by the T-invariant divisors $D_1, \dots, D_n$. More precisely, we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{L} \mathbb{Z}^n \longrightarrow \mathrm{Pic}(X) \longrightarrow 0, \tag{1}$$

where $L = (l_1, \dots, l_n)$, i.e. the l_i are considered as linear forms on M. The i-th element of the standard basis of $\mathbb{Z}^n$ maps to the rational equivalence class of the divisor D_i. There is no canonical choice of coordinates for $\mathrm{Pic}(X)$, but there is a very natural and convenient representation for toric divisors if considered as elements in the group of numerical equivalence classes of curves $N_1(X)$. Consider the natural pairing on X:

$$N_1(X) \otimes \mathrm{Pic}(X) \longrightarrow \mathbb{Z}, \qquad (C, D) \mapsto C.D,$$

which is a non-degenerate bilinear form. The pairing is completely specified by the intersection products of the D_i among each other, which are given by

$$D_i.D_j = \begin{cases} a_i & \text{if } i = j, \\ 1 & \text{if } j \in \{i-1, i+1\}, \\ 0 & \text{else.} \end{cases}$$

Denote $\mathfrak{D} := (D_i.D_j)_{i,j=1,\dots,n}$ the corresponding matrix. Then we have a linear map $\mathbb{Z}^n \xrightarrow{\mathfrak{D}} \mathbb{Z}^n$ whose kernel is M, the group of numerically trivial T-invariant divisors. Given a T-invariant divisor $D := \sum_{i \in [n]} c_i D_i$, its image $\mathfrak{D}(D)$ is a tuple of the form $(d_1, \dots, d_n)$, where $d_i := d_i(D) := c_{i-1} + a_i c_i + c_{i+1} = D.D_i$. If we dualize sequence (1), we get

$$0 \longrightarrow \mathrm{Pic}(X)^* \longrightarrow \mathbb{Z}^n \xrightarrow{L^T} N \longrightarrow 0, \tag{2}$$

where L^T denotes the transpose of L. The kernel of L^T coincides with the image of $\mathfrak{D}$, so that we can identify $\mathrm{Pic}(X)^*$ with $N_1(X)$ in a natural way as subgroups of $\mathbb{Z}^n$. So, if considered as a curve, the tuple $(d_1, \dots, d_n)$ is a natural representation of D which does not depend on the choice of a T-invariant representative. Moreover, by sequence (2) we have for any tuple $(d_1, \dots, d_n) \in N_1(X)$ that

$$\sum_{i \in [n]} d_i l_i = 0.$$

By this we can identify $N_1(X)$ with the set of closed polygonal lines in $N_{\mathbb{Q}}$ whose segments are given by some multiple of every l_i. We will make use of this and give some more detail in section 7. Note that to determine whether some D is nef, it suffices to test this on the T-invariant divisors. We have:

Proposition 2.4: *Let D a T-invariant divisor on X, then*

(i) for every $i \in [n]$ we have $d_i = \deg \mathcal{O}(D)|_{D_i}$;
(ii) D is nef iff $d_i \geq 0$ for every $i \in [n]$.

In particular:

Proposition 2.5: *Denote $K_X = -\sum_{i=1}^n D_i$ the canonical divisor on X. Then $d_i(K_X) = -a_i - 2$ for all i. Then $-K_X$ is nef iff $a_i \geq -2$ for all i and $-K_X$ is ample iff $a_i \geq -1$ for i.*

Note that on a smooth toric surface an invertible sheaf is ample if and only if it is very ample. There are precisely 16 smooth complete toric surface whose anti-canonical divisor is nef (including the 5 del Pezzo surfaces which admit a toric structure). These are shown in table 1 in terms of the self-intersection numbers a_i. In this table, the first four surfaces are given their standard names, the other labels just reflect the length of the sequence $a_1, \dots, a_n$.

The short exact sequence (1) is an example for a *Gale* transform. By general properties of Gale transforms, for any subset I of $\{1, \dots, n\}$, the set $L_I := \{l_i \mid i \in I\}$ forms a basis of N iff the complementary set $\{D_i \mid i \notin I\}$ forms a basis of $\mathrm{Pic}(X)$, and L_I is a minimal linearly dependent set iff the complementary set is a maximal subset of the D_i which is contained in a hyperplane in $\mathrm{Pic}(X)$. Moreover, we can invert

Name	self-intersection numbers $a_1, \ldots, a_n$
$\mathbb{P}^2$	1, 1, 1
$\mathbb{P}^1 \times \mathbb{P}^1$	0, 0, 0, 0
$\mathbb{F}_1$	0, 1, 0, -1
$\mathbb{F}_2$	0, 2, 0, -2
5a	0, 0, -1, -1, -1
5b	0, -2, -1, -1, 1
6a	-1, -1, -1, -1, -1, -1
6b	-1, -1, -2, -1, -1, 0
6c	0, 0, -2, -1, -2, -1
6d	0, 1, -2, -1, -2, -2
7a	-1, -1, -2, -1, -2, -1, -1
7b	-1, -1, 0, -2, -1, -2, -2
8a	-1, -2, -1, -2, -1, -2, -1, -2
8b	-1, -2, -2, -1, -2, -1, -1, -2
8c	-1, -2, -2, -2, -1, -2, 0, -2
9	-1, -2, -2, -1, -2, -2, -1, -2, -2

TABLE 1. The 16 complete smooth toric surfaces whose anti-canonical divisor is nef.

any Gale transform by considering the dual short exact sequence. So by the sequence (2) we get back the l_i from the D_i.

Definition 2.6: Let P be a free $\mathbb{Z}$-module of rank $n-2$ together with a integral symmetric bilinear form $\langle\, , \rangle$. A sequence of elements $A_1, \ldots, A_n$ in P is called an *abstract toric system* iff it satisfies the following conditions:

(i) $\langle A_i, A_{i+1}\rangle = 1$ for $i \in [n]$;
(ii) $\langle A_i, A_j\rangle = 0$ for $i \neq j$ and $\{i, j\} \neq \{k, k+1\}$ for all $k \in [n]$;
(iii) $\sum_{i=1}^n \langle A_i, A_i\rangle = 12 - 3n$.

Clearly, for any given smooth complete toric surface X, the divisors $D_1, \ldots, D_n$ form an abstract toric system in $\mathrm{Pic}(X)$ with respect to the intersection form. We show that the data specifying an abstract toric system is equivalent to defining a toric surface.

Proposition 2.7: *Let P, $\langle\, , \rangle$ as in definition 2.6, $A_1, \ldots, A_n$ an abstract toric system and consider the Gale duals $l_1, \ldots, l_n$ in $N := \mathbb{Z}^n/P$ of the A_i. Then $N \cong \mathbb{Z}^2$ and the $l_1, \ldots, l_n$ generate the fan of a smooth complete toric surface X with T-invariant irreducible divisors $D_1, \ldots, D_n$ such that $D_i^2 = \langle A_i, A_i\rangle$ for every $1 \leq i \leq n$. In particular, we can identify P with $\mathrm{Pic}(X)$ and $\langle\, , \rangle$ with the intersection form on $\mathrm{Pic}(X)$.*

Proof. For $n < 3$ there is nothing to prove and for $n = 3$ the statement is easy to see. So we assume without loss of generality that $n \geq 4$. We first show that $\{A_j \mid j \neq i, i+1\}$ forms a basis of $\mathrm{Pic}(X)$ for every $i \in [n]$. This implies that $N \cong \mathbb{Z}^2$ and, by Gale duality, that the complementary pairs of l_i are bases of N. Up to cyclic renumbering, it suffices to show that $A_1, \ldots, A_{n-2}$ is a basis of $\mathrm{Pic}(X)$. We have $\langle A_1, A_2\rangle = 1$, $\langle A_n, A_1\rangle = 1$ and $\langle A_n, A_2\rangle = 0$. As $\langle\, , \rangle$ is integral, this implies that A_1, A_2 generate a subgroup of rank two of P. This subgroup is saturated, i.e. every element in P which can be represented by a rational linear combination of A_1 and A_2, can also be represented by an integral linear combination of A_1 and A_2. We proceed by induction. Assume that $i < n-2$ and that $A_1, \ldots, A_i$ are linearly independent and span a saturated subgroup of rank i of P. For any linear combination $B := \sum_{j=1}^i \alpha_j A_j$, we have $\langle B, A_{i+2}\rangle = 0$. But $\langle A_{i+1}, A_{i+2}\rangle = 1$ and therefore A_{i+1} cannot be such a linear combination and thus is linearly independent of $A_1, \ldots, A_i$. From integrality of the bilinear form it follows that $A_1, \ldots, A_{i+1}$ forms a saturated subgroup of P. So by induction $A_1, \ldots, A_{n-2}$ is a basis of P.

By Gale duality, we thus obtain a sequence of integral vectors $l_1, \ldots, l_n$ in $N \cong \mathbb{Z}^2$, where every pair l_i, l_{i+1} with $i \in [n]$ forms a basis of N. Consider the quotient $P/A_i^\perp \cong \mathbb{Z}$ for any i. By choosing an appropriate generator of $P/A_i^\perp$, we can identify the images of A_{i-1} and A_{i+1} with 1 and the image of

A_i with a_i. If we consider these elements as the Gale duals of l_{i-1}, l_i, l_{i+1} alone, we see that for every i we have a unique relation $l_{i-1} + a_i l_i + l_{i+1} = 0$ for $a_i = \langle A_i, A_i \rangle \in \mathbb{Z}$.

It only remains to show that for every l_k there do not exist l_i, l_{i+1} and $\alpha_i, \alpha_{i+1} \geq 0$ such that $l_k = \alpha_i l_i + \alpha_{i+1} l_{i+1}$. As the l_i, l_{i+1} form bases of N for every i, we see that the ordering (clockwise or counterclockwise) of the l_i might result in several "windings" until closing up with the final pair l_n, l_1. Assume that we partition the l_i according to such windings, i.e. we group them to $W_1 = \{l_1, \dots, l_{k_1}\}$, $W_2 = \{l_{k_1+1}, \dots, l_{k_2}\}, \dots, W_r = \{l_{k_{r-1}+1}, \dots, l_{k_r}\}$, where $k_r = n$. For every two windings W_j, W_{j+1}, we get that there exist α_j, α_{j+1} such that $l_1 = \alpha_j l_{k_j} + \alpha_{j+1} l_{k_{j+1}}$. We now add additional rays: first, we add $l_1^j = l_1$ for every W_j, second we add rays between l_{k_j} and l_1^j and between l_1^j and $l_{k_{j+1}}$ such that any two neighbouring rays are lattice bases of N. This way, we obtain a stack of r fans in N, each of which corresponds to a smooth toric surface. We denote n' the total number of rays after performing this process and a_i' the new intersection numbers; then we get by Propositions 2.2 and 2.3:

$$\sum_i a_i' = \sum_k a_k - 3(n' - n) = 12r - 3n' \Rightarrow \sum_i a_i = 12r - 3n = 12 - 3n,$$

hence $r = 1$. □

So we define:

Definition 2.8: Let $\mathcal{A} = A_1, \dots, A_n$ be an abstract toric system, then we write $Y(\mathcal{A})$ for the associated toric surface.

As we have seen, toric systems provide an alternative way to describe toric surfaces. Assume X is a toric surface, specified by lattice vectors $l_1, \dots, l_n$ in N and $D_1, \dots, D_n$ the associated torus invariant divisors, which form a toric system. Then an equivariant blow-down $X \to X'$ is described by removing some l_i with $l_i = l_{i-1} + l_{i+1}$. This induces an embedding of $\operatorname{Pic}(X')$ in $\operatorname{Pic}(X)$ as a hyperplane such that $D_i.D = 0$ for all $D \in \operatorname{Pic}(X')$. This corresponds to removing D_i and projecting $D_1, \dots, \widehat{D_i}, \dots D_n$ to $\operatorname{Pic}(X')$. More explicitly, for abstract toric systems this can be formulated as:

Lemma 2.9: *Let $A_1, \dots, A_n$ be an abstract toric system in P and i such that $\langle A_i, A_i \rangle = -1$. Then $A_1, \dots, A_{i-2}, A_{i-1} + A_i, A_{i+1} + A_i, A_{i+2}, \dots A_n$ is a toric system as well which is contained in the hyperplane $A_i^\perp$ with intersection product $\langle\, , \rangle|_{A_i^\perp}$.*

Proof. Denote $L := (l_1, \dots, l_n)$ the matrix whose columns are the l_i, $L' := (l_1, \dots, \widehat{l_i}, \dots, l_n)$, and consider $A := (A_1, \dots, A_n)$ as n-tuple of linear forms on P^*. Then the statement is equivalent to describing the map A' with respect to in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (P')^* & \xrightarrow{A'} & \mathbb{Z}^{n-1} & \xrightarrow{L'} & \mathbb{Z}^2 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & P^* & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{L} & \mathbb{Z}^2 & \longrightarrow & 0, \end{array}$$

which is a straightforward computation. □

So we denote:

Definition 2.10: Let $A_1, \dots, A_n$ be an abstract toric system and i such that $\langle A_i, A_i \rangle = -1$. Then we call $A_1, \dots, A_{i-2}, A_{i-1} + A_i, A_{i+1} + A_i, A_{i+2}, \dots, A_n$ its *blow-down.*

For a given abstract toric system $\mathcal{A}$, the sum $\sum_i A_i$ corresponds to the anti-canonical divisor of $Y = Y(\mathcal{A})$. A small computation shows that the Euler characteristics of the $-A_i$ vanishes:

Lemma 2.11: *Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be an abstract toric system, then for all i:*

$$\chi_{Y(\mathcal{A})}(-A_i) = 1 + \frac{1}{2}(\langle A_i, A_i \rangle - \langle \sum_j A_j, A_i \rangle) = 0.$$

Proof. We just note that $\langle \sum_j A_j, A_i \rangle = \langle A_{i-1}, A_i \rangle + \langle A_i, A_i \rangle + \langle A_{i+1}, A_i \rangle = 2 + \langle A_i, A_i \rangle$. □

Note that in general for two given toric systems $\mathcal{A}$ and $\mathcal{A}'$ the sums $\sum_{i=1}^n A_i$ and $\sum_{i=1}^n A'_i$ do not coincide. This can most trivially be seen in the case where $\mathcal{A}' = -\mathcal{A}$. Any integral orthogonal transformation maps toric systems to toric systems and in general such transformations do not leave $\sum_{i=1}^n A_i$ invariant, as we show in the following example.

Example 2.12: As in the introduction, consider X to be a t-fold blow-up of $\mathbb{P}^2$ with $H, R_1, \ldots, R_t$ a basis of $\mathrm{Pic}(X)$. Denote

$$\mathfrak{R}^i = \{E \in \mathrm{Pic}(X) \mid \chi(-E) = 0 \text{ and } -K_X.E = i\}$$

for every $i \in \mathbb{Z}$. It follows from the Riemann-Roch formula that $E^2 = i - 2$ for every $E \in \mathfrak{R}^i$ (compare Lemma 3.3 below). Now, for any $i, s \in \mathbb{Z}$ with $(i-2)s = -2$, and any $E \in \mathfrak{R}^i$ we can define a reflection r_E on $\mathrm{Pic}(X)$ by setting

$$r_E(D) = s(E.D)E + D$$

for any $D \in \mathrm{Pic}(X)$. Such a reflection clearly respects the intersection product. However, by definition, such a reflection preserves the anti-canonical divisor if and only if $E \in \mathfrak{R}^0$. If we take the abstract toric system

$$R_1 - R_2, R_2 - R_3, \ldots, R_{t-1} - R_t, R_t, H - \sum_{i=1}^t R_i, H, H - R_1$$

from the introduction and apply, say, r_{R_1} to it, where $R_1 \in \mathfrak{R}^1$, then we obtain

$$-R_1 - R_2, R_2 - R_3, \ldots, R_{t-1} - R_t, R_t, H + R_1 - \sum_{i=2}^t R_i, H, H + R_1.$$

These divisors add up to $r_{R_1}(-K_X) = 3H + R_1 - \sum_{i=2}^t R_i = -K_X + 2R_1$.

For constructing and analyzing abstract toric systems, we will need a weaker version:

Definition 2.13: Let P be a free $\mathbb{Z}$-module of rank $n-2$ together with a integral symmetric bilinear form $\langle\ ,\ \rangle$. A sequence of elements $A_1, \ldots, A_r$ with $r < n$ in P is called a *short toric system* if it satisfies the following conditions:

(i) $\langle A_i, A_{i+1}\rangle = 1$ for $1 \leq i < r$ and $\langle A_1, A_r\rangle = 1$;
(ii) $\langle A_i, A_j\rangle = 0$ for $i \neq j$, $\{i,j\} \neq \{1, r\}$, and $\{i,j\} \neq \{k, k+1\}$ for all $k \in [r-1]$.

There are two natural ways for constructing short toric systems from abstract toric systems:

Example 2.14: Let $A_1, \ldots, A_n$ be an abstract toric system, $t > 1$ and $I_1, \ldots, I_t \subset [n]$ a partition of $[n]$ into cyclic intervals such that $I_j \cup I_{j+1}$ ($I_1 \cup I_t$, respectively) form a cyclic interval for every j. Let $A'_j = \sum_{k \in I_j} A_k$, then $A'_1, \ldots, A'_t$ is a short toric system.

Example 2.15: Let X be a smooth complete rational surface and $b : X' \to X$ a blow-up. If $A_1, \ldots, A_n$ is an abstract toric system in $\mathrm{Pic}(X)$ with respect to the intersection form, then $b^*A_1, \ldots, b^*A_n$ is a short toric system in $\mathrm{Pic}(X')$.

3. Rational surfaces and toric systems

Let X be a smooth complete rational surface. From now on we fix $n := \mathrm{Pic}(X) + 2$. Recall that on a rational surface every invertible sheaf is exceptional. For any two divisors D, E on X, we have natural isomorphisms $\mathrm{Ext}^i_{\mathcal{O}_X}\big(\mathcal{O}(D), \mathcal{O}(E)\big) \cong H^i\big(X, \mathcal{O}(E-D)\big)$. Let $E_1, \ldots, E_n \in \mathrm{Pic}(X)$ such that $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ form an exceptional sequence, then $H^k\big(X, \mathcal{O}(E_i - E_j)\big) = 0$ for all $i > j$ and every $k \geq 0$. If, moreover, the sequence is strongly exceptional, we additionally get $H^k\big(X, \mathcal{O}(E_i - E_j)\big) = 0$ for all i, j and all $k > 0$. This leads to the following definition:

Definition 3.1: Let $D \in \mathrm{Pic}(X)$, then D is called

(i) *numerically left-orthogonal to* $\mathcal{O}_X$ if $\chi(-D) = 0$,
(ii) *left-orthogonal to* $\mathcal{O}_X$ if $h^i(-D) = 0$ for all i, and
(iii) *strongly left-orthogonal to* $\mathcal{O}_X$ if it is left-orthogonal to $\mathcal{O}_X$ and $h^i(D) = 0$ for all $i > 0$.

We will usually omit the reference to $\mathcal{O}_X$ and simply say that D is, e.g. left-orthogonal. The strength of above conditions is completely determined by h^1-vanishing:

Lemma 3.2: *Let $D \in \operatorname{Pic}(X)$ be numerically left-orthogonal. Then D is left-orthogonal iff $h^1(-D) = 0$. If $-K_X$ is effective, then D is strongly left-orthogonal iff $h^1(-D) = h^1(D) = 0$.*

Proof. By assumption $\chi(-D) = 0$. Then clearly $h^1(-D) = 0$ iff $h^0(-D) = h^2(-D) = 0$ iff D is left-orthogonal. It remains to show the "strongly" part for $h^1(D) = 0$. For this we have to show that $h^2(D) = 0$. By Serre duality, we have $h^2(D) = h^0(K_X - D)$. If $h^0(K_X - D) \neq 0$, we get an inclusion $h^0(-K_X) \subset h^0(-D)$, but this is impossible, because $h^0(-D) = 0$ and $-K_X$ is effective. □

By Riemann-Roch we have $\chi(D) = 1 + \frac{1}{2}(D^2 - K_X.D)$ for any divisor D, by which we get by symmetrization and anti-symmetrization:

$$\chi(D) + \chi(-D) = 2 + D^2 \quad \text{and}$$
$$\chi(D) - \chi(-D) = -K_X.D.$$

By numerical left-orthogonality we have $\chi(-D) = 1 + \frac{1}{2}(D^2 + K_X.D) = 0$ (compare this also to Lemma 2.11), which directly implies:

Lemma 3.3: *Let $D, E \in \operatorname{Pic}(X)$ numerically left-orthogonal, then*

(i) $\chi(D) = -K_X.D$;
(ii) $D^2 = \chi(D) - 2$; *in particular, if D is strongly left-orthogonal, then $D^2 = h^0(D) - 2$;*
(iii) $D + E$ *is numerically left-orthogonal iff $E.D = 1$ iff $\chi(D) + \chi(E) = \chi(D + E)$; in particular, if $D, E, E + D$ are strongly left-orthogonal, then $h^0(D + E) = h^0(D) + h^0(E)$;*
(iv) $E - D$ *is numerically left-orthogonal iff $D.E = \chi(D) - 1$; in particular, if $D, E, E - D$ are strongly left-orthogonal, then $h^0(D) \leq h^0(E)$ and $D.E = h^0(D) - 1$.*

Clearly, if $\mathcal{O}(E_1), \ldots \mathcal{O}(E_n)$ is a full exceptional sequence, then $n = \operatorname{rk} K_0(X) = \operatorname{rk} \operatorname{Pic}(X) + 2$ and all the differences $E_j - E_i$ for $i > j$ are left-orthogonal and in particular numerically left-orthogonal. We set $A_i := E_{i+1} - E_i$ for $1 \leq i < n$ and $A_n := -K_X - \sum_{i=1}^{n-1} A_i$. Then by Lemma 3.3 we get:

(1) $A_i.A_{i+1} = 1$ for $i \in [n]$;
(2) $A_i.A_j = 0$ for $i \neq j$ and $\{i, j\} \neq \{k, k+1\}$ for some $k \in [n]$;
(3) $\sum_{i=1}^n A_i = -K_X$.

