

Smooth orbifold K -Theory

Ulrich Bunke* Thomas Schick †

May 26, 2009

Abstract

We construct smooth equivariant K -theory of representable smooth orbifolds as a ring valued functor with the usual properties of a smooth extension of a cohomology theory. For proper submersions (with smooth fibres) we construct a push-forward map in smooth equivariant K -theory. Finally, we construct a non-degenerate intersection pairing for the subclass of smooth orbifolds which can be written as global quotients by a finite group action.

Contents

1	Introduction	2
2	Definition of smooth K-theory via cycles and relations	3
2.1	Equivariant forms and orbifold K -theory	3
2.2	Cycles	6
2.3	Relations	10
2.4	Smooth orbifold K -theory	16
2.5	Natural transformations and exact sequences	17
2.6	Calculations for $[*/G]$	20
3	Push-forward and \cup-product	21
3.1	Equivariant K -orientation	21
3.2	Definition of the Push-forward	24
3.3	Functoriality	29

*NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, GERMANY, ulrich.bunke@mathematik.uni-regensburg.de

†Mathematisches Institut, Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, GERMANY, schick@uni-math.gwdg.de; supported by Courant Research Center “Higher order structures in mathematics” via the German Initiative of Excellence

3.4	The cup product	31
3.5	Localization	33
4	The intersection pairing	35
4.1	The intersection pairing as an orbifold concept	35
4.2	The flat part and homotopy theory	39
4.3	Non-degeneracy of the intersection pairing	48
5	Examples	53
5.1	The smooth K -theory class of a mapping torus	53
5.2	Bordism	56
5.3	The intersection pairing for $[\mathbb{C}\mathbb{P}^1/(\mathbb{Z}/k\mathbb{Z})]$	57

1 Introduction

In this paper we give the construction of a model of smooth K -theory for orbifolds. It generalizes the model for smooth manifolds [BS07]. Major features are the cup-product, the push-forward, and the localization isomorphism.

Our construction includes a model of equivariant smooth K -theory for Lie group actions with finite stabilizers. The additional information coming with a construction in the realm of orbifolds is the independence from the choice of presentations. In equivariant terms, these are the induction and descend isomorphisms.

One of the motivations for the consideration of smooth K -theory came from mathematical physics, in particular from type-II superstring theory. Here it was used to capture the topological background of certain fields with differential form field strength, see e.g. [Wit98], [MM97]. For the corresponding theory on orbifolds one needs the corresponding generalization of smooth K -theory [SV07]. To serve this goal is one of the main motivation of this paper. As explained in [SV07], the intersection pairing in smooth K -theory on compact K -oriented orbifolds is an important aspect of the theory. In the present paper we construct a non-degenerated \mathbb{C}/\mathbb{Z} -valued pairing. It restricts to a non-degenerated \mathbb{R}/\mathbb{Z} -valued pairing on certain real subfunctors.

We now describe the contents of the paper. In Section 2 we construct the model of smooth K -theory and verify its basic properties. We first review the relevant orbifold notation. Then we define smooth K -theory for orbifolds by cycles and relations as a direct generalization of the smooth case [BS07].

Section 3 is devoted to the cup-product, the push-forward, and the localization theorem. These are again direct generalizations of the corresponding constructions in [BS07].

In Section 4 we prove two results. The first is Theorem 4.7 which identifies the flat part of smooth K -theory. It is a generalization of [BS07, Prop. 2.25]. Finally we show in

Theorem 4.13 that the intersection pairing is non-degenerate.

The final Section 5 contains some interesting explicit calculations.

Very recently the preprint [Ort] appeared. It gives another construction of smooth equivariant K -theory for finite group actions along the lines of [HS05]. It defines a push-forward to a point. The main difference between the two approaches is that our constructions are mainly analytical, whereas his are mainly homotopy theoretic.

Acknowledgement: A great part of the material of the present paper has been worked out around 2003. Motivated by [SV07] and fruitful personal discussion with R. Szabo and A. Valentino we transferred the theory to the case of orbifolds and worked out the details of the intersection pairing.

2 Definition of smooth K -theory via cycles and relations

2.1 Equivariant forms and orbifold K -theory

2.1.1 In the present paper we consider smooth K -theory for orbifolds. By definition an orbifold is a stack X in smooth manifolds which admits an orbifold atlas $A \rightarrow X$. An orbifold atlas is a representable smooth and surjective map from a smooth manifold such that the groupoid $A \times_X A \rightrightarrows A$ is étale. For the language of stacks we refer to [BSS07], [Hei05].

2.1.2 A major source of orbifolds are actions of discrete groups on smooth manifolds. Let G be a discrete group which acts on a smooth manifold M . The action $\mu : G \times M \rightarrow M$ is called proper, if the map $(\mu, \text{id}_M) : G \times M \rightarrow M \times M$ is proper. If the action is proper, then the quotient stack $[M/G]$ is an orbifold. The map $M \rightarrow [M/G]$ is an orbifold atlas. The associated groupoid is the action groupoid $G \times M \rightrightarrows M$.

Definition 2.1 *An orbifold of the form $[M/G]$ for a proper action of a discrete group on a smooth manifold is called good.*

Another source of examples arises from actions of compact Lie groups G on smooth manifolds M with finite stabilizers. In this case the quotient stack $[M/G]$ is a smooth stack with an atlas $M \rightarrow [M/G]$, but this atlas is not an orbifold atlas since the groupoid $G \times M \rightrightarrows M$ is not étale. In order to find an orbifold atlas we choose for every point $m \in M$ a transversal slice $T_m \subset M$ such that $G \times_{G_m} T_m \rightarrow M$ is a tubular neighbourhood of the orbit of m , where $G_m \subseteq G$ is the finite stabilizer of m . Then the composition $\sqcup_{m \in M} T_m \rightarrow M \rightarrow [M/G]$ is an orbifold atlas.

Definition 2.2 *An orbifold of the form $[M/G]$ for an action of a compact Lie group G with finite stabilizers on a smooth manifold M is called presentable.*

Note that, by definition, a presentation $[M/G]$ involves a *compact* group G .

Let X be an orbifold with orbifold atlas $A \rightarrow X$. It gives rise to the étale groupoid $\mathcal{A} : A \times_X A \rightrightarrows A$. To a manifold M we can associate the frame bundle $\mathbf{Fr}(M)$ in a functorial way. Therefore, the frame bundle $\mathbf{Fr}(A) \rightarrow A$ is \mathcal{A} -equivariant. The frame bundle of X is defined as the quotient stack $\mathbf{Fr}(X) := [\mathbf{Fr}(A)/\mathcal{A}]$. It does not depend on the choice of the atlas.

Definition 2.3 *An orbifold M is called effective if the total space of its frame bundle $\mathbf{Fr}(X) \rightarrow X$ is equivalent to a smooth manifold.*

It is known that an effective orbifold is presentable. On the other hand it is an open problem whether all orbifolds are presentable, see [HM04].

2.1.3 For an orbifold X let LX denote the inertia orbifold [BSS08]. In the case of a good orbifold of the form $[M/G]$ with a discrete group G , the inertia orbifold $L[M/G]$ is the quotient stack $[\hat{M}/G]$, where $\hat{M} := \sqcup_{g \in G} M^g$, $M^g \subseteq M$ is the smooth submanifold of fixed points of g , and the element $h \in G$ maps $M^g \rightarrow M^{hgh^{-1}}$ in the natural way.

2.1.4 For an orbifold X let Ω_X denote the sheaf of complexes of smooth complex differential forms. Its space of global sections will be denoted by $\Omega(X) := \Omega_X(X)$.

In particular we can consider the complex $\Omega(LX)$. Its cohomology is the delocalized orbifold de Rham cohomology $H_{dR,deloc}(X) := H(LX) := H^*(\Omega(LX))$ (see [BSS08]). In the case of a good orbifold we have

$$\Omega(L[M/G]) = \Omega(\hat{M})^G .$$

2.1.5 Let $E \rightarrow X$ be a complex vector bundle over an orbifold X . We choose a connection ∇^E . For self-containedness let us explain the notion of a connection using an orbifold atlas $A \rightarrow X$. We consider the associated étale groupoid $\mathcal{A} : A \times_X A \rightrightarrows A$. The vector bundle gives rise to an \mathcal{A} -equivariant vector bundle $E_A := E \times_X A \rightarrow A$, where the action is a fibrewise linear map

$$\begin{array}{ccc} (A \times_X A) \times_{\text{pr}_2, A} E_A & \xrightarrow{\quad} & E_A \\ & \searrow \text{pr}_1 & \swarrow \\ & & A \end{array} .$$

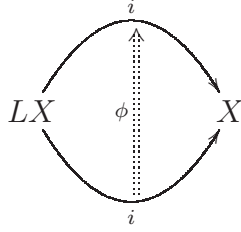
To choose a connection on $E \rightarrow X$ is equivalent to choose an \mathcal{A} -equivariant connection on E_A . For existence, first chose an arbitrary connection on E_A , and then average over \mathcal{A} .

2.1.6 Let $E_L \rightarrow LX$ be defined by the pull-back

$$\begin{array}{ccc} E_L & \longrightarrow & E \\ \downarrow & & \downarrow \\ LX & \xrightarrow{i} & X \end{array}$$

with the induced connection ∇^{E_L} .

Note that the map of stacks $i : LX \rightarrow X$ comes with a natural two-automorphism



induced by the two-automorphism in the pull-back square of the definition of the inertia stack LX

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & \nearrow \text{diag} & \downarrow \text{diag} \\ X & \longrightarrow & X \times X \end{array} .$$

It induces an automorphism of E_L

$$\begin{array}{ccc} E_L & \xrightarrow{\rho} & E_L \\ & \searrow & \swarrow \\ & LX & \end{array} .$$

Using the curvature $R^{\nabla^{E_L}} \in \Omega(LX, \text{End}(E_L))$ of the connection ∇^{E_L} we define the Chern form

$$\mathbf{ch}(\nabla^E) := \text{Tr} \rho e^{-\frac{1}{2\pi i} R^{\nabla^{E_L}}} \in \Omega(LX) .$$

This form is closed and represents the Chern character of E in delocalized cohomology $H_{dR, \text{deloc}}(X)$.

2.1.7 The inertia orbifold $i : LX \rightarrow X$ is a group-object in stacks over X , see [BSS08]. If $X = [M/G]$ so that $LX \cong [\bigsqcup_{g \in G} M^g/G]$, then the multiplication and inversion are given by $(x, g)(x, h) = (x, gh)$ for $x \in M^g \cap M^h$, and $I(x, g) = (x, g^{-1})$. If $\phi : i \Rightarrow i$ is the natural two-automorphism of i , then

$$\phi^{-1} = \phi \circ I \tag{1}$$

as 2-morphisms from $i \circ I = i$ to $i \circ I = i$. We define a real structure Q on $\Omega(LX)$ by $Q(\omega) := I^* \bar{\omega}$ and let $\Omega_{\mathbb{R}}(LX) \subseteq \Omega(LX)$ be the subcomplex of real forms for Q . It follows from (1) that

$$\begin{array}{ccc} E_L & \xrightarrow{\rho} & E_L \\ \cong \downarrow \text{can} & & \cong \downarrow \text{can} \\ I^* E_L & \xrightarrow{I^* \rho^{-1}} & I^* E_L \end{array}$$

commutes, too. Therefore we have $\mathbf{ch}(\nabla^E) \in \Omega_{\mathbb{R}}(LX)$.

2.1.8 Using the methods of [TXLG04] or [FHT07] one can define complex K -theory for local quotient stacks. Let us explain, for example, the idea of [FHT07]. It is based on the notion of a universal bundle of separable Hilbert spaces $H \rightarrow X$. Here universality is the property that for every other bundle of separable Hilbert spaces $H_1 \rightarrow X$ we have an isomorphism $H \oplus H_1 \cong H$. Let $\mathbf{Fred}(H) \rightarrow X$ be the associated bundle of Fredholm operators. Then one defines $K^{-*}(X)$ as the homotopy group $\pi_*(\Gamma(X, \mathbf{Fred}(H)))$ of the space of sections of $\mathbf{Fred}(H) \rightarrow X$. One can also directly define the group $K^{-1}(X)$ as the group $\pi_0(\mathbf{Fred}^*(H))$ of sections of selfadjoint Fredholm operators with infinite positive and negative spectrum.

For the present paper this set-up is too general since we want to do local index theory. In our case we want to represent K -theory classes by indices of families of Dirac operators, or in the optimal case, by vector bundles. For presentable orbifolds a construction of K -theory in terms of vector bundles has been given in [AR03]. Let X be an orbifold and consider a presentation $[M/G] \cong X$. Then the category of vector bundles over X is equivalent to the category of G -equivariant vector bundles over M . The Grothendieck group of the latter is $K_G^0(M)$, and we have $K^0(X) \cong K_G^0(M)$. The isomorphism $K(X) \cong K_G(M)$ can be taken as a definition since independence of the presentation follows e.g. from [PS07].

2.2 Cycles

2.2.1 In this paper we construct the smooth K -theory of compact presentable orbifolds. The restriction to compact orbifolds is due to the fact that we work with absolute K -groups. One could in fact modify the constructions in order to produce compactly supported smooth K -theory or relative smooth K -theory. But in the present paper, for simplicity, we will not discuss relative smooth cohomology theories.

We restrict our attention to presentable orbifolds since we want to use equivariant techniques. We do not know if our approach extends to general compact orbifolds, see 2.3.2.

2.2.2 We define the smooth K -theory $\hat{K}(B)$ as the group completion of a quotient of a semigroup of isomorphism classes of cycles by an equivalence relation. We start with the description of cycles.

Definition 2.4 *Let B be a compact presentable orbifold, possibly with boundary. A cycle for a smooth K -theory class over B is a pair (\mathcal{E}, ρ) , where \mathcal{E} is a geometric family, and $\rho \in \Omega(LB)/\mathbf{im}(d)$ is a class of differential forms.*

2.2.3 In the smooth case the notion of a geometric family has been introduced in [Bun] in order to have a short name for the data needed to define a Bismut super-connection [BGV04, Prop. 10.15]. In the present paper we need the straightforward generalization of this notion to orbifolds. Let B be an orbifold.

Definition 2.5 *A geometric family over B consists of the following data:*

1. a proper representable submersion with closed fibres $\pi : E \rightarrow B$,
2. a vertical Riemannian metric $g^{T^v\pi}$, i.e. a metric on the vertical bundle $T^v\pi \subset TE$, defined as $T^v\pi := \ker(d\pi : TE \rightarrow \pi^*TB)$.
3. a horizontal distribution $T^h\pi$, i.e. a bundle $T^h\pi \subseteq TE$ such that $T^h\pi \oplus T^v\pi = TE$.
4. a family of Dirac bundles $V \rightarrow E$,
5. an orientation of $T^v\pi$.

Here, a family of Dirac bundles consists of

1. a hermitean vector bundle with connection (V, ∇^V, h^V) on E ,
2. a Clifford multiplication $c : T^v\pi \otimes V \rightarrow V$,
3. on the components where $\dim(T^v\pi)$ has even dimension a $\mathbb{Z}/2\mathbb{Z}$ -grading z .

We require that the restrictions of the family Dirac bundles to the fibres $E_b := \pi^{-1}(b)$, $b \in B$, give Dirac bundles in the usual sense (see [Bun, Def. 3.1]):

1. The vertical metric induces the Riemannian structure on E_b ,
2. The Clifford multiplication turns $V|_{E_b}$ into a Clifford module (see [BGV04, Def.3.32]) which is graded if $\dim(E_b)$ is even.
3. The restriction of the connection ∇^V to E_b is a Clifford connection (see [BGV04, Def.3.39]).

The condition that the projection $E \rightarrow B$ is representable in particular implies that the fibres E_b are smooth manifolds. If $f : A \rightarrow B$ is an orbifold atlas, then we can form an étale groupoid $A \times_B A \rightrightarrows A$ which represents the stack B . The pull-back of the geometric family along f is a geometric family in the smooth setting which in addition carries an action of the groupoid. In the other direction, an equivariant geometric family over this groupoid determines uniquely a geometric family over the stack B .

Let $[M/G] \cong B$ be a presentation and \mathcal{E} be a geometric family over B . Then $M \times_B \mathcal{E} \rightarrow M$ is the underlying bundle of a G -equivariant geometric family $M \times_B \mathcal{E}$ over M . Vice versa, a G -equivariant geometric family \mathcal{F} over M induces a geometric family $\mathcal{E} := [\mathcal{F}/G]$ over B . If $F \rightarrow M$ is the underlying G -equivariant bundle, then the underlying bundle of \mathcal{E} is the map of quotient stacks $[F/G] \rightarrow [M/G] \cong B$.

A geometric family is called even or odd, if $T^v\pi$ is even-dimensional or odd-dimensional, respectively.

2.2.4 Let \mathcal{E} be an even geometric family over B . It gives rise to a bundle of graded separable Hilbert spaces $H_1 \rightarrow B$ with fibre $H_{1,b} \cong L^2(E_b, V_{|E_b})$. We furthermore have an associated family of Dirac operators which gives rise to a section $F_1 := D^+(D^2 + 1)^{-\frac{1}{2}} \in \mathbf{Fred}(H_1^+, H_1^-)$. Let $H \rightarrow B$ be the universal Hilbert space bundle as in 2.1.8. We choose isomorphisms $H_1^\pm \oplus H \cong H$. Extending F by the identity we get a section $F \in \mathbf{Fred}(H)$. Its homotopy class represents the index

$$\mathbf{index}(\mathcal{E}) \in K^0(B)$$

of the geometric family.

Alternatively we can use a presentation $[M/G] \cong B$. Then $M \times_B \mathcal{E}$ is a G -equivariant geometric family over M . The index of the associated equivariant family of Dirac operators $\mathbf{index}(M \times_B \mathcal{E}) \in K_G^0(M)$ represents $\mathbf{index}(\mathcal{E}) \in K^0(B)$ under the isomorphism $K^0(B) \cong K_G^0(M)$.

The index of an odd geometric family can be understood in a similar manner.

2.2.5 Here is a simple example of a geometric family with zero-dimensional fibres. Let $\pi : V \rightarrow B$ be a complex $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle. Note that the projection of a vector bundle π is by definition representable so that the fibres V_b for $b \in B$ are complex vector spaces.

Assume that V comes with a hermitean metric h^V and a hermitean connection ∇^V which are compatible with the $\mathbb{Z}/2\mathbb{Z}$ -grading. The geometric bundle (V, h^V, ∇^V) will usually be denoted by \mathbf{V} .

Using a presentation of B it is easy to construct a metric and a connection on a given vector bundle $V \rightarrow B$. Indeed, let $[M/G] \cong B$ be a presentation. Then $M \times_B V \rightarrow V$ is a G -equivariant vector bundle over M . We now can choose some metric or connection (by glueing local choices using a partition of unity). Then we can average these choices in order to get G -equivariant structures. These induce corresponding structures on the quotient $V \cong [M \times_B V/G]$.

Alternatively one could use an orbifold atlas $A \rightarrow B$ and choose a metric or connection on the bundle $A \times_B V \rightarrow V$. Again we can average these objects with respect to the action of the groupoid $A \times_B A \rightrightarrows A$ in order to get equivariant geometric structures. These induce corresponding structures on $V \rightarrow B$.

We consider the submersion $\pi := \mathbf{id}_B : B \rightarrow B$. In this case the vertical bundle is the zero-dimensional bundle which has a canonical vertical Riemannian metric $g^{T^v\pi} := 0$, and for the horizontal bundle we must take $T^h\pi := TB$. Furthermore, there is a canonical orientation of p . The geometric bundle \mathbf{V} can naturally be interpreted as a family of Dirac bundles on $B \rightarrow B$. In this way \mathbf{V} gives rise to a geometric family over B which we will usually denote by \mathcal{V} .

This construction shows that we can realize every class in $K^0(B)$ for a presentable B as the index of a geometric family. Indeed, let $x \in K^0(B)$. We choose a presentation

$B \cong [M/G]$ so that $K^0(B) \cong K_G^0(M)$. There exists a G -equivariant $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle $W \rightarrow M$ which represents the image of x in $K_G^0(M)$. Let $V := [W/G] \rightarrow B$ be the induced vector bundle over B and \mathcal{V} be the associated geometric family. Then we have $\text{index}(\mathcal{V}) = x$.

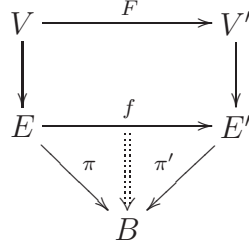
2.2.6 In order to define a representative of the negative of the smooth K -theory class represented by a cycle (\mathcal{E}, ρ) we introduce the notion of the opposite geometric family.

Definition 2.6 *The opposite \mathcal{E}^{op} of a geometric family \mathcal{E} is obtained by reversing the signs of the Clifford multiplication and the grading (in the even case) of the underlying family of Clifford bundles, and of the orientation of the vertical bundle.*

2.2.7 Our smooth K -theory groups will be $\mathbb{Z}/2\mathbb{Z}$ -graded. On the level of cycles the grading is reflected by the notions of even and odd cycles.

Definition 2.7 *A cycle (\mathcal{E}, ρ) is called even (or odd, resp.), if \mathcal{E} is even (or odd, resp.) and $\rho \in \Omega^{odd}(LB)/\text{im}(d)$ (or $\rho \in \Omega^{ev}(LB)/\text{im}(d)$, resp.).*

2.2.8 Let \mathcal{E} and \mathcal{E}' be two geometric families over B . An isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ consists of the following data:



where

1. f is a diffeomorphism over B ,
2. F is a bundle isomorphism over f ,
3. f preserves the horizontal distribution, the vertical metric and the orientation.
4. F preserves the connection, Clifford multiplication and the grading.

Compared with the smooth case the new ingredient is the two-isomorphism filling the triangle which is part of the data.

Definition 2.8 *Two cycles (\mathcal{E}, ρ) and (\mathcal{E}', ρ') are called isomorphic if \mathcal{E} and \mathcal{E}' are isomorphic and $\rho = \rho'$. We let $G^*(B)$ denote the set of isomorphism classes of cycles over B of parity $* \in \{ev, odd\}$.*

2.2.9 Given two geometric families \mathcal{E} and \mathcal{E}' we can form their sum $\mathcal{E} \sqcup_B \mathcal{E}'$ over B . The underlying proper submersion with closed fibres of the sum is $\pi \sqcup \pi' : E \sqcup E' \rightarrow B$. The remaining structures of $\mathcal{E} \sqcup_B \mathcal{E}'$ are induced in the obvious way.