Therefore we get an abstract toric system from an exceptional sequence. Note that in general not every abstract toric system can be of this form, as $\sum_{i=1}^n A_i = -K_X$ implies $(\sum_{i=1}^n A_i)^2 = 12 - 3n$, but not vice versa, as example 2.12 shows. But with this stronger condition, we pass from abstract toric systems to actual toric systems:

Definition 3.4: Let X a smooth complete rational surface. Then a *toric system* (on X) is an abstract toric system $A_1, \ldots, A_n \in \operatorname{Pic}(X)$ such that $\sum_{i=1}^n A_i = -K_X$.

Note that after passing from the $E_1, \ldots, E_n$ to $\mathcal{A} = A_1, \ldots, A_n$, the construction of the toric surface $Y(\mathcal{A})$ is entirely canonical. In particular, we conclude the following remarkable observation:

Theorem 3.5: *Let X be a smooth complete rational surface. Then to any full exceptional sequence of invertible sheaves on X with associated toric system $\mathcal{A}$ we can associate in a canonical way a smooth complete toric surface $Y(\mathcal{A})$ with torus invariant prime divisors $D_1, \ldots, D_n$ such that $D_i^2 = A_i^2$ for every $i \in [n]$.*

A toric system generates an infinite sequence of invertible sheaves

$$\ldots, \mathcal{O}(-A_n), \mathcal{O}_X, \mathcal{O}(A_1), \mathcal{O}(A_1 + A_2), \ldots, \mathcal{O}(\sum_{i=1}^{n-1} A_i), \mathcal{O}(-K_X), \mathcal{O}(-K_X + A_1), \ldots$$

If some subsequence of length n of this sequence is a strongly exceptional sequence, we will follow the convention that the toric system is enumerated such that this sequence can be written as $\mathcal{O}_X$, $\mathcal{O}(A_1)$, $\mathcal{O}(A_1 + A_2)$, $\ldots$, $\mathcal{O}(\sum_{i=1}^{n-1} A_i)$. In particular, $\sum_{i \in I} A_i$ is strongly left-orthogonal for every interval $I \subset [n-1]$. In general we will assume nothing about the strong left-orthogonality of A_n. If the toric system gives rise to a cyclic strongly exceptional sequence, then $\sum_{i\in I} A_i$ is strongly left-orthogonal for every *cyclic* interval $I \subset [n]$.

Definition 3.6: We say that a toric system $A_1, \ldots, A_n$ is (cyclic, strongly) *exceptional* if the associated sequence of invertible sheaves $\mathcal{O}_X$, $\mathcal{O}(A_1)$, $\ldots$, $\mathcal{O}(\sum_{i=1}^{n-1} A_i)$ generates a (cyclic, strongly) exceptional sequence.

Note that a priori a toric system and the conditions on cohomology vanishing do not completely determine the ordering of the A_i. In particular, if $A_1, \ldots, A_n$ is a cyclic (strongly) exceptional toric system, then so is $A_n, \ldots, A_1$. If $A_1, \ldots, A_n$ is a (strongly) exceptional toric system, then so is $A_{n-1}, \ldots, A_1, A_n$.

4. Left-orthogonal divisors on rational surfaces

Any smooth complete rational surface X can be obtained by a sequence of blow-ups $X = X_t \xrightarrow{b_t} X_{t-1} \xrightarrow{b_{t-1}} X_{t-2} \xrightarrow{b_{t-2}} \cdots \xrightarrow{b_1} X_0$, where X_0 is either $\mathbb{P}^2$ or some Hirzebruch surface $\mathbb{F}_a$. If we fix the sequence of morphisms $b_t, \ldots, b_1$, we obtain a natural basis of $\mathrm{Pic}(X)$ with respect to this sequence as follows. If $X_0 = \mathbb{P}^2$, we denote as before H the hyperplane class of $\mathbb{P}^2$, and for every b_i, we denote R_i the class of the associated exceptional divisor in $\mathrm{Pic}(X_i)$. For simplicity, we identify H and the R_i with their pullbacks in $\mathrm{Pic}(X)$. Every blow-up increases the rank of the Picard group by one and the pullback yields an inclusion of $\mathrm{Pic}(X_{i-1})$ into $\mathrm{Pic}(X_i)$ as a hyperplane. Then R_i is additional generator, which is orthogonal to $\mathrm{Pic}(X_{i-1})$ with respect to the intersection product. We have the following relations:

$$H^2 = 1, \quad R_i^2 = -1, \quad H.R_i = 0 \text{ for all } i, \quad \text{ and } R_i.R_j = 0 \quad \text{ for all } i \neq j.$$

In particular, we have $t = \mathrm{rk}\,\mathrm{Pic}(X) - 1$. So, in the case where X is a blow-up of $\mathbb{P}^2$, we easily get a basis of $\mathrm{Pic}(X)$ which diagonalizes the intersection product. In the case where $X_0 = \mathbb{F}_a$ for some $a \geq 0$, we start with a basis P, Q of $\mathrm{Pic}(\mathbb{F}_a)$ as before, and by the same process, we obtain a basis $P, Q, R_1, \ldots, R_t$ of $\mathrm{Pic}(X)$, where $t = \mathrm{rk}\,\mathrm{Pic}(X) - 2$. Here, the most convenient choice for our purpose is P, Q to be the integral generators of the nef cone in $\mathrm{Pic}(\mathbb{F}_a)_{\mathbb{Q}}$ such that $P^2 = 0$ and $Q^2 = a$. So we get

$$\begin{aligned} &P^2 = 0, \quad Q^2 = a, \quad P.Q = 1, \quad R_i^2 = -1, \\ &P.R_i = Q.R_i = 0 \text{ for all } i, \quad \text{and } R_i.R_j = 0 \ \text{ for all } i \neq j. \end{aligned}$$

Often our arguments below do not depend on the choice of X_0, and for simplicity we will often leave this choice implicit and assume that $t = n - 3$ or $t = n - 4$ as it fits.

Definition 4.1: Let $D \in \mathrm{Pic}(X)$, then we denote the projection of D to $\mathrm{Pic}(X_i)$ by $(D)_i$.

The projection $(D)_i$ just is 'forgetting' the coordinates R_t, R_{t-1}, $\ldots$, R_{i+1}, i.e. if $D = \alpha P + \beta Q + \sum_{j=1}^t \gamma_j R_j$ or $D = \beta H + \sum_{j=1}^t \gamma_j R_j$, respectively, then $(D)_i = \alpha P + \beta Q + \sum_{j=1}^i \gamma_j R_j$ or $(D)_i = \beta H + \sum_{j=1}^i \gamma_j R_j$, respectively.

By Lemma 3.2, left-orthogonality is determined by numerical left-orthogonality and h^1-vanishing. Our strategy to understand (strongly) left-orthogonal divisors will be to start with h^1-vanishing and then to establish numerical left-orthogonality. For this, we first need a couple of lemmas related to h^0- and h^1-vanishing.

Lemma 4.2: *Let E be an irreducible (-1)-divisor and X' the surface obtained from blowing down E. If D is the pullback to X of some divisor on X', then for every $k \in \mathbb{Z}$, we have* $\deg \mathcal{O}(D + kE)|_E = -k$.

Proof. For $k \in \mathbb{Z}$ consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(D + (k-1)E) \longrightarrow \mathcal{O}(D + kE) \longrightarrow \mathcal{O}_E(D + kE) \longrightarrow 0.$$

Then, for the Euler characteristics, we get $\chi\big(\mathcal{O}_E(D + kE)\big) = \chi(D + kE) - \chi(D + (k-1)E) = 1 - k$, where the latter equality follows from Riemann-Roch and $D.E = 0$. Hence $\mathcal{O}_E(D + kE) \cong \mathcal{O}_E(-k)$ and the assertion follows. □

We use this to investigate h^0- and h^1-vanishing. If a divisor has nonzero h^1, then so has its preimage under blow-up. For h^0 and h^2, we have the opposite picture:

Lemma 4.3: *Let D and E as in Lemma 4.2.*

(i) *If* $h^0(D) = 0$, *then* $h^0(D + kE) = 0$ *for all* $k \in \mathbb{Z}$.
(ii) *If* $h^1(D) \neq 0$, *then* $h^1(D + kE) \neq 0$ *for all* $k \in \mathbb{Z}$.
(iii) *If* $h^2(D) = 0$, *then* $h^2(D + kE) = 0$ *for all* $k \in \mathbb{Z}$.

Proof. For $k = 0$ there is nothing to prove. If $k > 0$, we do induction on k. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(D + (k-1)E) \longrightarrow \mathcal{O}(D + kE) \longrightarrow \mathcal{O}_E(D + kE) \longrightarrow 0.$$

By lemma 4.2, we have $\deg \mathcal{O}(D + kE)|_E = -k$ and therefore $h^0\big(\mathcal{O}(D + kE)\big) = 0$. So by the long exact cohomology sequence we get $h^0(D + (k-1)E) = h^0(D + kE)$, $h^1(D + (k-1)E) \leq h^1(D + kE)$, and $h^2(D + (k-1)E) \geq h^2(D + kE)$. For (i), we have by induction assumption $h^0(D + (k-1)E) = 0$ and so $h^0(D + kE) = 0$. For (ii), we have by induction assumption $h^1(D + (k-1)E) > 0$ and so $h^1(D + kE) > 0$. For (iii), we have by induction assumption $h^2(D + (k-1)E) = 0$ and so $h^2(D + kE) = 0$.

For $k < 0$, we do induction from $k + 1$ to k. In this case, we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(D + kE) \longrightarrow \mathcal{O}(D + (k+1)E) \longrightarrow \mathcal{O}_E(D + (k+1)E) \longrightarrow 0.$$

So $\deg \mathcal{O}(D + (k+1)E)|_E = -k - 1 \geq 0$ and therefore $h^1\big(\mathcal{O}(D + (k+1)E)\big) = 0$. Then by the long exact cohomology sequence, we get $h^0(D + kE) \leq h^0(D + (k+1)E)$, $h^1(D + kE) \geq h^1(D + (k+1)E)$, and $h^2(D + kE) = h^2(D + (k+1)E)$. For (i), we have by induction assumption $h^0(D + (k+1)E) = 0$ and so $h^0(D + kE) = 0$. For (ii), we have by induction assumption $h^1(D + (k+1)E) > 0$ and so $h^1(D + kE) > 0$. For (iii), we have by induction assumption $h^2(D + (k+1)E) = 0$ and so $h^2(D + kE) = 0$. □

Definition 4.4: Let $D \in \operatorname{Pic}(X)$ with $(D)_0 \neq 0$. Then we call $D \in \operatorname{Pic}(X)$ *pre-left-orthogonal* with respect to X_0 iff $h^0\big((-D)_0\big) = h^1(-D) = 0$, and *strongly pre-left-orthogonal* if it is pre-left-orthogonal and $h^1(D) = 0$.

Note the little twist that for pre-left-orthogonality we do not just require h^0-vanishing, but instead have conditions on X_0. This makes the following an immediate consequence of Lemma 4.3:

Corollary 4.5: *If* D *is pre-left-orthogonal, then so is* $(D)_i$ *for* $i = 1, \dots, t$.

If D is a pre-left-orthogonal divisor on X_{t-1}, then in general $D + \gamma_t R_t$ will only be pre-left-orthogonal for a few possible values of γ_t. The following lemma gives some sufficient conditions.

Lemma 4.6: *Let* $D \in \operatorname{Pic}(X)$ *and* $k \geq l \geq 0$. *If* $D - kR_t$ *is pre-left-orthogonal, then* $D - lR_t$ *is also pre-left-orthogonal. If* $D - kR_t$ *is strongly pre-left-orthogonal, then so is* $D - lR_t$.

Proof. We do both cases by induction on l, starting with $l = k$. For $l = k$, there is nothing to show. Also $(D - kR_t)_0 = (D - lR_t)_0$, so there is nothing to show for h^0. Assume now that $k > l > 0$ and $D - lR_t$ is pre-left-orthogonal. We consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(-D + (l-1)R_t) \longrightarrow \mathcal{O}(-D + lR_t) \longrightarrow \mathcal{O}_{R_t}(-D + lR_t) \longrightarrow 0.$$

By lemma 4.2 we have $\deg \mathcal{O}_E(-D + lR_t) = -l < 0$, and thus $h^0\big(\mathcal{O}_{R_t}(-D + lR_t)\big) = 0$. Then by the long exact cohomology sequence $h^1(-D + (l-1)R_t) \leq h^1(-D + lR_t) = 0$ and the first assertion follows by induction. If $D - lE$ is strongly pre-left-orthogonal, we consider the following short exact sequence

$$0 \longrightarrow \mathcal{O}(D - lR_t) \longrightarrow \mathcal{O}(D - (l-1)R_t) \longrightarrow \mathcal{O}_{R_t}(D - (l-1)R_t) \longrightarrow 0.$$

Again, by lemma 4.2 he have $\deg \mathcal{O}_{R_t}(D - (l-1)R_t) = l - 1 \geq 0$ and therefore $h^1\big(\mathcal{O}_{R_t}(D - (l-1)R_t)\big) = 0$. Then by the long exact cohomology sequence, we have $0 = h^1(D - lR_t) \geq h^1(D - (l-1)R_t) \geq 0$ and the second assertion follows by induction. □

Now we classify (strongly) pre-left-orthogonal divisors on $\mathbb{P}^2$ and on the $\mathbb{F}_a$. Denote H the class of a line on $\mathbb{P}^2$. As the condition of h^1-vanishing is vacuous for invertible sheaves on $\mathbb{P}^2$, we trivially observe:

Proposition 4.7: *A divisor on* $\mathbb{P}^2$ *is (pre-)left-orthogonal iff it is strongly (pre-)left-orthogonal. The pre-left-orthogonal divisors are given by* kH, *where* $k > 0$, *and the left-orthogonal divisors are* $H, 2H$.

For the case of a Hirzebruch surface $\mathbb{F}_a$, we choose P, Q as before and the following statements can be seen rather straightforwardly, for instance by using toric methods as in [HP06], [Per07].

Proposition 4.8: *The pre-left-orthogonal divisors on a Hirzebruch surface are:*

(i) *on* $\mathbb{F}_0$*:* $P + kQ$, $kP + Q$ *for* $k \in \mathbb{Z}$, $kP + lQ$ *for* $k, l \geq 2$*;*
(ii) *on* $\mathbb{F}_a$*, with* $a > 0$*:* P, $kP + Q$ *for* $k \in \mathbb{Z}$, $kP + lQ$ *for* $k \geq 1 - a$ *and* $l \geq 2$*;*

A pre-left-orthogonal divisors is strongly pre-left-orthogonal iff it is not of the type $P + kQ$ *or* $kP + Q$ *for* $k < -1$ *or of type* $kP + lQ$ *for* $l \geq 2$ *and* $k < \max\{-1, 1 - a\}$*.*

Proposition 4.9: *Let* $\mathbb{F}_a$ *be a Hirzebruch surface.*

(i) *If* $a = 0$*, then the left-orthogonal divisors are given by* $P + kQ$, $kP + Q$ *for* $k \in \mathbb{Z}$*.*
(ii) *If* $a > 0$*, then the left-orthogonal divisors are given by* P, $kP + Q$ *for* $k \in \mathbb{Z}$*, and* $(1 - a)P + 2Q$*.*
(iii) *Left-orthogonal divisors of type* $kP + Q$ *or* $P + kQ$ *are strongly left-orthogonal iff* $k \geq -1$*. Divisors of type* $(1 - a)P + 2Q$ *are strongly left-orthogonal iff* $a \leq 2$*.*

In coordinates chosen with respect to a minimal model X_0, the anti-canonical divisor on X can be written as

$$-K_X = 3H - \sum_{i=1}^{t} R_i \quad \text{or}$$

$$-K_X = (2 - a)P + 2Q - \sum_{i=1}^{t} R_i, \text{ respectively.}$$

For $X_0 = \mathbb{P}^2$ and some divisor $D = \beta H + \sum_{i=1}^{t} \gamma_i R_i$, we get by Riemann-Roch the following formulas for the Euler characteristics of D:

(3) $$\chi(D) = \binom{\beta + 2}{2} - \sum_i \binom{\gamma_i}{2}$$

(4) $$\chi(-D) = \binom{\beta - 1}{2} - \sum_i \binom{\gamma_i + 1}{2},$$

where we write $\binom{x}{2} = \frac{1}{2}x(x - 1)$ for any $x \in \mathbb{Z}$. For $X_0 = \mathbb{F}_a$ and $D = \alpha P + \beta Q + \sum_{i=1}^{t} \gamma_i R_i$, we get:

(5) $$\chi(D) = (\alpha + 1)(\beta + 1) + a\binom{\beta + 1}{2} - \sum_i \binom{\gamma_i}{2}$$

(6) $$\chi(-D) = (\alpha - 1)(\beta - 1) + a\binom{\beta}{2} - \sum_i \binom{\gamma_i + 1}{2}$$

If $\chi(-D) = 0$, we obtain linear equations for $\chi(D) = -K_X D$ in either coordinates:

$$\chi(D) = 3\beta + \sum_i \gamma_i$$

$$\chi(D) = 2\alpha + (2 + a)\beta + \sum_i \gamma_i.$$

We now look at the case where $(D)_0 = 0$. In this case, we have to take into account the relative configuration of R_i and R_j.

Definition 4.10: Assume $i, j > 0$ and denote x_j and x_i the points on X_{j-1} and X_{i-1}, respectively, which are blown up by the maps b_j and b_i. We define a partial order $\geq$ on the set $\{R_1, \ldots, R_t\}$ by setting $R_i \geq R_i$ for every i and $R_j \geq R_i$ if $j > i$ and $b_i \circ \cdots \circ b_{j-1}(x_j) = x_i$.

Now we get:

Proposition 4.11: *Let* $D \in \operatorname{Pic}(X)$ *such that* $(D)_0 = 0$*. Then* D *is left-orthogonal if there exists* $i \in [t]$ *and* $S \subset [t] \setminus \{i\}$ *such that* $D = R_i - \sum_{j \in S} R_j$ *and* $R_i \not\geq R_j$ *for all* $j \in S$*. Moreover,* D *is strongly left-orthogonal iff it is of the form* R_i *for some* $i \in [t]$ *or of the form* $R_i - R_j$ *such that* R_i *and* R_j *are incomparable with respect to the partial order* $\geq$*.*

Proof. Note that for $(D)_0 = 0$, by Lemma 4.3 (iii), we can always assume that $h^2(D) = h^2(-D) = 0$. Let $D = \sum_i \gamma_i R_i$, then $\chi(-D) = 0$ by formula (4) or (6) yields:

$$\sum_j \binom{\gamma_j + 1}{2} = 1.$$

But then there is precisely one $i \in [t]$ with $\gamma_i \in \{1, -2\}$ and $\gamma_j \in \{0, -1\}$ for all other j. If $\gamma_i = -2$, we consider R_i as irreducible divisor on X_i and we consider the following part of a long exact cohomology sequence:

$$H^1\big(X_i, \mathcal{O}_{X_i}(\sum_{j\in S} R_j)\big) \longrightarrow H^1\big(X_i, \mathcal{O}_{X_i}(2R_i + \sum_{j \in S} R_j)\big) \longrightarrow H^1\big(X_i, \mathcal{O}_{2R_i}(2R_i + \sum_{j\in S} R_j)\big) \longrightarrow 0$$

for some $S \subset [i]$. As $\chi(\sum_{j\in S} R_j) = 1 = h^0(\sum_{j \in S} R_j)$, we get $h^1\big(\mathcal{O}_{X_i}(\sum_{j\in S} R_j)\big) = 0$ and thus $h^1\big(\mathcal{O}_{X_i}(2R_i + \sum_{j\in S} R_j)\big) = h^1\big(\mathcal{O}_{2R_i}(2R_i + \sum_{j\in S} R_j)\big)$. By lemma 4.3 we can assume without loss of generality that $i \geq j$ for all $j \in S$. Then we get $\mathcal{O}_{2R_i}(2R_i + \sum_{j\in S} R_j) \cong \mathcal{O}_{2R_i}(2R_i)$ and we compute $\chi\big(\mathcal{O}_{2R_i}(2R_i)\big) = \chi\big(\mathcal{O}(2R_i)\big) - 1 = -1$ and thus $h^1\big(\mathcal{O}_{X_0}(2R_i + \sum_{j\in S} R_j)\big) \neq 0$.

So we are left with divisors of the form $R_i - \sum_{j\in S} R_j$ for some $S \subset [t]$. By Serre duality, we have $h^2(-R_i + \sum_{j\in S} R_j) = h^0(K_X + R_i - \sum_{j\in S} R_j) \leq h^0\big((K_X + R_i - \sum_{j \in S} R_j)_0\big) = h^0\big((K_X)_0\big) = 0$. If there exists $k \in S$ such that $R_i \geq R_k$, then $R_k - R_i$ is effective, and $-R_i + \sum_{j\in S} R_j$ is a sum of effective divisors and therefore $h^0(-R_i + \sum_{j\in S} R_j) \neq 0$. If there exists $k \in S$ such that R_i and R_k are incomparable, then we may assume that this k is minimal with respect to $\geq$. Then $h^0(R_k - R_i) = 0$, and by lemma 4.3 we can conclude that $h^0(-R_i + \sum_{j\in S} R_j) = 0$. The remaining possibility is that $R_j \geq R_i$ for all $j \in S$. In that case, denote E_i the strict transform on X of the exceptional divisor of the blow-up b_i. Then there exists $T_i \subset [t]$ such that E_i is rationally equivalent to $R_i - \sum_{j\in T_i} R_j$. Then $-R_i + \sum_{j\in S} R_j$ is rationally equivalent to $\sum_{j\in S\setminus T_i} R_j - \sum_{j \in T_i \setminus S} R_j - E_i$. If any of $S \setminus T_i$ or $T_i \setminus S$ are empty, we have $h^0(-R_i + \sum_{j\in S} R_j) = 0$. Otherwise, if any R_k, R_l wit $k \in S \setminus T_i$ and $l \in T_i \setminus S$ are incomparable, then $h^0(R_k - R_l) = h^0(-R_i + \sum_{j\in S} R_j) = 0$. If not, we choose $k \in T_i \setminus S$. Then there exists $T_k \subset [t]$ such that $E_k = R_k - \sum_{l\in T_k} R_l$ and we iterate our previous argument until we get the difference of two incomparable R_j or we can write $-R_i + \sum_{j\in S} R_j$ as the inverse of an effective divisor.

So, unless there exists $j \in S$ with $R_i \geq R_j$, we can now conclude together with $\chi(-R_i + \sum_{j\in S} R_j) = 0$ that $h^i(-R_i + \sum_{j\in S} R_j) = 0$ for all i. This shows the first assertion. For strong left-orthogonality, we necessarily need $\chi(R_i - \sum_{j\in S} R_j) \geq 0$, which is the case iff S is empty or $S = \{j\}$ for some $j \neq i$. A divisor R_i always is strongly left-orthogonal. For $R_i - R_j$ we have $\chi(R_i - R_j) = 0$, and $h^1(R_i - R_j) = 0$ is equivalent to $h^0(R_i - R_j) = 0$. But this is in turn is equivalent to incomparability of R_i and R_j. ☐

For $(D)_0 \neq 0$, we have the following statement:

Proposition 4.12: *If $D = (D)_0 + \sum_i \gamma_i R_i$ is left-orthogonal and $(D)_0$ is pre-left-orthogonal, then $\gamma_i \leq 0$ for all i.*

Proof. Assume $\chi(-D) = 0$ and $\gamma_k > 0$ for some k, then $\chi(-D + \gamma_k R_k) = \binom{\gamma_k + 1}{2} > 0$. As $h^0\big((-D)_0\big) = 0$, we also have $h^0(-D + \gamma_k R_k) = 0$. Therefore we have $\chi(-D + \gamma_k R_k) = h^2(-D + \gamma_k R_k) - h^1(-D + \gamma_k R_k) > 0$, hence $h^2(-D + \gamma_k R_k) > 0$. But by Serre duality, $h^2(-D) = h^0(K_X + D) \geq h^0(K_X + D - \gamma_k R_k) = h^2(-D + \gamma_k R_k) > 0$, which is a contradiction to the left-orthogonality of D, and the assertion follows. ☐

Remark 4.13: Note that in the case where D is strongly left-orthogonal but $(D)_0$ is not strongly pre-left-orthogonal, this implies that $h^0\big((D)_0\big) = 0$ and therefore $h^0(D) = 0$. But then $-D$ is left-orthogonal, too, and $(-D)_0$ is strongly pre-left-orthogonal.

We now consider some special cases concerning proposition 4.12.

Lemma 4.14: *Let X be a smooth complete rational surface, D a very ample and strongly left-orthogonal divisor on X. Consider a blow-up $b : \tilde{X} \to X$ in four points x_1, x_2, x_3, x_4, where x_1 and x_2 are on X and x_3 and x_4 are infinitesimal points lying over x_1 and x_2, respectively. Denote $R_1, \ldots, R_4$ the pullbacks of the exceptional divisors of b to $\operatorname{Pic}(\tilde{X})$, then the divisors $D - R_i$ and $D - R_i - R_j$ with $i \neq j$ are strongly left-orthogonal on $\tilde{X}$.*

Proof. It follows directly from our previous discussions that the divisors $D - R_i$ and $D - R_i - R_j$ are left-orthogonal. It remains to show that $h^1(D - R_i) = h^1(D - R_i - R_j) = 0$. By lemma 3.3 (iii) we know $\chi(D - R_i) = \chi(D) - 1$ and $\chi(D - R_i - R_j) = \chi(D) - 2$. So it suffices to show that $h^0(D - R_i - R_j) < h^0(D - R_i) < h^0(D)$ for any $i \neq j$. But this is an immediate consequence of [Har77], V.4, Remark 4.0.2 and preceding remarks. □

5. Exceptional sequences of invertible sheaves on rational surfaces

We first show that cyclicity for exceptional sequences of invertible sheaves is no additional condition:

Proposition 5.1: *Let X be a smooth complete rational surface. Then every exceptional sequence of invertible sheaves is cyclic.*

Proof. Let $A_1, \ldots, A_n$ be an exceptional toric system. Then for every interval $I \subset [n-1]$ we have $h^i(-A_I) = 0$ for every i. By Serre duality, we get $h^i(-A_I) = h^i(K_X + A_{[n]\setminus I}) = h^{2-i}(-A_{[n]\setminus I}) = 0$ for every i. So A_J is left-orthogonal for every cyclic interval J of $[n]$ and $A_1, \ldots, A_n$ corresponds to a cyclic exceptional sequence. □

On $\mathbb{P}^2$, there is a unique toric system which gives rise to a cyclic strongly exceptional sequence, but, as we will see for the case of Hirzebruch surfaces, Proposition 5.1 does not hold for *strongly* exceptional sequences in general. Recall that P, Q are generators of the nef cone of the Hirzebruch surface $\mathbb{F}_a$, where $P^2 = 0$, $Q^2 = a$, and $P.Q = 1$.

Proposition 5.2: *On a Hirzebruch surface $\mathbb{F}_a$ there are the following toric systems:*

(i) $P, sP + Q, P, -(a+s)P + Q$ for $s \in \mathbb{Z}$ for any a;
(ii) $-\frac{a}{2}P + Q, P + s(-\frac{a}{2}P + Q), -\frac{a}{2}P + Q, P - s(-\frac{a}{2}P + Q)$ for $s \in \mathbb{Z}$ and a even.

Toric systems of type (i) are always exceptional. They are strongly exceptional for $s \geq -1$, where $A_4 = -(a+s)P + Q$. They are cyclic strongly exceptional iff $s \geq -1$ and $a + s \leq 1$.

Toric systems of type (ii) are almost never exceptional. The exceptions are for $a = 0$, where type (ii) is symmetric to type (i) by exchanging P and Q, and for $a = 2$ and $s = 0$, which then coincides with a toric system of type (i) and is cyclic strongly exceptional.

Proof. Any toric system must represent a Hirzebruch surface. Therefore, for any toric system A_1, A_2, A_3, A_4 we can assume that $A_1^2 = A_3^2 = 0$ and $A_2^2 = -A_4^2 = -b$ for some $b \in \mathbb{Z}$. So for a general element $\alpha P + \beta Q$ with $\alpha, \beta \in \mathbb{Z}$, the equations $(\alpha P + \beta Q)^2 = 0$ and $\chi(-\alpha P - \beta Q) = 0$ have always the solution $\alpha = 1$, $\beta = 0$. If a is even, we get a second solution, $\alpha = -\frac{a}{2}$ and $\beta = 1$. The condition $A_1.A_3 = 0$ can only be fulfilled if $A_1 = A_3 = P$, or if $A_1 = A_3 = -\frac{a}{2}P + Q$.

In the first case, using $A_1.A_2 = A_1.A_4 = 1$ and $A_2.A_4 = 0$, we get that $A_2 = sP + Q$ and $A_4 = -(a+s)P + Q$ for some $s \in \mathbb{Z}$ which indeed form a toric system way for every $s \in \mathbb{Z}$.

In the second case with a even, we similarly compute that $A_2 = P + s(-\frac{a}{2}P + Q)$ and $A_4 = P - s(-\frac{a}{2}P + Q)$ for some $s \in \mathbb{Z}$.

The classification of exceptional sequences (cyclic or strong) among these follows by inspection of the classification of (strongly) left-orthogonal divisors of proposition 4.9. □

Remark 5.3: From Proposition 5.2 follows that for a toric system $\mathcal{A}$ on a Hirzebruch surface $\mathbb{F}_a$, the associated Hirzebruch surface $Y(\mathcal{A})$ is isomorphic to $\mathbb{F}_b$, where $b - a$ is even.

As in the previous section, we assume that a sequence of blowups $X = X_t \longrightarrow \cdots \longrightarrow X_0$ is fixed, where X_0 is $\mathbb{P}^2$ or some $\mathbb{F}_a$, together with a corresponding basis of $\operatorname{Pic}(X)$, either $H, R_1, \ldots, R_t$ if $X_0 \cong \mathbb{P}^2$, or $P, Q, R_1, \ldots, R_t$ if $X_0 \cong \mathbb{F}_a$. Any toric system $\mathcal{A} = A_1, \ldots, A_{n-t+i}$ on some X_i pulls back to a short toric system on X in the sense of Definition 2.13 (see Example 2.15). Such a short toric system can easily be extended to a toric system by using the $R_{i+1}, \ldots, R_t$ as follows. For any $i + 1 \leq j_1 \leq t$ we denote $\mathcal{A}_1$ the sequence

$$A_1, \ldots, A_{s-1}, A_s - R_{j_1}, R_{j_1}, A_{s+1} - R_{j_1}, A_{s+2}, \ldots, A_{n-t+i},$$

which augments $\mathcal{A}$ at some position s. Note that this augmentation is understood in the cyclic sense, i.e. we do not exclude $s = n - t + i$. If $i = t - 1$, then this sequence is a toric system on X; otherwise, it is again a short toric system. Inductively, for $1 < k < t - i$ we can in the same way augment $\mathcal{A}_{k-1}$ to a short toric system $\mathcal{A}_k$ by some R_{j_k} for $j_k \in \{i+1, \ldots, t\} \setminus \{j_l \mid 1 \leq l < k\}$ and finally we arrive at a toric system $\mathcal{A}_{t-i}$. Of course, $\mathcal{A}_{t-i}$ also depends on the positions at which the $\mathcal{A}_k$ have been augmented. A toric system obtained this way in general cannot be interpreted as successive augmentation via pullbacks from the X_j with $i < j < t$ as we have not imposed any condition on the ordering of the j_k. We will see below that the interesting augmentations which are obtained this way are precisely those which are augmentations via pullbacks.

Definition 5.4: We call an exceptional toric system on $\mathbb{P}^2$ or $\mathbb{F}_a$ a *standard toric system*. On a smooth complete rational surface X, we call a toric system which is the augmentation of a standard toric system a *standard augmentation*. A standard augmentation is *admissible* if it contains no element of the form $R_i - \sum_{j \in S} R_j$ such that $R_j \leq R_i$ for some $j \in S$.

Note that the condition of admissibility is precisely the condition of Proposition 4.11 on left-orthogonality of divisors of the form $R_i - \sum_{j \in S} R_j$. This condition implies that a standard augmentation is admissible iff there exists a bijection $j : [t] \to [t]$, $k \mapsto j_k$ such that $R_k \geq R_l$ iff $R_{j_k} \geq R_{j_l}$. Then we can rearrange the ordering of the blow-ups accordingly such that $X = X_{j_t} \to \cdots \to X_{j_1} \to X_0$ and the augmentation then can be considered as an successive augmentation along these blow-ups. The following proposition shows that this way we get many exceptional sequences in the form of standard augmentations.

Proposition 5.5: *Every standard augmentation yields a full exceptional sequence on X iff it is admissible.*

Proof. Let $\mathcal{A} = A_1, \ldots, A_n$ be the augmented sequence. If $X_0 = \mathbb{P}^2$, we can renumber this sequence such that A_n is of the form $H - \sum_{i \in S} R_i$ for some subset S of $[t]$. We claim that $\mathcal{A} = A_1, \ldots, A_{n-1}$ yield an exceptional sequence iff it is admissible. That is, every $A_I := \sum_{i \in I} A_i$ for some non-cyclic interval $I \subset [n-1]$ is left-orthogonal iff $\mathcal{A}$ is admissible. Clearly, every such A_I is numerically left-orthogonal. We have two cases. First, $lH - \sum_{i \in T} R_i$ with $T \subset [t]$ and $l \in \{1, 2\}$. By Serre duality we get $h^2(-lH + \sum_{i \in T} R_i) = h^0(-(3-l)H + \sum_{i \notin T} R_i) = 0$ and thus $lH - \sum_{i \in T} R_i$ is left-orthogonal (without any condition on admissibility). Second, we have $A_I = R_i - \sum_{i \in T} R_i$ with $T \subset [t]$, which is left-orthogonal by proposition 4.11 iff $R_j \not\leq R_i$ for all $j \in T$. In particular, all A_I are of this form iff $\mathcal{A}$ is admissible.

If $X_0 = \mathbb{F}_a$, we can renumber the sequence such that A_n is of the form $Q - (a+n)P - \sum_{i \in S} R_i$ for some subset S of $[t]$. Then for A_I we have three cases. First, $P - \sum_{i \in T} R_i$ with $T \subset [T]$. By Serre duality we get $h^2(-P + \sum_{i \in T} R_i) = h^0(-2Q - (1-a)P - \sum_{i \notin T} R_i) = 0$ and so $P - \sum_{i \in T} R_i$ is left-orthogonal. Second, we have $Q + nP - \sum_{i \in T} R_i$ with $T \subset [T]$ and $n \in \mathbb{Z}$. Again, by Serre duality, we get $h^2(-Q - nP + \sum_{i \in T} R_i) = h^0(-Q - (2-n-a)P - \sum_{i \notin T} R_i) = 0$ and thus $Q + nP - \sum_{i \in T} R_i$ is left-orthogonal. Third, we have $A_I = R_i - \sum_{i \in T} R_i$ with $T \subset [t]$, which is left-orthogonal by proposition 4.11 iff $R_j \not\leq R_i$ for all $j \in T$. In particular, all A_I are of this form iff $\mathcal{A}$ is admissible.

We have seen now that a standard augmentation is admissible iff all A_I are left-orthogonal. It follows directly from the results of [Orl93] that standard augmentations are full. □

So, by observing that we can lift any standard sequence on some X_0 to an admissible standard augmentation on X, the following is an immediate consequence of Proposition 5.5:

Theorem 5.6: *Every smooth complete rational surface has a full exceptional sequence of invertible sheaves.*

Let us denote $b_i : X_i \longrightarrow X_{i-1}$ the i-th blow-up in the sequence $X = X_t \to \cdots \to X_0$. We assume that b_i can be partitioned into two sets $S_1 := \{b_1, \ldots, b_s\}$ and $S_2 := \{b_{s+1}, \ldots b_t\}$ for $1 < s \leq t$ such that the b_i within S_l for $l \in \{1, 2\}$ commute. In other words, we assume that X can be obtained from $\mathbb{P}^2$ or $\mathbb{F}_a$ by two times simultaneously blowing up (possibly) several points.

Theorem 5.7: *With above assumptions on X and $X_0 = \mathbb{P}^2$, the following is a full strongly exceptional toric system:*

$$R_s, R_{s-1} - R_s, \ldots, R_1 - R_2, H - R_1, H - R_{s+1}, R_{s+1} - R_{s+2}, \ldots, R_{t-1} - R_t, R_t, H - \sum_{i=1}^{t} R_i.$$

Proof. We have to check that $\sum_{i \in I} A_i$ is strongly left-orthogonal for every interval $I \subset [n-1]$. Here we have $A_1 = R_s$ and $A_n = H - \sum_{i=1}^t R_i$. There are precisely four types of divisors which can be represented in this way, namely R_i, $R_i - R_j$ for R_i, R_j incomparable, H, $2H$, $H - R_i$ and $2H - R_i - R_j$ for $i \neq j$. The divisors H, $2H$ are clearly strongly left-orthogonal. The left-orthogonality of R_i and $R_i - R_j$ follows from proposition 4.11, the left-orthogonality of $H - R_i$ and $2H - R_i - R_j$ from Lemma 4.14. The toric system clearly is an admissible standard augmentation and so from Proposition 5.5 it follows that the resulting exceptional sequence is full. □

Analogously, we get:

Theorem 5.8: *With above assumptions on X and $X_0 = \mathbb{F}_a$ for some $a \geq 0$ and $n \geq -1$, the following is a full strongly exceptional toric system:*

$$R_s, R_{s-1} - R_s, \ldots, R_1 - R_2, P - R_1, nP + Q, P - R_{s+1}, R_{s+1} - R_{s+2}, \ldots,$$
$$R_{t-1} - R_t, R_t, -(a+n)P + Q - \sum_{i=1}^t R_i.$$

Proof. Here, $\sum_{i \in I} A_i$ is of the form R_i, $R_i - R_j$ for R_i, R_j incomparable, P, $nP + Q$ with $n \geq -1$, $P - R_i$, $nP + Q - R_i$ for $n \geq 0$, and $nP + Q - R_i - R_j$ for $n \geq 1$. The divisors P, $nP + Q$ clearly are strongly left-orthogonal (see Lemma 4.9). The left-orthogonality of R_i and $R_i - R_j$ follows from Proposition 4.11, the left-orthogonality of $nP + Q - R_i$ and $nP + Q - R_i - R_j$ from Lemma 4.14. The cases $P - R_i$ and $Q - R_i$ are clear because P and Q are globally generated. Also, the toric system is an admissible augmentation of a standard sequence and so from proposition 5.5 it follows that the resulting exceptional sequence is full. □

The following theorem is an immediate consequence of Theorem 5.8.

Theorem 5.9: *Any smooth complete rational surface which can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step) has a full strongly exceptional sequence of invertible sheaves.*

Remark 5.10: Note that for the existence of strongly exceptional sequences it suffices to consider $X_0 = \mathbb{F}_a$ for some $a \geq 0$, as every blow-up of $\mathbb{P}^2$ factorizes through a blow-up of $\mathbb{F}_1$. Nevertheless, as we will see later on, for cyclic strongly exceptional sequences it will be advantageous also to consider augmentations coming from $\mathbb{P}^2$.

The converse of Theorem 5.9 is true for strongly exceptional sequences coming from standard augmentations:

Theorem 5.11: *Let $\mathbb{P}^2 \neq X$ be a smooth complete rational surface which admits a full strongly exceptional standard augmentation then X can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step).*

We prove this theorem in section 6.

Remark 5.12: We will see in Theorem 8.1 that in the toric case every full strongly exceptional sequence of invertible sheaves is equivalent to a strongly exceptional standard augmentation which implies (Theorem 8.2) that a toric surface different from $\mathbb{P}^2$ admits such a sequence iff it can be obtained by blowing up a Hirzebruch surface at most two times. So, in a sense, the existence of a full strongly exceptional sequence of invertible sheaves can be considered as a geometric characterization of a surface. Presumably, Theorem 8.1 should generalize to all rational surfaces, but at present it is not clear to us whether the procedure of sections 7 to 10 can be generalized in an effective way.

The following theorem gives a strong constraint on the existence of cyclic strongly exceptional sequences of invertible sheaves on rational surfaces in general:

Theorem 5.13: *Let X be a smooth complete rational surface on which a full cyclic strongly exceptional sequence of invertible sheaves exists. Then* $\operatorname{rk}\operatorname{Pic}(X) \leq 7$.

Proof. Let $\mathcal{A} = A_1, \dots, A_n$ be the associated toric system. As every A_i is strongly left-orthogonal, it follows that $\chi(A_i) \geq 0$ for every i. Therefore by Proposition 2.5 the anti-canonical bundle of the associated toric surface $Y(\mathcal{A})$ must be nef. From the classification of such toric surfaces (see table 1) it follows that $\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Pic}\big(Y(\mathcal{A})\big) \leq 7$. □

In particular, Theorem 5.13 implies that not even every del Pezzo surface has a cyclic strongly exceptional sequence of invertible sheaves. However, if $\operatorname{rk}\operatorname{Pic}(X) \leq 7$, we have the following positive result:

Theorem 5.14: *Let X be a del Pezzo surface with $\operatorname{rk}\operatorname{Pic}(X) \leq 7$, then there exists a full cyclic strongly exceptional sequence of invertible sheaves on X.*

Proof. Recall that a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ in at most 8 points (see [Dem80]). The case $\mathbb{P}^1 \times \mathbb{P}^1$ is clear from Proposition 5.2. For the other cases, by our assumptions it suffices to assume that X is a blow-up of $\mathbb{P}^2$ in at most 6 points $x_1, \dots, x_6$. Moreover, it suffices to only consider the maximal case, i.e. $\operatorname{rk}\operatorname{Pic}(X) = 7$ and the cases of smaller rank will follow immediately. We first give an example for a cyclic exceptional toric system and then show that it is cyclic strongly exceptional. We fix a blow-down $X \to \mathbb{P}^2$ and denote $R_1, \dots, R_6$ the exceptional divisors and H the class of a line on $\mathbb{P}^2$. Then by Proposition 5.5 the following is a full cyclic exceptional sequence:

$$H - R_1 - R_2 - R_5, R_2, R_1 - R_2, H - R_1 - R_3 - R_4,$$
$$R_4, R_3 - R_4, H - R_3 - R_5 - R_6, R_6, R_5 - R_6.$$

To show that a toric system $A_1, \dots, A_6$ is cyclic strongly exceptional, we have to show that for every cyclic interval $I \subset [6]$ the sum $A_I := \sum_{i \in I} A_i$ is strongly left-orthogonal. There are several possible cases what A_I can be. First, if $A_I = R_i$ for some $i \in [6]$ or $A_I = R_i - R_j$ for some $i \neq j \in [6]$, strong left-orthogonality follows from Proposition 4.11. The next cases are of the form $H - R_i$, $H - R_i - R_j$ and $H - R_i - R_j - R_k$, respectively, where i, j, k pairwise distinct. Analogous to the arguments in the proof of 4.14, we have to discuss the existence of base points. As H is very ample, its associated complete linear system does not have base points. So $h^0(H - R_i) < h^0(H)$, and we conclude as in the proof of 4.14 that $H - R_i$ is strongly left-orthogonal. For any two x_i, x_j, we can find a line on $\mathbb{P}^2$ which does pass through x_i but not through x_j. So, the linear system $|H - R_i|$ is base point free and $H - R_i - R_j$ is strongly left-orthogonal for any $i \neq j$. The divisor $H - R_i - R_j$ is not base point free. Its base points lie on the line connecting x_i and x_j. But as X is del Pezzo, none of the other x_k lie on this line. So we have $h^0(H - R_i - R_j - R_k) < h^0(H - R_i - R_j)$ and thus $H - R_i - R_j - R_k$ is strongly left-orthogonal. Similarly, using [Har77], V.4, Corollary 4.2, we see that $2H - \sum_{i \in S} R_i$ is strongly left-orthogonal for any $S \subsetneq [6]$. The next cases are of the form $3H - \sum_{i \in S} R_i$, where $S \subseteq [6]$ and $|S| \geq 4$. As $|S| < 7$, it follows from [Har77], V.4, Proposition 4.3, that these are strongly left-orthogonal, too. The remaining cases are of the form $3H - 2R_i - \sum_{k \neq i,j} R_k$ with $i \neq j \in [6]$. By [Har77], V.4, Proposition 4.3, $3H - \sum_{k \neq j} R_k$ has no base points, therefore $h^0(3H - 2R_i - \sum_{k \neq i,j} R_k) < h^0(3H - \sum_{k \neq j} R_k)$ and $3H - 2R_i - \sum_{k \neq i,j} R_k$ is strongly left-orthogonal □

Remark 5.15: Note that for a del Pezzo surface X with $\operatorname{rk}\operatorname{Pic}(X) \leq 7$ the toric system of the type as given in the proof of Theorem 5.14 in general is not the only possibility. It is an exercise to write down all possible admissible standard augmentations for $X_0 = \mathbb{P}^2$ and to check the conditions whether the resulting toric system is cyclic and strong. For example, for X del Pezzo, the strongly exceptional toric systems as given in Theorem 5.7 are cyclic iff $t \leq 3$. Moreover, it follows from the proof of Theorem 5.14 that the conditions on X can be weakened in general. Though the toric system given in the proof does require that no three points are collinear, it admits a configuration of 6 points lying on a conic and certain configurations of infinitely near points. We will see in Theorems 8.5 and 8.6 that at least in the toric case the existence of such sequences is equivalent to $-K_X$ nef.