Definition 2.9 *The sum of two cycles (\mathcal{E}, ρ) and (\mathcal{E}', ρ') is defined by*

$$(\mathcal{E}, \rho) + (\mathcal{E}', \rho') := (\mathcal{E} \sqcup_B \mathcal{E}', \rho + \rho') .$$

The sum of cycles induces on $G^*(B)$ the structure of a graded abelian semigroup. The identity element of $G^*(B)$ is the cycle $0 := (\emptyset, 0)$, where \emptyset is the empty geometric family.

2.3 Relations

2.3.1 In this subsection we introduce an equivalence relation \sim on $G^*(B)$. We show that it is compatible with the semigroup structure so that we get a semigroup $G^*(B)/\sim$. We then define the smooth K -theory $\hat{K}^*(B)$ as the group completion of this quotient.

In order to define \sim we first introduce a simpler relation "paired" which has a nice local index-theoretic meaning. The relation \sim will be the equivalence relation generated by "paired".

2.3.2 The main ingredients of our definition of "paired" are the notions of a taming of a geometric family \mathcal{E} introduced in [Bun, Def. 4.4], and the η -form of a tamed family [Bun, Def. 4.16].

In this paragraph we shortly review the notion of a taming and the construction of the eta forms. In the present paper we will use η -forms as a black box with a few important properties which we explicitly state at the appropriate places below.

If \mathcal{E} is a geometric family over B , then we can form a family of Hilbert spaces $H(\mathcal{E}) \rightarrow B$ with fibre $H_b := L^2(E_b, V|_{E_b})$. If \mathcal{E} is even, then this family is in addition $\mathbb{Z}/2\mathbb{Z}$ -graded.

A pre-taming of \mathcal{E} is smooth section $Q \in \Gamma(B, B(H(\mathcal{E})))$ such that $Q_b \in B(H_b)$ is self-adjoint given by a smooth integral kernel $Q \in C^\infty(E \times_B E, V \boxtimes V^*)$. In the even case we assume in addition that Q_b is odd, i.e. that it anticommutes with the grading z . The geometric family \mathcal{E} gives rise to a family of Dirac operators $D(\mathcal{E})$, where $D(\mathcal{E}_b)$ is an unbounded selfadjoint operator on H_b , which is odd in the even case.

The pre-taming is called a taming if $D(\mathcal{E}_b) + Q_b$ is invertible for all $b \in B$.

The family of Dirac operators $D(\mathcal{E})$ has a K -theoretic index which we denote by

$$\mathbf{index}(\mathcal{E}) \in K(B) .$$

Let $B \cong [M/G]$ be a presentation. The pull-back of the geometric family along $f : M \rightarrow B$ gives a G -equivariant family $f^*\mathcal{E}$ over M . The equivariant family of Dirac operators $D(f^*\mathcal{E})$ has an index $\mathbf{index}_G(f^*\mathcal{E}) \in K_G(M)$ which corresponds to $\mathbf{index}(\mathcal{E})$ under the isomorphism $K_G(M) \cong K(B)$.

If the geometric family \mathcal{E} admits a taming, then the associated family of Dirac operators admits an invertible compact perturbation, and hence $\text{index}(\mathcal{E}) = 0$. In the smooth case the converse is also true. If $\text{index}(\mathcal{E}) = 0$ and \mathcal{E} is not purely zero-dimensional, then \mathcal{E} admits a taming. The argument is as follows. The bundle of Hilbert spaces $H(\mathcal{E}) \rightarrow B$ is universal. If $\text{index}(\mathcal{E}) = 0$, then the section of unbounded Fredholm operators $D(\mathcal{E})$ admits an invertible compact perturbation $D(\mathcal{E}) + \tilde{Q}$. We can approximate \tilde{Q} in norm by pretamings. A sufficiently good approximation of \tilde{Q} by a pretaming is a taming. In the orbifold case the situation is more complicated. In general, the bundle $H(\mathcal{E}) \rightarrow B$ is not universal. Therefore we may have to stabilize. It is at this point that we use the assumption that the orbifold is presentable.

Lemma 2.10 *If $\text{index}(\mathcal{E}) = 0$, then there exists a geometric family \mathcal{G} (of the same parity of \mathcal{E}) such that $\mathcal{E} \sqcup_B \mathcal{G} \sqcup_B \mathcal{G}^{op}$ has a taming.*

Proof. We first consider the even case. Let $[M/G] \cong B$ be a presentation and $\mathcal{F} := M \times_B \mathcal{E}$ be the corresponding equivariant geometric family. Let H^+ be a universal G -Hilbert space, i.e. a G -Hilbert space isomorphic to $l^2 \otimes L^2(G)$. We consider the $\mathbb{Z}/2\mathbb{Z}$ -graded space $H := H^+ \oplus \Pi H^+$, where for a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space U the symbol ΠU denotes the same underlying vector space equipped with the opposite grading. The sum $H(\mathcal{F}) \oplus H$ is now universal. Since $\text{index}(\mathcal{E}) = 0$, the extension $D(\mathcal{F}) \oplus 1$ of $D(\mathcal{F})$ to $H(\mathcal{F}) \oplus H$ has an equivariant compact selfadjoint odd invertible perturbation $D(\mathcal{F}) \oplus 1 + \tilde{Q}$.

In the next step we cut down H to a finite-dimensional subspace. Let (P_n^+) be sequence of invariant projections on H^+ such that $P_n^+ \rightarrow \text{id}_{H^+}$ strongly. We set $P_n := P_n^+ \oplus P_n^+$ on $H = H^+ \oplus \Pi H^+$. Then for sufficiently large n the operator $(1 \oplus P_n)((D(\mathcal{F}) \oplus 1) + \tilde{Q})(1 \oplus P_n)$ is invertible on $\text{im}(1 \oplus P_n)$. Hence we have found a finite-dimensional G -representation $V := P_n H$ of the form $V = V^+ \oplus \Pi V^+$ such that the perturbation $D(\mathcal{F}) \oplus 1 + \hat{Q}$ of $D(\mathcal{F}) \oplus 0$ by the equivariant compact odd selfadjoint $\hat{Q} := 1 \oplus P_n + (1 \oplus P_n)\tilde{Q}(1 \oplus P_n)$ is invertible on $H(\mathcal{F}) \oplus V$. Finally we approximate \hat{Q} by a family Q represented by a smooth integral kernel.

Let \mathcal{V}^+ be an equivariant zero-dimensional geometric family based on the trivial bundle $M \times V \rightarrow M$. Then we set $\mathcal{G} := [\mathcal{V}^+/G]$. The operator Q constructed above provides the taming of $\mathcal{E} \sqcup_B \mathcal{G} \sqcup_B \mathcal{G}^{op}$.

In the odd-dimensional case we argue as follows. We again choose a presentation $[M/G] \cong B$ and form $\mathcal{F} := M \times_B \mathcal{E}$ as above. In this case we let $H := H^+$ be an ungraded universal G -Hilbert space.

Since $\text{index}(\mathcal{E}) = 0$ the extension $D(\mathcal{F}) \oplus 1$ of $D(\mathcal{F})$ to $H(\mathcal{E}) \oplus H$ admits an equivariant compact selfadjoint invertible perturbation $D(\mathcal{F}) \oplus 1 + \tilde{Q}$. We can again find a finite-dimensional projection P_n on H such that $(1 \oplus P_n)(D(\mathcal{F}) \oplus 1 + \tilde{Q})(1 \oplus P_n)$ is still invertible. We get the invertible operator $D(\mathcal{F}) \oplus 1 + \hat{Q}$ on $H(\mathcal{E}) \oplus V$ with $V := P_n H$ and $\hat{Q} := 1 \oplus P_n + (1 \oplus P_n)\tilde{Q}(1 \oplus P_n)$. We again approximate \hat{Q} by an operator Q with smooth

kernel.

We now choose an odd geometric family \mathcal{X} over a point such that $\dim \ker(D(\mathcal{X})) = 1$ and form the G -equivariant family $\mathcal{Y} := p^* \mathcal{X} \otimes V$, where $p : M \rightarrow *$. The kernel of $D(\mathcal{Y})$ is isomorphic to $M \times V$. Using this identification we can define Q on $H(\mathcal{F}) \oplus \ker(D(\mathcal{Y}))$. Its extension by zero on $H(\mathcal{F}) \oplus H(\mathcal{Y}) = H(\mathcal{F} \sqcup_M \mathcal{Y})$ is a taming of $\mathcal{F} \sqcup_M \mathcal{Y}$.

Let R be the projection onto $\ker(D(\mathcal{Y}))$. The operator $D(\mathcal{Y}) + R$ is invertible so that we can consider R as a taming of \mathcal{Y} . All together, $Q \oplus R$ defines a taming of $\mathcal{F} \sqcup_M \mathcal{Y} \sqcup_M \mathcal{Y}^{op}$. We now let $\mathcal{G} := [\mathcal{Y}/G]$ and get a taming of $\mathcal{E} \sqcup_B \mathcal{G} \sqcup_B \mathcal{G}^{op}$. \square

Definition 2.11 *A geometric family \mathcal{E} together with a taming will be denoted by \mathcal{E}_t and called a tamed geometric family.*

Let \mathcal{E}_t be a taming of the geometric family \mathcal{E} by the family $(Q_b)_{b \in B}$.

Definition 2.12 *The opposite tamed family \mathcal{E}_t^{op} is given by the taming $-Q \in \Gamma(B, B(H(\mathcal{E})))$ of \mathcal{E}^{op} .*

Note that the bundle of Hilbert spaces $H(\mathcal{E}) \rightarrow B$ and $H(\mathcal{E}^{op}) \rightarrow B$ associated to \mathcal{E} and \mathcal{E}^{op} are canonically isomorphic (up to reversing the grading in the even case) so that this formula makes sense.

2.3.3 The local index form $\Omega(\mathcal{E}) \in \Omega(LB)$ is a closed differential form canonically associated to a geometric family. It represents the Chern character of the index of \mathcal{E} . Let $A_t(\mathcal{E})$ denote the family of rescaled Bismut super-connections on $H(\mathcal{E}) \rightarrow B$. We define $H(\mathcal{E})_L \rightarrow LB$ as the pull-back

$$\begin{array}{ccc} H(\mathcal{E})_L & \longrightarrow & H(\mathcal{E}) \\ \downarrow & & \downarrow \\ LB & \longrightarrow & B \end{array} .$$

Let $A_t(\mathcal{E})_L$ denote the pull-back of the super-connection. As explained in 2.1.5 the bundle $H(\mathcal{E})_L$ comes with a canonical automorphism $\rho_{H(\mathcal{E})_L}$. For $t > 0$ the form

$$\Omega(\mathcal{E})_t := \varphi \text{Tr}_s \rho_{H(\mathcal{E})_L} e^{-A_t^2(\mathcal{E})_L} \in \Omega_{\mathbb{R}}(LB)$$

is real by the argument given in 2.1.7, closed, and represents $\mathbf{ch}(\text{index}(\mathcal{E})) \in H_{dR, \text{deloc}}(B)$. Here φ is a normalization operator. It acts on $\Omega(LG)$ and is defined by

$$\varphi := \begin{cases} \left(\frac{1}{2\pi i}\right)^{\text{deg}/2} & \text{even case} \\ \frac{-1}{\sqrt{\pi}} \left(\frac{1}{2\pi i}\right)^{\frac{\text{deg}-1}{2}} & \text{odd case} \end{cases} .$$

The methods of local index theory show that $\Omega(\mathcal{E})_t$ has a limit as $t \rightarrow \infty$.

Definition 2.13 We define the local index form $\Omega(\mathcal{E}) \in \Omega_{\mathbb{R}}(LB)$ of the geometric family \mathcal{E} over B as the limit

$$\Omega(\mathcal{E}) := \lim_{t \rightarrow 0} \Omega(\mathcal{E})_t .$$

It is clear from the construction that

Theorem 2.14

$$\mathbf{ch}_{dR}(\mathbf{index}(\mathcal{E})) = [\Omega(\mathcal{E})] \in H_{dR,deloc}(B) .$$

In the following we give a differential geometric description of $\Omega(\mathcal{E})$. The automorphism $\rho_{H(\mathcal{E})_L}$ comes from the canonical automorphism $\rho_{\mathcal{E}}$ of the pull-back $\mathcal{E}_L := LB \times_B \mathcal{E}$. The usual finite-progagation speed estimates show that as t tends to zero the super-trace $\mathrm{Tr}_s \rho_{H(\mathcal{E})_L} e^{-A_t^2(\mathcal{E})_L}$ localizes at the fixed points of $\rho_{\mathcal{E}}$.

Let $\pi : E \rightarrow B$ be the underlying fibre bundle of \mathcal{E} , and let $V \rightarrow E$ be the Dirac bundle. If we apply the loops functor to the projection π we get a diagram

$$\begin{array}{ccc} LE & \longrightarrow & E \\ \downarrow L\pi & & \downarrow \pi \\ LB & \longrightarrow & B \end{array} .$$

The fibre bundle $LE \rightarrow LB$ is exactly the bundle of fixed points of $\rho_{\mathcal{E}}$. Therefore the local index form is given as an integral

$$\Omega(\mathcal{E}) = \int_{LE/LB} I(\mathcal{E})$$

for some $I(\mathcal{E}) \in \Omega_{\mathbb{R}}(LE)$. Let \mathcal{U} be a tubular neighbourhood of the map $i : LE \rightarrow E$. We let $V_L := LE \times_E V \rightarrow LE$ be the pull-back of $V \rightarrow E$. Similarly, we let $T^v \pi_L \rightarrow LE$ be the pull-back of the vertical bundle $T^v \pi \rightarrow E$. Both bundles come with canonical automorphisms (see 2.1.5)

$$\rho_{T^v \pi_L} : T^v \pi_L \rightarrow T^v \pi_L , \quad \rho_{V_L} : V_L \rightarrow V_L .$$

The automorphism $\rho_{T^v \pi_L}$ preserves the orthogonal decomposition

$$T^v \pi_L \cong T^v L\pi \oplus N ,$$

where $T^v L\pi = \ker(1 - \rho_{T^v \pi_L})$. We let ρ^N denote the restriction of $\rho_{T^v \pi_L}$ to the normal bundle.

Then we have

$$\lim_{t \rightarrow 0} \mathrm{Tr}_s \rho_{H(\mathcal{E})_L} e^{-A_t^2(\mathcal{E})_L} = \lim_{t \rightarrow 0} \int_{LE/LB} \int_{\mathcal{U}/LE} \mathrm{tr}_s \rho_{V_L} K_{e^{-A_t^2(\mathcal{E})}}((x, \rho^N n), (x, n)) ,$$

where tr_s the the local super-trace of the integral kernel $K_{e^{-A_t^2(\mathcal{E})}}((x, n), (x', n'))$ of $e^{-A_t^2(\mathcal{E})_L}$, $x \in LE$, and $n \in \mathcal{U}_x$. The form $I(\mathcal{E})$ is thus given by

$$I(\mathcal{E})(x) = \lim_{t \rightarrow 0} \int_{\mathcal{U}/LE} \mathrm{tr}_s \rho_{V_L} K_{e^{-A_t^2(\mathcal{E})}}((x, \rho^N n), (x, n)) .$$

In the following we describe this form $I(\mathcal{E})$ in local geometric terms.

The vertical metric $T^v\pi$ and the horizontal distribution $T^h\pi$ together induce a connection $\nabla^{T^v\pi}$ on $T^v\pi$ (see 3.1.2 for more details). Locally on E we can assume that $T^v\pi$ has a spin structure. We let $S(T^v\pi)$ be the associated spinor bundle. Then we can write the family of Dirac bundles V as $V = S(T^v\pi) \otimes \mathbf{W}$ for a twisting bundle $\mathbf{W} = (W, h^W, \nabla^W, z^W)$, where $W := \mathrm{Hom}_{\mathrm{Cliff}(T^v\pi)}(S(T^v\pi), V)$ with induced geometric structures. Since ρ_{V_L} commutes with the action of $\mathrm{Cliff}(T^v\pi_L)$ this automorphism has values in

$$\mathrm{Cliff}(N) \otimes \mathrm{End}(W)_L \subseteq \mathrm{Cliff}(T^v\pi_L) \otimes \mathrm{End}(W)_L \cong \mathrm{End}(V_L) .$$

Therefore we can choose a decomposition $\rho_{V_L} = \rho_{V_L}^N \otimes \rho^W$ (or better as a sum of such terms, but we will omit this additional summation for simplification). We use ρ^W in order to define

$$\mathbf{ch}_{\rho^W}(W) := \mathrm{Tr}_s \rho^W e^{-\frac{1}{2\pi i} R^W} .$$

Let $\sigma : \mathrm{Cliff}(N) \rightarrow \Lambda^* N$ be the symbol map and $T^N : \Lambda^* N \rightarrow \mathbb{R}$ be the Beresin integral with respect to the canonical orientation of N . With these building blocks we can write out the form I as

$$I(\mathcal{E}) = c_{\dim(E/B)} \left(\frac{1}{4\pi} \right)^{\dim(T^v L\pi)/2} \frac{T^N(\sigma(\rho_{V_L}^N)) \hat{\mathbf{A}}(T^v L\pi) \mathbf{ch}_{\rho^W}(\mathbf{W})}{\det^{1/2}(1 - \rho^{N,-1}) \det^{1/2}(1 - \rho^N \exp(-R^N/2))} , \quad (2)$$

where

$$c_n := \begin{cases} (-2i)^{n/2} & n \text{ is even} \\ (-2i)^{\frac{n+1}{2}} & n \text{ is odd} \end{cases}$$

This form is independent of choices. The explicit form of the local index density will not be needed in rest of the present paper.

2.3.4 Let \mathcal{E}_t be a tamed geometric family (see Definition 2.11) over B . The taming is used to modify the Bismut super-connection $A_\tau(\mathcal{E})$ for $\tau > 1$ in order to make the zero form degree part invertible. For $\tau \geq 2$ we have $A_\tau(\mathcal{E}_t) = A_\tau(\mathcal{E}) + \tau Q$, for $\tau \in (0, 1)$ we have $A_\tau(\mathcal{E}_t) = A_\tau(\mathcal{E})$, and on the interval $\tau \in (1, 2)$ we interpolate smoothly between these two families. The taming has the effect that the integral kernel of $e^{-A_\tau(\mathcal{E}_t)^2}$ vanishes exponentially for $\tau \rightarrow \infty$ in the C^∞ -sense. The η -form $\eta(\mathcal{E}_t) \in \Omega_{\mathbb{R}}(LB)$ is defined by

$$\eta(\mathcal{E}_t) := \tilde{\varphi} \int_0^\infty \mathrm{Tr}_s \rho_{\mathcal{E}} \partial_\tau A_\tau(\mathcal{E}_t)_L e^{-A_\tau(\mathcal{E}_t)_L^2} d\tau , \quad (3)$$

where $\tilde{\varphi}$ again acts on $\Omega(LB)$ and is defined by

$$\tilde{\varphi} = \begin{cases} (2\pi i)^{-\frac{\text{deg}+1}{2}} & \text{even case} \\ \frac{-1}{\sqrt{\pi}}(2\pi i)^{-\text{deg}/2} & \text{odd case} \end{cases}$$

Note the even and odd refers to the dimension of the fibre. The corresponding η -form has the opposite parity.

Convergence at $\tau \rightarrow \infty$ is due to the taming. The convergence at $\tau \rightarrow 0$ follows from the standard equivariant local index theory for the Bismut super-connection. The same methods imply

$$d\eta(\mathcal{E}_t) = \Omega(\mathcal{E}) . \quad (4)$$

2.3.5 Now we can introduce the relations "paired" and \sim .

Definition 2.15 *We call two cycles (\mathcal{E}, ρ) and (\mathcal{E}', ρ') paired if there exists a taming $(\mathcal{E} \sqcup_B \mathcal{E}'^{op})_t$ such that*

$$\rho - \rho' = \eta((\mathcal{E} \sqcup_B \mathcal{E}'^{op})_t) .$$

We let \sim denote the equivalence relation generated by the relation "paired".

Lemma 2.16 *The relation "paired" is symmetric and reflexive.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

Lemma 2.17 *The relations "paired" and \sim are compatible with the semigroup structure on $G^*(B)$.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

We get an induced semigroup structure on $G^*(B)/\sim$.

Lemma 2.18 *If $(\mathcal{E}_0, \rho_0) \sim (\mathcal{E}_1, \rho_1)$, then there exists a cycle (\mathcal{E}', ρ') such that $(\mathcal{E}_0, \rho_0) + (\mathcal{E}', \rho')$ is paired with $(\mathcal{E}_1, \rho_1) + (\mathcal{E}', \rho')$.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

2.4 Smooth orbifold K -theory

2.4.1 In this Subsection we define the assignment $B \rightarrow \hat{K}(B)$ from compact presentable orbifolds to $\mathbb{Z}/2\mathbb{Z}$ -graded abelian groups. Recall Definition 2.9 of the semigroup of isomorphism classes of cycles. By Lemma 2.17 we can form the semigroup $G^*(B)/\sim$.

Definition 2.19 *We define the smooth K -theory $\hat{K}^*(B)$ of B to be the group completion of the abelian semigroup $G^*(B)/\sim$.*

If (\mathcal{E}, ρ) is a cycle, then let $[\mathcal{E}, \rho] \in \hat{K}^*(B)$ denote the corresponding class in smooth K -theory.

We now collect some simple facts which are helpful for computations in $\hat{K}(B)$ on the level of cycles.

Lemma 2.20 *We have $[\mathcal{E}, \rho] + [\mathcal{E}^{op}, -\rho] = 0$.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

Lemma 2.21 *Every element of $\hat{K}^*(B)$ can be represented in the form $[\mathcal{E}, \rho]$.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

Lemma 2.22 *If $[\mathcal{E}_0, \rho_0] = [\mathcal{E}_1, \rho_1]$, then there exists a cycle (\mathcal{E}', ρ') such that $(\mathcal{E}_0, \rho_0) + (\mathcal{E}', \rho')$ is paired with $(\mathcal{E}_1, \rho_1) + (\mathcal{E}', \rho')$.*

Proof. We can copy the argument of the corresponding Lemma in [BS07]. □

2.4.2 In this paragraph we extend $B \mapsto \hat{K}^*(B)$ to a contravariant functor from compact orbifolds to $\mathbb{Z}/2\mathbb{Z}$ -graded groups. Let $f: B_1 \rightarrow B_2$ be a morphism of orbifolds. Then we define

$$f^* : \hat{K}^*(B_2) \rightarrow \hat{K}^*(B_1)$$

by

$$f^*[\mathcal{E}, \rho] := [f^*\mathcal{E}, Lf^*\rho],$$

where $f^*\mathcal{E} = B_1 \times_{B_2} \mathcal{E}$ and $Lf: LB_1 \rightarrow LB_2$ is obtained from f by an application of the loops functor. For the details of the construction of the pull-back of geometric families

we refer to [BS07]. It is easy to check that the construction is well-defined and additive. At this point we use in particular the relation

$$\eta(f^*\mathcal{E}_t) = f^*\eta(\mathcal{E}_t) . \quad (5)$$

If $g : B_0 \rightarrow B_1$ is the second morphisms of compact presentable orbifolds, then we have the relation

$$f^* \circ g^* = (f \circ g)^* : \hat{K}(B_2) \rightarrow \hat{K}(B_0) .$$

2.5 Natural transformations and exact sequences

2.5.1 In this subsection we introduce the transformations R, I, a , and we show that they turn the functor \hat{K} into a smooth extension of $(K, \mathbf{ch}_{\mathbb{C}})$ in the sense of the natural generalization of the definition [BS07, Def. 1.1] to the orbifold case.