We conclude this section with some more technical properties of strongly exceptional sequences. As before, we assume that a sequence of blow-downs to a minimal surface X_0 is chosen. First we consider parts of a toric system which are "vertical" with respect to X_0:

Lemma 5.16: *Let $A_1, \ldots, A_k \in \mathrm{Pic}(X)$ such that $A_i.A_{i+1} = 1$ for $1 \leq i < k$ and $A_i.A_j = 0$ else such that $A_I := \sum_{i \in I} A_i$ is strongly left-orthogonal and $(A_I)_0 = 0$ for every interval $I \subset [k]$. Then this system is, up to reversing the order of the A_i, of one of the following shapes:*

(i) $A_1 = R_{i_1} - R_{i_2}, A_2 = R_{i_2} - R_{i_3}, \ldots, A_k = R_{i_k} - R_{i_{k+1}}$,
(ii) $A_1 = R_{i_1} - R_{i_2}, A_2 = R_{i_2} - R_{i_3}, \ldots, A_{k-1} = R_{i_{k-1}} - R_{i_k}, A_k = R_{i_k}$,

where the R_{i_l} are pairwise incomparable.

Proof. By proposition 4.11 every A_I must be of the form R_i or $R_i - R_j$ for some $i, j \in [t]$ such that R_i and R_j are incomparable. Moreover, $(R_{i_p} - R_{i_q}).(R_{i_s} - R_{i_t}) = 1$ iff either $q = s$ and $p \neq t$ or $q \neq s$ and $p = t$. Moreover, $(R_{i_p} - R_{i_q}).(R_{i_s} - R_{i_t}) = 0$ iff $\{p, q\} \cap \{s, t\} = \emptyset$. This readily implies that the sequence $A_1, \ldots, A_k$ must be of one of the above forms. □

For the parts of a toric system which are not vertical to $\mathrm{Pic}(X_0)$, we would like to have a normal form. Let $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ be a strongly exceptional sequence and $A_1, \ldots, A_n$ its associated toric system. One of the requirements is that $\mathrm{Hom}\big(\mathcal{O}(E_i), \mathcal{O}(E_j)\big) = H^0\big(X, \mathcal{O}(-\sum_{k=j}^{i-1} A_i)\big) = 0$ for $i > j$. So, clearly, for any $1 \leq i < n$ with $\chi(A_i) = 0$, we can exchange E_i and E_{i+1} such that $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_{i+1}), \mathcal{O}(E_i), \ldots, \mathcal{O}(E_n)$ also forms a strongly exceptional sequence. The toric system then becomes:

$$A_1, \ldots, A_{i-2}, A_{i-1} + A_i, -A_i, A_{i+1} + A_i, A_{i+2}, \ldots, A_n.$$

We introduce the following notion with this operation in mind.

Definition 5.17: Let $\mathcal{A} = A_1, \ldots, A_n$ be a toric system. If $\mathcal{A}$ gives rise to a (cyclic) strongly exceptional sequence, we say that $\mathcal{A}$ is in *normal form* with respect to X_0 if $(A_i)_0$ is either zero or strongly pre-left-orthogonal for every $1 \leq i < n$ (for every $1 \leq i \leq n$, respectively).

Assume that $\mathcal{A}$ gives rise to a strongly exceptional sequence and is not in normal form, i.e. there exists some A_i with $1 \leq i < n$ such that $(A_i)_0$ is non-trivial and not strongly pre-left-orthogonal. This implies that $\chi(A_i) = h^0(A_i) = 0$ and there are no homomorphisms between $\mathcal{O}(D_{i-1})$ and $\mathcal{O}(D_i)$. In fact, there exists a maximal interval $I \subset [n-1]$ containing i such that $\mathrm{Hom}\big(\mathcal{O}(D_k), \mathcal{O}(D_l)\big) = 0$ for every $k, l \in I$. Clearly, any reordering of the D_k with $k \in I$ is a strongly exceptional sequence, too. We are going to show that every strongly exceptional sequence comes, up to such reordering, from a toric system in normal form.

Proposition 5.18: *Let X be a smooth complete rational surface and X_0 a minimal model for X. Then any (cyclic) strongly exceptional sequence of invertible sheaves on X can be reordered such that the associated toric system is in normal form with respect to X_0.*

Proof. Let $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ be a strongly exceptional sequence and $\mathcal{A} = A_1, \ldots, A_n$ its associated toric system. As remarked above, for any interval $[k, \ldots, l+1] \subset [n]$ such that $\chi(A_i) = 0$ for every $k \leq i \leq l$, we can exchange the positions of any two $\mathcal{O}(E_i)$, $\mathcal{O}(E_j)$ with $i, j \in I$. In particular, if we want to move $\mathcal{O}(E_{l+1})$ to the leftmost position, it is easy to see that the toric system becomes

$$\ldots, A_{k-1}, \sum_{i=k}^{l} A_i, -\sum_{i=k+1}^{l} A_i, A_{k+1}, \ldots, A_{l-1}, A_l + A_{l+1}, A_{l+2}, \ldots$$

Let $1 \leq l < n$ be minimal such that $(A_l)_0$ is non-trivial and not strongly pre-left-orthogonal. Then exchanging $\mathcal{O}(E_{l+1})$ with $\mathcal{O}(E_l)$ changes the toric system to $\ldots, A_{i-2}, A_{i-1}+A_i, -A_i, A_{i+1}+A_i, A_{i+2}, \ldots$ such that $(-A_l)_0$ is strongly pre-left-orthogonal. Now possibly $(A_{i-1} + A_i)_0$ is no longer strongly pre-left-orthogonal. In this case we iterate moving $\mathcal{O}(E_{l+1})$ to the left. This process eventually stops, because of one of two reasons. First, $\mathcal{O}(E_{l+1})$ ends up at the most left position and we are getting $-\sum_{i=1}^{l} A_i, A_1, \ldots, A_{l-1}, A_l+A_{l+1}, A_{l+2}, \ldots, A_n+\sum_{i=1}^{l} A_i$. Second, $\mathcal{O}(E_{l+1})$ is at $(k+1)$-th position, but $(\sum_{i=k}^{l} A_i)_0$ is strongly pre-left-orthogonal. Consequently, after moving $\mathcal{O}(E_{l+1})$, the smallest $1 \leq l' < n$ such that $(A_{l'})_0$ is non-trivial and not strongly pre-left-orthogonal is strictly bigger than l. So, by iterating this exchange process, we end up with a toric system in normal form.

If $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_n)$ is a cyclic strongly exceptional sequence, we are free to cyclically change the enumeration of the A_i. In particular, from the general classification of toric surfaces, it follows that there cannot be a cyclic interval $I \subset [n]$ of length bigger than $n - 3$ such that $h^0(A_i) = 0$ for every $i \in I$. Moreover, if $\mathcal{A}$ is not in normal form, we can choose the enumeration of the A_i the way that, if $(A_l)_0$

is non-trivial and not strongly pre-left-orthogonal, then $h^0(A_1) > 0$ and we choose $l < k < n$ minimal such that $h^0(A_k) > 0$. This implies that $(\sum_{i=1}^{p} A_i)_0$ and $(\sum_{i=q}^{k} A_i)_0$ are strongly pre-left-orthogonal for every $1 \leq p < k$ and every $1 < q \leq k$. So the part $A_1, \ldots, A_k$ is in normal form. We iterate this and eventually all of $\mathcal{A}$ will be in normal form. □

6. Proof of Theorem 5.11

Assume first that $X_0 = \mathbb{P}^2$ and $\mathcal{A}_0 = H, H, H$. If we blow up $X_1 \to X_0$, then there is, up to cyclic change of enumeration, only one possible augmentation $\mathcal{A}_1 = H - R_1, R_1, H - R_1, H$. But X_1 is isomorphic to $\mathbb{F}_1$ and if we choose the usual generators P, Q of the nef cone of X_1 as a basis of $\operatorname{Pic}(X_1)$, we get $P = H - R_1$, $Q = H$. In these coordinates we have $\mathcal{A}_1 = P, Q - P, P, Q$, which is the unique cyclic strongly exceptional standard toric system on $\mathbb{F}_1$. So, to prove the theorem it suffices to consider standard toric systems coming from Hirzebruch surfaces according to the classification of Proposition 5.2. We assume that X is obtained by a sequence of blow-ups $X = X_t \to \cdots \to X_0$ of a Hirzebruch surface $X_0 \cong \mathbb{F}_a$ and denote $P, Q, R_1, \ldots, R_t$ the corresponding basis of $\operatorname{Pic}(X)$.

For any divisor D we denote $\operatorname{bs}(D)$ the base locus of the complete linear system $|D|$. Note that for any effective divisor D the condition $\chi(-D) = 0$ is equivalent to the arithmetic genus of D being zero. It is straightforward to check that in this case the underlying reduced divisor D_{red} also has arithmetic genus zero. So, because $h^0(R_i) = 1$ for every i, the divisor class R_i is represented by a unique, possibly non-reduced, effective divisor of arithmetic genus zero and $\operatorname{bs}(R_i)$ coincides with the support of this divisor whose arithmetic genus is also zero. The image of $\operatorname{bs}(R_i)$ in X_0 is contained in some fiber of the ruling $\mathbb{F}_a \to \mathbb{P}^1$ which we denote by f_i and which represents the divisor class P.

For any R_i we denote E_i the strict transform on X of the corresponding exceptional divisor on X_i. By abuse of notation we also use E_i for the strict transforms on the X_j with $j \geq i$. Any effective divisor D whose support contains E_i can be written $D = D' + n_i E_i$ where D' is effective and does not have any component with support E_i. We call n_i the *multiplicity* of E_i with respect to D. By abuse of notion we will also sometimes call n_i the multiplicity of R_i.

We recall that the R_i form a partially ordered set. The maximal elements have the property that $E_i^2 = -1$. Any maximal chain of R_i contains precisely one maximal element. All maximal elements are incomparable and can be blown down simultaneously. In the nicest cases we will see that the maximal length of maximal chains will be at most two and that X can be blown down to X_0 in two steps. However, the most part of our analysis in this section will be concerned with the cases where there exist maximal chains of bigger length. In general there will be only very few of these and, if such chains exist, we will have to look for some other way to blow down to some minimal model X_0' which might not coincide with X_0. For this we will need exceptional divisors which do not coincide with one of the E_i. These exceptional divisors can be the strict transform of some fiber f_i or, in the case $X_0 \cong \mathbb{F}_1$, of the unique divisor on X_0 with self-intersection -1. Note in the sequel we will consider the case where blow-ups are only over a fixed fiber f. This will be without loss of generality, because in our conclusion at the end of this section we will make use of the fact that $f_i \neq f_j$ implies that R_i and R_j are incomparable.

Lemma 6.1: *We use notation as before.*

- *(i) For any i, the divisor class $P - R_i$ is strongly left-orthogonal and* $\operatorname{bs}(P - R_i)$ *contains* $\operatorname{bs}(R_j)$ *for every R_j with $f_j = f_i$ and $R_i \not\leq R_j$.*
- *(ii) If the multiplicity of R_i with respect to the total transform of f_i is greater than* 1*, then* $\operatorname{bs}(P - R_i)$ *contains* $\operatorname{bs}(R_j)$ *for every R_j with $f_j = f_i$.*
- *(iii) For any $i \neq j$, the divisor class $P - R_i - R_j$ is strongly left-orthogonal iff either $f_i \neq f_j$ or R_i and R_j are comparable (say, $R_i \leq R_j$) and* $\operatorname{bs}(P - R_i)$ *does not contain* $\operatorname{bs}(R_j)$*.*

Proof. Clearly we have $\chi(-P + R_i) = 0$, $h^k(-P + R_i) = 0$ for all k, and $\chi(P - R_i) = 1$. To show that $P - R_i$ is strongly left-orthogonal we need only to show that $h^0(P - R_i) = 1$. But this follows from the fact that the divisor class P is nef and therefore base-point free and hence $h^0(P - R_i) = h^0(P) - 1 = 1$. The divisor class $P - R_i$ is nontrivial and its base locus is a curve of arithmetic genus zero which projects to f_i. The total transform of f_i is a representative of P in $\operatorname{Pic}(X)$ and contains the base loci of all the R_j with $f_j = f_i$. By subtracting R_i from P, we at most (but not necessarily) cancel the base loci of those R_j with $R_i \leq R_j$ and (i) follows. If the multiplicity of R_i with respect to the total transform of f_i is greater than 1, then the multiplicities of all E_j with $R_i \leq R_j$ with respect to P is strictly smaller than their multiplicities with respect to R_i. Therefore $\mathrm{bs}(P - R_i)$ also contains $\mathrm{bs}(R_j)$ for $R_i \leq R_j$ and (ii) follows. From (i) it follows that statement (iii) essentially is a case distinction for determining when $\mathrm{bs}(R_j)$ is not contained in the base locus of $P - R_i$. □

Lemma 6.2: *Consider the divisor Q on $X_0 \cong \mathbb{F}_1$ and some strongly left-orthogonal divisor class $Q - R_i - R_j$ on X with R_i, R_j incomparable and $f := f_i = f_j$. Then $\mathrm{bs}(Q - R_i - R_j)$ contains the total transform of f.*

Proof. The class Q is the pullback of the class of lines in $\mathbb{P}^2$. Denote p the image of R_i in X_0, then we can identify the linear system $|Q - R_i|$ with the set of lines passing through the image of p in $\mathbb{P}^2$. If R_j lies over some other point of f than p, then $\mathrm{bs}(Q - R_i - R_j)$ fixes two points on f and thus contains f. If R_j also lies over P, then we first observe that $\mathrm{bs}(Q - R_i)$ contains $\mathrm{bs}(R_k)$ for all $k \neq i$ and $R_k \leq R_i$. So, the condition that $Q - R_i - R_j$ is strongly left-orthogonal implies that R_i is minimal and hence $R_i \leq R_j$, which is a contradiction. □

Lemma 6.3: *Let $A_1, \ldots, A_n$ a strongly exceptional toric system on X which is a standard augmentation of $P, sP + Q, P, -(a + s)P + Q$ with $s \geq -1$ for some choice of $X_0 \cong \mathbb{F}_a$ such that $f_i = f_j$ for all i, j. Assume that $A_k, A_{k+1}, \ldots A_l$ for some $1 \leq k < l < n$ is a subsequence of the toric system which contains the two slots around one of the P, i.e. $(A_{k-1})_0, (A_{l+1})_0 \notin \{0, P\}$, $(A_p)_0 = P$ for one $k \leq p \leq l$, and $(A_q)_0 = 0$ for all $k \leq q \leq l$ with $q \neq p$. Then $A_k, \ldots, A_l$ is, up to possible order inversions, of one of these forms:*

(i) $R_{i_{l-k}}, R_{i_{l-k-1}} - R_{i_{k-l}}, \ldots, R_{i_2} - R_{i_3}, R_{i_1} - R_{i_2}, P - R_{i_1}$, *where the R_{i_j} are pairwise incomparable;*
(ii) $R_{i_1}, P - R_{i_1} - R_{i_2}, R_{i_2} - R_{i_3}, \ldots, R_{i_{l-k-1}} - R_{i_{k-l}}, R_{i_{k-l}}$, *where $R_{i_1} \leq R_{i_j}$ and the R_{i_j} are pairwise incomparable for $j > 1$.*

Proof. After the first augmentation we get $R_{i_1}, P - R_{i_1}$. In the second step, we can extend this sequence in the middle, or to the left, or to the right. By extending in the middle, we get $R_{i_1} - R_{i_2}, R_{i_2}, P - R_{i_1} - R_{i_2}$ which implies that $\mathrm{bs}(R_{i_2}) \notin \mathrm{bs}(R_{i_1}) \cup \mathrm{bs}(P - R_{i_1})$, where the right hand side coincides with the total transform of a fiber f_i on X, which is not possible. By extending to the left, we get $R_{i_2}, R_{i_1} - R_{i_2}, P - R_{i_1}$ with the necessary condition that $\mathrm{bs}(R_{i_1}) \cap \mathrm{bs}(R_{i_2}) = \emptyset$ and therefore $R_{i_1}, \ldots, R_{i_2}$ are incomparable. By iterating to the left, we obtain that the R_{i_j} are pairwise incomparable and therefore we arrive at the form (i). If we extend to the right instead, we get $R_{i_1}, P - R_{i_1} - R_{i_2}, R_{i_2}$ and by Lemma 6.1 (iii) R_{i_1}, R_{i_2} must be comparable. In the next step, we extend without loss of generality to the right and get $R_{i_1}, P - R_{i_1} - R_{i_2}, R_{i_2} - R_{i_3}, R_{i_3}$ where R_{i_1}, R_{i_3} are comparable and R_{i_2}, R_{i_3} are incomparable. If we extend to the left in the next step, this implies that the pairs R_{i_1}, R_{i_4} and R_{i_2}, R_{i_3} are incomparable, but R_{i_2} and R_{i_3} are comparable to R_{i_1} and R_{i_4} respectively, which is not possible. So, we can continue extending only to the right and we inductively obtain that the R_{i_j} are pairwise incomparable for $j > 1$ and R_{i_1} is comparable with every R_{i_j}. If $l - k > 2$, this implies that $R_{i_1} \leq R_{i_j}$ for every $j > 1$. □

Now we consider standard augmentations starting from a standard sequence $P, sP+Q, P, -(s+a)P+Q$ with $s \geq -1$ on X_0. For this, we have four "slots", in which we can insert the R_i successively. The augmented toric system is of the form $A_1, \ldots, A_n$, where possibly A_n is only left-orthogonal but not strongly left-orthogonal. For $(A_n)_0$, there are four possibilities, namely $(A_n)_0 = 0$, $(A_n)_0 = P$, $(A_n)_0 = sP + Q$ and $(A_n)_0 = -(s + a)P + Q$. We will first consider the last case.

Proposition 6.4: *Let $\mathcal{A} = A_1, \ldots, A_n$ be a strongly exceptional toric system which is a standard augmentation of the toric system $P, sP + Q, P, -(s + a)P + Q$ with $s \geq -1$ on $X_0 \cong \mathbb{F}_a$ such that $(A_n)_0 = -(s + a)P + Q$ and $f_i = f_j$ for all i, j. Then X can be obtained from blowing up a Hirzebruch surface two times (in possibly several points in each step).*

Proof. We denote f the distinguished fiber such that $f = f_i$ for all $i \in [t]$. Because $(A_n)_0 = -(s+a)P+Q$ the toric system has two subsequences which are of the form as stated in Lemma 6.3. This implies that there is a partition of the set $\{R_1, \ldots, R_t\}$ into two subsets $S_1 := \{R_{i_1}, \ldots, R_{i_r}\}$, $S_2 := \{R_{j_1}, \ldots, R_{j_s}\}$ such that, if nonempty, the elements in each of these subsets either are (i) incomparable or (ii) $R_{i_1} \leq R_{i_k}$ and the R_{i_k} incomparable for all $k > 1$ ($R_{j_1} \leq R_{j_k}$ and the R_{j_k} incomparable for all $k > 1$, respectively). If both S_1 and S_2 are empty, there is nothing to prove. If one of S_1, S_2 is empty, then the length of a maximal chain of comparable elements among the R_i is at most two and the proposition follows. So we assume that S_1 and S_2 both are nonempty. If S_1 and S_2 both satisfy property (i), then again the length of a maximal chain of comparable elements among the R_i is at most two and the proposition follows. If both satisfy property (ii), then we have two cases. The first is that R_{i_1}, R_{j_1} both are minimal. Then again the length of a maximal chain of comparable elements among the R_i is at most two. The second case is that only one of these, say R_{i_1}, is minimal and $R_{i_1} = R_1$ without loss of generality. On X_1 we have $f^2 = -1$ and we can choose to either blow down R_1 or f. If we choose f, then we obtain another of basis for $\operatorname{Pic}(X_1)$ given by P', Q', R'_1, where $P' = P$, $R'_1 = P - R_1$ and $Q' = Q + \delta P - R_1$, where $\delta = 1$ if R_1 corresponds to a blow-up of a point on the zero section of the fibration $\mathbb{F}_a \to \mathbb{P}^1$, and $\delta = 0$ otherwise. If we complete this basis to a basis of $\operatorname{Pic}(X)$ by using the R_i with $i > 1$, the sequence $R_{i_1}, P - R_{i_1} - R_{i_2}, R_{i_2} - R_{i_3}, \ldots, R_{i_{r-1}} - R_{i_r}, R_{i_r}$ becomes $P' - R'_1, R'_1 - R_{i_2}, R_{i_2} - R_{i_3}, \ldots, R_{i_{r-1}} - R_{i_r}, R_{i_r}$ with $R'_1, R_{i_2}, \ldots R_{i_r}$ pairwise incomparable. So we have reduced to the case that S_1 satisfies property (i) and S_2 satisfies property (ii). Moreover, we can assume that R_{j_1} is not minimal as otherwise we can choose another basis as we did before and reduce to the case that both S_1 and S_2 satisfy property (i).