2.5.2 We first define the natural transformation

$$I : \hat{K}(B) \rightarrow K(B)$$

by

$$I[\mathcal{E}, \rho] := \mathbf{index}(\mathcal{E}) .$$

The proof that this is well-defined can be copied from [BS07]. The relation $\mathbf{index}(f^*\mathcal{E}) = f^*\mathbf{index}(\mathcal{E})$ shows that I is a natural transformation of functors from presentable compact orbifolds to $\mathbb{Z}/2\mathbb{Z}$ -graded abelian groups.

We consider the functor $B \mapsto \Omega^*(LB)/\mathbf{im}(d)$, $*$ $\in \{ev, odd\}$ as a functor from orbifolds to $\mathbb{Z}/2\mathbb{Z}$ -graded abelian groups. We construct a parity-reversing natural transformation

$$a : \Omega^*(LB)/\mathbf{im}(d) \rightarrow \hat{K}^*(B)$$

by

$$a(\rho) := [\emptyset, -\rho] .$$

Let $\Omega_{d=0}^*(LB)$ be the group of closed forms of parity $*$ on B . Again we consider $B \mapsto \Omega_{d=0}^*(LB)$ as a functor from orbifolds to $\mathbb{Z}/2\mathbb{Z}$ -graded abelian groups. We define a natural transformation

$$R : \hat{K}(B) \rightarrow \Omega_{d=0}(LB)$$

by

$$R([\mathcal{E}, \rho]) = \Omega(\mathcal{E}) - d\rho .$$

The map R is well-defined by the same argument as in [BS07]. It follows from $\Omega(f^*\mathcal{E}) = f^*\Omega(\mathcal{E})$ that R is a natural transformation.

2.5.3 The natural transformations satisfy the following relations:

Lemma 2.23 1. $R \circ a = d$

2. $\mathbf{ch}_{dR} \circ I = [\dots] \circ R$.

Proof. The first relation is an immediate consequence of the definition of R and a . The second relation is the local index Theorem 2.14. \square

2.5.4 Via the embedding $H_{deloc}(B) = H(LB) \subseteq \Omega(LB)/\mathbf{im}(d)$, the Chern character $\mathbf{ch}_{dR} : K(B) \rightarrow H_{deloc}(B)$ can be considered as a natural transformation

$$\mathbf{ch}_{dR} : K(B) \rightarrow \Omega(LB)/\mathbf{im}(d) .$$

Proposition 2.24 *The following sequence is exact:*

$$K(B) \xrightarrow{\mathbf{ch}_{dR}} \Omega(LB)/\mathbf{im}(d) \xrightarrow{a} \hat{K}(B) \xrightarrow{I} K(B) \rightarrow 0 .$$

Proof.

2.5.5 We start with the surjectivity of $I : \hat{K}(B) \rightarrow K(B)$. The main point is the fact that every element $x \in K(B)$ can be realized as the index of a geometric family over B . Here we use again that the orbifold is presentable. Let $[M/G] \cong B$ be a presentation. Given a class in $K(B)$ let $x \in K_G(M)$ be the corresponding class under the isomorphism $K(B) \cong K_G(M)$. It suffices to show that x can be realized as the index of a G -equivariant geometric family \mathcal{E} over M . We first consider the even case. Then x can be represented by a $\mathbb{Z}/2\mathbb{Z}$ -graded G -vector bundle $V \rightarrow M$. As in 2.2.5 we construct a G -equivariant geometric family with zero-dimensional fibre $\mathcal{V} \rightarrow M$ such that $\mathbf{index}(\mathcal{V}) = x$.

In the odd case we let $y \in K_G^0(S^1 \times M, \{1\} \times M)$ be the the class corresponding to x under the suspension isomorphism $K_G^0(S^1 \times M, \{1\} \times M) \cong K_G^1(M)$. As above we can find an equivariant geometric family \mathcal{V} over $S^1 \times M$ such that $\mathbf{index}(\mathcal{V}) \in K_G^0(S^1 \times M)$ is the image of y under $K_G^0(S^1 \times M, \{1\} \times M) \rightarrow K_G^0(S^1 \times M)$. Using the standard metric on S^1 and the canonical horizontal bundle $TM \subset T(S^1 \times M)$ for $p : S^1 \times M \rightarrow M$ we can define a G -equivariant geometric family $p_!(\mathcal{V})$ over M such that $\mathbf{index}(p_!\mathcal{V}) = x$.

2.5.6 Next we show exactness at $\hat{K}(B)$. For $\rho \in \Omega(LB)/\mathbf{im}(d)$ we have $I \circ a(\rho) = I([\emptyset, -\rho]) = \mathbf{index}(\emptyset) = 0$, hence $\hat{I} \circ a = 0$. Consider a class $[\mathcal{E}, \rho] \in \hat{K}(B)$ which satisfies $I([\mathcal{E}, \rho]) = 0$. Using Lemma 2.10 and Lemma 2.20 we can replace \mathcal{E} by $\mathcal{E} \sqcup_B (\tilde{\mathcal{E}} \sqcup_B \tilde{\mathcal{E}}^{op})$ for some geometric family $\tilde{\mathcal{E}}$ without changing the smooth K -theory class such that \mathcal{E} admits a taming \mathcal{E}_t . Therefore, (\mathcal{E}, ρ) is paired with $(\emptyset, \rho - \eta(\mathcal{E}_t))$. It follows that $[\mathcal{E}, \rho] = a(\eta(\mathcal{E}_t) - \rho)$.

2.5.7 In order to prepare the proof of exactness at $\Omega(B)/\mathbf{im}(d)$ we need some facts about the classification of tamings of a geometric family \mathcal{E} . As in [BS07] we introduce the notion of boundary taming and will use an index theorem for boundary tamed families in order to compare tamings. Let \mathcal{F} be a geometric family with boundary \mathcal{E} over B and \mathcal{E}_t be a taming. Then we have a boundary tamed family \mathcal{F}_{bt} and can consider $\mathbf{index}(\mathcal{F}_{bt}) \in K(B)$.

Theorem 2.25 *In $H_{dR,deloc}(B)$ we have the following equality:*

$$\mathbf{ch}_{dR}(\mathbf{index}(\mathcal{F}_{bt})) = [\Omega(\mathcal{F}) + \eta(\mathcal{E}_t)] .$$

Proof. We can repeat the proof given in [Bun]. The only modifications are

1. We consider the pull-back of $(\mathcal{F}, \mathcal{E}_t)$ to LB which comes with canonical automorphisms $(\rho_{\mathcal{F}}, \rho_{\mathcal{E}_t})$.
2. We replace $\mathrm{Tr}_s \dots$ by $\mathrm{Tr}_s \rho_{\mathcal{F}}$ (or $\mathrm{Tr}_s \rho_{\mathcal{E}_t}$, respectively).
3. The small time analysis of this trace takes the localization of the heat kernel at the fibrewise fixed points of the canonical automorphisms into account.

In view of this theorem we can argue as in [BS07] that if \mathcal{E}_t and \mathcal{E}'_t are two tamings of a geometric family, then the difference of the associated η -forms is closed and we have

$$[\eta(\mathcal{E}_t) - \eta(\mathcal{E}'_t)] \in \mathbf{ch}_{dR}(K(B)) \subset H_{dR,deloc}(B) .$$

We now show exactness at $\Omega(LB)/\mathbf{im}(d)$. Let $\rho \in \Omega(LB)/\mathbf{im}(d)$ be such that $a(\rho) = [\emptyset, -\rho] = 0$. Then by Lemma 2.22 there exists a cycle $(\hat{\mathcal{E}}, \hat{\rho})$ such that $(\hat{\mathcal{E}}, \hat{\rho} - \rho)$ pairs with $(\hat{\mathcal{E}}, \hat{\rho})$. Using Lemma 2.17 we can add a copy $\hat{\mathcal{E}}^{op}$ and see that $(\mathcal{E}, \hat{\rho} - \rho)$ is paired with $(\emptyset, \hat{\rho})$, where $\mathcal{E} = \hat{\mathcal{E}} \sqcup_B \hat{\mathcal{E}}^{op}$. The taming which induces this relation will be denoted by \mathcal{E}'_t . We have $\eta(\mathcal{E}'_t) = -\rho$. Because of the odd $\mathbb{Z}/2\mathbb{Z}$ -symmetry the family \mathcal{E} admits another taming \mathcal{E}_t with vanishing η -form. Therefore

$$\rho = [\eta(\mathcal{E}_t)] \in \mathbf{ch}_{dR}(K(B)) .$$

2.5.8 It remains to show that for $x \in K(B)$ we have $a \circ \mathbf{ch}_{dR}(x) = 0$. Note that $a \circ \mathbf{ch}_{dR}(x) = [\emptyset, -\mathbf{ch}_{dR}(x)]$. The proof is accomplished by showing that there exists a geometric family $\mathcal{E} = \hat{\mathcal{E}} \sqcup_B \hat{\mathcal{E}}^{op}$ which admits tamings \mathcal{E}_t and \mathcal{E}'_t such that $\eta(\mathcal{E}_t) - \eta(\mathcal{E}'_t) = \mathbf{ch}_{dR}(x)$, i.e. $\mathbf{index}((\mathcal{E} \times I)_{bt}) = x$, where the boundary taming $(\mathcal{E} \times I)_{bt}$ is induced by \mathcal{E}_t and \mathcal{E}'_t .

To this end we modify the corresponding argument given in [BS07]. To be specific, let us consider the even case. First of all, using a presentation $B \cong [M/G]$, we will actually consider the equivariant problem. Let H be a universal G -Hilbert space. Then the G -space $GL_1(H)$ has the homotopy type of the classifying space of K_G^1 . Let $x \in K_G^1(M)$

be represented by an equivariant map $x : M \rightarrow GL_1(H)$. If (P_n) is a strong equivariant approximation of the identity of H for sufficiently large n , then the G -map

$$(1 - P_n) + P_n x P_n : M \rightarrow GL_1(H)$$

is G -homotopic to x . Let \mathcal{V} be the equivariant geometric family on M constructed from the $\mathbb{Z}/2\mathbb{Z}$ -graded G -vector bundle $V := \text{im}(P_n)$. The matrices

$$Q := \begin{pmatrix} 0 & P_n x^* P_n \\ P_n x P_n & 0 \end{pmatrix}, \quad Q' := \begin{pmatrix} 0 & \text{id}_V \\ \text{id}_V & 0 \end{pmatrix}$$

represent tamings of $\mathcal{E} := \mathcal{V} \sqcup_M \mathcal{V}^{op}$. We use Q and Q' at $\mathcal{E} \times \{0\}$ and $\mathcal{E} \times \{1\}$ in order to define $(\mathcal{E} \times I)_{bt}$. As in [BS07] we can now show that $\text{index}((\mathcal{E} \times I)_{bt}) = x$.

The odd case is similar. \square

2.5.9 We define a real structure \hat{Q} on $\hat{K}(B)$ by $Q([\mathcal{E}, \rho]) := [\mathcal{E}, Q(\rho)]$, where $Q(\rho) = I^*(\bar{\rho})$ is as in 2.1.7. Since the local index forms and eta forms are real, \hat{Q} is well-defined. We define the real subfunctor

$$\hat{K}_{\mathbb{R}}(B) := \{x \in \hat{K}(B) \mid \hat{Q}(x) = x\}.$$

By restriction we get natural transformations

$$R : \hat{K}_{\mathbb{R}}(B) \rightarrow \Omega_{\mathbb{R}}(LB), \quad a : \Omega_{\mathbb{R}}^*(LB)/\text{im}(d) \rightarrow \hat{K}_{\mathbb{R}}(B)$$

so that

$$K(B) \xrightarrow{\text{ch}_{dR}} \Omega_{\mathbb{R}}(LB)/\text{im}(d) \xrightarrow{a} \hat{K}_{\mathbb{R}}(B) \xrightarrow{I} K(B) \rightarrow 0$$

is exact.

2.6 Calculations for $[*/G]$

2.6.1 Let G be a finite group. We consider the orbifold $[*/G]$. Note that $K_G^0([*/G]) \cong K_G^0(*) \cong R(G)$ as rings, where $R(G)$ denotes the representation ring of G . Moreover, $K_G^1([*/G]) \cong 0$. We have $L[*/G] = [G/G]$, where G acts on itself by conjugation. Therefore

$$\Omega(L[*/G]) \cong \mathbb{C}[G]^G \cong H_{dR, \text{deloc}}([*/G])$$

is the ring of conjugation invariant functions. The Chern character fits into the diagram

$$\begin{array}{ccc} K^0([*/G]) & \xrightarrow{\text{ch}} & H_{dR, \text{deloc}}^*([*/G]) \\ \downarrow \cong & & \downarrow \cong \\ R(G) & \xrightarrow{\text{Tr}} & \mathbb{C}[G]^G \end{array} .$$

Lemma 2.26 *We have*

$$\hat{K}^*([*/G]) \cong \begin{cases} R(G) & * = 0 \\ \mathbb{C}[G]^G/R(G) & * = 1 \end{cases} .$$

Proof. We use the exact sequence given by Proposition 2.24. □

Note that $\text{Tr} : R(G) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \mathbb{C}[G]^G$ is an isomorphism so that

$$\mathbb{C}[G]^G/R(G) \cong R(G) \otimes_{\mathbb{Z}} \mathbb{T} .$$

It restricts to an isomorphism $R(G)_{\mathbb{R}} := R(G) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \Omega_{\mathbb{R}}(L[*/G]) \subset \mathbb{C}[G]^G$.

Corollary 2.27 *We have*

$$\hat{K}_{\mathbb{R}}^*([*/G]) \cong \begin{cases} R(G) & * = 0 \\ R(G)_{\mathbb{R}}/R(G) \cong \mathbb{R}(G) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} & * = 1 \end{cases} .$$

3 Push-forward and \cup -product

3.1 Equivariant K -orientation

3.1.1 The notion of a $Spin^c(n)$ -reduction of an $SO(n)$ -principal bundle extends directly from the smooth case to the orbifold case using the appropriate notions of principal bundles in the realm of stacks.

Let $p : W \rightarrow B$ be a proper submersion between orbifolds with vertical bundle $T^v p$. We assume that $T^v p$ is oriented. A choice of a vertical metric $g^{T^v p}$ gives an SO -reduction $SO(T^v p)$ of the frame bundle $\text{Fr}(T^v p)$, the bundle of oriented orthonormal frames.

A map between smooth manifolds is called K -oriented if its stable normal bundle is equipped with a K -theory Thom class. It is a well-known fact [ABS64] that this is equivalent to the choice of a $Spin^c$ -structure on the stable normal bundle. Finally, isomorphism classes of choices of $Spin^c$ -structures on $T^v p$ and the stable normal bundle of p are in bijective correspondence.

In the equivariant situation the situation is more complicated. For the purpose of the present paper we will work with vertical structures along the morphisms $p : W \rightarrow B$. A $Spin^c$ -reduction of an $SO(n)$ -principal bundle $P \rightarrow E$ over some stack E is a pair (Q, ϕ) , where $Q \rightarrow E$ is a $Spin^c(n)$ -principal bundle and $\phi : Q \times_{Spin^c(n)} SO(n) \rightarrow P$ is an isomorphism of $SO(n)$ -principal bundles. Note that this includes a choice for the fibre product and a two-isomorphism filling the corresponding triangle.

Definition 3.1 *A topological K -orientation of a morphism between orbifolds $p : W \rightarrow B$ is a $Spin^c$ -reduction of $SO(T^v p)$.*

3.1.2 If we choose a horizontal distribution $T^h p$, then we get a connection $\nabla^{T^v p}$ which restricts to the Levi-Civita connection along the fibres. The connection $\nabla^{T^v p}$ can be considered as an $SO(n)$ -principal bundle connection on the frame bundle $SO(T^v p)$. Given a topological K -orientation of p

$$\begin{array}{ccc} Q & \xrightarrow{\phi} & SO(T^v p) \\ & \searrow & \swarrow p \\ & W & \end{array}$$

we can choose a $Spin^c$ -reduction $\tilde{\nabla}$ of $\nabla^{T^v p}$, i.e. a connection on the $Spin^c$ -principal bundle Q which reduces to $\nabla^{T^v p}$. If we think of the connections $\nabla^{T^v p}$ and $\tilde{\nabla}$ in terms of horizontal distributions $T^h SO(T^v p)$ and $T^h Q$, then we say that $\tilde{\nabla}$ reduces to $\nabla^{T^v p}$ if $d\phi(T^h Q) = \phi^*(T^h SO(T^v p)) \subset \phi^*(TSO(T^v p))$. Observe, that in contrast to the $Spin$ -case $\tilde{\nabla}$ is not unique.

3.1.3 The $Spin^c$ -reduction of $\text{Fr}(T^v p)$ determines a spinor bundle $S^c(T^v p)$, and the choice of $\tilde{\nabla}$ turns $S^c(T^v p)$ into a family of Dirac bundles.

In this way the choices of the $Spin^c$ -structure and $(g^{T^v p}, T^h p, \tilde{\nabla})$ turn $p : W \rightarrow B$ into a geometric family \mathcal{W} . We define the closed form

$$\hat{\mathbf{A}}_\rho^c(\tilde{\nabla}) := I(\mathcal{W}) \in \Omega_{\mathbb{R}}(LW) . \quad (6)$$

Its cohomology class will be denoted by $\hat{\mathbf{A}}_\rho(LW) \in H(LW)$.

3.1.4 Though we will not need the explicit formula let us for completeness specialize formula (2) to the family \mathcal{W} .

Locally on W we can choose a $Spin$ -structure on $T^v p$ with associated spinor bundle $S(T^v p)$. Then we can write $S^c(T^v p) = S(T^v p) \otimes L$ for a hermitean line bundle L with connection. The spin structure is given by a $Spin$ -reduction $q : R \rightarrow SO(T^v p)$ which can actually be considered as a subbundle of Q . Since q is a double covering and thus has discrete fibres, the connection $\nabla^{T^v p}$ (in contrast to the $Spin^c$ -case) has a unique lift to a $Spin(n)$ -connection on R . The spinor bundle $S(T^v p)$ is associated to R and has an induced connection. The square of the locally defined line bundle L is the globally defined bundle $L^2 \rightarrow W$ associated to the $Spin^c$ -bundle Q via the canonical representation $\lambda : Spin^c(n) \rightarrow U(1)$. The connection $\tilde{\nabla}$ thus induces a connection on ∇^{L^2} , and hence a connection on the locally defined square root L . Note that vice versa, ∇^{L^2} and $\nabla^{T^v p}$ determine $\tilde{\nabla}$ uniquely.

3.1.5 Let $L_L^2 \rightarrow LW$ be the pull-back under the canonical map $LW \rightarrow W$, and let $\rho_{L_L^2}$ denote the corresponding automorphism of L_L^2 (see 2.1.5). Since L^2 is a line bundle, it acts as multiplication by a complex number which we will denote by the same symbol.

We introduce the form

$$c_1(\tilde{\nabla}) := \frac{1}{2} R^{L_L^2} \quad (7)$$

which would be the (unnormalized) Chern form of the bundle L_L in case of a global *Spin*-structure.

We actually must lift the automorphism $\rho_{L_L^2}$ to the bundle L_L , i.e. we must choose a square root ρ_{L_L} . Once we have fixed this root we get an induced element

$$\rho^N \in \mathbf{End}(S(T^v\pi)) \cong \mathbf{Cliff}(N) .$$

If $T(\sigma(\rho^N))$ denotes the Beresin integral, then the combination

$$\rho_{L_L} T(\sigma(\rho^N))$$

is independent of the choices.

The pull-back $T^v p_L \rightarrow LW$ of the vertical bundle has a decomposition $T^v p_L = T^v LP \oplus N$ such that $\rho_{T^v p_L} = \mathbf{id}_{T^v LP} \oplus \rho^N$ which is preserved by $\nabla_L^{T^v p}$, i.e. $\nabla_L^{T^v p} = \nabla^{T^v LP} \oplus \nabla^N$. We consider the induced action of ρ^N on the spinor module $S(N)$.

Let $R^{\nabla^{T^v LP}} \in \Omega^2(W, \mathbf{End}(T^v LP))$ denote the curvature of $\nabla^{T^v LP}$ and form

$$\hat{\mathbf{A}}(\nabla^{T^v LP}) := \mathbf{det}^{1/2} \left(\frac{\frac{R^{\nabla^{T^v LP}}}{2}}{\sinh \left(\frac{R^{\nabla^{T^v LP}}}{2} \right)} \right) .$$

Lemma 3.2 *In this context, the relevant differential form (6) for local index theory is*

$$\hat{\mathbf{A}}_\rho^c(\tilde{\nabla}) = c_{\dim(LW/LB)} \left(\frac{1}{4\pi} \right)^{\dim(LW/LB)/2} \frac{\rho_{L_L} T(\sigma(\rho^N)) \hat{\mathbf{A}}(\nabla^{T^v LP}) \wedge e^{-c_1(\tilde{\nabla})}}{\mathbf{det}^{1/2}(1 - \rho^{N,-1}) \mathbf{det}^{1/2}(1 - \rho^N e^{-\frac{1}{2}R^{\nabla^N}})} \in \Omega(LW) .$$

3.1.6 The dependence of the form $\hat{\mathbf{A}}^c(\tilde{\nabla})$ on the data is described in terms of the transgression form. Let $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i)$, $i = 0, 1$, be two choices of geometric data. Then we can choose geometric data $(\hat{g}^{T^v p}, \hat{T}^h p, \hat{\tilde{\nabla}})$ on $\hat{p} = \mathbf{id}_{[0,1]} \times p : [0, 1] \times W \rightarrow [0, 1] \times B$ (with the induced *Spin*^c-structure on $T^v \hat{p}$) which restricts to $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i)$ on $\{i\} \times B$. The class

$$\tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0) := \int_{[0,1] \times LW/LW} \hat{\mathbf{A}}_\rho^c(\hat{\tilde{\nabla}}) \in \Omega_{\mathbb{R}}(LW)/\mathbf{im}(d)$$

is independent of the extension and satisfies

$$d\tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = \hat{\mathbf{A}}_\rho^c(\tilde{\nabla}_1) - \hat{\mathbf{A}}_\rho^c(\tilde{\nabla}_0) . \quad (8)$$

Definition 3.3 *The form $\tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0)$ is called the transgression form.*

Note that we have the identity

$$\tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_2, \tilde{\nabla}_1) + \tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = \tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_2, \tilde{\nabla}_0) . \quad (9)$$

As a consequence we get the identities

$$\tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}, \tilde{\nabla}) = 0 , \quad \tilde{\hat{\mathbf{A}}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0) = -\hat{\mathbf{A}}_\rho^c(\tilde{\nabla}_0, \tilde{\nabla}_1) . \quad (10)$$

3.1.7 We can now introduce the notion of a smooth K -orientation of a proper submersion $p : W \rightarrow B$ between orbifolds. We fix an underlying topological K -orientation of p (see Definition 3.1) which is given by a $Spin^c$ -reduction of $SO(T^v p)$ after choosing an orientation and a metric on $T^v p$.