In the remaining case, the length of a maximal chain of comparable R_i is either two or three. If it is two, the proposition follows. In the case where it is three, we assume without loss of generality that $\mathcal{A}$ is an augmentation of a strongly exceptional toric system on X_3 with $R_1 \leq R_2 \leq R_3$ such that $R_1 \in S_1$ and $R_2, R_3 \in S_2$. Then the divisor $P - R_2 - R_3$ is strongly left-orthogonal and by Lemma 6.1 (ii) it follows that the multiplicity of R_2 with respect to the total transform of f is one. In particular, R_2 does not come from a blow-up of the intersection of f with E_1 on X_1. If we now go back to X, then the R_{i_k} are incomparable with R_1 and hence with R_2, because $R_1 \leq R_2$. Thus the R_{i_k} are minimal. So, by blowing down simultaneously all E_i with $E_i^2 = -1$ (which includes E_3) on X, we arrive at the surface X_2. Here, we have $f^2 = -1$ and $E_2^2 = -1$. So, we can blow-down these two divisors simultaneously and arrive at some Hirzebruch surface X'_0. □

If $s + a > 1$, it follows by Lemma 4.3 that necessarily $(A_n)_0 = -(s + a)P + Q$. If $s + a \leq 1$, then possibly $(A_n)_0 \in \{0, P\}$ and the standard toric system $P, sP + Q, P, -(s + a)P + Q$ must be cyclic strongly exceptional on $\mathbb{F}_a$ for which, by Proposition 5.2, there are only four possibilities. Our first step will be to reduce these to one.

Proposition 6.5: *Let $\mathcal{A} = A_1, \ldots, A_n$ a toric system on X with $(A_n)_0 \neq -(s + a)P + Q$. Then there exists a sequence of blow-downs $X = X'_t \to \cdots X'_0$ such that $X'_0 \cong \mathbb{F}_1$ and $\mathcal{A}$ is an augmentation of the toric system $P', Q', P', Q' - P'$ on X'_0.*

Proof. As argued before, $\mathcal{A}$ necessarily is an augmentation of a cyclic strongly exceptional standard sequence. In particular, $X_0 \cong \mathbb{F}_a$ with $0 \leq a \leq 2$. If $a = 1$, there is nothing to prove. If $a = 0$, there are, up to symmetry by exchanging P and Q, two such toric systems, P, Q, P, Q and $P, P + Q, P, -P + Q$. If we consider the blow-up $X_1 \to X_0$, then in the first case, there exists, up to cyclic reordering and order inversion, only one possible augmentation which is given by $P - R_1, R_1, Q - R_1, P, Q$ which is a cyclic strongly exceptional toric system on X_1. If we consider some projection $X_0 \to \mathbb{P}^1$ such that P represents a general fiber, then the divisor $P - R_1$ is rationally equivalent to the strict transform under the blow-up and has self-intersection (-1). If we blow down this divisor, we obtain $X_1 \to X'_0 \cong \mathbb{F}_1$. If we denote P', Q' the corresponding divisors in $\operatorname{Pic}(\mathbb{F}_1)$, then we get a change of coordinates in $\operatorname{Pic}(X_1)$ via $P = P'$, $Q = Q' - R'_1$, and $R_1 = P' - R'_1$. In this basis the toric system is given as $R'_1, P' - R'_1, Q' - P', P', Q' - R'_1$ and the assertion follows in this case. We proceed similarly in the second case. As $h^0(P - Q) = 0$ and $(A_n)_0 \neq P - Q$ by assumption, the only possible augmentation (up to cyclic reordering and order inversion) on X_1 is given by $P - R_1, R_1, P + Q - R_1, P, -P + Q$. By the same change of coordinates as before we get $R'_1, P' - R'_1, Q', P', Q' - P' - R'_1$ and the assertion follows for this case.

Now assume that $a = 2$. Then the only cyclic strongly exceptional toric system is given by $P, Q - P, P, Q - P$ and the only possible augmentation on X_1 is given by $P - R_1, R_1, Q - P - R_1, P, Q - P$. The base locus of the complete linear system of the divisor $Q - P$ consists of one fixed component which is the zero section of the fibration $\mathbb{F}_2 \to \mathbb{P}^1$. Therefore, if X_1 is a blow-up on the zero section, we have $h^0(Q - P - R_1) = h^0(Q - P) = 2$ and $Q - P - R_1$ is not strongly left-orthogonal and thus necessarily $(A_n)_0 = Q - P$ which is a contradiction to our assumptions. So we can assume without loss of generality that X_1 is a blow-up of X_0 at some point which is not on the zero section. In this case we can conclude as before that there exists a blow-down to $X'_0 \cong \mathbb{F}_1$ and a corresponding change of coordinates $P = P' - R_1'$, $Q = Q' + P' - R_1'$, and $R_1 = P' - R_1'$ such that the toric system is represented as $R_1', P' - R_1', Q' - P', P', Q' - R_1'$ which is of the required form. $\square$

Proposition 6.6: *Let $\mathcal{A} = A_1, \ldots, A_n$ be a strongly exceptional toric system which is a standard augmentation of the toric system $P, Q, P, -P + Q$ on $X_0 \cong \mathbb{F}_1$. Then X can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step).*

Proof. We will only consider the case $(A_n)_0 \in \{0, P\}$. Otherwise, the result follows from Proposition 6.4. We will denote b the zero section (respectively its strict transform) of the $\mathbb{P}^1$-fibration $X_0 \to \mathbb{P}^1$ with $b^2 = -1$ on X_0. Note that in some steps below we will have to blow-down b to arrive at some convenient minimal model X_0'. Strictly speaking, this would require us not only to consider blow-ups of a fixed fiber f but rather the general case. However, in these few cases this would only increase the number of case distinctions without changing the arguments. So we will keep the assumption that all blow-ups lie above one distinguished fiber f.

First note that $h^0(-P + Q) = 1$ and therefore any divisor of the form $-P + Q - R_i - R_j$ cannot be strongly left-orthogonal. This together with the condition $(A_n)_0 \neq -P + Q$ implies that we can use at most one of the two slots around $-P + Q$ in the toric system $P, Q, P, -P + Q$ for augmentations. Moreover, for any $(A_n)_0$, we can assume that the augmentations in the two slots surrounding one of the P's are strongly left-orthogonal and therefore we get there a subsequence of $\mathcal{A}$ which corresponds to one of the two shapes given in Lemma 6.3. The slot between P and Q can be augmented at most three times because $h^0(Q) = 3$. Because of our general assumption that all blow-ups lie over the same fiber, we can even conclude by Lemma 6.2 that this slot even can be extended at most two times. For the same reason, if this slot has been extended two times, then the other slot neighbouring Q cannot be extended any more. Denote S_1 the subset of the R_i used for augmenting the two slots around P. We have seen that S_1 can consist of at most three elements. In the maximal case, we have $S_1 = \{R_{i_1}, R_{i_2}, R_{i_3}\}$ such that $R_{i_1} \leq R_{i_2}, R_{i_3}$ and R_{i_2}, R_{i_3} incomparable. In this case, the remaining two slots cannot be augmented without violating our condition on $(A_n)_0$ and thus the assertion follows. So we assume from now that S_1 consists of at most two elements, which may be comparable or not. Also note that the base locus of $P - Q$ coincides with the support of the total transform of b on X. Therefore, in the cases where either R_{i_1} and R_{i_2} are comparable, or $S_1 = \{R_{i_1}\}$ and R_{i_1} is used for augmentation in the slot between $-P+Q$ and P, these divisors cannot come from blowing up points on or above b.

If $(A_n)_0 = P$, then the content of the two slots neighbouring this "bad" P must be of the form as given in Lemma 5.16 (ii). That is, we have a partition of the set of the R_i into three sets, S_1, S_2, S_3, where the latter two each consist of pairwise incomparable elements. If both S_2, S_3 are empty, the assertion follows. If S_1 consists of two elements, then only one of S_2, S_3 can be nonempty, say S_2. If the two elements in S_1 are incomparable, we have thus a partition into two subsets of incomparable elements and the assertion follows. If the two elements in S_1 are comparable, i.e. $R_{i_1} \leq R_{i_2}$, then we have (up to order inversion) the subsequence $R_{i_1}, P - R_{i_1} - R_{i_2}, R_{i_2}$ in $\mathcal{A}$. By Lemma 6.1 the divisor R_{i_1} must have multiplicity 1 with respect to P and R_{i_2} cannot come from blowing up a point on the fiber f. By this, after blowing down all R_i with $E_i^2 = -1$ we are left with at most one chain of length 2, containing at least one E_i with $E_i^2 = -1$ and we have $f^2 = -1$. So, by simultaneously blowing down these two divisors we arrive at some minimal surface X_0 and the assertion follows. If S_1 consists of only one element, then we have two possibilities. If R_{i_1} is used for augmentation in the slot between $-P + Q$ and P, then the other slot neighbouring $-P + Q$ is blocked for further augmentation and only one more slot is free for augmentation by incomparable R_i. So we can blow down X to X_0 in at most two steps. If R_{i_1} is used for augmentation in the slot between P and Q, then we can have two nonempty sets S_2, S_3. Let us assume that the elements in S_2 are used for augmentation between Q and P, and the elements in S_3 for augmentation between $-P + Q$ and P. As the base locus of $-P + Q$ contains the support of the total transform of b on X, none of the $R_i \in S_3$ are lying over any point of b. So, if R_{i_0} lies over some point in b, then it can be part of a chain of comparable R_i of length two. Hence, the maximal such length is at most two for all R_i. Hence the assertion follows. If R_{i_0} does not lie over some point of b, then the maximal length of a chain of comparable R_i which lie over some point of b is one, and after simultaneously blowing down the E_i with $E_i^2 = -1$, there is no such R_i left. But then R_{i_0} might still be part of a chain of length 2, which will be the only such chain and another simultaneous blow-down will leave one of the components of this chain. However, now we can additionally blow down b instead and we will arrive at some other X_0' within two steps and the assertion follows.

If $(A_n)_0 = 0$, then A_n is located in one of the slots and the subsequence of $\mathcal{A}$ in this slot can be of one of the following forms:

$$(7) \qquad Q - P - R_{j_1}, R_{j_1} - \sum_{k=2}^{r} R_{j_k}, R_{j_r}, R_{j_{r-1}} - R_{j_r}, \ldots, R_{j_2} - R_{j_3}, P - R_{j_1} - R_{j_2} - F,$$

$$(8) \quad Q - R_{j_1} - R_{j_2} - G, R_{j_2} - R_{j_3}, \ldots, R_{j_{r-1}} - R_{j_r}, R_{j_r}, R_{j_1} - \sum_{i=2}^{r} R_{j_i} - \sum_{i=1}^{s} R_{k_i},$$
$$R_{k_s}, R_{k_{s-1}} - R_{k_s}, \ldots, R_{k_1} - R_{k_2}, P - R_{j_1} - R_{k_1} - H,$$

where F, G, H denote some possible additional summands coming from augmentations in the neighbouring slots. We denote $S_2 := \{R_{j_2}, \ldots, R_{j_r}\}$ and $S_2 := \{R_{k_1}, \ldots, R_{k_s}\}$. In the case 7, we have $R_{j_1} \leq R_{j_i}$ and the R_{j_i} incomparable for all $i > 1$. In the case 7, we have $R_{k_1} \leq R_{j_i}$ and the R_{j_i} incomparable for all i. If both S_2 and S_3 are empty, then $\mathcal{A}$ is an augmentation by the elements of S_1 and by R_{j_1} and one possible augmentation by some R_i in the remaining slot. Then R_i and R_{j_1} must be comparable. These can form a chain of length at most three which cannot lie over b. Therefore we can conclude as before that we can blow-down the surface X to a surface X_0' in at most two steps. If S_2 consists of two incomparable elements, then the other neighbouring slot of Q is blocked for augmentations and the remaining augmentation must be of the form (7) with R_{j_1} (and thus all the R_{j_i}) not lying over b. So, if there exists a chain of length three, this chain cannot lie over b and again we can blow-down in two steps to some X_0'. If S_1 consists of two comparable elements then the remaining augmentation must be of the form (7) where $S_2 = \emptyset$, as $G \neq 0$. Then we possibly have a maximal chain of length four, where at least one of the elements in S_1 and one of R_{j_1} and R_{k_i} involved have multiplicity one, and all the R_{k_i} incomparable. With similar arguments as before, we can always blowing down X to some X_0' in two steps by possibly contracting the fiber f.

In the remaining cases we have to consider S_1 consisting of one or two elements. The arguments are completely analogous to the previous arguments and we leave these to the reader. □

We conclude that Theorem 5.11 follows from Propositions 6.5 and 6.6 in the case $(A_n)_0 \neq -(a+s)P + Q$. For the case $(A_n)_0 = -(a+s)P + Q$ we note that if $f_i \neq f_j$ then R_i and R_j are incomparable. Moreover, from the proof of Proposition 6.4 we see that in order to blow-down to some X_0' we may have to blow-down the strict transform of some fiber. But any such choices can be made simultaneously. This proves Theorem 5.11.

7. Divisorial cohomology vanishing on toric surfaces

Let X be a smooth complete toric surface whose associated fan is generated by lattice vectors $l_1, \ldots, l_n$ and recall that $\mathrm{Pic}(X)$ is generated by the T-invariant divisors $D_1, \ldots, D_n$. Recall from section 2 that, besides the coordinates associated to a minimal model X_0, we have two further coordinatizations for $\mathrm{Pic}(X)$. The first is given by choosing for a given divisor D a T-invariant representative $D \sim \sum_{i=0}^n c_i D_i$ such that we can identify this representative with a tuple $(c_1, \ldots, c_n)$ in $\mathbb{Z}^n$. The second coordinatization is given by tuples $(d_1, \ldots, d_n) \in \mathbb{Z}$ such that $\sum_{i \in [n]} d_i l_i = 0$. The d_i are uniquely determined by the c_i by the relations $d_i = c_{i-1} + a_i c_i + c_{i+1}$ for $i \in [n]$. The c_i are determined by the d_i up to a character $m \in M$, that is, $\sum_{i=0}^n c_i D_i \sim \sum_{i=0}^n c_i' D_i$ iff there exists some $m \in M$ such that $c_i' = c_i + l_i(m)$ for all $i \in [n]$. In what follows, we will use all of these coordinatizations for the classification of strongly left-orthogonal divisors on X.

Now assume that for a given divisor D, a T-invariant representative $D \sim \sum_{i=0}^n c_i D_i$ is chosen. Then we can associate to D a hyperplane arrangement $\{H_i\}_{i \in [n]}$ in $M_{\mathbb{Q}}$ which is given by hyperplanes

$$H_i := \{m \in M_{\mathbb{Q}} \mid l_i(m) = -c_i\}.$$

The twist $c_i \mapsto c_i + l_i(m)$ for some $m \in M$ then corresponds to a translation of the hyperplane arrangement by the lattice vector $-m$. The action of T induces an eigenspace decomposition of the space of global sections of $\mathcal{O}(D)$:

$$H^0\big(X, \mathcal{O}(D)\big) \cong \bigoplus_{m \in M} H^0\big(X, \mathcal{O}(D)\big)_m.$$

The nontrivial isotypical components $H^0\big(X, \mathcal{O}(D)\big)_m$ are one-dimensional and we have

$$H^0\big(X, \mathcal{O}(D)\big)_m \neq 0 \quad \text{iff} \quad l_i(m) \geq -c_i \;\; \text{for all } i \in [n]$$

for $m \in M$, i.e. the non-vanishing isotypical components correspond to the characters which are contained in a distinguished chamber of the hyperplane arrangement.

Definition 7.1: Let $D = \sum_{i=1}^n c_i D_i$ be a torus invariant divisor, then we denote $G_D := \{m \in M \mid H^0(X, \mathcal{O}(D))_m \neq 0\} = \{m \in G_D \mid l_i(m) \geq -c_i \text{ for all } i \in [n]\}$ and $G_D^\circ := \{m \in G_D \mid l_i(m) > -c_i \text{ for all } i \in [n]\}$.

As the set G_D counts the global sections of a T-invariant divisor D, by Serre duality, the set G_D° can naturally be associated with a T-eigenbasis of $H^2\big(X, \mathcal{O}(-D)\big)$. Namely, the canonical divisor on X is given by $K_X = -\sum_{i=1}^n D_i$ and $h^2(-D) = h^0(K_X + D) = |G_D^\circ|$. We want to interpret strong (pre-)left-orthogonality as a problem of counting lattice points, starting from G_D for some strongly pre-left-orthogonal divisor D on $\mathbb{P}^2$ or $\mathbb{F}_a$ as classified in propositions 4.7 and 4.9. In general, the region containing G_D is not quite a lattice polytope, but rather close to being one, as we will see in Proposition 7.12. This is illustrated in the following example.

Example 7.2: Figure 1 shows examples for strongly pre-left-orthogonal divisors on $\mathbb{P}^2$ and $\mathbb{F}_a$. The dots indicate the set G_D, the white dots the subset G_D°.

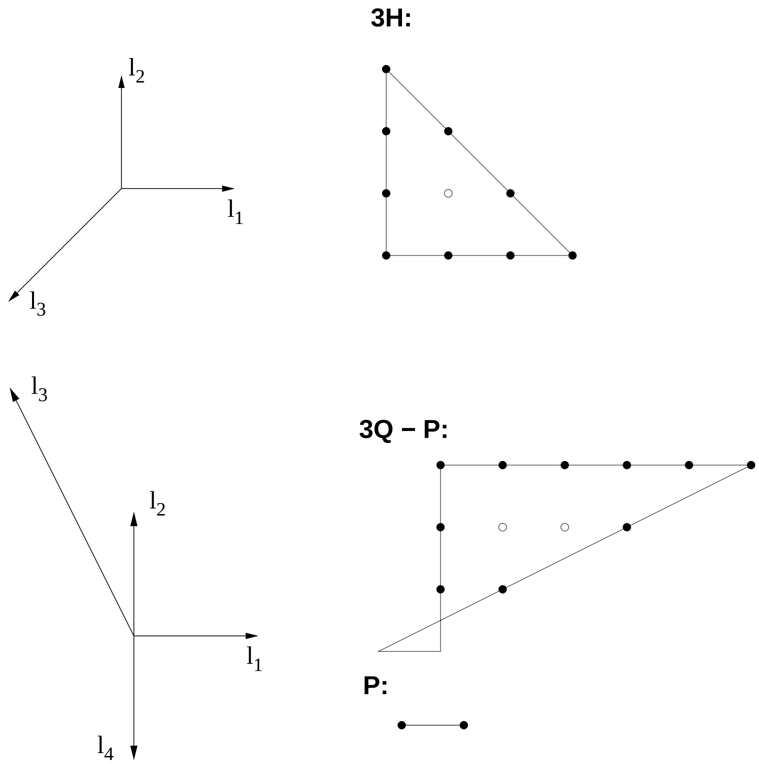

FIGURE 1. The fans for $\mathbb{P}^2$ and $\mathbb{F}_2$ and the regions in M containing G_D, for the cases $D = 3H$ on $\mathbb{P}^2$ and $D = P$, $D = 3Q - P$ on $\mathbb{F}_2$, respectively.

Consider any pre-left-orthogonal divisor βH, where $\beta > 0$, on $\mathbb{P}^2$. Then it is easy to see that formulas (3) and (4),

$$\chi(\beta H) = \binom{\beta + 2}{2}, \qquad \chi(-\beta H) = \binom{\beta - 1}{2},$$

count $G_{\beta H}$ and $G_{\beta H}^0$, respectively. Similarly, formulas (5) and (6),

$$\chi(\alpha P + \beta Q) = (\alpha + 1)(\beta + 1) + a\binom{\beta + 1}{2}, \qquad \chi(-\alpha P - \beta Q) = (\alpha - 1)(\beta - 1) + a\binom{\beta}{2},$$

count $G_{\alpha P+\beta Q}$ and $G^{\circ}_{\alpha P+\beta Q}$, respectively.

For the γ_i, there is a similar interpretation. Assume we have fixed a sequence of blow-ups $b_1, \ldots, b_t$ as in the previous section, where every b_k is toric. For some $k \in [t]$, there are $p, q, r \in [n]$ such that l_p and l_q span a cone in the fan of X_{k-1} and $l_r = l_p + l_q$ represents the toric blow-up b_k. We have:

Lemma 7.3: *Let $p, q \in S \subset [n]$ such that the l_j with $j \in S$ span the fan of X_{k-1} and denote $D = \sum_{i \in S} c_i D_i$ a T-invariant divisor. Then $b_k^* D \sim \sum_{i \in S} c_i D_i + (c_p + c_q) D_r$ and $\gamma_i R_i \sim c_r D_r$ on X_k.*

Proof. Only the first assertion needs a proof. Let L the matrix whose rows are the l_i with $i \in S$ and L' the matrix consisting of the same rows as L but with the additional row $l_p + l_q$ added between l_p and l_q. The assertion follows form the commutativity of the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{L} & \mathbb{Z}^{|S|} & \longrightarrow & \mathrm{Pic}(X_{i-1}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{L'} & \mathbb{Z}^{|S|+1} & \longrightarrow & \mathrm{Pic}(X_i) & \longrightarrow & 0. \end{array}$$

□

For given $\gamma_k \leq 0$, we consider the lattice triangle which is inscribed by the lines H_p, H_q, H_r and whose lattice points we can count:

Definition 7.4: Let l_p, l_q, l_r be as before and $\gamma_k \leq 0$, then we denote

(i) $T_{\gamma_k} := \{m \in M \mid l_p(m) \geq -c_p, l_q(m) \geq -c_q, l_r(m) \leq -c_r\}$,
(ii) $T^-_{\gamma_k} := \{m \in M \mid l_p(m) \geq -c_p, l_q(m) \geq -c_q, l_r(m) < -c_r\}$,
(iii) $T^+_{\gamma_k} := \{m \in M \mid l_p(m) > -c_p, l_q(m) > -c_q, l_r(m) \leq -c_r\}$.

As l_p and l_q form a basis of N, by translation by some $m \in M$ we can assume without loss of generality that $c_p = c_q = 0$. Then, using Lemma 7.3, we can directly see that the lattice points of T_{γ_k}, $T^+_{\gamma_k}$, $T^-_{\gamma_k}$ are counted by binomial coefficients. We have $|T_{\gamma_k}| = \binom{\gamma_k - 1}{2}$, $|T^-_{\gamma_k}| = \binom{\gamma_k}{2}$, and $|T^+_{\gamma_k}| = \binom{\gamma_k + 1}{2}$. This is illustrated in the following example.