We consider the set \mathcal{O} of tuples $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ where the first three entries have the same meaning as above (see 3.1.2), and $\sigma \in \Omega^{odd}(LW)/\mathbf{im}(d)$. We introduce a relation $o_0 \sim o_1$ on \mathcal{O} : Two tuples $(g_i^{T^v p}, T_i^h p, \tilde{\nabla}_i, \sigma_i)$, $i = 0, 1$ are related if and only if $\sigma_1 - \sigma_0 = \hat{\mathbf{A}}_\rho^c(\tilde{\nabla}_1, \tilde{\nabla}_0)$. We claim that \sim is an equivalence relation. In fact, symmetry and reflexivity follow from (10), while transitivity is a consequence of (9).

Definition 3.4 *The set of smooth K -orientations which refines a fixed underlying topological K -orientation of $p : W \rightarrow B$ is the set of equivalence classes \mathcal{O}/\sim .*

Note that $\Omega^{odd}(LW)/\mathbf{im}(d)$ acts on the set of smooth K -orientations. If $\alpha \in \Omega^{odd}(LW)/\mathbf{im}(d)$ and $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ represents a smooth K -orientation, then the translate of this orientation by α is represented by $(g^{T^v p}, T^h p, \tilde{\nabla}, \sigma + \alpha)$. As a consequence of (9) we get:

Corollary 3.5 *The set of smooth K -orientations refining a fixed underlying topological K -orientation is a torsor over $\Omega^{odd}(LW)/\mathbf{im}(d)$.*

If $o = (g^{T^v p}, T^h p, \tilde{\nabla}, \sigma) \in \mathcal{O}$ represents a smooth K -orientation, then we will write

$$\hat{\mathbf{A}}^c(o) := \hat{\mathbf{A}}_\rho^c(\tilde{\nabla}), \quad \sigma(o) := \sigma.$$

3.2 Definition of the Push-forward

3.2.1 We consider a proper submersion between orbifolds $p : W \rightarrow B$ with a choice of a topological K -orientation. Assume that p has closed fibres. Let $o = (g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ represent a smooth K -orientation which refines the given topological one. To every geometric family \mathcal{E} over W we associate a geometric family $p_! \mathcal{E}$ over B .

Let $\pi : E \rightarrow W$ denote the underlying proper submersion with closed fibres of \mathcal{E} which comes with the geometric data $g^{T^v \pi}$, $T^h \pi$ and the family of Dirac bundles (V, h^V, ∇^V) .

The underlying proper submersion with closed fibres of $p_! \mathcal{E}$ is

$$q := p \circ \pi : E \rightarrow B.$$

The horizontal bundle of π admits a decomposition $T^h \pi \cong \pi^* T^v p \oplus \pi^* T^h p$, where the isomorphism is induced by $d\pi$. We define $T^h q \subseteq T^h \pi$ such that $d\pi : T^h q \cong \pi^* T^h p$. Furthermore we have an identification $T^v q = T^v \pi \oplus \pi^* T^v p$. Using this decomposition we define the vertical metric $g^{T^v q} := g^{T^v \pi} \oplus \pi^* g^{T^v p}$. These structures give a connection $\nabla^{T^v q}$ which in general differs from the sum $\nabla^{T^v \pi} \oplus \pi^* \nabla^{T^v p}$.

The orientations of $T^v\pi$ and T^vp induce an orientation of T^vq .

Finally we must construct the Dirac bundle $p_!\mathcal{V} \rightarrow E$. Locally on E we can choose a $Spin^c$ -structure on $T^v\pi$ with spinor bundle $S^c(T^v\pi)$, and with a $Spin^c$ -connection $\tilde{\nabla}_\pi$ which refines the connection $\nabla^{T^v\pi}$. We define the twisting bundle

$$Z := \text{Hom}_{\text{Cliff}(T^v\pi)}(S^c(T^v\pi), V) .$$

The connections $\tilde{\nabla}_\pi$ and ∇^V induce a connection ∇^Z .

The local $Spin^c$ -structure of $T^v\pi$ together with the $Spin^c$ -structure of T^vp induce a $Spin^c$ -structure on $T^vq \cong T^v\pi \oplus \pi^*T^vp$. It has an induced connection $\tilde{\nabla}^\oplus$ which refines the direct sum connection $\nabla^{T^v\pi} \oplus \pi^*\nabla^{T^vp}$. We let $\tilde{\nabla}_q$ be the $Spin^c$ -connection on T^vq which refines $\tilde{\nabla}^\oplus$ and has the same central curvature as $\tilde{\nabla}^\oplus$. Locally we can define the family of Dirac bundles $p_!V := S(T^vq) \otimes Z$. It is easy to see that this bundle is well-defined independent of the choices of local $Spin^c$ -structure and therefore a globally defined family of Dirac bundles.

Remark: Note that the notion of locality in the realm of orbifolds is more complicated than it might appear at the first glance. To say that we choose a local $Spin^c$ -structure means that we use an orbifold atlas $A \rightarrow B$ and consider an open subset $U \subset A$, and that we choose a $Spin^c$ -structure after pulling the family back to U . Thus in particular we do not (and can not) require that it is equivariant with respect to the local automorphism groupoid $U \times_B U \rightrightarrows U$. Therefore our twisting bundle Z is not equivariant, too. On the other hand, the tensor product $S(T^vq) \otimes Z$ is completely canonical and thus is equivariant.

Definition 3.6 *Let $p_!\mathcal{E}$ denote the geometric family given by $q : E \rightarrow B$ and $p_!V \rightarrow E$ with the geometric structures defined above.*

3.2.2 Let $p : W \rightarrow B$ be a proper submersion with a smooth K -orientation represented by o . In 3.2.1 we have constructed for each geometric family \mathcal{E} over W a push-forward $p_!\mathcal{E}$. Now we introduce a parameter $a \in (0, \infty)$ into this construction.

Definition 3.7 *For $a \in (0, \infty)$ we define the geometric family $p_!^a\mathcal{E}$ as in 3.2.1 with the only difference that the metric on $T^vq = T^v\pi \oplus \pi^*T^vp$ is given by $g_a^{T^vq} = a^2g^{T^v\pi} \oplus \pi^*g^{T^vp}$.*

The family of geometric families $p_!^a\mathcal{E}$ is called the adiabatic deformation of $p_!\mathcal{E}$. There is a natural way to define a geometric family \mathcal{F} on $(0, \infty) \times B$ such that its restriction to $\{a\} \times B$ is $p_!^a\mathcal{E}$. In fact, we define $\mathcal{F} := (\text{id}_{(0, \infty)} \times p)_!((0, \infty) \times \mathcal{E})$ with the exception that we take the appropriate vertical metric.

Although the vertical metrics of \mathcal{F} and $p_!^a\mathcal{E}$ collapse as $a \rightarrow 0$ the induced connections and the curvature tensors on the vertical bundle T^vq converge and simplify in this limit. This fact is heavily used in local index theory, and we refer to [BGV04, Sec 10.2] for details. In particular, the integral

$$\tilde{\Omega}(a, \mathcal{E}) := \int_{(0, a) \times LB/LB} \Omega(\mathcal{F})$$

converges, and we have

$$\lim_{a \rightarrow 0} \Omega(p_!^a \mathcal{E}) = \int_{LW/LB} \hat{\mathbf{A}}^c(o) \wedge \Omega(\mathcal{E}), \quad \Omega(p_!^a \mathcal{E}) - \int_{LW/LB} \hat{\mathbf{A}}^c(o) \wedge \Omega(\mathcal{E}) = d\tilde{\Omega}(a, \mathcal{E}). \quad (11)$$

3.2.3 Let $p : W \rightarrow B$ be a proper submersion with closed fibres between orbifolds with a smooth K -orientation represented by o . We now start with the construction of the push-forward $p_! : \hat{K}(W) \rightarrow \hat{K}(B)$. For $a \in (0, \infty)$ and a cycle (\mathcal{E}, ρ) we define

$$\hat{p}_!^a(\mathcal{E}, \rho) := [p_!^a \mathcal{E}, \int_{LW/LB} \hat{\mathbf{A}}^c(o) \wedge \rho + \tilde{\Omega}(a, \mathcal{E}) + \int_{LW/LB} \sigma(o) \wedge R([\mathcal{E}, \rho])] \in \hat{K}(B). \quad (12)$$

Since $\hat{\mathbf{A}}^c(o)$ and $R([\mathcal{E}, \rho])$ are closed the maps

$$\begin{aligned} \Omega(LW)/\text{im}(d) \ni \rho &\mapsto \int_{LW/LB} \hat{\mathbf{A}}^c(o) \wedge \rho \in \Omega(LB)/\text{im}(d), \\ \Omega(LW)/\text{im}(d) \ni \sigma(o) &\mapsto \int_{LW/LB} \sigma(o) \wedge R([\mathcal{E}, \rho]) \in \Omega(LB)/\text{im}(d) \end{aligned}$$

are well-defined. It immediately follows from the definition that $p_!^a : G(W) \rightarrow \hat{K}(B)$ is a homomorphism of semigroups.

3.2.4 The homomorphism $p_!^a : G(W) \rightarrow \hat{K}(B)$ commutes with pull-back. More precisely, let $f : B' \rightarrow B$ be a morphism of orbifolds. Then we define the submersion $p' : W' \rightarrow B'$ by the cartesian diagram

$$\begin{array}{ccc} W' & \xrightarrow{F} & W \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array} .$$

The differential $dF : TW' \rightarrow F^*TW$ induces an isomorphism $dF : T^v W' \xrightarrow{\sim} F^*T^v W$. Therefore the metric, the orientation, and the $Spin^c$ -structure of $T^v \pi$ induce by pull-back corresponding structures on $T^v p'$. We define the horizontal distribution $T^h p'$ such that $dF(T^h p') \subseteq F^*T^h p$. Finally we set $\sigma' := LF^* \sigma$. The representative of a smooth K -orientation given by these structures will be denoted by $o' := f^*o$. An inspection of the definitions shows:

Lemma 3.8 *The pull-back of representatives of smooth K -orientations preserves equivalence and hence induces a pull-back of smooth K -orientations.*

Recall from 3.1.3 that the representatives o and o' of the smooth K -orientations enhance p and p' to geometric families \mathcal{W} and \mathcal{W}' . We have $f^* \mathcal{W} \cong \mathcal{W}'$.

Note that we have $LF^* \hat{\mathbf{A}}^c(o) = \hat{\mathbf{A}}^c(o')$. If \mathcal{E} is a geometric family over W , then an inspection of the definitions shows that $f^* p_!(\mathcal{E}) \cong p'_!(F^* \mathcal{E})$. The following Lemma now follows immediately from the definitions

Lemma 3.9 *We have $f^* \circ \hat{p}_!^a = \hat{p}'_!^a \circ F^* : G(W) \rightarrow \hat{K}(B')$.*

3.2.5

Lemma 3.10 *The class $\hat{p}_!^a(\mathcal{E}, \rho)$ does not depend on $a \in (0, \infty)$.*

Proof. The proof can be copied from [BS07, Lemma 3.11]. \square

In view of this Lemma we can omit the superscript a and write $p_!(\mathcal{E}, \rho)$ for $p_!^a(\mathcal{E}, \rho)$.

3.2.6 Let \mathcal{E} be a geometric family over W which admits a taming \mathcal{E}_t . Recall that the taming is given by a family of smoothing operators $(Q_w)_{w \in W}$.

The family of operators along the fibres of $p_!\mathcal{E}$ induced by Q is not a taming of $p_!\mathcal{E}_t$ since it is not given by a smooth integral kernel but rather by a family of fibrewise smoothing operators. Nevertheless it can be used in the same way as a taming in order to define e.g. the η -forms which we will denote by $\eta(p_!\mathcal{E}_t)$. To be precise, we add the term $\chi(ua^{-1})ua^{-1}Q$ to the rescaled super-connection $A_u(p_!\mathcal{E})$, where χ vanishes near zero and is equal to 1 on $[1, \infty)$. This means that we switch on the taming at time $u \sim a$, and we rescale it in the same way as the vertical part of the Dirac operator. In this situation we will speak of a generalized taming. We can control the behaviour of $\eta(p_!\mathcal{E}_t)$ in the adiabatic limit $a \rightarrow 0$.

Theorem 3.11

$$\lim_{a \rightarrow 0} \eta(p_!\mathcal{E}_t) = \int_{LW/LB} \hat{\mathbf{A}}^c(o) \wedge \eta(\mathcal{E}_t) .$$

Proof. The proof of this theorem can be obtained by combining standard methods of equivariant local index theory with the adiabatic techniques developed by the school of Bismut. We will give the details elsewhere. \square

Since the geometric family $p_!\mathcal{E}$ admits a generalized taming it follows that $\text{index}(p_!\mathcal{E}) = 0$. Hence we can also choose a taming $(p_!\mathcal{E})_t$. The latter choice together with the generalized taming induce a generalized boundary taming of the family $p_!\mathcal{E} \times [0, 1]$ over B . We have, as in [BS07, Lemma 3.13],

Lemma 3.12 *The difference of η -forms $\eta((p_!\mathcal{E})_t) - \eta(p_!\mathcal{E}_t)$ is closed. Its de Rham cohomology class satisfies*

$$[\eta((p_!\mathcal{E})_t) - \eta(p_!\mathcal{E}_t)] \in \mathbf{ch}_{dR}(K(B)) .$$

3.2.7 We now show that $p_! : G(W) \rightarrow \hat{K}(B)$ passes through the equivalence relation \sim . Since $p_!$ is additive it suffices by Lemma 2.18 to show the following assertion.

Lemma 3.13 *If (\mathcal{E}, ρ) is paired with $(\tilde{\mathcal{E}}, \tilde{\rho})$, then $\hat{p}_!(\mathcal{E}, \rho) = \hat{p}_!(\tilde{\mathcal{E}}, \tilde{\rho})$.*

Proof. The proof can be copied from [BS07, Lemma 3.14]. \square

3.2.8 We let

$$\hat{p}_! : \hat{K}(W) \rightarrow \hat{K}(B) \tag{13}$$

denote the map induced by the construction (12).

Definition 3.14 We define the integration of forms $p_!^o : \Omega(LW) \rightarrow \Omega(LB)$ by

$$p_!^o(\omega) = \int_{LW/LB} (\hat{\mathbf{A}}^c(o) - d\sigma(o)) \wedge \omega$$

Since $\hat{\mathbf{A}}^c(o) - d\sigma(o)$ is closed we also have a factorization

$$p_!^o : \Omega(LW)/\text{im}(d) \rightarrow \Omega(LB)/\text{im}(d) .$$

Our constructions of the homomorphisms

$$\hat{p}_! : \hat{K}(W) \rightarrow \hat{K}(B) , \quad p_!^o : \Omega(LW) \rightarrow \Omega(LB)$$

involve an explicit choice of a representative $o = (g^{T^v p}, T^h p, \tilde{\nabla}, \sigma)$ of the smooth K -orientation lifting the given topological K -orientation of p . But both push-forward maps are actually independent of the choice of the representative.

Lemma 3.15 *The homomorphisms $\hat{p}_! : \hat{K}(W) \rightarrow \hat{K}(B)$ and $p_!^o : \Omega(W) \rightarrow \Omega(B)$ only depend on the smooth K -orientation represented by o .*

Proof. It can be copied from [BS07, Lemma 3.17] □

3.2.9 Let $p : W \rightarrow B$ be a proper submersion between orbifolds with closed fibres with a smooth K -orientation represented by o . We now have constructed the homomorphism (13). In the present paragraph we study the compatibility of this construction with the curvature $R : \hat{K} \rightarrow \Omega_{d=0}$.

Lemma 3.16 *For $x \in \hat{K}(W)$ we have*

$$R(\hat{p}_!(x)) = p_!^o(R(x)) .$$

Proof. The proof can be copied from [BS07, Lemma 3.16].

3.2.10 Let $p : W \rightarrow B$ be a proper submersion between orbifolds with closed fibres with a topological K -orientation. We choose a smooth K -orientation which refines the topological K -orientation. In this case we say that p is smoothly K -oriented.

Definition 3.17 We define the push-forward $\hat{p}_! : \hat{K}(W) \rightarrow \hat{K}(B)$ to be the map induced by (12) for some choice of a representative of the smooth K -orientation

We also have well-defined maps

$$p_!^o : \Omega(LW) \rightarrow \Omega(BL) , \quad p_!^o : \Omega(LW)/\text{im}(d) \rightarrow \Omega(LB)/\text{im}(d) .$$

Let us state the result about the compatibility of $p_!$ with the structure maps of smooth K -theory as follows.

Proposition 3.18 *The following diagrams commute:*

$$\begin{array}{ccccccc} K(W) & \xrightarrow{\text{ch}_{dR}} & \Omega(LW)/\text{im}(d) & \xrightarrow{a} & \hat{K}(W) & \xrightarrow{I} & K(W) \\ \downarrow p_! & & \downarrow p_!^o & & \downarrow \hat{p}_! & & \downarrow p_! \\ K(B) & \xrightarrow{\text{ch}_{dR}} & \Omega(LB)/\text{im}(d) & \xrightarrow{a} & \hat{K}(B) & \xrightarrow{I} & K(B) \end{array} \quad (14)$$

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{R} & \Omega_{d=0}(LW) \\ \downarrow \hat{p}_! & & \downarrow p_!^o \\ \hat{K}(B) & \xrightarrow{R} & \Omega_{d=0}(LB) \end{array} \quad (15)$$

Proof. We copy the proof of [BS07, Prop. 3.19]. □

3.3 Functoriality

3.3.1 We now discuss the functoriality of the push-forward with respect to iterated fibre bundles. Let $p : W \rightarrow B$ be as before together with a representative of a smooth K -orientation $o_p = (g^{T^v p}, T^h p, \tilde{\nabla}_p, \sigma(o_p))$. Let $r : B \rightarrow A$ be another proper submersion between orbifolds with closed fibres with a topological K -orientation which is refined by a smooth K -orientation represented by $o_r := (g^{T^v r}, T^h r, \tilde{\nabla}_r, \sigma(o_r))$.

We can consider the geometric family $\mathcal{W} := (W \rightarrow B, g^{T^v p}, T^h p, S^c(T^v p))$ and apply the construction 3.2.2 in order to define the geometric family $r_!^a(\mathcal{W})$ over A . The underlying submersion of the family is $q := r \circ p : W \rightarrow A$. Its vertical bundle has a metric $g_a^{T^v q}$ and a horizontal distribution $T^h q$. The topological $Spin^c$ -structures of $T^v p$ and $T^v r$ induce a topological $Spin^c$ -structure on $T^v q = T^v p \oplus p^* T^v r$. The family of Clifford bundles of $p_! \mathcal{W}$ is the spinor bundle associated to this $Spin^c$ -structure.

In order to understand how the connection $\tilde{\nabla}_q^a$ behaves as $a \rightarrow 0$ we choose local spin structures on $T^v p$ and $T^v r$. Then we write $S^c(T^v p) \cong S(T^v p) \otimes L_p$ and $S^c(T^v r) \cong S(T^v r) \otimes L_r$ for one-dimensional twisting bundles with connection L_p, L_r . The two local spin structures induce a local spin structure on $T^v q \cong T^v p \oplus p^* T^v r$. We get $S^c(T^v q) \cong S(T^v q) \otimes L_q$ with $L_q := L_p \otimes p^* L_r$. The connection $\nabla_q^{a, T^v q}$ converges as $a \rightarrow 0$. Moreover, the twisting connection on L_q does not depend on a at all. Since $\nabla_q^{a, T^v q}$ and ∇_q^L determine $\tilde{\nabla}_q^a$ (see 3.1.3) we conclude that the connection $\tilde{\nabla}_q^a$ converges as $a \rightarrow 0$. We introduce the following notation for this adiabatic limit:

$$\tilde{\nabla}^{adia} := \lim_{a \rightarrow 0} \tilde{\nabla}_q^a .$$

3.3.2 We keep the situation described in 3.3.1.

Definition 3.19 We define the composite $o_q^a := o_r \circ_a o_p$ of the representatives of smooth K -orientations of p and r by

$$o_q^a := (g_a^{T^v q}, T^h q, \tilde{\nabla}_q^a, \sigma(o_q^a)) ,$$

where

$$\sigma(o_q^a) := \sigma(o_p) \wedge p^* \hat{\mathbf{A}}_\rho^c(o_r) + \hat{\mathbf{A}}_\rho^c(o_p) \wedge p^* \sigma(o_r) - \tilde{\mathbf{A}}_\rho^c(\tilde{\nabla}^{adia}, \tilde{\nabla}_q^a) - d\sigma(o_p) \wedge p^* \sigma(o_r) .$$

Lemma 3.20 This composition of representatives of smooth \hat{K} -orientations preserves equivalence and induces a well-defined composition of smooth K -orientations which is independent of a .

Proof. The proof is the same as the one of [BS07, Lemma 3.22]. \square

3.3.3 We consider the composition of proper K -oriented submersions

$$\begin{array}{ccc} W & \xrightarrow{p} & B \xrightarrow{r} A \\ & & \searrow q \swarrow \end{array}$$

with representatives of smooth K -orientations o_p of p and o_r of r . We let $o_q := o_p \circ_p r$ be the composition. These choices define push-forwards $\hat{p}_!, \hat{r}_!$ and $\hat{q}_!$ in smooth K -theory.

Theorem 3.21 We have the equality of homomorphisms $\hat{K}(W) \rightarrow \hat{K}(A)$

$$\hat{q}_! = \hat{r}_! \circ \hat{p}_! .$$

Proof. The proof only depends on the formal properties of transgression forms. It can be copied from [BS07, Thm. 3.23]. \square

3.3.4 We call a representative $o = (g^{T^v p}, T^h p, \tilde{\nabla}_p, \sigma(o_p))$ of a smooth K -orientation of $p : W \rightarrow B$ real, if and only if $\sigma(o_p) \in \Omega_{\mathbb{R}}^{odd}(LW)/\text{im}(d)$. Furthermore, we observe that being real is a property of the equivalence class of o . If o is real, then it immediately follows from (12) that the associated push-forward preserves the real subfunctors, i.e. that by restriction we get integrations

$$\hat{p}_! : \hat{K}_{\mathbb{R}}(W) \rightarrow \hat{K}_{\mathbb{R}}(B) , \quad \hat{p}_!^o : \Omega_{\mathbb{R}}(LW) \rightarrow \Omega_{\mathbb{R}}(B).$$

3.4 The cup product

3.4.1 In this section we define and study the cup product

$$\cup : \hat{K}(B) \otimes \hat{K}(B) \rightarrow \hat{K}(B) .$$

It turns smooth K -theory into a functor on compact presentable orbifolds with values in $\mathbb{Z}/2\mathbb{Z}$ -graded rings.

3.4.2 Let \mathcal{E} and \mathcal{F} be geometric families over B . The formula for the product involves the product $\mathcal{E} \times_B \mathcal{F}$ of geometric families over B . The detailed description of the product is easy to guess, but let us employ the following trick in order to give an alternative definition.

The underlying proper submersions of \mathcal{E} and \mathcal{F} give rise to a diagram

$$\begin{array}{ccc} E \times_B F & \xrightarrow{\delta} & F \\ \downarrow & & \downarrow p \\ E & \longrightarrow & B \end{array} .$$

Let us for the moment assume that the vertical metric, the horizontal distribution, and the orientation of p are complemented by a topological $Spin^c$ -structure together with a $Spin^c$ -connection $\tilde{\nabla}$ as in 3.2.1. The Dirac bundle \mathcal{V} of \mathcal{F} has the form $\mathcal{V} \cong W \otimes S^c(T^v p)$ for a twisting bundle W with a hermitean metric and unitary connection (and $\mathbb{Z}/2\mathbb{Z}$ -grading in the even case), which is uniquely determined up to isomorphism. Let $p^* \mathcal{E} \otimes W$ denote the geometric family which is obtained from $p^* \mathcal{E}$ by twisting its Dirac bundle with $\delta^* W$. Then we have

$$\mathcal{E} \times_B \mathcal{F} \cong p_!(p^* \mathcal{E} \otimes W) .$$

In the description of the product of geometric families we could interchange the roles of \mathcal{E} and \mathcal{F} .

If the vertical bundle of \mathcal{E} does not have a global $Spin^c$ -structure, then it has at least a local one. In this case the description above again gives a complete description of the local geometry of $\mathcal{E} \times_B \mathcal{F}$ (see the Remark in 3.2.1).

3.4.3 We now proceed to the definition of the product in terms of cycles. In order to write down the formula we assume that the cycles (\mathcal{E}, ρ) and (\mathcal{F}, θ) are homogeneous of degree e and f , respectively.

Definition 3.22 *We define*

$$(\mathcal{E}, \rho) \cup (\mathcal{F}, \theta) := [\mathcal{E} \times_B \mathcal{F}, (-1)^e \Omega(\mathcal{E}) \wedge \theta + \rho \wedge \Omega(\mathcal{F}) - (-1)^e d\rho \wedge \theta] .$$

Proposition 3.23 *The product is well-defined. It turns $B \mapsto \hat{K}(B)$ into a functor from compact presentable orbifolds to unital graded-commutative rings. By restriction it induces a ring structure on the real subfunctor $\hat{K}_{\mathbb{R}}(B)$.*

Proof. The proof can be copied from [BS07, Prop. 4.2]. That the product preserves the real subspace immediately follows from the definitions. \square

3.4.4 In this paragraph we study the compatibility of the cup product in smooth K -theory with the cup product in topological K -theory and the wedge product of differential forms.

Lemma 3.24 *For $x, y \in \hat{K}(B)$ we have*

$$R(x \cup y) = R(x) \wedge R(y) , \quad I(x \cup y) = I(x) \cup I(y) .$$

Furthermore, for $\alpha \in \Omega(LB)/\text{im}(d)$ we have

$$a(\alpha) \cup x = a(\alpha \wedge R(x)) .$$

Proof. Straightforward calculation using the definitions and that $\text{index}(\mathcal{E} \times_B \mathcal{F}) = \text{index}(\mathcal{E}) \cup \text{index}(\mathcal{F})$ and $\Omega(\mathcal{E} \times_B \mathcal{F}) = \Omega(\mathcal{E}) \wedge \Omega(\mathcal{F})$.

3.4.5 Let $p: W \rightarrow B$ be a proper submersion with closed fibres with a smooth K -orientation. In 3.2.7 we defined the push-forward $\hat{p}_!: \hat{K}(W) \rightarrow \hat{K}(B)$. The explicit formula in terms of cycles is (12). The projection formula states the compatibility of the push-forward with the \cup -product.

Proposition 3.25 *Let $x \in \hat{K}(W)$ and $y \in \hat{K}(B)$. Then*

$$\hat{p}_!(\hat{p}^* y \cup x) = y \cup \hat{p}_!(x) .$$

The proof can be copied from [BS07, Prop. 4.5]. \square

3.5 Localization

3.5.1 In the present subsection we show that a version of Segal's localization theorem [Seg68] holds for smooth K -theory. Let $B = [M/G]$ be an orbifold represented by the action of a *finite* group on a manifold M . Then we have the projection $\pi : [M/G] \rightarrow [*/G]$. For $g \in G$ let $[g] = \{hgh^{-1} | h \in G\}$ denote the conjugacy class g . Note that M^g is a smooth submanifold of M , and for $l \in G$ we have a diffeomorphism $h : M^l \rightarrow M^{hgh^{-1}}$. We choose, G -equivariantly, tubular neighbourhoods $M^h \subseteq \tilde{M}^h$ for all $h \in G$, set $\tilde{M}^{[g]} := \bigcup_{h \in [g]} \tilde{M}^h \subseteq M$, and we consider the open suborbifold $B^g := [\tilde{M}^{[g]}/G] \subseteq B$. We let $i : B^g \rightarrow B$ denote the inclusion. Note that B^g is considered as an orbifold approximation of the orbispace $[\bigcup_{h \in [g]} M^h/G]$ in the homotopy category of orbispaces.

3.5.2 Note that $\hat{K}^0([*/G]) \cong R(G)$, see 2.26. Therefore $\hat{K}(B)$ and $\hat{K}(B^g)$ becomes a $R(G)$ -modules via π^* , π_g^* , and the cup-product. In this way $i^* : \hat{K}(B) \rightarrow \hat{K}(B^g)$ is a map of $R(G)$ -modules.

If we identify, using the character, $R(G)$ with a subalgebra of the algebra of class functions on G ,

$$R(G) \subset R(G)_{\mathbb{C}} \cong \mathbb{C}[G]^G,$$

we see that $[g]$ gives rise to an ideal $I([g]) \subset R(G)$ consisting of all class functions which vanish at $[g]$.

For an $R(G)$ -module M we denote by $M_{I([g])}$ its localization at the ideal $I([g])$.

3.5.3

Theorem 3.26 *The restriction $i^* : \hat{K}(B) \rightarrow \hat{K}(B^g)$ induces, after localization at $I([g])$, an isomorphism*

$$i^* : \hat{K}(B)_{I([g])} \rightarrow \hat{K}(B^g)_{I([g])}.$$

Proof. We first observe that there is a natural $R(G)$ -module structure on $\Omega(LB)/\text{im}(d)$ such that the sequence

$$\rightarrow K(B) \xrightarrow{\text{ch}} \Omega(LB)/\text{im}(d) \xrightarrow{a} \hat{K}(B) \xrightarrow{I} K(B) \rightarrow$$

becomes an exact sequence of $R(G)$ -modules. Then we prove the analog of the localization theorem for equivariant forms. If we combine this with the topological case,

$$i^* : K(B)_{I([g])} \xrightarrow{\sim} K(B^g)_{I([g])},$$

proven in [Seg68] the result then follows from the Five Lemma.

Let us start with the $R(G)$ -module structure on $\Omega(LB)$. The map $\pi : B \rightarrow [*/G]$ induces a homomorphism $L\pi^* : \Omega(L[*/G]) \rightarrow \Omega(LB)$. We now use the identification $\Omega(L[*/G]) \cong \mathbb{C}[G]^G \cong R(G)_{\mathbb{C}}$.

3.5.4

Lemma 3.27 *The natural map*

$$Li^* : (\Omega(LB)/\mathbf{im}(d))_{I([g])} \rightarrow (\Omega(LB^g)/\mathbf{im}(d))_{I([g])}$$

is an isomorphism.

Proof. Since localization is an exact functor it commutes with taking quotients. Therefore it suffices to show that

$$Li^* : \ker(d_0(LB))_{I([g])} \rightarrow \ker(d_0(LB^{[g]}))_{I([g])}, \quad Li^* : \Omega(LB)_{I([g])} \rightarrow \Omega(LB^{[g]})_{I([g])}$$

are isomorphisms, where $d_k : \Omega^k \rightarrow \Omega^{k+1}$. We give the argument for the second case. The argument for the first isomorphism is similar.

Let CG denote the set of conjugacy classes in G . For $[h] \in CG$ we define the G -manifold $M^{[h]} := \bigsqcup_{l \in [h]} M^l$. Then

$$LB \cong \bigsqcup_{[h] \in CG} [M^{[h]}/G]$$

is a decomposition into a disjoint union of orbifolds. Accordingly we obtain a decomposition

$$\Omega(LB) \cong \bigoplus_{[h] \in CG} \Omega(M^{[h]})^G.$$

Let now $h \in G$ and $\langle h \rangle$ be the subgroup generated by h . If $\langle h \rangle \cap [g] = \emptyset$, then there exists an element $x \in R(G)$ with $x(g) \neq 0$ and $x_{\langle h \rangle} = 0$. Therefore, we get

$$\Omega(LB)_{I([g])} \cong \bigoplus_{[h] \in CG, \langle h \rangle \cap [g] \neq \emptyset} \Omega(M^{[h]})_{I([g])}^G.$$

A similar reasoning applies to B^g in place of B :

$$\Omega(LB^g)_{I([g])} \cong \bigoplus_{[h] \in CG, \langle h \rangle \cap [g] \neq \emptyset} \Omega((\tilde{M}^{[g]})^{[h]})_{I([g])}^G.$$

If $\langle h \rangle \cap [g] \neq \emptyset$, then the restriction $\Omega(M^{[h]})^G \rightarrow \Omega((\tilde{M}^{[g]})^{[h]})^G$ is an isomorphism. In fact, the map $(\tilde{M}^{[g]})^{[h]} \rightarrow M^{[h]}$ is a G -diffeomorphism. \square

This finishes the proof of the localization theorem. \square

4 The intersection pairing

4.1 The intersection pairing as an orbifold concept

4.1.1 We start with the definition of a trace on the complex representation ring $R(G)$ for a compact group G . Note that the underlying abelian group of $R(G)$ is the free \mathbb{Z} -module generated by the set \hat{G} of equivalence classes of irreducible complex representations of G . The unit $1 \in R(G)$ is represented by the trivial representation of G on \mathbb{C} .

We define

$$\mathrm{Tr}_G : R(G) \rightarrow \mathbb{Z}, \quad \mathrm{Tr}_G\left(\sum_{\pi \in \hat{G}} n_\pi \pi\right) := n_1.$$

The bilinear form

$$(\cdot, \cdot) : R(G) \otimes R(G) \rightarrow \mathbb{Z}, \quad (x, y) = \mathrm{Tr}_G(xy)$$

is non-degenerated. In fact

$$(\pi, \pi') = \begin{cases} 1 & \pi' = \pi^* \\ 0 & \text{else} \end{cases}, \quad (16)$$

where π^* denotes the dual representation of π .

The map Tr_G extends to the complexifications $R_{\mathbb{C}}(G) := R(G) \otimes \mathbb{C}$, the map

$$\mathrm{Tr}_G : R_{\mathbb{C}}(G) \rightarrow \mathbb{C}$$

will be denoted by the same symbol.

4.1.2 Let G be a finite group. We identify

$$R_{\mathbb{C}}(G) \cong \mathbb{C}[G]^G$$

(G acts by conjugations on itself) via

$$\sum_{\pi \in \hat{G}} n_\pi \pi \mapsto \sum_{\pi \in \hat{G}} n_\pi \chi_\pi,$$

where $\chi_\pi \in \mathbb{C}[G]^G$ denotes the character of π , then

$$\mathrm{Tr}_G(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Indeed, if π is non-trivial, then $\frac{1}{|G|} \sum_{g \in G} \chi_\pi(g) = 0$, and $\frac{1}{|G|} \sum_{g \in G} \chi_1(g) = 1$ by the orthogonality relations for characters.

4.1.3 Let G be finite. Note that $L[* / G] = [G / G]$, where G acts on itself by conjugation. We have $\Omega([G / G]) \cong \mathbb{C}[G]^G$. We thus define

$$\mathrm{Tr}_G : \Omega(L[* / G]) \rightarrow \mathbb{C}, \quad \mathrm{Tr}_G(f) := \frac{1}{|G|} \sum_{g \in G} f(g) .$$

Observe that for $x \in K([* / G]) \cong R(G)$ and $f = \mathbf{ch}(x)$ we have $\mathrm{Tr}_G(f) \in \mathbb{Z}$. Therefore we get an induced map

$$\mathrm{Tr}_G : \Omega(L[* / G]) / \mathrm{Im}(\mathbf{ch}) \rightarrow \mathbb{C} / \mathbb{Z} =: \mathbb{T} . \quad (17)$$

4.1.4 Let G be a compact Lie group and consider a compact G -manifold M with a G -equivariant K -orientation. In this situation we have a push-forward $f_! : K_G(M) \rightarrow K_G(*)$ along the projection $f : M \rightarrow *$. We define the intersection form

$$(\cdot, \cdot) : K_G(M) \otimes K_G(M) \xrightarrow{\cup} K_G(M) \xrightarrow{f_!} K_G(*) \cong R(G) \xrightarrow{\mathrm{Tr}_G} \mathbb{Z} . \quad (18)$$

4.1.5 In certain special cases this intersection form is compatible with induction. Let $G \hookrightarrow H$ be an inclusion of finite groups. Then $H \times_G M$ has an induced H -equivariant K -orientation.

Proposition 4.1 *If $G \hookrightarrow H$ is an inclusion of finite groups, then the following diagram commutes:*

$$\begin{array}{ccc} K_G(M) \otimes K_G(M) & \xrightarrow{(\cdot, \cdot)} & \mathbb{Z} \\ \downarrow \mathrm{ind}_G^H \otimes \mathrm{ind}_G^H & & \parallel \\ K_H(H \times_G M) \otimes K_H(H \times_G M) & \longrightarrow & \mathbb{Z} \end{array} .$$

Proof. The cup product and the integration are defined on the level of orbifolds. Hence they are compatible with induction, i.e.

$$\begin{array}{ccccc} K_G(M) \otimes K_G(M) & \xrightarrow{\cup} & K_G(M) & \xrightarrow{f_!^G} & R(G) \\ \downarrow \mathrm{ind}_G^H \otimes \mathrm{ind}_G^H & & \downarrow \mathrm{ind}_G^H & & \downarrow \mathrm{ind}_G^H \\ K_H(H \times_G M) \otimes K_H(H \times_G M) & \xrightarrow{\cup} & K_H(H \times_G M) & \xrightarrow{f_!^H} & R(H) \end{array}$$

commutes. We thus must show that the following diagram commutes

$$\begin{array}{ccc} R(G) & \xrightarrow{\mathrm{Tr}_G} & \mathbb{Z} \\ \downarrow \mathrm{ind}_G^H & & \parallel \\ R(H) & \xrightarrow{\mathrm{Tr}_H} & \mathbb{Z} \end{array} .$$

If $\pi \in \hat{G}$, then

$$\text{ind}_G^H(\pi) = [\mathbb{C}[H] \otimes V_\pi]^G ,$$

where we use the right G -action on $\mathbb{C}[H]$ in order to define the invariants. The H -action is induced by the left action. Since $\text{Res}_H^G 1 = 1$, by Frobenius reciprocity

$$\text{Tr}_H \text{ind}_G^H(\pi) = \text{Tr}_G V_\pi ,$$

as $\text{Tr}_H(V)$ counts the multiplicity of 1 in V . \square

If G/H is not zero-dimensional, then an H -equivariant K -orientation of M does not necessarily induce a G -equivariant K -orientation of $G \times_H M$. The problem is that G/H does not have, in general, an H -equivariant K -orientation.

4.1.6 Let $H \subseteq G$ be a normal subgroup of a finite group which acts freely on a closed equivariantly K -oriented G -manifold N with quotient $M := N/H$. Then the group $K := G/H$ acts on the closed equivariantly K -oriented G -manifold M . We have an equivalence of orbifolds $[N/G] \xrightarrow{\sim} [M/K]$ induced by the projection $\pi : N \rightarrow M$. Let $f^K : M \rightarrow *$ and $f^G : N \rightarrow *$ denote the corresponding projections to the point.

If V is a representation of G , then K acts on the subspace $\text{inv}^H(V) := V^H$ of H -invariants. We therefore get an induced homomorphism $\text{inv}^H : R(G) \rightarrow R(K)$.

Proposition 4.2 *The following diagram commutes:*

$$\begin{array}{ccccc} K_G(N) & \xrightarrow{f_!^G} & R(G) & \xrightarrow{\text{Tr}_G} & \mathbb{Z} \\ \cong \uparrow \pi^* & & \downarrow \text{inv}^H & & \parallel \\ K_K(M) & \xrightarrow{f_!^K} & R(K) & \xrightarrow{\text{Tr}_K} & \mathbb{Z} \end{array} .$$

Proof. We give an analytic argument. It follows from the relation $\text{inv}^G = \text{inv}^K \circ \text{inv}^H$, that the right square commutes. We now show that the left square commutes, too. Let $x \in K_K(M)$ be represented by a K -equivariant geometric family \mathcal{E} . Then $\pi^* \mathcal{E}$ is a G -equivariant geometric family over N . Then $f_!^K(x)$ is represented by the K -equivariant geometric family $f_!^K \mathcal{E}$ over the point $*$. The corresponding element in $R(K)$ is the representation of K on $\ker(D(f_!^K \mathcal{E}))$. Similarly, $f_!^G(x)$ is represented by the representation of G on $\ker(D(f_!^G \pi^* \mathcal{E}))$. The projection $f_!^G \pi^* \mathcal{E} \rightarrow f_!^K \mathcal{E}$ is a regular covering with covering group H , respecting all the geometric structure. In particular, we have $H(f_!^K \mathcal{E}) = H(f_!^G \pi^* \mathcal{E})^H$ and $\ker(D(f_!^K \mathcal{E})) = \ker(D(f_!^G \pi^* \mathcal{E}))^H$ as representations of K . This implies the commutativity of the left square. \square

4.1.7 In the following theorem we show that the intersection pairing is a well-defined concept at least for orbifolds which admit a presentation as a quotient of a closed equivariantly K -oriented G -manifold for a finite group G .

Theorem 4.3 *If B is an orbifold which admits a presentation $B \cong [M/G]$ for a finite group G such that $[M/G] \rightarrow [*/G]$ is K -oriented, then (18) induces a well-defined intersection pairing*

$$K(B) \otimes K(B) \rightarrow \mathbb{Z} .$$

Proof. We choose a presentation $B \cong [M/G]$ and define the pairing such that

$$\begin{array}{ccc} K(B) \otimes K(B) & \xrightarrow{(\dots)} & \mathbb{Z} \\ \downarrow \cong & & \parallel \\ K_G(M) \otimes K_G(M) & \xrightarrow{(\dots)} & \mathbb{Z} \end{array}$$

commutes. We must show that this construction does not depend on the choice of the presentation. Let $B \cong [M'/G']$ be another presentation.

We use the set-up of [PS07] where the 2-category of orbifolds is identified with a localization of a full subcategory of Lie groupoids.

Let $G \rtimes M$ and $G' \rtimes M'$ be the action groupoids. Since they represent the same orbifold B the isomorphism $G \rtimes M \cong G' \rtimes M'$ in this localization is represented by a diagram

$$\begin{array}{ccc} & \mathcal{K} & \\ v \swarrow & & \searrow u \\ G \rtimes M & & G' \rtimes M' \end{array}$$

where \mathcal{K} is a Lie groupoid and v and u are essential equivalences. By [PS07, Prop. 6.1] this diagram is isomorphic (in the category of morphisms) between $G \rtimes M$ and $G' \rtimes M'$ to a diagram of the form

$$\begin{array}{ccc} & (G \times G') \rtimes N & \\ u \swarrow & & \searrow v \\ G \rtimes M & & G' \rtimes M' \end{array} ,$$

where now $u : N \rightarrow M$ and $N \rightarrow M'$ are equivariant maps over the projections $G \times G' \rightarrow G$ and $G \times G' \rightarrow G'$.

For $\bar{x}, \bar{y} \in K(B)$ let $x, y \in K_G(M)$ and $x', y' \in K_{G'}(M')$ be the corresponding elements under $K(B) \cong K_G(M) \cong K_{G'}(M')$. We have $u^*x = v^*x'$ and $u^*y = v^*y'$. The subgroups $G', G \subseteq G \times G'$ are normal and act freely on N . By Proposition 4.2 we get

$$\mathrm{Tr}_G(f_!^G(x \cup y)) = \mathrm{Tr}_{G \times G'}(f_!^{G \times G'}(u^*x \cup u^*y)) = \mathrm{Tr}_{G \times G'}(f_!^{G \times G'}(v^*x' \cup v^*y')) = \mathrm{Tr}_{G'}(f_!^{G'}(x' \cup y')) ,$$

where $f^G, f^{G \times G'}$ and $f^{G'}$ are the corresponding projections to the point. \square

4.2 The flat part and homotopy theory

4.2.1 If B is an orbifold, then we can consider the flat part $U(B) := \ker(R : \hat{K}(B) \rightarrow \Omega(LB))$ of the smooth K -theory of B . The functor $B \mapsto U(B)$ is homotopy invariant. The main goal of the present section is to identify this functor in homotopy-theoretic terms. In the language of [BS09], we are going to show that U is topological.