Example 7.5: With notation as before, figure 2 shows the local configuration of l_p, l_q, l_r and the relative positions of H_p, H_q, H_r for $\gamma_k = -3$. The dots indicate the $\binom{-3-1}{2} = 10$ lattice points in T_{γ_k}, the gray

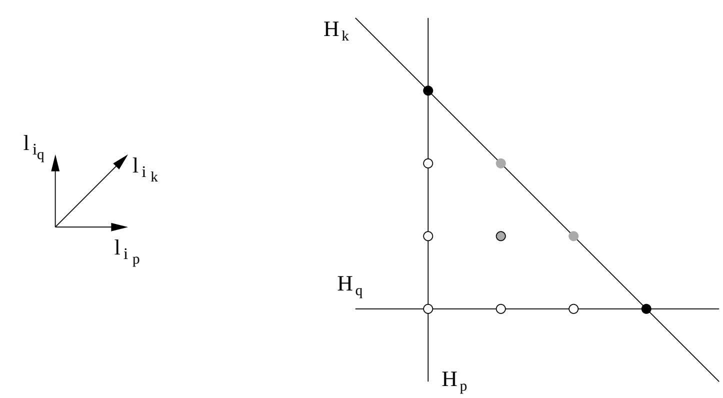

FIGURE 2. Three primitive vectors l_p, l_q, l_r which pairwise generate N and the corresponding orthogonal hyperplane arrangement for $\gamma_k = -3$.

dots the $\binom{-3+1}{2} = 3$ lattice points in $T^+_{\gamma_k}$ and the circled dots the $\binom{-3}{2} = 6$ lattice points in $T^-_{\gamma_k}$, with one lattice point in the intersection $T^+_{\gamma_k} \cap T^-_i$.

By Proposition 4.12, a pre-left-orthogonal divisor D is of the form $(D)_0 + \sum_{i=1}^t \gamma_i R_i$ with $(D)_0$ pre-left-orthogonal on X_0 and $\gamma_i \leq 0$ for every i. The following proposition shows that strong pre-left-orthogonality is equivalent to that the T_{γ_i} cut out the lattice points of $G_{(D)_0}$ in a well-behaved manner.

Proposition 7.6: *Let $k > 0$ and consider a blow-up $b_k : X_k \longrightarrow X_{k-1}$ with notation as before. Let D be a pre-left-orthogonal divisor on X_{k-1} and $\gamma_k \leq 0$. Then $b_k^* D + \gamma_k R_k$ is pre-left-orthogonal iff $T^+_{\gamma_k} \subset G^\circ_D$. If D is strongly pre-left-orthogonal, then $b_k^* D + \gamma_k R_k$ is strongly pre-left-orthogonal iff $T^+_{\gamma_k} \subset G^\circ_D$ and $T^-_{\gamma_k} \subset G_D$.*

Proof. By Riemann-Roch we get $\chi(\gamma_k R_k) = 1 - \binom{\gamma_k}{2}$ and $\chi(-\gamma_k R_k) = 1 - \binom{\gamma_k + 1}{2}$. Moreover, we get $\chi(b_k^* D + \gamma_k R_k) = \chi(D) + \chi(\gamma_k R_k) - 1 = \chi(D) - \binom{\gamma_k}{2}$ and $\chi(-b_k^* D - \gamma_k R_k) = \chi(-D) + \chi(-\gamma_k R_k) - 1 = \chi(-D) - \binom{\gamma_k+1}{2}$. So we see that $h^1(-b_k^* D - \gamma_k R_k) = 0$ iff $T^+_{\gamma_k}$ precisely cuts $\binom{\gamma_k+1}{2}$ lattice points out of G°_D and $h^1(b_k^* D + \gamma_k R_k) = 0$ iff $T^+_{\gamma_k}$ precisely cuts $\binom{\gamma_k}{2}$ lattice points out of G_D and the assertion follows. □

Consequently, we get:

Corollary 7.7: *Let $D = (D)_0 + \sum_i \gamma_i R_i$ pre-left-orthogonal. Then D is left-orthogonal iff $G_{(D)_0} \setminus G_D = \coprod_{k=1}^t T^-_{\gamma_k}$. Moreover, D is strongly left-orthogonal iff $G_{(D)_0} \setminus G_D = \coprod_{k=1}^t T^-_{\gamma_k}$ and $G^\circ_{(D)_0} = \coprod_{k=1}^t T^+_{\gamma_k}$.*

In terms of lattice figures in M, strong left-orthogonality can be understood by proposition 7.6 and corollary 7.7 as follows. We start with an almost lattice polytope associated to a strongly pre-left-orthogonal divisor $(D)_0$ on X_0 and successively cut out lattice points of $G_{(D)_0}$ and $G^\circ_{(D)_0}$ by moving in hyperplanes H_r until $G^\circ_{(D)_0 + \sum_k \gamma_k R_k}$ is empty and the sets $\{T^+_{\gamma_k} \mid k \in [t]\}$ and $\{T^-_{\gamma_k} \mid k \in [t]\}$ form a "tiling" of $G_{(D)_0} \setminus G_{(D)_0 + \sum_k \gamma_k R_k}$ and $G^\circ_{(D)_0}$, respectively. We illustrate this in the following example.

Example 7.8: Figure 3 shows on the left the fan of $\mathbb{F}_2$ from figure 1 blown up three times by successively adding the primitive vectors l_1, l_3, and l_2. Note that the numbering of the R_j does not match with the numbering of the l_i, but rather the order in which the l_i were added to the fan. The right side shows the hyperplane arrangements for five examples of divisors D all of which have $(D)_0 = 3Q - P$, with G_{3Q-P} and G°_{3Q-P} shown in figure 1. In a) the hyperplanes H_1, H_2, H_3 are indicated. The dark gray area indicates T_{γ_1}, the medium gray indicates T_{γ_2}, and the light gray T_{γ_3}. In a) we have $D = 3Q - P - 2R_1$; here $T^-_{\gamma_1}$ cuts out three elements of G_{3Q-P} and $T^+_{\gamma_1}$ cuts out one of G°_{3Q-P}. Therefore D is pre-left-orthogonal in this case. In b) we have $D = 3Q - P - 2R_1$ and $T^-_{\gamma_3}$ cuts out only one of G_{3Q-P} and $T^+_{\gamma_1}$ none of G°_{3Q-P}. Therefore D is not strongly pre-left-orthogonal. Note that R_1 and R_3 behave differently because l_{i_1} does form a basis of N with either of the two primitive vectors which belong to the fan of $\mathbb{F}_2$ and in whose positive span l_{i_1} is contained, whereas l_{i_3} does not. In the cases c), d), e), all $T^-_{\gamma_i}$ and $T^+_{\gamma_i}$ cut out the correct number of lattice points of G_{3Q-P} and G°_{3Q-P}, respectively, such that precisely the two elements in G°_{3Q-P} are cut out. So in all these cases D is strongly left-orthogonal.

We will also need to know how we can pass from the coordinates associated to a minimal model X_0 to the d_i-coordinates. For this we first illustrate the correspondence between divisors of the form $\alpha P + \beta Q + \sum_{i=1}^t \gamma_i R_i$ and polygonal lines of the form $\sum_{i \in [n]} d_i l_i = 0$ in the following example.

Example 7.9: It is convenient to interpret the relation $\sum_{i \in [n]} d_i l_i = 0$ as closed polygonal lines. If we successively place the vectors $d_i l_i$ end to end in $N_\mathbb{Q}$, we obtain a figure which can be viewed as a polygonal line complex embedded in the arrangement $\{H_i\}_{i \in [n]}$, rotated by 90 degrees. Figure 4 shows examples of divisors on the surface shown in figure 3. Note that the order in which the $d_i l_i$ are placed end to end is not canonical, but there are the two obvious choices (clockwise or counterclockwise) by which the line complex can be interpreted as being embedded in the corresponding hyperplane arrangement.

To change from coordinates associated to X_0 to d_i-coordinates, by linearity it suffices to consider $\big(d_1(D), \ldots, d_n(D)\big)$, where D is one of P, Q, H, R_i, $i \in [t]$. For the following lemma we assume that the fan of X_0 is generated by l_b, l_c, l_d, l_e if $X_0 \cong \mathbb{F}_a$ or by l_b, l_c, l_d if $X_0 \cong \mathbb{P}^2$. In the first case we assume that $l_b + l_d = a l_c$. With respect to R_k, we choose l_p, l_q, l_r as above. The following lemma is just an observation:

Lemma 7.10: *(i) If $X_0 \cong \mathbb{P}^2$, then $d_i(H) = 1$ if $i \in \{b, c, d\}$ and $d_i(H) = 0$ otherwise.*
(ii) If $X_0 \cong \mathbb{F}_a$, then $d_i(P) = 1$ if $i \in \{c, e\}$ and $d_i(P) = 0$ otherwise.
(iii) If $X_0 \cong \mathbb{F}_a$, then $d_c(Q) = a$, $d_i(Q) = 1$ if $i \in \{b, c\}$, and $d_i(Q) = 0$ otherwise.

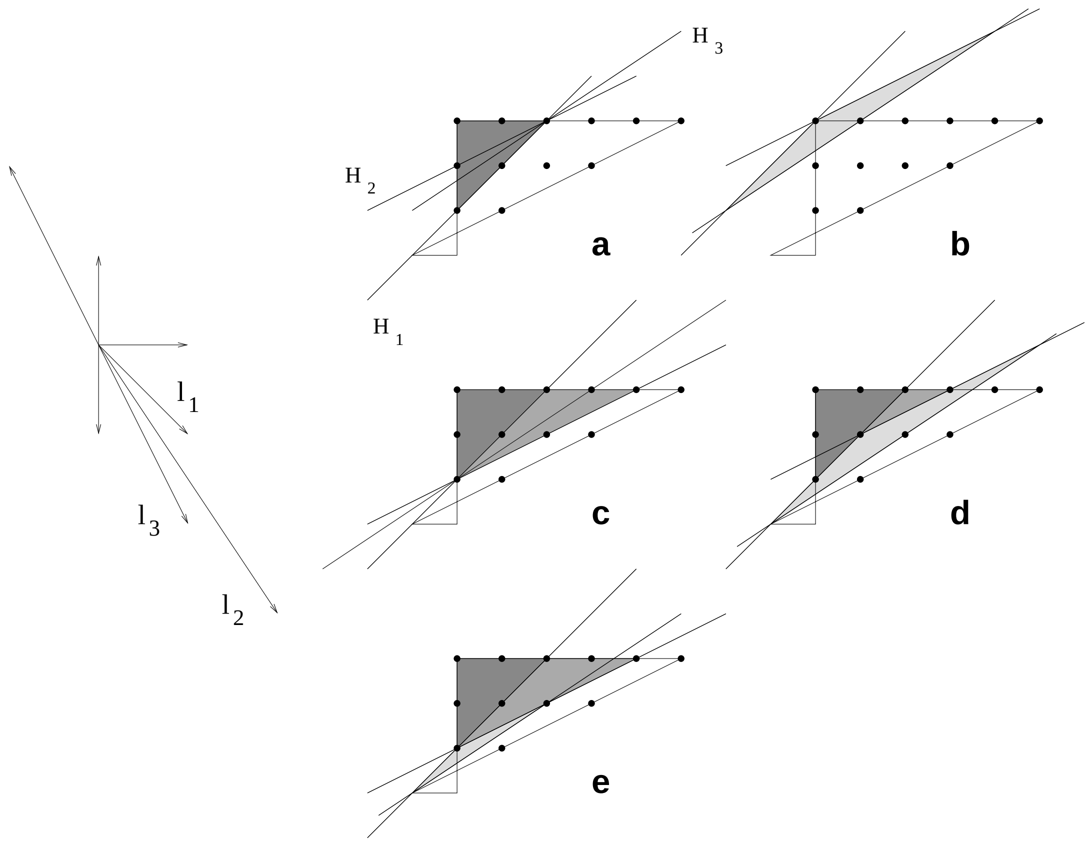

FIGURE 3. The fan of $\mathbb{F}_2$ blown up three times and the hyperplane arrangements corresponding to the divisors a) $3Q - P - 2R_1$, b) $3Q - P - 2R_3$, c) $3Q - P - 2R_1 - 2R_2$, d) $3Q - P - 2R_1 - R_2 - 2R_3$, e) $3Q - P - 2R_1 - 2R_2 - R_3$.

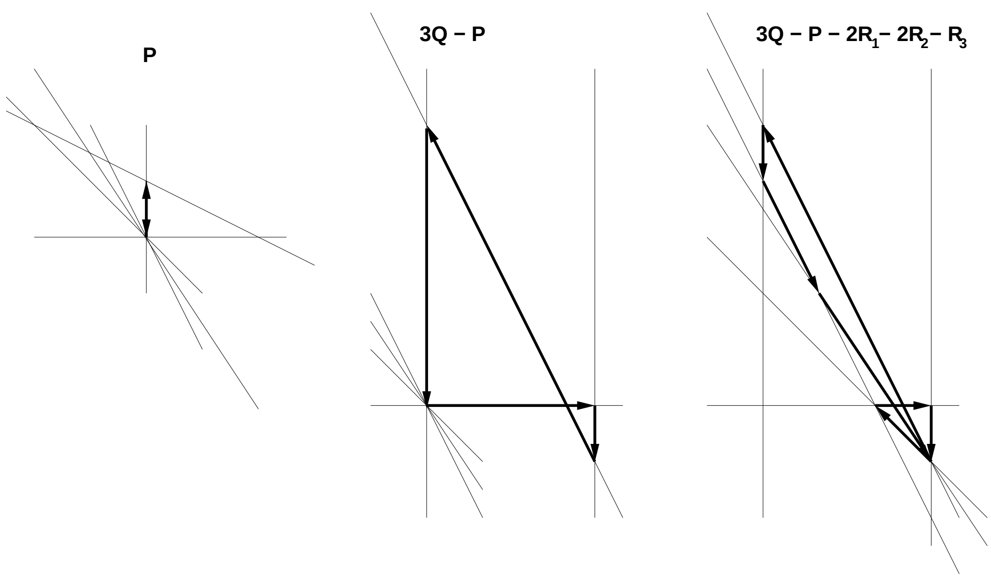

FIGURE 4. The fan of figure 3 and the polygonal lines associated to the divisors P, $3Q - P$, and $3Q - P - 2R_1 - 2R_2 - R_3$. The picture shows the hyperplane arrangements associated to these divisors rotated by 90 degrees and the polygonal lines embedded into them.

(iv) *Without assumptions on X_0 we have $d_p(R_k) = d_q(R_k) = 1$, $d_r(R_k) = -1$, and $d_i(R_k) = 0$ otherwise.*

If we compare figure 4 with figure 3, we see that in these examples, for strongly left-orthogonal D, the associated polygonal line contains G_D. More generally, we get:

Lemma 7.11: *Let $D = (d_1, \dots, d_n) \in N_1(X)$ be a T-invariant curve on a smooth complete toric surface X. If, as a divisor, D is numerically left-orthogonal, then $\chi(D) = \sum_i d_i$.*

Proof. Let $E = \sum_{i\in[n]} c_i' D_i$, then it follows from the discussion in section 2 that $D.E = \sum_{i\in[n]} d_i c_i'$. We apply this to $E = -K_X = \sum_{i\in[n]} D_i$ and use Lemma 3.3 (i). □

Of course, if D is strongly left-orthogonal, then it follows that $h^0(D) = \sum_i d_i$. If moreover, $(D)_0$ is strongly pre-left-orthogonal, it follows by induction, starting from the classification of propositions 4.7 and 4.8, that $G_D \subset \bigcup_{d_{i_k}\geq 0} H_k$, i.e. the positive d_i attribute to the global sections not only numerically, but the associated line segments bounding G_D actually contain G_D.

By Proposition 2.4 a divisor D is nef iff $d_i \geq 0$ for every i. Then the associated polygonal line complex is the boundary of a lattice polytope in $M_{\mathbb{Q}}$. The figures of example 7.9 show that these strongly left-orthogonal divisors are almost nef, as in every case $d_i \geq -1$ for every $i \in [n]$. This also holds in general:

Proposition 7.12: *Let D be a strongly left-orthogonal divisor on X. Then $\sum_{i\in I} d_i(D) \geq -1$ for every cyclic interval $I \subset [n]$.*

Proof. We choose some sequence of equivariant blow-downs to some minimal model X_0. Assume first that $(D)_0 = 0$. Then by Lemma 4.11 $D = R_k$ for some k or $D = R_k - R_l$ for $k \neq l \in [t]$ and R_k, R_l incomparable. For p, q, r as above, we have by Lemma 7.10 that $d_i(R_k) = -1$ for $i = r$, $d_i(R_k) = -1$ for $i \in \{p, q\}$ and $d_i(R_k) = 0$ else. So the assertion follows immediately for $D = R_k$. For $D = R_k - R_l$ we have just to take into account that the R_k and R_l are incomparable. If $(D)_0 \neq 0$ we can assume without loss of generality that $(D)_0$ is strongly pre-left-orthogonal. Otherwise, we have necessarily $h^0(D) = h^0\big((D)_0\big) = 0$ and $-(D)_0$ is strongly pre-left-orthogonal. Then if the statement is true for the case $-(D)_0$ strongly pre-left-orthogonal, we have $d_i \leq 1$ for every i and therefore by above discussion that $-d_i \geq -1$ for every i.

We show by induction on $(D)_k$, $k = 0, \dots, t$ that the assertion is true for a strongly pre-left-orthogonal divisor D. For $k = 0$, the assertion is true by inspection of the classification of strongly pre-left-orthogonal divisors on $\mathbb{P}^2$ (proposition 4.7) and $\mathbb{F}_a$ (proposition 4.8). It also follows that $d_j = |G_{(D)_0} \cap H_i|$ if l_i belongs to fan associated to X_0, i.e. the d_i count the lattice length plus one of the bounding faces of the polygonal line inscribing $G_{(D)_0}$. In the induction step we will show that this is still true for all triples p, q, r and all $k > 0$. For $k > 0$, let $(D)_k - (D)_{k-1} = \gamma_k R_k$. Consider the triple l_p, l_q, l_r as before, by Proposition 7.6 it is a necessary condition that H_p and H_q intersect in some $m \in G_{(D)_k} \setminus G^{\circ}_{(D)_k}$. Moreover, necessarily $d_p, d_q \geq -\gamma_k - 1$ and the result follows from above characterization of $d_i(R_k)$. □

Remark 7.13: If $a_i = -1$ for some i, then we can find a basis of $\mathrm{Pic}(X)$ with respect to some minimal model X_0 such that $R_t = D_i$. For any strongly pre-left-orthogonal divisor D it follows that $D = (D)_{t-1} + \gamma_t R_t$ for some $\gamma_t \leq 0$. Therefore, we have $d_i \geq 0$. If $a_i \geq 0$, the divisor D_i necessarily is the strict transform of some torus invariant divisor on X_0. So by the classification 4.7 and 4.8, the only cases with $a_i \geq 0$ and $d_i(D) = -1$ is where $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and D is the pullback of $P - Q$ or $Q - P$. Otherwise, if $a_i > 0$, then $d_i \geq 0$.

8. Strongly exceptional sequences of invertible sheaves on toric surfaces

The following results give a full classification of strongly exceptional sequences of invertible sheaves on smooth complete toric surfaces.

Theorem 8.1: *Let X be a smooth complete toric surface, then for every strongly exceptional toric system $\mathcal{A}$ there exists a sequence of blow-downs $X = X_t \to \cdots \to X_0$, where $X_0 = \mathbb{P}^2$ or $X_0 = \mathbb{F}_a$ for some $a \geq 0$ such that the normal form of $\mathcal{A}$ is a standard augmentation from X_0.*

As a corollary of Theorems 5.11 and 8.1 we thus obtain:

Theorem 8.2: *Let $X \neq \mathbb{P}^2$ be a smooth complete toric surface. Then there exists a full strongly exceptional sequence of invertible sheaves on X if and only if X can be obtained by equivariantly blowing up a Hirzebruch surface two times (in possibly several points in each step).*

We will prove Theorem 8.1 in the remaining sections. In this section we will state and prove some of its direct consequences.

Corollary 8.3: *Let X be a smooth complete toric surface. If there exists a strongly exceptional sequence of invertible sheaves on X, then* $\operatorname{rk}\operatorname{Pic}(X) \leq 14$.

Proof. A Hirzebruch surface $\mathbb{F}_a$ has four torus fixed points. So, after blowing up some of these points, the resulting toric surface has up to 8 fixed points. After blowing up these, we get a toric surface X whose fan is generated by at most 16 lattice vectors and thus $\operatorname{rk}\operatorname{Pic}(X) \leq 14$, and the statement follows from Theorem 8.2. □

Example 8.4: Consider the toric surface which is given by the sequence of self-intersection numbers $-2, -2, -1, -3, -2, 0, 1$. It is easy to see that there is no way to blow-down this surface to any Hirzebruch surface in only two steps. So by Theorem 8.2 there does not exist a strongly exceptional sequence of invertible sheaves on this surface. This is the counterexample which has been verified by explicit computations in [HP06]. Now consider the blow-up of this surface given by $-2, -2, -1, -3, -2, -1, -1, 0$. This surface can be blown-down to a $\mathbb{F}_1$ in two steps by simultaneously blowing down two divisors in each step. Therefore by Theorem 5.9 there exist strongly exceptional sequences of invertible sheaves on this surface. More concretely, if the $\mathbb{F}_1$ is spanned by lattice vectors l_1, l_2, l_3, l_6 with $l_3 = l_2 + l_6$, we subsequently add $l_7 = l_1 + l_6$, $l_8 = l_1 + l_7$, $l_4 = l_3 + l_6$ and $l_5 = l_4 + l_6$. Then, for example, we get a family of strongly exceptional toric systems by

$$R_1, R_3 - R_1, P - R_3, sP + Q, P - R_2, R_2 - R_4, R_4, -(s+1)P + Q - R_1 - R_2 - R_3 - R_4$$

for $s \geq -1$.