4.2.2 A G -equivariant K -orientation provides a G -equivariant fundamental class $[M] \in K_{\dim M}^G(M)$. Let us represent K -homology in the KK -picture, i.e.

$$K_n^G(M) := KK^G(C(M), \mathbf{Cliff}(\mathbb{R}^n)) .$$

The equivariant fundamental class is represented by the equivariant Kasparov module $(L^2(M, E), D)$, where $E = P \times_{Spin^c(n)} \mathbf{Cliff}(\mathbb{R}^n)$ is the Dirac bundle associated to the $Spin^c(n)$ -principal bundle P determined by the K -orientation. Note that the Dirac operator D of E commutes with the action of $\mathbf{Cliff}(\mathbb{R}^n)$ from the right.

Let $x \in M$ and G_x be the stabilizer group. Then $T_x M \cong T_x Gx \oplus N$, where G_x acts non-trivially on N . A tubular neighbourhood of Gx can be identified with $U_x := G \times_{G_x} V_x$, where $V_x \subset N$ is a disc. The restriction of the fundamental class to U_x gives an element

$$[M]_{U_x} \in K_n^G(U_x, \partial U_x) \cong K_n^{G_x}(V_x, \partial V_x) .$$

Note that V_x admits a G -equivariant $Spin^c$ -structure. It is uniquely determined by the K -orientation of M upto a choice of a G_x -equivariant $Spin^c$ -structure on the vector space $T_{G_x}(G/G_x)$. The $Spin^c$ -structure gives an equivariant Thom class and the Thom isomorphism

$$R(G_x) \cong K_n^{G_x}(V_x, \partial V_x)$$

of $R(G_x)$ -modules. The characterizing property of a fundamental class is that $[M]_{U_x}$ is a generator of the $R(G_x)$ -module $K_n^{G_x}(V_x, \partial V_x)$ for every $x \in M$. This condition does not depend on the choice of the $Spin^c$ -structure on $T_{G_x}(G/G_x)$.

The equivariant K -theory fundamental class induces a Poincaré duality isomorphism

$$P : K_G^*(M) \xrightarrow{\cap [M]} K_{n-*}^G(M) .$$

Note that the intersection pairing can be written in the form

$$K_G(M) \otimes K_G(M) \xrightarrow{1 \otimes P} K_G(M) \otimes K^G(M) \xrightarrow{eval} R(G) \xrightarrow{\mathrm{Tr}_G} \mathbb{Z} .$$

4.2.3 Recall that $\mathbb{T} := \mathbb{C}/\mathbb{Z}$. We define a new G -equivariant cohomology theory which associates to a G -space M the group

$$k_G^{\mathbb{T}}(M) := \text{Hom}_{\mathbb{Z}}(K^G(M), \mathbb{T}) .$$

In fact, since \mathbb{T} is a divisible and hence injective abelian group, the long exact sequences for K^G induce long exact sequences for $k_G^{\mathbb{T}}$.

Complex conjugation in \mathbb{T} induces a natural involution on $k_G^{\mathbb{T}}(M)$. Its fixed points will be denoted by $k_G^{\mathbb{R}/\mathbb{Z}}(M)$. In other words, $k_G^{\mathbb{R}/\mathbb{Z}}(M) \cong \text{Hom}_{\mathbb{Z}}(K^G(M), \mathbb{R}/\mathbb{Z}) \subseteq \text{Hom}_{\mathbb{Z}}(K^G(M), \mathbb{T})$. If M is equivariantly K -oriented, then we have a natural pairings

$$K_G(M) \otimes k_G^{\mathbb{T}}(M) \xrightarrow{\text{ev}} \mathbb{T} , \quad K_G(M) \otimes k_G^{\mathbb{R}/\mathbb{Z}}(M) \xrightarrow{\text{ev}} \mathbb{R}/\mathbb{Z} \quad (19)$$

given by

$$x \otimes \phi \mapsto \phi(P(x)) .$$

Since P is an isomorphism, by Pontryagin duality this pairing is non-degenerated in the sense that it induces a monomorphism

$$K_G(M) \hookrightarrow \text{Hom}_{\text{Ab}}(k_G^{\mathbb{T}}(M), \mathbb{T})$$

and isomorphisms

$$k_G^{\mathbb{T}}(M) \cong \text{Hom}_{\text{Ab}}(K_G(M), \mathbb{T}) , \quad K_G(M) \cong \text{Hom}_{\text{Ab}}(k_G^{\mathbb{R}/\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z}) .$$

For the latter, we use only continuous homomorphisms and the usual topology on $k_G^{\mathbb{R}/\mathbb{Z}}(M)$ as a dual of a discrete group.

4.2.4 Let us represent G -equivariant homology theories h^G by G -spectra \underline{h}^G . Since $R_{\mathbb{C}}(G) := R(G) \otimes \mathbb{C}$ is a flat $R(G)$ -module we get a new homology theory

$$K_{\mathbb{C}}^G(M) := K^G(M) \otimes_{R(G)} R_{\mathbb{C}}(G) \cong K^G(M) \otimes_{\mathbb{Z}} \mathbb{C} .$$

The natural transformation $K^G \rightarrow K_{\mathbb{C}}^G$ extends to a fibre sequence of spectra

$$\underline{K}^G \rightarrow \underline{K}_{\mathbb{C}}^G \rightarrow \underline{K}_{\mathbb{T}}^G$$

which defines the spectrum $\underline{K}_{\mathbb{T}}^G$. We let $K_{\mathbb{T}}^{\mathbb{T}}$ denote the cohomology theory represented by $\underline{K}_{\mathbb{T}}^G$. It is a K_G -module theory. In a similar manner, if we set $K_{\mathbb{R}}^G(M) := K^G(M) \otimes \mathbb{R}$ and consider $\underline{K}^G \rightarrow \underline{K}_{\mathbb{R}}^G \rightarrow \underline{K}_{\mathbb{R}/\mathbb{Z}}^G$, then we define a K_G -module cohomology theory $K_G^{\mathbb{R}/\mathbb{Z}}$.

4.2.5 The evaluation extends to the complexifications

$$\text{eval}_{\mathbb{C}} : K_G^{\mathbb{C}}(M) \otimes K^G(M) \rightarrow R_{\mathbb{C}}(G) ,$$

and there is a natural transformation

$$K_G^{\mathbb{C}}(M) \rightarrow k_G^{\mathbb{T}}(M) , \quad K_G^{\mathbb{C}}(M) \ni x \mapsto \{K^G(M) \ni z \mapsto [\text{Tr}_G(\text{eval}(x \otimes z))]_{\mathbb{C}/\mathbb{Z}} \in \mathbb{T}\} .$$

The composition

$$K_G(M) \rightarrow K_G^{\mathbb{C}}(M) \rightarrow k_G^{\mathbb{T}}(M)$$

vanishes. On the level of spectra we thus have a factorization i in

$$\begin{array}{ccc} \underline{K_G} & & \\ \downarrow & \dashrightarrow^0 & \\ \underline{K_G^{\mathbb{C}}} & \longrightarrow & \underline{k_G^{\mathbb{T}}} \\ \downarrow & \dashrightarrow^i & \\ \underline{K_G^{\mathbb{T}}} & & \end{array}$$

Since all our spectra are in fact even, i.e. their homotopy groups vanish in odd degrees, this factorization is unique.

In a similar manner we define a transformation $i : K_G^{\mathbb{R}/\mathbb{Z}} \rightarrow k_G^{\mathbb{R}/\mathbb{Z}}$.

4.2.6 Let $H \subseteq G$ be a closed subgroup. Then we have

$$K_G(G/H) \cong K_H(*) \cong R(H) , \quad K_G^{\mathbb{C}}(G/H) \cong K_H^{\mathbb{C}}(*) \cong R_{\mathbb{C}}(H) ,$$

and hence, again as our spectra are even,

$$K_G^{\mathbb{T}}(G/H) \cong R_{\mathbb{C}}(H)/R(H) \cong R(H) \otimes \mathbb{T} .$$

Furthermore

$$k_G^{\mathbb{T}}(G/H) \cong \text{Hom}_{\text{Ab}}(K^G(G/H), \mathbb{T}) \cong \text{Hom}_{\text{Ab}}(K^H(*), \mathbb{T}) \cong \text{Hom}_{\text{Ab}}(R(H), \mathbb{T}) .$$

Let $x \in K_G^{\mathbb{C}}(G/H) \cong R_{\mathbb{C}}(H)$ and $[x] \in K_G^{\mathbb{T}}(G/H) \cong R_{\mathbb{C}}(H)/R(H)$ be the induced class. Then we have for $i : K_G^{\mathbb{C}}(G/H) = R_{\mathbb{C}}(H) \rightarrow k_G^{\mathbb{T}}(G/H) = \text{Hom}_{\text{Ab}}(R(H), \mathbb{T})$

$$i([x]) \in k_G^{\mathbb{T}}(G/H) \cong \text{Hom}_{\text{Ab}}(R(H), \mathbb{T}) , \quad i(x)(y) = [\text{Tr}_G(yx)] = [(y, x)] , \quad \forall y \in R(H) .$$

Because of (16) the map i is injective. It is surjective if and only if $R(H)$ is finitely generated, i.e. if H is finite.

Lemma 4.4 *If G is finite, then $i : K_G^{\mathbb{T}} \rightarrow k_G^{\mathbb{T}}$ and $i : K_G^{\mathbb{R}/\mathbb{Z}} \rightarrow k_G^{\mathbb{R}/\mathbb{Z}}$ are equivalences. If G is compact and acts on a compact manifold with finite stabilizers, then $i : K_G^{\mathbb{T}}(M) \rightarrow k_G^{\mathbb{T}}(M)$ and $i : K_G^{\mathbb{R}/\mathbb{Z}}(M) \rightarrow k_G^{\mathbb{R}/\mathbb{Z}}(M)$ are isomorphisms.*

Proof. We only discuss the complex case. The real is similar. The first statement follows from the discussion above. The second is proved by induction over G -cells (which are of the form $G/H \times D^n$ with finite $H \subset G$) using Mayer-Vietoris and again that $i : K_G^{\mathbb{T}}(G/H) = K^{\mathbb{T}}(H) \rightarrow k_G^{\mathbb{T}}(G/H) \cong \text{Hom}_{\text{Ab}}(R(H), \mathbb{T})$ is an isomorphism for finite subgroups $H \subset G$. \square

Corollary 4.5 *If G is a compact group which acts on a G -equivariantly K -oriented closed manifold with finite stabilizers, then the pairing*

$$\langle \cdot, \cdot \rangle : K_G(M) \otimes K_G^{\mathbb{T}}(M) \xrightarrow{\cup} K_G^{\mathbb{T}}(M) \xrightarrow{f_!} R(G) \xrightarrow{\text{Tr}_G} \mathbb{T}$$

is a non-degenerated pairing in the sense that the induced map

$$K_G(M) \rightarrow \text{Hom}_{\text{Ab}}(K_G^{\mathbb{T}}(M), \mathbb{T})$$

is a monomorphism, and that

$$K_G^{\mathbb{T}}(M) \rightarrow \text{Hom}_{\text{Ab}}(K_G(M), \mathbb{T}) , \quad K_G^{\mathbb{R}/\mathbb{Z}}(M) \rightarrow \text{Hom}_{\text{Ab}}(K_G(M), \mathbb{R}/\mathbb{Z})$$

are isomorphisms.

Proof. Indeed, under the isomorphism $i : K_G^{\mathbb{T}}(M) \xrightarrow{\sim} k_G^{\mathbb{T}}(M)$ the pairing $\langle \cdot, \cdot \rangle$ is identified with the evaluation pairing (19). \square

4.2.7 Let B be a compact orbifold.

Definition 4.6 *We define the flat K -theory of B (or its real part, respectively,) as the kernel of the curvature morphisms:*

$$U(B) := \ker(R : \hat{K}(B) \rightarrow \Omega(LB)) , \quad U^{\mathbb{R}}(B) := \ker(R : \hat{K}_{\mathbb{R}}(B) \rightarrow \Omega_{\mathbb{R}}(LB)) .$$

If $B = [M/G]$ for a compact Lie group G acting on a compact manifold with finite stabilizers, then we will also write

$$U_G(M) := U([M/G]) , \quad U_G^{\mathbb{R}}(M) := U^{\mathbb{R}}([M/G]) .$$

Note that, as always for smooth cohomology theories, $U(B)$ fits into a long exact sequence

$$\dots \rightarrow K^{n-1}(B) \rightarrow H^{n-1}(LB) \rightarrow U^n(B) \rightarrow K^n(B) \rightarrow H^n(LB) \rightarrow \dots .$$

Using the notation $H_G(M) := H(L[M/G])$, $K_G(M) = K([M/G])$ we get a sequence

$$\dots \rightarrow K_G(M) \rightarrow H_G(M) \rightarrow U_G(M) \rightarrow K_G(M) \rightarrow H_G(M) \rightarrow \dots .$$

4.2.8 We want to define maps

$$j : U_G(M) \rightarrow K_G^{\mathbb{T}}(M) , \quad j : U_G^{\mathbb{R}}(M) \rightarrow K_G^{\mathbb{R}/\mathbb{Z}}(M)$$

by constructing the lower horizontal map in the diagrams

$$\begin{array}{ccc} & K_G^{\mathbb{T}}(M) & \\ j \nearrow & & \searrow i \\ U_G(M) & \xrightarrow{j_G} & k_G^{\mathbb{T}}(M) \end{array} \quad \cong \quad \begin{array}{ccc} & K_G^{\mathbb{R}/\mathbb{Z}}(M) & \\ j \nearrow & & \searrow i \\ U_G^{\mathbb{R}}(M) & \xrightarrow{j_G} & k_G^{\mathbb{R}/\mathbb{Z}}(M) \end{array} .$$

Its construction involves integration $\int_{[M/G]/[*G]}^{\hat{K}}$ of flat classes along the map $[M/G] \rightarrow [*G]$ with respect some choice of a smooth refinement of the topological K -orientation. The integral does not depend on that choice.

In order to stay in the category of orbifolds we must assume that G is a *finite* group. We set for $\xi \in K^G(M)$, $u \in U_G(M)$

$$j_G(u)(\xi) := \text{Tr}_G \left(\int_{[M/G]/[*G]}^{\hat{K}} u \cup \widehat{P^{-1}(\xi)} \right) \in \mathbb{T} .$$

Here $\widehat{P^{-1}(\xi)} \in \hat{K}_G(M)$ denotes a smooth refinement of the Poincaré dual of ξ (since it is copped with a flat class the construction will not depend on this choice),

$$\int_{[M/G]/[*G]}^{\hat{K}} u \cup \widehat{P^{-1}(\xi)} \in U([*G]) \cong \Omega([G/G]) / \mathbf{im}(\mathbf{ch}) ,$$

and we use the factorization (17) of the trace map. We indicate by a superscript in which theory the integration is understood. It follows from the compatibility (15) that the integral of a flat class is again a flat class. It is easy to see that j_G restricts to the real parts.

Theorem 4.7 *Assume that G is a finite group, and that M is an equivariantly K -oriented closed G -manifold. Then the maps*

$$j : U_G(M) \rightarrow K_G^{\mathbb{T}}(M) , \quad j : U_G^{\mathbb{R}}(M) \rightarrow K_G^{\mathbb{R}/\mathbb{Z}}(M)$$

are isomorphisms.

Proof. We discuss the complex case. The real case is similar. Since $[M/G]$ is an orbifold, the Chern character induces an isomorphism

$$\mathbf{ch}_G : K_G^{\mathbb{C}}(M) \xrightarrow{\sim} H_G(M) .$$

We consider the following diagram with exact horizontal sequences

$$\begin{array}{ccccccccc}
K_G(M) & \xrightarrow{\mathbf{ch}_G} & H_G(M) & \xrightarrow{a} & U_G(M) & \xrightarrow{\beta} & K_G(M) & \xrightarrow{\mathbf{ch}_G} & H_G(M) & . \\
\parallel & & \uparrow \mathbf{ch}_G \cong & & \downarrow j & & \parallel & & \uparrow \mathbf{ch}_G \cong & \\
K_G(M) & \longrightarrow & K_G^{\mathbb{C}}(M) & \longrightarrow & K_G^{\mathbb{T}}(M) & \xrightarrow{\delta} & K_G(M) & \longrightarrow & K_G^{\mathbb{C}}(M) &
\end{array} \quad (20)$$

Lemma 4.8 *The diagram commutes.*

If we assume this lemma it follows by the Five Lemma that j is an isomorphism. \square

Note that all terms in (20) are $K_G(M)$ -modules, and all transformations are $K_G(M)$ -module maps. Moreover, all transformations are compatible with integration. We will use these facts later.

The basic idea of our proof of Lemma 4.8 is the following: we show how to realize all relevant classes as push-forwards of classes on $M_n \times M$, where M_n is the Moore space for $\mathbb{Z}/n\mathbb{Z}$ (a smooth manifold model). It therefore suffices to check commutativity for $M_n \times M$.

The reduced cohomology of $M_n \times M$ is only torsion, so that the vector spaces $K_G^{\mathbb{C}}(M_n \times M)$ are trivial and the diagram (20) simplifies accordingly. Moreover, we will show that because of the non-degeneracy of the pairing, $\delta \circ j = \beta$ follows if $\int_{M \times M_n / M_n} \delta \circ j = \int_{M \times M_n / M_n} \delta \circ \beta$. The pairing is of course between K_G and $K_G^{\mathbb{T}}$ and is given by cup product and integration. But, because of the triviality of $K_G^{\mathbb{C}}(M_n \times M)$, for $M_n \times M$ the functor $K_G^{\mathbb{T}}$ can be replaced by K_G .

By compatibility with cup product and integration, it suffices therefore to check the commutativity for M_n . But this is actually a statement in non-equivariant theory and already known to be correct.

4.2.9 We now show Lemma 4.8 in detail. It is clear that the first and the fourth square commute. Next we show that the second square commutes. Let $x \in K_G^{\mathbb{C}}(M)$. Then we must show that $j_G(a(\mathbf{ch}_G(x))) = \phi(x)$, where $\phi : K_G^{\mathbb{C}}(M) \rightarrow k_G^{\mathbb{C}}(M) \rightarrow k_G^{\mathbb{T}}(M)$ is the

natural map. To this end let $\xi \in K^G(M)$. Then we have

$$\begin{aligned}
\phi(x)(\xi) &= \mathrm{Tr}_G \left[\int_M^{K_G} x \cup P^{-1}(\xi) \right]_{\mathbb{C}/\mathbb{Z}} \\
&= \mathrm{Tr}_G \left[\int_{L[M/G]/L[*]/G}^{H_G} \hat{\mathbf{A}}_\rho^c(LM) \cup \mathbf{ch}_G(x) \cup \mathbf{ch}_G(P^{-1}(\xi)) \right]_{\mathbb{C}/\mathbb{Z}} \\
&= \mathrm{Tr}_G \left[\int_{[M/G]/[*]/G}^{\hat{K}} a(\mathbf{ch}_G(x)) \cup \mathbf{ch}_G(P^{-1}(\xi)) \right]_{\mathbb{C}/\mathbb{Z}} \\
&= \mathrm{Tr}_G \left[\int_{[M/G]/[*]/G}^{\hat{K}} a(\mathbf{ch}_G(x)) \cup \widehat{P^{-1}(\xi)} \right]_{\mathbb{C}/\mathbb{Z}} \\
&= j_G(a(\mathbf{ch}_G(x)))(\xi)
\end{aligned}$$

4.2.10 Finally we show that the third square in (20) commutes. The argument is surprisingly complicated. First of all note that $\mathrm{im}(\beta) = K_G^{\mathrm{tors}}(M) \subseteq K_G(M)$ is the torsion subgroup. Let $t \in K_G^{\mathrm{tors}}(M)$. Then there exists an integer $n \in \mathbb{N}$ such that $nt = 0$. Let $f : S^1 \rightarrow S^1$ be the degree n -covering. We form the mapping cone sequence

$$\begin{array}{ccc}
S^1 & \xrightarrow{f} & S^1 & \longrightarrow & C(f) & , \\
& & & \searrow \pi & \uparrow \sim & \\
& & & & M_n &
\end{array} \tag{21}$$

where M_n is a compact manifold with boundary which approximates the cone $C(f)$. It is a smooth model of the Moore space of $\mathbb{Z}/n\mathbb{Z}$. Using the long exact sequences of reduced cohomology and K -theory groups

$$\begin{aligned}
\tilde{H}(S^1) &\xleftarrow{n} \tilde{H}(S^1) \xleftarrow{\pi^*} \tilde{H}(M_n) \xleftarrow{\delta} \\
\tilde{K}(S^1) &\xleftarrow{n} \tilde{K}(S^1) \xleftarrow{\pi^*} \tilde{K}(M_n) \xleftarrow{\delta}
\end{aligned}$$

we get

$$\tilde{H}^*(M_n) \cong 0, \quad H^*(M_n) \cong \begin{cases} \mathbb{C} & * = 0 \\ 0 & * \geq 1 \end{cases} .$$

and

$$\tilde{K}^*(M_n) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases}, \quad K^*(M_n) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases} .$$

This implies that

$$U^0(M_n) \cong \mathbb{Z}/n\mathbb{Z}, \quad U^1(M_n) \cong \mathbb{T} .$$

In particular, we see that $\beta : U^0(M_n) \rightarrow K^{0,\mathrm{tors}}(M_n)$ is an isomorphism.

We now analyse the map $\pi^* : U^0(M_n) \rightarrow U^0(S^1) \cong \mathbb{T}$. We know from [BS07] that the map $j : U \rightarrow K^{\mathbb{T}}$ induces an isomorphism of reduced cohomology theories (i.e. the non-equivariant version of the Theorem 4.7 holds true). Since U is a reduced cohomology theory we have a mapping cone sequence

$$U^0(S^1) \xleftarrow{n} U^0(S^1) \xleftarrow{\pi^*} U^0(M_n) \xleftarrow{\delta} U^1(S^1) \xleftarrow{1} U^1(S^1) ,$$

where we use the known actions of f^* on $U^0(S^1) \cong H^1(S^1, \mathbb{Z}) \otimes \mathbb{T}$ and $U^1(S^1) \cong H^0(S^1, \mathbb{Z}) \otimes \mathbb{T}$. We get

$$\begin{array}{ccccccc} 0 & \longrightarrow & U^0(M_n) & \longrightarrow & U^0(S^1) & \longrightarrow & U^0(S^1) \ . \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mathbb{T} & \xrightarrow{n} & \mathbb{T} \end{array}$$

In particular we see that

$$I_n := \int_{S^1}^U \circ \pi^* \circ \beta^{-1} : \mathbb{Z}/n\mathbb{Z} \cong K^0(M_n) \rightarrow \mathbb{T} \cong U^{-1}(*)$$

is the usual embedding. Note that in the non-equivariant case we have $j \circ \delta = \beta$. Therefore, we also have $I_n = \int_{S^1}^{K^{\mathbb{T}}} \circ \pi^* \circ \delta^{-1}$.

The product of the mapping cone sequence (21) with M induces a long exact sequence

$$K_G(S^1 \times M, * \times M) \xleftarrow{(f \times \text{id})^*} K_G(S^1 \times M, * \times M) \xleftarrow{(\pi \times \text{id})^*} K_G(M_n \times M, * \times M) \xleftarrow{\delta_1} K_G(S^1 \times M, * \times M)$$

in equivariant K -theory. Note that $K_G(M_n \times M, * \times M)$ is a torsion group which is a summand

$$K_G(M_n \times M) \cong K_G(M_n \times M, * \times M) \oplus K_G(M) .$$

Further note that

$$K_G(S^1 \times M, * \times M) \oplus K_G(M) \cong K_G(S^1 \times M) .$$

We now consider

$$\text{or}_{S^1} \times t \in K_G(S^1 \times M, * \times M) .$$

Since

$$(f \times \text{id})^*(\text{or}_{S^1} \times t) = n \text{or}_{S^1} \times t = \text{or}_{S^1} \times nt = 0$$

we can choose a class $z \in K_G(M_n \times M, * \times M)$ such that $(\pi \times \text{id})^*(z) = \text{or}_{S^1} \times t$. Since $K_G(M_n \times M, * \times M)$ is a torsion group, we can further find an element $\hat{z} \in U_G(M_n \times M)$ such that $\beta(\hat{z}) = z$. Since β is natural we have

$$\beta \circ (\pi \times \text{id})^*(\hat{z}) = \text{or}_{S^1} \times t .$$

Furthermore, we know that β commutes with $\int_{S^1}^{U_G}$. Therefore we have

$$\beta \circ \int_{[S^1 \times M/G]/[M/G]}^{U_G} \circ (\pi \times \text{id})^*(\hat{z}) = t .$$

We define

$$\hat{t} := \int_{[S^1 \times M/G]/[M/G]}^{U_G} \circ (\pi \times \text{id})^*(\hat{z}) \in U_G(M) .$$

If we let t run over all torsion classes in $K_G(M)$, then the set of corresponding $\hat{t} \in U_G(M)$ generates $U_G(M)/\text{im}(a)$. Therefore in order to show that the third square in (20) commutes it suffices to show that $\beta(\hat{t}) = \delta(j(\hat{t}))$ for all these classes.

From now on we assume that the degree of t has the opposite parity as $\dim(M)$. We calculate using the projection formula

$$\begin{aligned} \text{Tr}_G \circ \int_{[M/G]/[*G]}^{U_G} \hat{t} &= \text{Tr}_G \circ \int_{[S^1 \times M/G]/[*G]}^{U_G} (\pi \times \text{id})^*(\hat{z}) \\ &= \text{Tr}_G \circ \int_{[S^1/G]/[*G]}^{U_G} \circ \int_{[S^1 \times M/G]/[S^1/G]}^{U_G} (\pi \times \text{id})^*(\hat{z}) \\ &= \int_{S^1}^U \circ \pi^* \circ \text{Tr}_G \circ \int_{[M_n \times M]/[M_n/G]}^{U_G} \hat{z} \\ &= \int_{S^1}^U \circ \pi^* \circ \beta^{-1} \circ \text{Tr}_G \circ \beta \circ \int_{[M_n \times M/G]/[M_n/G]}^{U_G} \hat{z} \\ &= \int_{S^1}^U \circ \pi^* \circ \beta^{-1} \circ \text{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} \beta(\hat{z}) \\ &= \int_{S^1}^U \circ \pi^* \circ \beta^{-1} \circ \text{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} z \\ &= I_n(\text{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} z) . \end{aligned}$$

We also know that $\text{im}(\delta)$ is the torsion subgroup. Therefore we can find $\tilde{z} \in K_G^{\mathbb{T}}(M_n \times M)$ such that $\delta(\tilde{z}) = z$. We have

$$\delta \circ (\pi \times \text{id})^*(\tilde{z}) = \text{or}_{S^1} \times t .$$

Furthermore, we have

$$\delta \circ \int_{S^1 \times M/M}^{K_G^{\mathbb{T}}} \circ (\pi \times \text{id})^*(\tilde{z}) = t .$$

We define

$$\tilde{t} := \int_{S^1 \times M/M}^{K_G^{\mathbb{T}}} \circ (\pi \times \text{id})^*(\tilde{z}) .$$

Then we have

$$\begin{aligned}
\mathrm{Tr}_G \circ \int_M^{K_G^\mathbb{T}} \tilde{t} &= \mathrm{Tr}_G \circ \int_{S^1 \times M}^{K_G^\mathbb{T}} (\pi \times \mathrm{id})^*(\tilde{z}) \\
&= \mathrm{Tr}_G \circ \int_{S^1}^{K_G^\mathbb{T}} \circ \int_{S^1 \times M/S^1}^{K_G^\mathbb{T}} (\pi \times \mathrm{id})^*(\tilde{z}) \\
&= \int_{S^1}^{K^\mathbb{T}} \circ \pi^* \circ \mathrm{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} \tilde{z} \\
&= \int_{S^1}^{K^\mathbb{T}} \circ \pi^* \circ \delta^{-1} \circ \mathrm{Tr}_G \circ \delta \circ \int_{M_n \times M/M_n}^{K_G} \tilde{z} \\
&= \int_{S^1}^{K^\mathbb{T}} \circ \pi^* \circ \delta^{-1} \circ \mathrm{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} \delta(\tilde{z}) \\
&= \int_{S^1}^U \circ \pi^* \circ \delta^{-1} \circ \mathrm{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} z \\
&= I_n(\mathrm{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} z) .
\end{aligned}$$

We now show that $\delta \circ j(\hat{t}) = t$. Because of the $K_G(M)$ -module structure, in the calculation above we can replace t by $t \cup \mathrm{pr}_M^*(P^{-1}(\xi))$ for $\xi \in K^G(M)$. Then \hat{t} , \tilde{t} and z get replaced by $\hat{t} \cup \widehat{P^{-1}(\xi)}$, $\tilde{t} \cup P^{-1}(\xi)$ and $z \cup \mathrm{pr}_M^*(P^{-1}(\xi))$. For all $\xi \in K^G(M)$ such that $\deg(\xi) + \deg(t) \equiv \dim(M) + 1$ we have

$$\begin{aligned}
j_G(\hat{t})(\xi) &= \mathrm{Tr}_G \circ \int_{[M/G]/[*G]}^{U_G} \hat{t} \cup \widehat{P^{-1}(\xi)} \\
&= I_n(\mathrm{Tr}_G \circ \int_{M_n \times M/M_n}^{K_G} (z \cup \mathrm{pr}_M^*(P^{-1}(\xi)))) \\
&= \mathrm{Tr}_G \circ \int_M^{K_G^\mathbb{T}} \tilde{t} \cup P^{-1}(\xi) \\
&= i(\tilde{t})(\xi) .
\end{aligned}$$

Therefore $\delta \circ j(\hat{t}) = \delta(\tilde{t}) = t = \beta(\hat{t})$. □

4.3 Non-degeneracy of the intersection pairing

4.3.1 In this subsection we construct intersection pairings

$$\hat{K}(B) \otimes \hat{K}(B) \rightarrow \mathbb{T} , \quad \hat{K}_\mathbb{R}(B) \otimes \hat{K}_\mathbb{R}(B) \rightarrow \mathbb{R}/\mathbb{Z}$$

for a K -oriented orbifold B (Definition 4.11, Proposition 4.12). The main result is Theorem 4.13 which states that the intersection pairing is non-degenerated.

4.3.2 In the following, for $x \in \hat{K}([*/G])$ let $x^1 \in \hat{K}^1([*/G])$ denote the degree-one component.

Fix G, H, K and M, N as in 4.1.6. In addition we assume that the map $[N/G] \rightarrow [*/G]$ has a smooth K -orientation. Then $[M/K] \rightarrow [*/K]$ has an induced smooth K -orientation, and the integration maps $\hat{f}_!^G$ and $\hat{f}_!^K$ along the projections $f^K : [M/K] \rightarrow [*/K]$ and $f^G : [N/G] \rightarrow [*/G]$ are defined.

4.3.3 We define the map $\text{av}^H : \Omega(L[*/G]) \rightarrow \Omega(L[*/K])$ as the average over H -orbits $\mathbb{C}[G]^G \rightarrow \mathbb{C}[K]^K$,

$$\text{av}^H(f)(Hg) := \frac{1}{|H|} \sum_{h \in H} f(hg).$$

If V is a representation of G with character χ_V , then $\text{av}^H(\chi_V)$ is the character of V^H as a representation of K . Therefore the left square in

$$\begin{array}{ccccccc} R(G) & \longrightarrow & \Omega(L[*/G]) & \longrightarrow & \hat{K}^1([*/G]) & \longrightarrow & 0 \\ & & \downarrow \text{inv}^H & & \downarrow \text{av}^H & & \downarrow \text{av}^H \\ R(K) & \longrightarrow & \Omega(L[*/K]) & \longrightarrow & \hat{K}^1([*/K]) & \longrightarrow & 0 \end{array}$$

commutes, and this gives the dotted arrow.

4.3.4

Proposition 4.9 *The diagram*

$$\begin{array}{ccccc} \hat{K}([M/K]) & \xrightarrow{(\hat{f}_!^K \dots)^1} & \hat{K}^1([*/K]) & \xrightarrow{\text{Tr}_K} & \mathbb{T} \\ & & \uparrow \text{av}^H & & \parallel \\ \hat{K}([N/G]) & \xrightarrow{(\hat{f}_!^G \dots)^1} & \hat{K}^1([*/G]) & \xrightarrow{\text{Tr}_G} & \mathbb{T} \end{array} \quad (22)$$

commutes.

Proof. Since Tr_K and Tr_G are given as averages over K and G , and the average in stages, first over H and then over K is equal to the average over G , we see that the right square commutes.

We now show that the left square commutes. Consider $\hat{x} = [\mathcal{E}, \rho] \in \hat{K}^1([M/K])$, where we actually think of \mathcal{E} as a K -equivariant geometric family over M . The class $f_!^K(\hat{x})$ is according to (12) represented by

$$[f_!^K \mathcal{E}, \int_{L[M/K]/L[*/K]} (\hat{\mathbf{A}}^c(o) \wedge \rho + \sigma(o) \wedge R(\hat{x})) + \tilde{\Omega}(1, \mathcal{E})].$$

The pull-back $\pi^*\mathcal{E}$ is a G -equivariant geometric family over N . The class $f_!^G(\pi^*\hat{x})$ is represented by

$$[f_!^G \pi^* \mathcal{E}, \int_{L[N/G]/L[*]/G} L\pi^* \left(\hat{\mathbf{A}}^c(o) \wedge \rho + \sigma(o) \wedge R(\hat{x}) \right) + \tilde{\Omega}(1, \pi^* \mathcal{E})].$$

4.3.5 We first show that the left square of (22) commutes on classes of the form $[\emptyset, \rho]$, i.e. we show that

$$\mathbf{av}^H \circ f_!^G \circ L\pi^* \rho = f_!^K(\rho).$$

To this end we make the isomorphism $\Omega(L[M/K]) \cong \Omega(L[N/G])$ explicit. First recall that $\Omega(L[M/K]) \cong [\bigoplus_{k \in K} \Omega(M^k)]^K$ and $\Omega(L[N/G]) \cong [\bigoplus_{g \in G} \Omega(N^g)]^G$. We write $\omega = \bigoplus_{g \in G} \omega_g$ with $\omega_g \in \Omega(N^g)$.

Let $\pi : \bigsqcup_{g \in G} N^g \rightarrow \bigsqcup_{k \in K} M^k$ be the map inducing $L[N/G] \rightarrow L[M/K]$. If $Hg \in K$ fixes an element $Hn \in M$, then $n \in N^{gh}$ for a suitable $h \in H$. Indeed, $ghn = \tilde{h}^{-1}n$ for suitable $\tilde{h}, \tilde{h} \in H$. Therefore $\tilde{h}ghn = g\tilde{h}g^{-1}\tilde{h}n = n$. We set $h := \tilde{h}g^{-1}\tilde{h}$.

On the other hand, if $n \in N^g$, then $Hn \in M^{Hg}$. Indeed, $HgHn = Hgn = Hn$. It follows that for $Hn \in M^{Hg}$ we have

$$\pi^{-1}(Hn) = \bigsqcup_{h \in H} (Hn \cap N^{gh}).$$

Assume that $n \in N^g$ and $\tilde{h}n \in N^g$. Then $gn = n$ and $g\tilde{h}n = \tilde{h}n = \tilde{h}gn$, hence $\tilde{h}n = g^{-1}\tilde{h}gn$. Since $g^{-1}\tilde{h}g \in H$ and H acts freely this implies that $\tilde{h} \in H_g$. Vice versa, if $\tilde{h} \in H_g$, then with $n \in N^g$ we have also $\tilde{h}n \in N^g$. We conclude that for $n \in N^g$ we have $Hn \cap N^g = H_g n$, so that

$$|Hn \cap N^g| = \begin{cases} |H_g| & |Hn \cap N^g| \neq 0 \\ 0 & \text{else} \end{cases}.$$

Therefore $N^g \rightarrow M^{gH}$ is a $|H_g|$ -fold covering. Moreover, if $Hn \in M^{Hg}$, then

$$|H| = \sum_{h \in H, |Hn \cap N^{gh}| \neq 0} |H_{gh}|. \quad (23)$$

We consider $g \in G$ such that $N^g \neq \emptyset$. Note that $\pi(N^g) \subseteq M^{gH}$ is an open and closed submanifold. If $\omega \in \Omega(L[M/K])$, then

$$\begin{aligned} f_!^G(L\pi^*\omega)(g) &= \int_{N^g} \pi_{H_g}^* \omega_{H_g} \\ &= |H_g| \int_{\pi(N^g)} \omega_{H_g|_{\pi(N^g)}} \end{aligned}$$

Alltogether

$$\begin{aligned}
\mathrm{av}^H f_!^G(L\pi^*\omega)(Hg) &= \frac{1}{|H|} \sum_{h \in H} f_!^G(L\pi^*\omega)(gh) \\
&= \frac{1}{|H|} \sum_{h \in H} |H_{gh}| \int_{\pi(N_{gh})} \omega_{H_{gh}|_{\pi(N_{gh})}} \\
&\stackrel{(23)}{=} \int_{M^{Hg}} \omega_{Hg} \\
&= f_!^K(\omega)(Hg) .
\end{aligned}$$

This calculation shows that the left square in (4.9) commutes on elements of the form $[\emptyset, \rho]$.

4.3.6 We now consider a geometric family \mathcal{E} over M . Note that $\tilde{\Omega}(\mathcal{E}, 1) = f_!^K(\alpha)$ for some $\alpha \in \Omega(L[M/K])$. It follows from the locality of α that $\tilde{\Omega}(\pi^*\mathcal{E}, 1) = f_!^G(\pi^*\alpha)$. Hence $\mathrm{av}^H(\tilde{\Omega}(\pi^*\mathcal{E}, 1)) = \tilde{\Omega}(\mathcal{E}, 1)$.

We continue with classes of the form $[\mathcal{E}, 0]$. As $K^1([*/K]) = 0$, and as we only consider odd classes, we can choose, after stabilization, a K -invariant taming $(f_!^K \mathcal{E})_t$. It lifts to a G -invariant taming $(f_!^G \pi^* \mathcal{E})_t$. Note that

$$[f_!^K \mathcal{E}, 0] = [\emptyset, -\eta((f_!^K \mathcal{E})_t)] , \quad [f_!^G \pi^* \mathcal{E}, 0] = [\emptyset, -\eta((f_!^G \pi^* \mathcal{E})_t)] .$$

We must show that

$$\mathrm{av}^H(\eta((f_!^G \pi^* \mathcal{E})_t)) = \eta((f_!^K \mathcal{E})_t) .$$

To this end we write out the definition (3) of the eta invariant. We have

$$\eta((f_!^G \pi^* \mathcal{E})_t)(g) = \frac{-1}{\pi} \int_0^\infty \mathrm{Tr} g \partial_t A_\tau e^{A_\tau^2} d\tau ,$$

where $A_\tau := A_\tau((f_!^G \pi^* \mathcal{E})_t)$ is the family of rescaled tamed Dirac operators on the G -Hilbert space $H(f_!^G \pi^* \mathcal{E})$. The important observation is now that $H(f_!^K \mathcal{E})$ can naturally be identified with the subspace of H -invariants $H(f_!^G \pi^* \mathcal{E})^H$, and the restriction of A_τ to this subspace is $A_\tau((f_!^K \mathcal{E})_t)$. Note that $\frac{1}{|H|} \sum_{h \in H} h$ acts as the projection onto the subspace of H -invariants. Therefore

$$\mathrm{av}^H(\eta((f_!^G \pi^* \mathcal{E})_t))(Hg) = \frac{1}{|H|} \sum_{h \in H} \eta((f_!^G \pi^* \mathcal{E})_t)(hg) = \eta((f_!^K \mathcal{E})_t)(Hg) .$$

Alltogether we thus have shown that

$$\mathrm{av}^H[f_!^G \pi^* \mathcal{E}, 0] = [f_!^K \mathcal{E}, 0] .$$

This finishes the proof of Proposition 4.9. □

4.3.7 Let B be an orbifold which admits a presentation $B \cong [M/G]$ with a finite group G . Assume that the map $[M/G] \rightarrow [*/G]$ is proper and smoothly K -oriented.

Proposition 4.10 *If $B \cong [M'/G']$ is another presentation of B with a finite group G' , then $[M'/G'] \rightarrow [*/G']$ has an induced smooth K -orientation. This correspondence preserves reality.*

Proof. We use the method and notation of the proof of Theorem 4.3. The smooth K -orientation of $[M/G] \rightarrow [*/G]$ is given by G -invariant data on M , see 3.1.7. It lifts to $G \times G'$ -equivariant data on N , and finally induces the G' -equivariant data on M' which gives the induced orientation of $[M'/G'] \rightarrow [*/G']$. This correspondence respects the equivalence relation between representatives of smooth K -orientations. \square

In view of Proposition 4.10 we can talk about a smooth K -orientation of an orbifold which admits a presentation $[M/G]$ with a finite group.

Definition 4.11 *A smooth K -orientation o of an orbifold B is represented by a smooth K -orientation of the map $[M/G] \rightarrow [*/G]$, where $B \cong [M/G]$ is a presentation of B for a finite group G .*

If o' is a smooth K -orientation represented by $[M'/G'] \rightarrow [/G']$, where $B \cong [M'/G']$ is a presentation of B for another finite group G' , then $o' = o$ if o' is equal to the smooth K -orientation induced on $[M'/G'] \rightarrow [*/G']$ by o according to Proposition 4.10. The smooth K -orientation of B is called real if it is represented by a real smooth K -orientation of $[M/G] \rightarrow [*/G]$.*

According to this definition, if an orbifold admits a smooth K -orientation, then it admits in particular a presentation as a quotient of a closed manifold by a finite group.

4.3.8

Proposition 4.12 *Let B be an orbifold with a smooth K -orientation represented by a smooth K orientation of $[M/G] \rightarrow [*/G]$. The pairing*

$$\hat{K}(B) \otimes \hat{K}(B) \xrightarrow{\cup} \hat{K}(B) \cong \hat{K}([M/G])^{\mathrm{Tr}_{G^o}(\int_{[M/G] \rightarrow [*/G]} \dots)^1} \mathbb{T}$$

is well-defined independent of the choice of the representative of the smooth K -orientation. If the orientation of B is real, then by restriction we get a well-defined pairing

$$\hat{K}_{\mathbb{R}}(B) \otimes \hat{K}_{\mathbb{R}}(B) \rightarrow \mathbb{R}/\mathbb{Z} .$$

Proof. We again use the technique of the proof of Theorem 4.3. If $B \cong [M/K]$ and $B \cong [M'/K']$ are two presentations, then there is a third presentation $B \cong [N/G]$ such

that $K, K' \subset G$ are normal subgroups and $M \cong N/K'$ and $M' \cong N/K$. We now use Proposition 4.9 which gives

$$\mathrm{Tr}_K f_1^K(x \cup y) = \mathrm{Tr}_G(f_1^G(\pi^*(x \cup y))) = \mathrm{Tr}_{K'} f_1^{K'}(x' \cup y'),$$

where $x, y \in \hat{K}([M/K])$ and $x', y' \in \hat{K}([M'/K'])$ are such that $\pi^*x = \mathrm{pr}'^*x'$ and $\pi^*y = \mathrm{pr}'^*y'$. \square

4.3.9

Theorem 4.13 *Let B be an orbifold with a smooth K -orientation. The intersection pairing*

$$\hat{K}(B) \otimes \hat{K}(B) \xrightarrow{(\cdot, \cdot)} \mathbb{T}$$

is non-degenerated. If the orientation of B is real, then the restriction

$$\hat{K}_{\mathbb{R}}(B) \otimes \hat{K}_{\mathbb{R}}(B) \xrightarrow{(\cdot, \cdot)} \mathbb{R}/\mathbb{Z}$$

is non-degenerated.

Proof. We can apply the argument of the proof of [FMS07, Proposition B6] using the fact that

$$\begin{aligned} U_G(M) \otimes K_G(M) &\xrightarrow{\cup} U_G(M)^{\mathrm{Tr}_G \circ (\int_{[M/G] \rightarrow [*/G]} \cdots)^1} \mathbb{T} \\ U_G^{\mathbb{R}}(M) \otimes K_G(M) &\xrightarrow{\cup} U_G^{\mathbb{R}}(M)^{\mathrm{Tr}_G \circ (\int_{[M/G] \rightarrow [*/G]} \cdots)^1} \mathbb{R}/\mathbb{Z} \end{aligned}$$

are non-degenerated pairings by Theorem 4.7 and Corollary 4.5. \square

5 Examples

5.1 The smooth K -theory class of a mapping torus

5.1.1 In this subsection we generalize the results of [BS07, 5.2]

5.1.2 Let G be a finite group. We consider a geometric $\mathbb{Z}/2\mathbb{Z}$ -graded G -bundle $\mathbf{V} := (V, h^V, \nabla^V, z)$ over S^1 , where we let G act trivially on S^1 . Let $1 \in S^1$ be the base point. The group G acts on the fibres V_1^{\pm} of the homogeneous components of V . We assume that $V_1^+ \cong V_1^-$ as representations of G . Let \mathcal{V} denote the corresponding G -equivariant geometric family over S^1 . Equivalently, we can consider the family $[\mathcal{V}/G]$ over $[S^1/G]$.