For a cyclic strongly exceptional toric system $\mathcal{A}$ on X the associated toric surface $Y(\mathcal{A})$ has a nef anti-canonical divisor. It turns out that this even is a necessary condition for X if X itself is a toric surface:

Theorem 8.5: *Let X be a smooth complete toric surface. If there exists a cyclic strongly exceptional sequence of invertible sheaves on X, then its anti-canonical divisor is nef.*

Proof. By Proposition 2.5 we have to show that $a_i \geq -2$ for every i. Assume that $\mathcal{A} = A_1, \ldots, A_n$ is a cyclic strongly exceptional toric system and assume that $a_i < -2$ for some i. We denote $d_i^j := d_i(A_j)$ for every $j \in [n]$. Then $\sum_{j \in [n]} d_i^j = a_i + 2 < 0$ by Proposition 2.5. Because $\mathcal{A}$ is cyclic and strong, every sum $\sum_{j \in I} A_j$ is strongly left-orthogonal for every proper cyclic interval $I \subset [n]$. In particular, $\sum_{j \in I} d_i^j \geq -1$ for every such I by Proposition 7.12. Now assume that there exists $j \in [n]$ such that $d_i^j = -1$. Without loss of generality, we can assume that $j = 1$. Then by choosing a decomposition $[n] \setminus \{1\} = I_1 \coprod I_2$, where I_1, I_2 are intervals, we can consider A_1, A_1', A_2', a short toric system of length 3 as in example 2.14. Then $d_i^1 + d_i(A_1') \geq -1$ and $d_i^1 + d_i(A_2') \geq -1$, hence $d_i(A_1') \geq 0$ and $d_i(A_2') \geq 0$, and we get $a_i \geq -3$. Now assume that $a_i = -3$. Then there exist at least two j such that $d_i^j = -1$; because otherwise, if there was only one j with $d_i^j = -1$, the condition that $\sum_{j=1}^n d_i^j = -1$ would imply that $d_i^k = 0$ for all $k \neq j$ and thus all the A_k with $k \neq j$ are contained in a hyperplane in $\operatorname{Pic}(X)$, which is not possible. Let j, k such that $d_i^k, d_i^j = -1$. Then $|k - j| > 1$, as $A_l + A_{l+1}$ must be strongly left-orthogonal for every $l \in [n]$. So we can consider a short toric system to periodicity 4: A_1', A_2', A_3', A_4' with $d_i(A_1') = d_i(A_3') = -1$. As $A_1' + A_2' + A_3'$ and $A_2' + A_3' + A_4'$ must be strongly left-orthogonal, this implies that $d_i(A_2'), d_i(A_4') \geq 1$ and so $a_i \geq -2$, a contradiction. □

The converse is also true in the toric case:

Theorem 8.6: *If X is a smooth complete toric surface with nef anti-canonical divisor, then there exists a full cyclic strongly exceptional sequence of invertible sheaves on X.*

Proof. The case of $\mathbb{P}^2$ is clear, and Hirzebruch surfaces are covered by 5.2. For the remaining two del Pezzo surfaces the existence follows from Theorem 5.14. For the other cases, we give in table 2 a list of examples, one for each surface. By construction, these toric systems are exceptional and to check

5b	$\underline{-1}, \underline{-2}, 0, 1, -1$	$H - R_1, R_1, H - R_1 - R_2, R_2, H - R_2$
6b	$\underline{-1}, \underline{-2}, -1, \underline{-1}, 1, -1$	$H - R_1 - R_3, R_1, H - R_1 - R_2, R_2, H - R_2 - R_3, R_3$
6c	$\underline{-1}, \underline{-2}, 0, 0, \underline{-1}, -2$	$H - R_1 - R_3, R_1, H - R_1 - R_2, R_2, H - R_2 - R_3, R_3$
6d	$\underline{-1}, \underline{-2}, -2, 0, 1, -2$	$P - R_1, R_1, Q - R_1 - R_2, R_2, P - R_2, Q - P$
7a	$\underline{-1}, -1, -1, \underline{-1}, \underline{-2}, -1, \underline{-2}$	$H - R_1 - R_2, R_2, R_1 - R_2, H - R_1 - R_3 - R_4,$ $R_4, R_3 - R_4, H - R_3$
7b	$\underline{-1}, \underline{-2}, 0, -1, \underline{-1}, \underline{-2}, -2$	$H - R_1 - R_3, R_3, R_1 - R_3, H - R_1 - R_2 - R_4,$ $R_4, R_2 - R_4, H - R_2$
8a	$\underline{-1}, -2, \underline{-1}, -2, \underline{-1}, -2, \underline{-1}, -2$	$P - R_1 - R_4, R_1, Q - R_1 - R_2, R_2, P - R_2 - R_3,$ $R_3, Q - R_3 - R_4, R_4$
8b	$\underline{-1}, -2, \underline{-1}, -1, \underline{-2}, \underline{-1}, -2, \underline{-2}$	$H - R_1 - R_2 - R_4, R_4, R_2 - R_4, R_1 - R_2, H - R_1 - R_3,$ $R_3 - R_5, R_5, H - R_3 - R_5$
8c	$\underline{-1}, \underline{-2}, -2, \underline{-2}, \underline{-1}, -2, 0, -2$	$P - R_1 - R_4, R_4, R_1 - R_4, P + Q - R_1 - R_3, R_3 - R_2,$ $R_2, P - R_2 - R_3, -P + Q$
9	$\underline{-1}, \underline{-2}, -2, \underline{-1}, \underline{-2}, -2, \underline{-1}, \underline{-2}, -2$	$H - R_1 - R_4 - R_5, R_4, R_1 - R_4, H - R_1 - R_3 - R_6,$ $R_6, R_3 - R_6, H - R_2 - R_3 - R_5, R_2, R_5 - R_2$

TABLE 2. Cyclic strongly exceptional toric systems on toric surfaces with nef anti-canonical divisor.

that these are indeed cyclic strongly exceptional is a direct application of Proposition 7.6 and Corollary 7.7. Note that for 8a and 8c we have given examples which are augmentations of cyclic strongly toric systems on $\mathbb{P}^1 \times \mathbb{P}^1$ and there is an ambiguity of assigning P and Q. For 8a, both cases are cyclic strongly exceptional. For 8c, we choose Q to be the class of the unique torus invariant prime divisor with self-intersection zero on 8c. □

9. STRAIGHTENING OF STRONGLY LEFT-ORTHOGONAL TORIC DIVISORS

In order to proof Theorem 8.1 we classify strongly left-orthogonal divisors on a given toric surface X. For this, we introduce in this section a procedure for simplifying a given strongly left-orthogonal divisor. We call this procedure a *straightening*. We will classify strongly left-orthogonal divisors up to straightening.

Lemma 9.1: *Let D be a T-invariant strongly left-orthogonal divisor on X and $i \in [n]$ such that $D_i^2 = -1$. If $d_i(D) < 0$, then either $h^0(D) = 0$ or $D = D_i$.*

Proof. We write $D = \gamma_t D_i + (D)_{t-1}$, where X_{t-1} is the blow-down of X along D_i. Then $d_i = -\gamma_t$ by Proposition 2.4 and Lemma 4.2. By Proposition 4.12 and Remark 4.13 this implies that $(D)_0$ is not pre-left-orthogonal with respect to the choice of any minimal model X_0 for X which factorizes through X_{t-1}. But then we either have $h^0(D) = 0$ or $(D)_0 = 0$ or both. If $(D)_0 = 0$, then by Proposition 4.11 we have $D = R_p - R_q$ for some $p, q \in [t]$ or $D = R_p$ for $p \in [t]$. In the first case, we also get $h^0(D) = 0$, in the second, we necessarily have $R_p = D_i$ by Lemma 7.10. □

So for any strongly left-orthogonal divisor D which is not a prime divisor D_j, we will assume without loss of generality that $d_i \geq 0$ for any $i \in [n]$ such that $D_i^2 = -1$. Otherwise, we will just take $-D$ instead of D. Let us write $D = \gamma_t D_i + (D)_{t-1}$ for $X \to X_{t-1}$ the blow-down of D_i. If $-1 \leq \gamma_t \leq 0$, then $T^+_{\gamma_t} = \emptyset$ and it follows from Lemma 4.6, Proposition 7.6, and Corollary 7.7 that $(D)_{t-1}$ is strongly left-orthogonal on X_{t-1}. By iterating, we obtain a sequence of blow-downs $X = X_t \to \cdots \to X_s$, where $s \geq 0$ and X_s lies over some (not necessarily completely specified yet) minimal model X_0. We can write $D = (D)_s + \sum_{i=s+1}^{t} \epsilon_i R_i$, where $\epsilon_i \in \{0, -1\}$ for every i and R_i is the total transform on X of the exceptional divisor of the blow-up $X_i \to X_{i-1}$. The divisor $(D)_s$ now has the property that either $(D)_s$ coincides with a prime divisor D_i on X_s with $D_i^2 = -1$ or $d_i\big((D)_s\big) \geq 2$ with respect to every T-invariant prime divisor D_i on X_s with $D_i^2 = -1$. It follows from Corollary 3.3 (iv) that $h^0(D) = h^0\big((D)_s\big) + s - t$.

Definition 9.2: Assume $(D)_s$ is constructed as above and does not coincide with a prime divisor D_i on X_s. Then we call $(D)_s$ a *straightening* of D. A divisor D is *straightened* if $D = (D)_s$ (and consequently $X = X_s$).

In the sequel we will keep the index 's' to denote that X_s has been chosen with respect to the straightening of some strongly left-orthogonal divisor. In general, $s \neq 0$ and a straightening $(D)_s$ is not unique. However, we will show that the existence of a straightened divisor imposes a strong condition on the geometry of X.

Proposition 9.3: *Let X be a smooth complete toric surface and D a straightened divisor on X. Then either $-K_X$ is nef or $X \cong \mathbb{F}_a$ with $a \geq 3$.*

To prove Proposition 9.3 we first show an auxiliary statement. Let $f \in [n]$ and denote $e_1, \ldots, e_r$, $g_1, \ldots, g_u \in [n]$ all indices i such that l_f and l_i form a basis of N, where the enumeration is as follows. Consider the line generated by l_f in $N_{\mathbb{Q}}$, Then all the e_i are contained in one half plane bounded by this line and all the g_j in the other. Moreover, we require that for any $i < j$, the vector l_{e_j} is contained in the cone generated by l_f and l_{e_i}, and l_{g_j} is contained in the cone generated by l_f and l_{g_i}, respectively. We denote $S \subset [n]$ all i such that l_i is contained in one of the cones σ_1, σ_2, where σ_1 is generated by l_{e_1} and l_f, and σ_2 is generated by l_{g_1} and l_f. Let $D = \sum_{i \in [n]} c_i D_i$ be a T-invariant divisor. We denote

$$\begin{aligned} Z_f &:= \{m \in M \mid l_i(m) = -c_f + 1\}, \\ Z'_f &:= \big\{m \in Z \mid l_i(m) > -c_i \text{ for all } i \in \{f, e_1, \ldots, e_j, g_1, \ldots, g_k\}\big\}, \\ Z''_f &:= \big\{m \in M \mid l_f(m) = 0 \text{ and } l_i(m) \geq 0 \text{ for } i \in \{e_1, \ldots, e_r, g_1, \ldots, g_u\}\big\}. \end{aligned}$$

Lemma 9.4: *If $a_f \leq -3$, then there exists $m \in Z_f$ such that $l_i(m) > -c_i$ for all $i \in S$.*

Proof. It follows from Proposition 2.3 that there exists a sequence of blow-downs $X = X_t \to \cdots \to X_p$ such that the cones generated by l_f, l_{e_1} and l_{g_1} do not contain any lattice vector which belongs to the fan associated to X_p. Correspondingly, we have injective maps $\phi : [r] \to [t]$, $\psi : [s] \to [t]$ such that R_{ϕ_i}, R_{ψ_j} are the total transforms the exceptional divisors associated to the primitive vectors l_{e_i} and l_{g_j}, respectively. Then for $i < j$, we have $R_{\phi_i} < R_{\phi_j}$ and $R_{\psi_i} < R_{\psi_j}$, respectively, and R_{ϕ_i}, R_{ψ_j} incomparable for all i, j. Note that we have the relations $0 = a_f l_f + l_{e_j} + l_{g_k}$, where $a_f = D_f^2$, and $0 = l_{e_1} + b l_f + l_{g_1}$ for some $b \geq a_f$. We write $D = (D)_p + \sum_{i=p+1}^{t} \gamma_i R_i$. Then the $T^+_{\gamma_{\phi_i}}$, $T^-_{\gamma_{\phi_i}}$ and $T^+_{\gamma_{\psi_i}}$, $T^-_{\gamma_{\psi_i}}$ have to fulfill the conditions of Proposition 7.6 and Lemma 7.7. In particular, we have $d_f = d_f(D) = c_{e_1} + b c_f + c_{g_1} + \sum_{i=1}^{j} \gamma_{\phi_i} + \sum_{i=1}^{k} \gamma_{\psi_i}$ with $d_f \geq -1$ by Proposition 7.12. Let $l_{e_r}(m) = -c_{e_r} + k_{e_r}$ and $l_{g_u}(m) = -c_{g_u} + k_{g_u}$ for some $m \in Z_f$ and $k_{e_r}, k_{g_u} \in \mathbb{Z}$. Then we have $k_{e_r} + k_{g_u} = c_{e_r} + a_f c_f + c_{k_u} - a_f = d_f - a_f$. The number of solutions such that $k_{e_r}, k_{g_u} > 0$ is given by $\max\{0, d_f - a_f - 1 \geq 1\}$, which is always nonzero for $a_f \leq -3$. We denote . We claim that if $a_f \leq -3$ then there exists $m \in Z'_f$ such that $l_i(m) > -c_i$ for all $i \in S$. Assume there exists $i \in S \setminus \{f, e_1, \ldots, e_r, g_1, \ldots, g_u\}$ such that $l_i(m) \leq -c_i$ for some $m \in Z'_f$. Without loss of generality, we assume that l_i is contained in σ_1. As l_i and l_f do not form a basis of N, then the fact that the hyperplane H_i cuts out lattice points in T' implies that H_i also cuts out at least the same number of lattice points m of Z''_f. But because $a_f \leq -3$, we have $|Z'_f| > |Z''_f|$ and the claim follows. □

Proof of Proposition 9.3. If there does not exist $f \in [n]$ such that $a_f < -2$, then $-K_X$ is nef by Proposition 2.5. So if there exists such an f we show that $X_s \cong \mathbb{F}_a$ for $a \geq 3$. With above notation there exists $m \in M$ such that $l_i(m) > -c_i$ for all $i \in S$ by Lemma 9.4. Assume first that there exists $u \in [n]$ such that $l_u = -l_f$. In this case there do not exist l_v which are contained in one of the cones generated by l_u and l_{e_1} or l_u and l_{g_1}, respectively, because any blow-up of one of these cones would require a lattice vector l_i which forms a basis of N together with l_u and therefore with l_f. This lattice vector then would be one of the l_{e_i} or l_{g_j}, which is excluded by assumption. But then the hyperplane H_u must pass through Z_f, as otherwise $h^2(-D) \neq 0$, and $(D)_0 = kP + Q$, where $k \geq -1$, with respect to the minimal model X_0 associated to the fan generated by l_{e_1}, l_{g_1}, l_f, l_u. But $G^\circ_{nP+Q} = \emptyset$ and thus $\gamma_i \in \{0, -1\}$ for all $p < i \leq t$ and in fact $\gamma_i = 0$, as D is straightened. This implies $X = X_0 \cong \mathbb{F}_{|b|}$, where $l_{e_1} + b l_f + l_{g_1} = 0$. Such an l_u necessarily exists in the following cases. If $a > 1$, then by the classification of toric surfaces l_f must belong to any minimal model for X which can be obtained by blowing down

X_p, and there necessarily exists $l_u = -l_f$. If $a = 1$, then l_{e_1} and l_{g_1} form a basis of N and the blow-up of the cone generated by these two just yields l_u. So either $X_0 = \mathbb{F}_1$ or $X_0 = \mathbb{P}^2$. If $a < -1$, then none of l_{e_1}, l_{g_1}, l_f can be blown-down and thus together with $-l_f$ must span the fan of a minimal model $\mathbb{F}_{|b|}$.

It remain to consider the cases $b \in \{0, -1\}$ and there is no $u \in [n]$ with $l_u = -l_f$. If $a = 0$, then $l_{e_1} = -l_{g_1}$ and l_{e_1}, l_{g_1}, l_f must be part of a fan of any minimal model X_0 which is a blow-down of X_p. Moreover, there exists l_{v_1} such that $l_f + bl_{v_1} + l_{e_1} = 0$, where without loss of generality $b > 0$ (and therefore $b > 1$), and all l_i in the fan associated to X_p for i different from e_1, g_1, f, v_1, are contained in the cone generated by l_{v_1} and l_{g_1}. Then we have $(D)_0 = kP + lQ$ with respect to the coordinates in $\mathrm{Pic}(X_0)$, where the fan of X_0 is generated by l_{e_1}, l_{g_1}, l_f, l_{v_1}. The divisor $(D)_0$ is strongly pre-left-orthogonal and for any $i \notin \{e_1, g_1, f, v_1\}$, the index of the subgroup of N generated by l_f and l_i is at least 3. Let $v_1, \ldots, v_w \subset [n]$ denote all elements such that l_{v_i} forms a basis of N together with l_{g_1} and denote $D = (D)_0 + \sum_{i=2}^{w} \gamma_{v_i} R_i +$ rest. Then $\sum_{i=2}^{w} \gamma_{v_i} \leq k + 1$ and because the index of the subgroup of N generated by l_f and one of the l_{v_i} with $i > 1$ is at least 3 and we have $\coprod_{i=2}^{w} T^+_{\gamma_{\eta_i}} \cap Z'_f = \emptyset$, where $\eta : \{2, \ldots, w\} \to [n]$ is the injective map which associates the R_i to the elements $v_2, \ldots, v_w$. Hence Z'_f must be empty and therefore $a_f \geq -2$.

In the last case, $a = -1$, for every $i \notin \{e_1, g_1, f\}$ with l_i part of the fan associated to X_p, by our assumptions the index of the subgroup of N generated by l_f and l_i is at least two and, similarly as in the previous case, we have $\coprod_{i \in K} T^+_{\gamma_{\eta_i}} \cap Z'_f = \emptyset$, where $K \subset [n]$ denotes those i such that l_i in the complement of σ_1 and σ_2. Hence we have $a_f \geq -2$. □

Using Corollary 7.7 and Proposition 9.3 it is a rather straightforward exercise to go through table 1 and to find all possible straightened divisors.

Proposition 9.5: *Table 3 shows a complete list of straightened divisors and their associated toric surfaces.*

$\mathbb{P}^2$	1 1 1	H, 2H
$\mathbb{P}^1 \times \mathbb{P}^1$	0 0 0 0	$P + sQ, Q + sP$, where $s \geq -1$
$\mathbb{F}_1$	0 -1 0 1	P, $Q + sP$, where $s \geq 1$
$\mathbb{F}_2$	0 -2 0 2	$P, 2Q - P, Q + sP$, where $s \geq -1$
$\mathbb{F}_a$, $a \geq 3$	0 -a 0 a	P, $Q + sP$, where $s \geq -1$
6d	$\underline{-1}$ $\underline{-2}$ $\underline{-2}$ 0 1 -2	$3H - 2R_1 - R_2 - R_3$
8a	$\underline{-1}$ -2 $\underline{-1}$ -2 $\underline{-1}$ $\underline{-2}$ -1 $\underline{-2}$	$4H - 2(R_1 + R_2 + R_3) - R_4 - R_5$
8c	$\underline{-1}$ $\underline{-2}$ $\underline{-2}$ -2 $\underline{-1}$ $\underline{-2}$ 0 -2	$4H - 2(R_1 + R_2 + R_4) - R_3 - R_5$
9	$\underline{-1}$ $\underline{-2}$ -2 $\underline{-1}$ $\underline{-2}$ -2 $\underline{-1}$ $\underline{-2}$ -2	$4H - 2(R_1 + R_3 + R_5) - R_2 - R_4 - R_6$

TABLE 3. Classification of straightened divisors. The first column of the table shows the name of the surface as given in table 1, the second column shows the self-intersection numbers of the toric divisors, and the third columns lists the straightened divisors on the surface. The underlined intersection numbers indicate which divisors are blown-down to obtain a minimal model and the numbering of the R_i is just the left-to-right order of the underlined divisors.

It turns out that there exist only four straightened divisors which are realized on toric surfaces different from $\mathbb{P}^2$ or $\mathbb{F}_a$. Their associated hyperplane arrangements and polygonal lines are shown in figure 5.

10. Proof of Theorem 8.1

Let $\mathcal{A} = A_1, \ldots, A_n$ be a strongly exceptional toric system on X. The first step for proving Theorem 8.1 is to consider the straightening of $A := \sum_{i=1}^{n-1} A_i$ and to find a preferred coordinate system for $\mathrm{Pic}(X)$ with respect to $\mathcal{A}$. The idea here is that by Proposition 9.5 there are only the few possibilities for X_s listed in table 3, which are already close to a minimal model X_0. It follows from Proposition 10.2 that every strongly exceptional sequence on X is an augmentation of a sequence on X_s. In the case where X_s is the projective plane or a Hirzebruch surface, we have $X_s = X_0$ and so by definition every augmentation of a strongly exceptional toric system on X_s is a standard augmentation. If If X_s is isomorphic to 6d, then the assertion of the theorem follows from Proposition 10.3. In remaining cases, i.e. X_s is one of 8a,

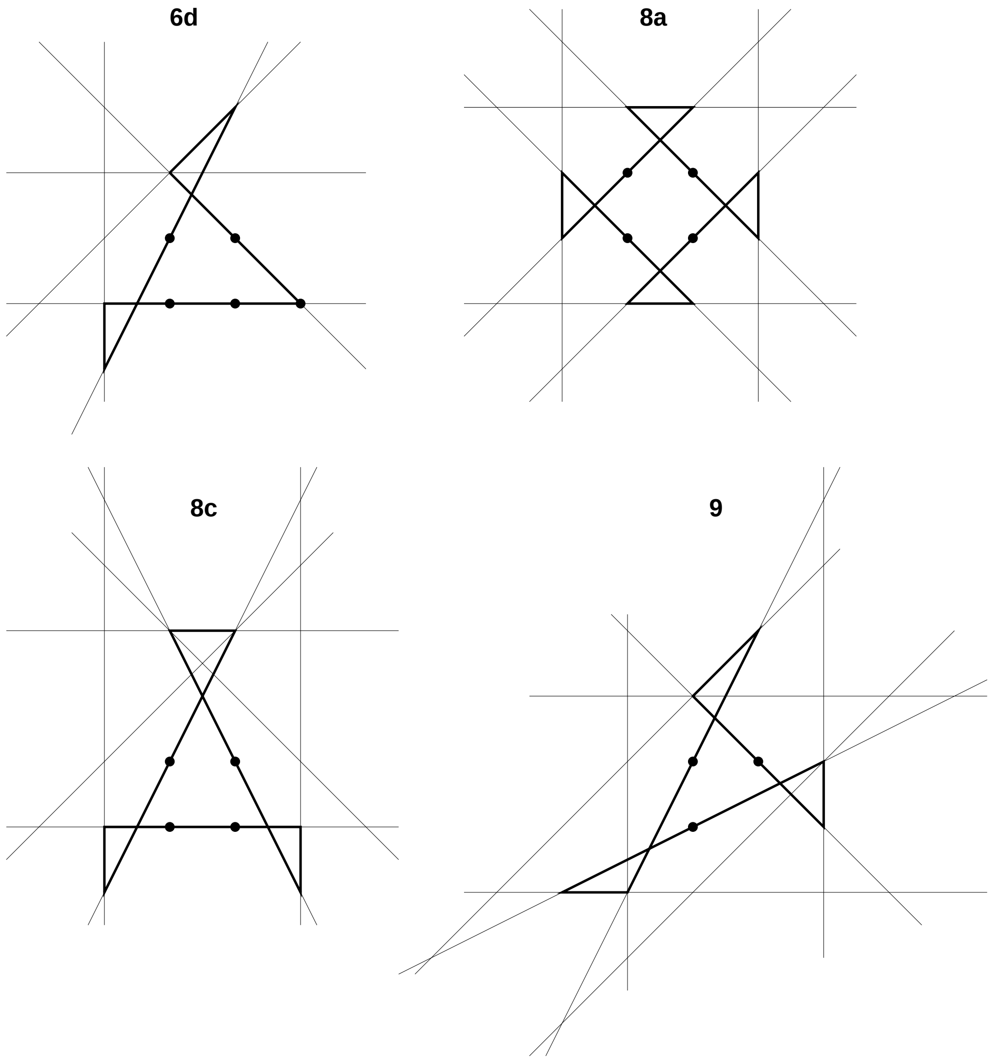

FIGURE 5. The hyperplane arrangements and the polygonal lines associated to the four straightened divisors which are not realized on $\mathbb{P}^2$ or a Hirzebruch surface. The dots indicate the global sections.