We have an exact sequence

$$K^1([S^1/G]) \xrightarrow{\mathbf{ch}} \Omega^1(L[S^1/G]/\mathbf{im}(d)) \xrightarrow{a} \hat{K}^0([S^1/G]) \xrightarrow{I} K^0([S^1/G]) \rightarrow 0 .$$

We identify

$$\Omega^1(L[S^1/G]/\mathbf{im}(d)) \cong R(G) \otimes \Omega^1(S^1)/\mathbf{im}(d) \cong R(G) \otimes \mathbb{C}$$

and

$$(\Omega^1(L[S^1/G]/\mathbf{im}(d))/\mathbf{ch}(K^1([S^1/G])) \cong R(G) \otimes \mathbb{T} .$$

The class $[\mathcal{V}, 0] \in \hat{K}^0(S^1)$ satisfies $I([\mathcal{V}, 0]) = 0$ and hence corresponds to an element of $R(G) \otimes \mathbb{T}$. This element is calculated in the following lemma.

For $g \in G$ we decompose $V^\pm = \bigoplus_{\theta \in U(1)} V^\pm(\theta)$ according to eigenvalues of g . Let $\phi^\pm(\theta) \in U(n_\theta)/\text{conj}$ denote the holonomies of $V^\pm(\theta)$ (well defined modulo conjugation in the group $U(n_\theta)$).

Lemma 5.1 *We have $[\mathcal{V}, 0] = a(\Phi)$, where $\Phi \in \Omega^1(L[S^1/G]/\mathbf{im}(d)) \cong \mathbb{C}[G]^G$ is given by*

$$\Phi(g) = \frac{1}{2\pi i} \sum_{\theta \in U(1)} \theta \log \frac{\det(\phi^+(\theta))}{\det(\phi^-(\theta))} .$$

Proof. We consider the map $q: [S^1/G] \rightarrow [* / G]$ with the canonical K -orientation given by the bounding *Spin*-structure of S^1 . By Proposition 3.18 we have a commutative diagram

$$\begin{array}{ccccc} R(G) \otimes \mathbb{C} & \xrightarrow{\sim} & \Omega^1(L[S^1/G]/(\mathbf{im}(d) + \mathbf{im}(\mathbf{ch}))) & \xrightarrow{a} & \hat{K}^0([S^1/G]) \\ \downarrow = & & \downarrow q_!^p & & \downarrow \hat{q}_! \\ R(G) \otimes \mathbb{C} & \xrightarrow{\sim} & \Omega^0(L[* / G]/\mathbf{im}(\mathbf{ch})) & \xrightarrow{a} & \hat{K}^1([* / G]) \end{array} .$$

In order to determine $[\mathcal{V}, 0]$ it therefore suffices to calculate $\hat{q}_!([\mathcal{V}, 0])$. Now observe that $q: S^1 \rightarrow *$ is the boundary of $p: D^2 \rightarrow *$. Since the underlying topological K -orientation of q is given by the bounding *Spin*-structure we can choose a smooth K -orientation of p with product structure which restricts to the smooth K -orientation of q . The bundle \mathbf{V} is topologically trivial. Therefore we can find a geometric G -bundle $\mathbf{W} = (W, h^W, \nabla^W, z)$, again with product structure, on D^2 which restricts to \mathbf{V} on the boundary. Let \mathcal{W} denote the corresponding geometric family over D^2 . Later we prove the bordism formula Proposition 5.4. It gives

$$\hat{q}_!([\mathcal{V}, 0]) = [\emptyset, p_!R([\mathcal{W}, 0])] = -a \left(\int_{L[D^2/G]/L[* / G]} \Omega^2(\mathcal{W}) \right) .$$

For $g \in G$ we have

$$\begin{aligned}
\Omega^2(\mathcal{W})(g) &= \frac{1}{2\pi i} \mathbf{ch}_2(\nabla^W)(g) \\
&= \frac{1}{2\pi i} \left(\mathbf{ch}_2(\nabla^{\det(W^+)}) (g) - \mathbf{ch}_2^G(\nabla^{\det(W^-)}) (g) \right) \\
&= \frac{-1}{2\pi i} \left[\mathrm{Tr} g R^{\nabla^{W^+}} - \mathrm{Tr} g R^{\nabla^{W^-}} \right] \\
&= \frac{-1}{2\pi i} \sum_{\theta} \theta [R^{\nabla^{\det W^+(\theta)}} - R^{\nabla^{\det W^-(\theta)}}] .
\end{aligned}$$

The holonomy $\det(\phi^\pm(\theta)) \in U(1)$ of $\det(\mathbf{V}^\pm(\theta))$ is equal to the integral of the curvature of $\det \mathbf{W}^\pm(\theta)$:

$$\log \det(\phi^\pm) = \int_{D^2} R^{\nabla^{\det(W^\pm)}} .$$

It follows that $\hat{q}_!([\mathcal{V}, 0]) = a(\Phi)$ with

$$\Phi(g) = \frac{1}{2\pi i} \sum_{\theta \in U(1)} \theta \log \frac{\det(\phi^+(\theta))}{\det(\phi^-(\theta))} .$$

□

5.1.3 Consider a finite group G and let \mathcal{E} be an equivariant geometric family over a point. We consider an additional automorphism ϕ of \mathcal{E} which commutes with the action of G . Then we can form the mapping torus $T(\mathcal{E}, \phi) := (\mathbb{R} \times \mathcal{E})/\mathbb{Z}$, where $n \in \mathbb{Z}$ acts on \mathbb{R} by $x \mapsto x + n$, and by ϕ^n on \mathcal{E} . The product $\mathbb{R} \times \mathcal{E}$ is a $G \times \mathbb{Z}$ -equivariant geometric family over \mathbb{R} (the pull-back of \mathcal{E} by the projection $\mathbb{R} \rightarrow *$). The geometric structures descend to the quotient by \mathbb{Z} and turn the mapping torus $T(\mathcal{E}, \phi)$ into a geometric family over $[S^1/G] = [(\mathbb{R}/\mathbb{Z})/G]$, where G acts trivially on S^1 . In the present subsection we study the class

$$[T(\mathcal{E}, \phi), 0] \in \hat{K}([S^1/G]) .$$

In the following we will assume that the parity of \mathcal{E} is even, and that $\mathbf{index}(\mathcal{E}) = 0$.

Let $\dim : K^0([S^1/G]) \rightarrow R(G)$ be the dimension homomorphism, which in this case is an isomorphism. Since $\dim I([T(\mathcal{E}, \phi), 0]) = \dim(\mathbf{index}(\mathcal{E})) = 0$ we have in fact

$$[T(\mathcal{E}, \phi), 0] \in \mathbf{im}(a) \cong (\Omega^1(L[S^1/G])/\mathbf{im}(d))/\mathbf{ch}(K^1([S^1/G])) \cong R(G) \otimes \mathbb{T} ,$$

see 5.1.2.

Let $V := \ker(D(\mathcal{E}))$. This graded G -vector space is preserved by the action of ϕ . We use the same symbol ϕ in order to denote the induced action on V .

We form the zero-dimensional family $\mathcal{V} := (\mathbb{R} \times V)/\mathbb{Z}$ over $[S^1/G]$. This bundle is isomorphic to the kernel bundle of $T(\mathcal{E}, \phi)$. The bundle of Hilbert spaces of the family $T(\mathcal{E}, \phi) \cup_{[S^1/G]} \mathcal{V}^{op}$ has a canonical subbundle of the form $\mathcal{V} \oplus \mathcal{V}^{op}$. We choose the taming $(T(\mathcal{E}, \phi) \cup_{[S^1/G]} \mathcal{V}^{op})_t$ which is induced by the isomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on this subbundle. Note that $[T(\mathcal{E}, \phi), 0] = [\mathcal{V}, \eta^1((T(\mathcal{E}, \phi) \cup_{[S^1/G]} \mathcal{V}^{op})_t)]$. Since

$$(T(\mathcal{E}, \phi) \cup_{[S^1/G]} \mathcal{V}^{op})_t$$

lifts to a product under the pull-back $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ we see that $\eta^1((T(\mathcal{E}, \phi) \cup_{[S^1/G]} \mathcal{V}^{op})_t) = 0$. It follows that $[T(\mathcal{E}, \phi), 0] = [\mathcal{V}, 0] \in R(G) \otimes \mathbb{T}$. This class has been calculated in terms of the action of ϕ on V in Lemma 5.1.

5.2 Bordism

5.2.1 Here we generalize the results of [BS07, 5.8].

5.2.2 A zero bordism of a geometric family \mathcal{E} over an orbifold B is a geometric family \mathcal{W} over B with boundary such that $\mathcal{E} = \partial\mathcal{W}$. The notion of a geometric family with boundary was discussed in detail in [Bun].

Proposition 5.2 *If \mathcal{E} admits a zero bordism \mathcal{W} , then in $\hat{K}^*(B)$ we have the identity*

$$[\mathcal{E}, 0] = [\emptyset, \Omega(\mathcal{W})].$$

Proof. Since \mathcal{E} admits a zero bordism we have $\mathbf{index}(\mathcal{E}) = 0$. In order to see this choose a presentation $B \cong [M/G]$. Then $M \times_B \mathcal{E}$ is a G -equivariant geometric family which admits a G -equivariant zero bordism $M \times_B \mathcal{W}$. By the equivariant bordism invariance of the index it follows that $\mathbf{index}(M \times_B \mathcal{E}) \in K_G(M)$ vanishes. This implies $\mathbf{index}(\mathcal{E}) = 0$ in $K(B)$.

It follows from Lemma 2.10 that after replacing \mathcal{E} by $\mathcal{E} \sqcup_B \tilde{\mathcal{E}} \sqcup_B \tilde{\mathcal{E}}^{op}$ and \mathcal{W} by $\mathcal{W} \sqcup_B (\mathcal{E} \times I)$ for a suitable geometric family $\tilde{\mathcal{E}}$ there exists a taming \mathcal{E}_t . This taming induces a boundary taming \mathcal{W}_{bt} . The obstruction against extending the boundary taming to a taming of \mathcal{W} is $\mathbf{index}(\mathcal{W}_{bt}) \in K(B)$. Using the method described in 2.5.8 we can adjust the taming \mathcal{E}_t such that $\mathbf{index}(\mathcal{W}_{bt}) = 0$. Here it might be necessary to add another family to $\tilde{\mathcal{E}}$. Then we extend the boundary taming \mathcal{W}_{bt} to a taming \mathcal{W}_t , possibly after a further stabilization, i.e. after adding a family $\mathcal{G} \sqcup_B \mathcal{G}^{op}$ with closed fibres.

We now apply

Theorem 5.3

$$\Omega(\mathcal{W}) = d\eta(\mathcal{W}_t) - \eta(\mathcal{E}_t) .$$

Proof. We adapt the proof of theorem [Bun, Thm. 4.13] using the remarks made in the proof of Theorem 2.25. \square

We see that $(\mathcal{E}, 0)$ is paired with $(\emptyset, \Omega(\mathcal{W}))$. This implies the assertion. \square

5.2.3 Let $p : W \rightarrow B$ be a proper representable submersion from an orbifold with boundary W which restricts to a submersion $q := p|_{\partial W} : V := \partial W \rightarrow B$. We assume that p has a topological K -orientation and a smooth K -orientation represented by o_p which refines the topological K -orientation. We assume that the geometric data of o_p have a product structure near V . In this case we have a restriction $o_q := o_p|_V$ which represents a smooth K -orientation of q . It is easy to see that this restriction of representatives (with product structure) preserves equivalence and gives a well-defined restriction of smooth K -orientations. We have the following version of bordism invariance of the push-forward in smooth K -theory.

Proposition 5.4 *For $y \in \hat{K}(W)$ we set $x := y|_V \in \hat{K}(V)$. Then we have*

$$\hat{q}_!(x) = [\emptyset, p_!^o R(y)] .$$

Proof. The proof can be copied from [BS07, 5.18]. \square

5.3 The intersection pairing for $[\mathbb{C}\mathbb{P}^1/(\mathbb{Z}/k\mathbb{Z})]$

5.3.1 For $k \in \mathbb{N}$ let $\Gamma := \mathbb{Z}/k\mathbb{Z}$. We fix a primitive k th root of unity ξ and let Γ act on \mathbb{C}^2 by $n(z_0, z_1) = (\xi^n z_0, z_1)$. This induces an action of Γ on $\mathbb{C}\mathbb{P}^1$. Let $X := [\mathbb{C}\mathbb{P}^1/\Gamma]$ be the corresponding orbifold.

We cover $\mathbb{C}\mathbb{P}^1$ by the standard charts $U := \{[u : 1] \mid u \in \mathbb{C}\}$ and $V := \{[1 : v] \mid v \in \mathbb{C}\}$. The transition is given by $v = \frac{1}{u}$. Therefore Γ acts on U by $[n]z := \xi^n z$, and on V by $[n]v = \xi^{-n}v$.

5.3.2 We calculate $K(X) \cong K_\Gamma(\mathbb{C}\mathbb{P}^1)$ using the Mayer-Vietoris sequence associated to the covering $U \cup V$. These spaces are equivariantly equivalent to points. Therefore $K_\Gamma(U) \cong K_\Gamma(V) \cong R(\Gamma) \cong \mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$. The latter is the free \mathbb{Z} -module generated by the classes $[l]$, $l \in 0, \dots, k-1$. Furthermore, $U \cap V \cong \mathbb{C}^*$ with a free Γ -action. Note that $\mathbb{C}^*/\Gamma \cong \mathbb{C}^*$. We therefore have

$$K_\Gamma^i(\mathbb{C}^*) \cong \mathbb{Z} \quad i = 0, 1 .$$

The Mayer-Vietoris sequence reads

$$\begin{array}{ccccc}
K^0(X) & \xrightarrow{\beta} & R(\Gamma) \oplus R(\Gamma) & \xrightarrow{\alpha} & \mathbb{Z} \\
\delta \uparrow & & \swarrow \sigma & & \downarrow \\
\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K^1(X)
\end{array}$$

The map α maps a pair of representations (χ, μ) of γ to the difference of their dimensions. In particular, it is surjective. Therefore $K_1^1(X) \cong 0$.

The map δ maps the integer $1 \in \mathbb{Z}$ to the class represented by the difference $L - 1$, where $1 \cong \mathbb{C}\mathbb{P}^1 \times \mathbb{C}$ with the trivial action of Γ on the fibres, and L is the bundle obtained from $U \times \mathbb{C}$ and $V \times \mathbb{C}$, again with trivial fibrewise action, glued with u^{-k} .

We now define a split σ as follows. Let $([l], [h]) \in R(\Gamma) \oplus R(\Gamma)$. Then $\alpha([l], [h]) = 0$. We define equivariant trivial bundles $L_U := U \times \mathbb{C}$ and $L_V := V \times \mathbb{C}$, where the actions on the fibres are given by $[l]$ and $[-h]$, respectively. We represent $l, h \in \{0, \dots, k-1\}$. Then we can glue the trivial bundles equivariantly using the transition function $\mathbb{C}^* \ni g \mapsto g^{h+l}$. The result is $L_{l,h} := \sigma([l], [h])$. Using this split and δ we get a decomposition

$$K^0(X) \cong \mathbb{Z} \oplus \ker(\alpha) .$$

5.3.3 The manifold $\mathbb{C}\mathbb{P}^1$ has an equivariant complex structure. It gives an equivariant $Spin^c$ -structure and therefore an equivariant K -orientation. In the following we calculate

$$\int_{[\mathbb{C}\mathbb{P}^1/\Gamma]} : K(X) \rightarrow R(\Gamma) .$$

The calculation is based on the explicit knowledge of the kernel and cokernel of the $Spin^c$ -Dirac operator twisted by suitable representatives of elements of $K(X)$. In fact, the $Spin^c$ -Dirac operator is the Dolbeault operator D . Therefore for a bundle $L \rightarrow \mathbb{C}\mathbb{P}^1$

$$\ker(D \otimes L) \cong H^0(\mathbb{C}\mathbb{P}^1, L) , \quad \text{coker}(D) \cong H^1(\mathbb{C}\mathbb{P}^1, L) \cong H^0(\mathbb{C}\mathbb{P}^1, K \otimes L^*)^* ,$$

where K denotes the canonical bundle. Let us start with the bundle L . The holomorphic sections on U are given by u^r with $r \geq 0$. Their transforms to V are given by v^{-k-r} . These are holomorphic, if $-r \geq k$. Therefore we have $H^0(\mathbb{C}\mathbb{P}^1, L) \cong 0$.

Note that $H^1(\mathbb{C}\mathbb{P}^1, L) \cong H^0(\mathbb{C}\mathbb{P}^1, K \otimes L^*)^*$. The holomorphic sections of U of the bundle $K \otimes L^*$ are $u^r du$ for $r \geq 0$. They are transformed to $-v^{k-2-r} dv$ on V and hence holomorphic for $0 \leq r \leq k-2$. Therefore,

$$H^1(\mathbb{C}\mathbb{P}^1, L) \cong \mathbb{C}[\Gamma] - [k-1] .$$

We see that

$$\text{Tr}_G \circ \int_{\mathbb{C}\mathbb{P}^1} L = -1 .$$

We now consider the case of $L_{l,h}$ with $h, l \in \mathbb{Z}$. The holomorphic sections on $L_{l,h}$ over U are given by u^s with $s \geq 0$. They are transformed to v^{-s-l-h} on V . These sections are holomorphic if $0 \leq s \leq -l-h$. We get

$$H^0(\mathbb{CP}^1, L_{l,h}) \cong \bigoplus_{i=0}^{-l-h} [i+l] .$$

The holomorphic sections on U of $K \otimes L^*$ are given by $u^s du$ with $s \geq 0$. They are transformed to $v^{-s-2+l+h} dv$ on V . For holomorphy we hence need $0 \leq s \leq l+h-2$.

Again we see that there is no cancelation between kernels and cokernels. As representations of Γ we have

$$H^1(\mathbb{CP}^1, L_{l,h}) \cong \bigoplus_{i=0}^{l+h-2} [i-l]$$

In this calculation $L = L_{0,k}$. We have $L_{l,h} \otimes L_{l',h'} = L_{l+l',h+h'}$.

5.3.4 Let us take $k = 2$. A basis of the \mathbb{Z} -module $K^0(X) \cong \mathbb{Z}^4$ is given by

$$(e_i)_{i=1}^4 := (L_{0,0}, L_{0,2}, [1]L_{0,0}, L_{1,0}) .$$

The matrix of the intersection pairing

$$A_{i,j} := (e_i, e_j)$$

is given by

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

which has determinant -1 . We furthermore have

$$\hat{K}^0(X) \cong K^0(X) , \quad \hat{K}^1(X) \cong H^0(LX)/\text{im}(\mathbf{ch}) \cong K^0(X)_{\mathbb{C}}/K^0(X) .$$

The pairing induced by

$$\begin{array}{ccc} K^0(X)_{\mathbb{C}} \otimes K^0(X) & \xrightarrow{(\dots)} & \mathbb{C} \\ \downarrow & & \downarrow \\ K^0(X)_{\mathbb{C}}/K^0(X) \otimes K^0(X) & \longrightarrow & \mathbb{T} \end{array}$$

is non-degenerated, as we already knew by Theorem 4.13.

References

- [ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl. 1):3–38, 1964.
- [AR03] Alejandro Adem and Yongbin Ruan. Twisted orbifold K -theory. *Comm. Math. Phys.*, 237(3):533–556, 2003.
- [BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.
- [BS07] Ulrich Bunke and Thomas Schick. Smooth K -theory, 2007. arXiv:0707.0046.
- [BS09] Ulrich Bunke and Thomas Schick. Uniqueness of smooth extensions of generalized cohomology theories, 2009. arXiv:0901.4423.
- [BSS07] Ulrich Bunke, Thomas Schick, and Markus Spitzweck. Sheaf theory for stacks in manifolds and twisted cohomology for S^1 -gerbes. *Algebr. Geom. Topol.*, 7:1007–1062, 2007.
- [BSS08] Ulrich Bunke, Thomas Schick, and Markus Spitzweck. Inertia and delocalized twisted cohomology. *Homology, Homotopy Appl.*, 10(1):129–180, 2008.
- [Bun] U. Bunke. Index theory, eta forms, and Deligne cohomology. arXiv:math.DG/0201112.
- [FHT07] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted k -theory i, 2007. arXiv:0711.1906.
- [FMS07] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. Heisenberg groups and noncommutative fluxes. *Annals Phys.*, 322:236, 2007.
- [Hei05] Jochen Heinloth. Survey on topological and smooth stacks. In *Mathematisches Institut Göttingen, WS04-05 (Y. Tschinkel, ed.)*, pages 1–31. 2005.
- [HM04] Andre Henriques and David S. Metzler. Presentations of noneffective orbifolds. *Trans. Amer. Math. Soc.*, 356(6):2481–2499 (electronic), 2004.
- [HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M -theory. *J. Differential Geom.*, 70(3):329–452, 2005.
- [MM97] Ruben Minasian and Gregory W. Moore. K -theory and ramond-ramond charge. *JHEP*, 11:002, 1997, hep-th/9710230.

- [Ort] Michael L. Ortiz. Differential equivariant k -theory, arXiv:0905.0476.
- [PS07] Dorette Pronk and Laura Scull. Translation groupoids and orbifold Bredon cohomology, 2007. arXiv:0705.3249.
- [Seg68] Graeme Segal. Equivariant K -theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.
- [SV07] Richard J. Szabo and Alessandro Valentino. Ramond-Ramond fields, fractional branes and orbifold differential K-theory, 2007. arXiv:0710.2773.
- [TXLG04] Jean-Louis Tu, Ping Xu, and Camille Laurent-Gengoux. Twisted K -theory of differentiable stacks. *Ann. Sci. École Norm. Sup. (4)*, 37(6):841–910, 2004.
- [Wit98] Edward Witten. D-branes and k -theory. *JHEP*, 12:019, 1998, hep-th/9810188.