8c, 9, we show in Proposition 10.4 that $X = X_s$. These three cases are analyzed in Propositions 10.5, 10.6, and 10.7, which show that in every case $\mathcal{A}$ is a standard augmentation on X_s. This completes the proof of Theorem 8.1.

Moreover, we draw the following corollary from Propositions 10.5, 10.6, and 10.7:

Corollary 10.1: *If X_s is one of 8a, 8c, 9, then $X = X_s$ and $\mathcal{A}$ is cyclic.*

Now we prove the statements mentioned above.

Proposition 10.2: *Every strongly exceptional toric system has a normal form which is an augmentation of a strongly exceptional toric system on X_s.*

Proof. Let $\mathcal{A} = A_1, \ldots, A_n$ be a strongly exceptional toric system and $A := \sum_{i=1}^{n-1} A_i$ and $(A)_s$ the straightening of A. We assume that $X \neq X_s$ and denote $R_t, \ldots, R_{s-1}$ the total transforms of the exceptional divisors of the blow-ups $b_1, \ldots, b_{s-1}$ and complete these to a basis of $\operatorname{Pic}(X)$ with respect to some X_0 which is a blow-down of X_s. We may now assume that $\mathcal{A}$ is in normal form. The divisor R_t represents a torus invariant prime divisor of self-intersection -1 on X. Then $A = (A)_{t-1} + \gamma_t R_t$, where $\gamma_t \in \{0, -1\}$, and $A_n = (A_n)_{t-1} + \delta_t R_t$, where $\gamma_t + \delta_t = -1$. There must be at least two of the A_i

which are not contained in the hyperplane $R_t^\perp$, as otherwise the projection $(A_1)_{t-1}, \ldots, (A_n)_{t-1}$ would also satisfy properties (i) and (ii) of Definition 2.6. But it is clear from the proof of Proposition 2.7 that this is not possible.

So, as $\mathcal{A}$ is in normal form, there must be some A_i such that $A_i = (A_i)_{t-1} + R_t$ and $(A_i)_0 = 0$. Let $i \in I = [i_1, i_2] \subset [n-1]$ be the maximal interval such that $(A_j)_0 = 0$ for every $j \in I$. Then the sequence $\mathcal{A}_I = A_{i_1}, \ldots, A_{i_2}$ must be of one of the forms (i) or (ii) of Lemma 5.16. Moreover, there cannot be any other $j \in [n] \setminus I$ such that $(A_j)_0 = 0$ and $A_j = (A_j)_{t-1} + R_t$ as this would necessarily contradict property (ii) of Definition 2.6. If $\mathcal{A}_I$ is of the form of Lemma 5.16 (ii), we have two possibilities.

First, $A_i = R_t$, which implies $A_{i-1} = (A_{i-1})_{t-1} - R_t$ (respectively $A_n = (A_n)_{t-1} - R_t$ if $i = 1$) and $A_{i+1} = (A_{i+1})_{t-1} - R_t$ and $(A_j)_{t-1} = 0$ for every other $j \in [n]$. Therefore we can consider the projection $(A_1)_{t-1}, \ldots, (A_{i-1})_{t-1}, (A_{i+1})_{t-1}, \ldots, (A_n)_{t-1}$ which is a strongly exceptional toric system in $\operatorname{Pic}(X_{t-1})$.

Second, $A_i = R_t - R_k$ for some $k < t$ and thus $\chi(A_i) = 0$, then, as in proposition 5.18, we can reorder the toric system by replacing A_i by $-A_i$, A_{i-1} by $A_{i-1} + A_i$ and A_{i+1} by $A_{i+1} + A_i$, respectively, such that it remains strongly exceptional. In particular, we can reorder it such that A_j becomes R_t for some $j \in I$ and apply the same argument as before.

If $\mathcal{A}_I$ is of the form of Lemma 5.16 (i), we can consider the divisors A_{i_1-1} and A_{i_2+1}, where we identify $i_1 - 1$ with n if $i_1 = 1$. Note that $i_2 - i_1 < t$, so that $i_1 - 1 \neq i_2 + 1$. Now again by reordering, we can change $\mathcal{A}$ such that either $A_{i_1} = (A_{i_1})_{t-1} - R_t$ and $A_{i_1-1} = (A_{i_1-1})_{t-1} + R_t$, or $A_{i_2} = (A_{i_2})_{t-1} - R_t$ and $A_{i_2+1} = (A_{i_2+1})_{t-1} + R_t$. But then by our assumption on I and $\mathcal{A}$ being of normal form, one of $i_1 - 1$, $i_2 + 1$ must be equal to n. But above we have seen that $\delta_t \leq 0$, which is a contradiction, and $\mathcal{A}_I$ cannot be of the form of Lemma 5.16 (i).

Altogether we have seen now that $\mathcal{A}$ is an extension of a strongly exact toric system on X_{t-1} and the proposition follows by induction. □

Proposition 10.3: *Let X be a toric surface isomorphic to* 6d *and $\mathcal{A} = A_1, \ldots, A_6$ a strongly exceptional toric system on X such that $A = (A)_s = \sum_{i=1}^5 A_i = 3H - 2R_1 - R_2 - R_3$ in the coordinates indicated in table 3. Then $\mathcal{A}$ is the augmentation of a standard sequence on X_2.*

Proof. Clearly $A_6 = R_1$, so $A_5 = (A_5)_2 - R_1$ and $A_1 = (A_1)_2 - R_1$. If we consider the projection $(A_1)_2, \ldots, (A_5)_2$ and denote $A_I := \sum_{i \in I} A_i$ for every interval $I \subset [4]$, then $(A_I)_2 = A_I$ if $1 \notin I$ and $(A_I)_2 - R_1 = A_I$ if $1 \in I$ and thus A_I is strongly left-orthogonal for every such I and thus $(A_1)_2, \ldots, (A_5)_2$ is a strongly exceptional toric system on X_2 and $\mathcal{A}$ an augmentation. □

Denote $P_{(A)_s} := \{m \in M_\mathbb{Q} \mid l_i(m) \geq -c_i\}$ the rational polytope containing $G_{(A)_s}$.

Lemma 10.4: *(i) Let X be a toric surface and $\mathcal{A} = A_1, \ldots, A_n$ a strongly exceptional toric system on X such that $A = (A)_s$ and P_{A_s} has no corners in M. Then $\mathcal{A}$ cannot be augmented to a strongly exceptional sequence on any toric blow-up of X.*

(ii) In the cases where X_s is one of 8a, 8c, 9, the polytope $P_{(A)_s}$ has no corners.

Proof. Write $(A)_s = \sum_{i=1}^n c_i D_i$. From 7.6 it follows that for $(A)_s - R_{i_1}$ to be strongly left-orthogonal, there must exist a lattice point $m \in G_D$ and l_i, l_j such that $l_i(m) = -c_i$ and $l_j(m) = -c_j$, i.e. m is a corner of $P_{(A)_s}$, and moreover, l_{i_1} must be contained in the positive span of l_i and l_j. So it follows that $(A)_s$ cannot be a straightening of a divisor living on some blow-up of X of the form $(A)_s - R_{i_1} - \cdots - R_{i_k}$, where $i_1, \ldots, i_k > t$. Now consider $\mathcal{A}' = A'_1, \ldots, A_{n+k}$ a toric system which is an augmentation of $\mathcal{A}$. As $(A)_s = (A')_{s'}$, where $s' = s + k$, the augmentation process can only happen between A_{n-1} and A_n, or between A_n and A_1. But then there exists $n' > l > n - 1$ such that $\sum_{i=1}^l A'_i = A_s - R_{i_l}$ with $i_l > t$, which cannot be strongly left-orthogonal, which proves (i). For (ii) we refer to figure 5. □

We observe that the condition of lemma 10.4 are fulfilled for the remaining three cases.

Proposition 10.5: *Let X be a toric surface isomorphic to 8a and $\mathcal{A} = A_1, \ldots, A_8$ a strongly exceptional toric system on X such that $A = (A)_s = \sum_{i=1}^7 A_i = 4H - 2(R_1 + R_2 + R_3) - R_4 - R_5$ in the coordinates indicated in table 3. Then $\mathcal{A}$ is cyclic strongly exceptional and its normal form is an extension of the standard toric system on $\mathbb{P}^2$. Without bringing it into normal form, the toric system cannot be extended to a strongly exceptional toric system on any toric blow-up of X.*

Proof. The latter assertion follows by Lemma 10.4. To prove the first claim, we have to check that for any nonempty cyclic interval $\emptyset \neq I \subsetneq [8]$ the divisor $A_I := \sum_{i \in I} A_i$ is strongly left-orthogonal. By assumption, this is true for every I which does not contain n, and it thus remains to check the complementary intervals $[n] \setminus I$ for $n \notin I$. For $A_8 = -H + R_1 + R_2 + R_3$ we have $\chi(A_8) = 0$ and with $K_X^2 = 4$ it follows that $\chi(A_I) \leq 4$ for every $\emptyset \neq I \subsetneq [8]$ by Lemma 3.3 (iii). Using Proposition 7.6 and Corollary 7.7 together with formulas (3) and (4), it is a straightforward exercise to determine all strongly left-orthogonal divisors with Euler characteristic at most 4. These are shown in table 4. We see that

$\chi(D)$	D
0	$R_i - R_j$ with $\{i, j\} \neq \{1, 5\}, \{3, 4\}$, $\pm(H - R_i - R_j - R_k)$ with i, j, k pairwise distinct and $\{i, j, k\} \neq \{1, 2, 5\}, \{2, 3, 4\}$
1	R_i for $i \in \{1, 2, 3, 4, 5\}$, $H - R_i - R_j$ for $i \neq j$, $2H - R_1 - R_2 - R_3 - R_4 - R_5$,
2	$H - R_i$ for $i \in \{1, 2, 3, 4, 5\}$, $2H - \sum_{i \neq j} R_j$ for $i \in \{1, 2, 3, 4, 5\}$
3	H, $2H - R_i - R_j - R_k$ with i, j, k pairwise distinct, $3H - 2R_i - \sum_{j \neq i} R_j$ for any i
4	$2H - R_i - R_j$ for $i \neq j$ $3H - 2R_i - \sum_{j \neq i,k} R_j$ for $k \neq i$ and $(i, k) \neq (1, 5), (3, 4)$, $4H - 2(R_i + R_j + R_k) - R_l - R_m$ for i, j, k, l, m pairwise distinct, $5H - 3R_i - 2(R_j + R_k + R_l) - R_m$ for i, j, k, l, m pairwise distinct and $i \in \{1, 4, 5\}$

TABLE 4. Strongly left-orthogonal divisors with Euler characteristic ≤ 4 on the variety 8a

almost all elements in this table can be paired, i.e. if some D is in the table, then also $-K_X - D$ is. So, because $-K_X = \sum_{i=1}^{8} A_i$, it follows that if A_I is in the table, then $A_{[n] \setminus I}$ is and the proposition follows. The only exceptions which cannot be completed to a strongly left-orthogonal pair are $2H - R_3 - R_4$, $2H - R_1 - R_5$, $3H - R_2 - R_3 - R_4 - 2R_5$, $3H - R_1 - R_2 - 2R_4 - R_5$, $4H - 2(R_1 + R_2 + R_5) - R_3 - R_4$, $4H - 2(R_2 + R_3 + R_4) - R_1 - R_5$, and $5H - 3R_i - 2(R_j + R_k + R_l) - R_m$. We show that these cannot be of the form A_I for $I \subset [n-1]$.

The case $5H+$ rest can be excluded at once, as by assumption $\mathcal{A}$ is in normal form with respect to X_0, hence we always have $(A_I)_0 = \beta H$ with $\beta < 4$. With respect to $\mathcal{A}$ and $I = [k, l]$ with $1 \leq k < l < n$, we consider the following four divisors: C_1, A_I, C_2, A_8, where A_I as before and $A_8 = -H + R_1 + R_2 + R_3$ as before, and $C_1 := \sum_{j=1}^{k-1} A_j$, $C_2 := \sum_{j=l+1}^{n-1} A_j$, where $C_1 = 0$ if $k = 1$ and $C_2 = 0$ if $l = n - 1$. Because of the properties of toric systems, we have that $A_8.(C_1 + C_2) = A_I.(C_1 + C_2) \in \{0, 1, 2\}$, depending on the C_i being nonzero or not.

Now let us assume that $A_I = 2H - R_3 - R_4$. Then $C_1 + C_2 = -K_X - A_8 - A_I = 2H - R_1 - 2(R_2 + R_3) - R_5$ and $A_8.(C_1 + C_2) = 3$, which is not possible.

If $A_I = 3H - R_2 - R_3 - R_4 - 2R_5$, we get $C_1 + C_2 = H + R_5 - 2R_1 - R_2 - R_3$ and $(C_1 + C_2).A_8 = 3$. Therefore this case is also excluded.

If $A_I = 4H - 2(R_1 + R_2 + R_5) - R_3 - R_5)$, then $(C_1 + C_2) = R_5 - R_3$ and $A_8.(C_1 + C_2) = -1$, which is not possible.

The remaining three cases differ only by enumeration from the first three and can be excluded analogously. Altogether, under the conditions of the proposition, the strongly exceptional toric system $\mathcal{A}$ is always cyclic. If we bring it into normal form by inverting A_8, we get that $A' = 2H - R_4 - R_5$ and $(A')_s = 2H$. So by Proposition 10.2 and the subsequent remark, the toric system is an extension of the toric system H, H, H on $\mathbb{P}^2$. □

Proposition 10.6: *Let X be a toric surface isomorphic to 8c and $\mathcal{A} = A_1, \ldots, A_8$ a strongly exceptional toric system on X such that $A = (A)_s = \sum_{i=1}^{7} A_i = 4H - 2(R_1 + R_2 + R_4) - R_3 - R_5$ in the coordinates indicated in table 3. Then $\mathcal{A}$ is cyclic strongly exceptional and its normal form is an extension of the standard toric system on $\mathbb{P}^2$. Without bringing it into normal form, the toric system cannot be extended to a strongly exceptional toric system on any toric blow-up of X.*

Proof. In this case the arguments are completely analogous to the proof of proposition 10.5. The only difference is the classification of strongly left-orthogonal divisors with Euler characteristic at most four, which is shown in table 5. In table 6 we list the divisors D from table 5 which are candidates for some A_I

$\chi(D)$	D
0	$\pm(R_i - R_j)$ with $i \in \{1,2,3\}$, $j \in \{4,5\}$, $\pm(H - R_i - R_j - R_k)$ with $i \neq j \in \{1,2,3\}$, $k \in \{4,5\}$
1	R_i for any i, $H - R_i - R_j$ for $i \neq j$, $2H - R_1 - R_2 - R_3 - R_4 - R_5$,
2	$H - R_i$ for any i, $2H - \sum_{i \neq j} R_j$ for any i
3	H, $2H - R_i - R_j - R_k$ with i, j, k pairwise distinct, $3H - 2R_i - \sum_{j \neq i} R_j$ for any i
4	$2H - R_i - R_j$ for $i \neq j$ $3H - 2R_i - \sum_{j \neq i,k} R_j$ for $k \neq i$ and $(i,k) \neq (4,5), (2,3), (1,3), (1,2)$, $4H - 2(R_i + R_j + R_k) - R_l - R_m$ for i, j, k, l, m pairwise distinct, $i, j \in \{1,2,3\}$, $k \in \{4,5\}$, $5H - 3R_i - 2(R_j + R_k + R_l) - R_m$ for i, j, k, l, m pairwise distinct and $i \in \{4,5\}$

TABLE 5. Strongly left-orthogonal divisors with Euler characteristic ≤ 4 on the variety 8c.

and do not have a strongly left-orthogonal partner together with $C := A - D$, and the intersection numbers $C.D$, $C.A_8$. As we can see, we get in every case that the intersection numbers are not compatible with A_I coming of a toric system. So, under the conditions of the proposition, the strongly exceptional toric

D	C	$C.D$	$C.A_8$
$2H - R_4 - R_5$	$2H - 2(R_1 + R_2) - R_3 - R_4$	3	3
$3H - 2R_5 - R_1 - R_2 - R_3$	$H + R_5 - R_1 - R_2 - 2R_4$	3	0
$3H - 2R_3 - R_1 - R_4 - R_5$	$H + R_3 - R_1 - 2R_2 - R_4$	3	2
$3H - 2R_3 - R_2 - R_4 - R_5$	$H + R_3 - 2R_1 - R_2 - R_4$	3	2
$3H - 2R_2 - R_3 - R_4 - R_5$	$H - 2R_1 - R_4$	2	1

TABLE 6. Testing intersection numbers of some divisors of table 5.

system $\mathcal{A}$ is always cyclic. If we bring it into normal form by inverting A_8, we get that $A' = 2H - R_3 - R_5$ and $(A')_s = 2H$. So by Proposition 10.2 and the subsequent remark, the toric system is an extension of the toric system H, H, H on $\mathbb{P}^2$. □

Proposition 10.7: *Let X be a toric surface isomorphic to 9 and $\mathcal{A} = A_1, \ldots, A_9$ a strongly exceptional toric system on X such that $A = (A)_s = \sum_{i=1}^{7} A_i = 4H - 2(R_1 + R_3 + R_5) - R_2 - R_4 - R_6$ in the coordinates indicated in table 3. Then $\mathcal{A}$ is cyclic strongly exceptional and its normal form is an extension of the standard toric system on $\mathbb{P}^2$. Without bringing it into normal form, the toric system cannot be extended to a strongly exceptional toric system on any toric blow-up of X.*

Proof. The proof is analogous to propositions 10.5 and 10.6. Here, we have $\chi(A) = 3$, and table 7 shows the strongly left-orthogonal divisors with Euler characteristic ≤ 3. The unpaired divisor $5H - 2(R_1 + R_2 + R_3 + R_4 + R_5 + R_6)$ can be excluded as once, as $\mathcal{A}$ is in normal form. For the other cases, we make use of the $\mathbb{Z}_3$-symmetry of the table and consider only three cases, and the others follow the same way by exchanging indices.

Assume first $A_I = 3H - 2R_2 - R_3 - R_4 - R_5 - R_6$, then $C := A - A_I = H - 2R_1 - R_3 - R_5$. Then $C.A_9 = C.(-H + R_1 + R_3 + R_5) = 3$, which is not possible.

The next case is $A_I = (2H - R_1 - R_2 - R_5)$. Then $C = 2H - R_1 - 2R_3 - R_5 - R_6$ and $C.A_9 = -1$, which is not possible.

The last case is $A_I = 4H - 2(R_1 + R_2 + R_5) - R_3 - R_4 - R_6$. Then $C = R_2 - R_3$ and $C.A_9 = -1$, and this case also is excluded.

$\chi(D)$	D
0	$R_i - R_j$ with $\{i,j\} \neq \{1,2\},\{3,4\},\{5,6\}$, $\pm(H - R_i - R_j - R_k)$ with i,j,k pairwise distinct, $\{i,j,k\} \setminus \{1,2\} \neq \{5\},\{6\}$; $\{i,j,k\} \setminus \{3,4\} \neq \{1\},\{2\}$; $\{i,j,k\} \setminus \{5,6\} \neq \{3\},\{4\}$, $2H - R_1 - R_2 - R_3 - R_4 - R_5 - R_6$
1	R_i for any i, $H - R_i - R_j$ for $i \neq j$, $2H - \sum_{j\neq i} R_j$ for any i,
2	$H - R_i$ for any i, $2H - \sum_{k\neq i,j} R_k$ for any $i \neq j$, $3H - 2R_i - \sum_{j\neq i} R_j$ for any i
3	H, $2H - R_i - R_j - R_k$ with i,j,k pairwise distinct, $3H - 2R_i - \sum_{k\neq i,j} R_j$ for any $i \neq j$ and $j \neq i+1$ if i odd, $4H - 2(R_i + R_j + R_k) - R_l - R_m - R_n$ with i,j,k,l,m,n pairwise distinct, $\{i,j,k\} \setminus \{1,2\} \neq \{5\},\{6\}$; $\{i,j,k\} \setminus \{3,4\} \neq \{1\},\{2\}$; $\{i,j,k\} \setminus \{5,6\} \neq \{3\},\{4\}$, $5H - 2(R_1 + R_2 + R_3 + R_4 + R_5 + R_6)$

TABLE 7. Strongly left-orthogonal divisors with Euler characteristic ≤ 3 on the variety 9

Again, altogether we get that under the conditions of the proposition, the strongly exceptional toric system $\mathcal{A}$ is always cyclic. If we bring it into normal form by inverting A_9, we get that $A' = 2H - R_2 - R_4 - R_6$ and $(A')_s = 2H$. So by Proposition 10.2 and the subsequent remark, the toric system is an extension of the toric system H, H, H on $\mathbb{P}^2$. □

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