THE $\widehat{\mathfrak{sl}}(n)_k$-WZNW FUSION RING: A COMBINATORIAL CONSTRUCTION AND A REALISATION AS QUOTIENT OF QUANTUM COHOMOLOGY

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ABSTRACT. A simple, combinatorial construction of the $\widehat{\mathfrak{sl}}(n)_k$-WZNW fusion ring, also known as Verlinde algebra, is given. As a byproduct of the construction one obtains an isomorphism between the fusion ring and a particular quotient of the small quantum cohomology ring of the Grassmannian $Gr_{k,k+n}$. We explain how our approach naturally fits into known combinatorial descriptions of the quantum cohomology ring, by establishing what one could call a 'Boson-Fermion-correspondence' between the two rings. We also present new recursion formulae for the structure constants of both rings, the fusion coefficients and the Gromov-Witten invariants.

CONTENTS

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1. INTRODUCTION

Given $k, n \in \mathbb{N}$ natural numbers, we can associate the following two rings:

The quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{k,n+k})$. Let $\mathrm{Gr}_{k,n+k}$ be the Grassmannian of k-planes in $\mathbb{C}^{k+n}$, considered as an algebraic variety. Then the *quantum cohomology ring* $qH^\bullet(\mathrm{Gr}_{k,n+k})$ is a deformation of the ordinary (integral) cohomology ring $H^\bullet(\mathrm{Gr}_{k,n+k})$ of $\mathrm{Gr}_{k,n+k}$. The latter has a basis given by the Schubert classes $[\Omega_\lambda]$, indexed by partitions λ whose Young diagram fits into a bounding box of size $k\times n$. The structure constants are just intersection numbers, the so-called Littlewood-Richardson coefficients. The quantum cohomology can be viewed as the free $\mathbb{Z}[q]$-module with the same basis, but the structure constants $\sum_d C^{\nu^*,d}_{\lambda,\mu} q^d$ are now the so-called 3-point genus 0 Gromov-Witten invariants, which count the number of rational curves of degree d passing through generic translates of the involved three Schubert cells. By a theorem of Siebert and Tian [40], there is an isomorphism of rings $qH^\bullet(\mathrm{Gr}_{k,k+n}) \cong \mathbb{Z}[q]\otimes_{\mathbb{Z}}\mathbb{Z}[e_1, e_2, \dots, e_k]/\left\langle h_{n+1}, \dots, h_{n+k-1}, h_{n+k} + (-1)^k q\right\rangle$, where e_i, $1 \le i \le k$ denotes the ordinary i^{th} elementary symmetric polynomial and h_r, $r \ge 1$, denotes the r^{th} complete symmetric function in k variables.

The fusion ring $\mathcal{F}(\widehat{\mathfrak{sl}}(n))_k$. The *fusion ring* $\mathcal{F} = \mathcal{F}(\widehat{\mathfrak{sl}}(n))_k$ of integrable highest weight representations of $\widehat{\mathfrak{sl}}(n)$ at level k has a natural basis indexed by integrable highest weights $\hat\lambda \in P_k^+$ of level k which can be identified with partitions λ whose Young diagram fits into a bounding box of size $(n-1)\times k$. The structure constants in this ring can be described in various ways, for instance via characters of the Lie algebra $\mathfrak{sl}(n)$ (Verlinde formula), in terms of combinatorics of the affine Weyl group (Kac-Walton formula), geometrically as dimensions of certain moduli spaces of generalised theta functions etc.

To motivate our discussion we recall the following theorem due to Witten [47], based on earlier work of Gepner [19] and Vafa [44] and Intrilligator [23] which states that there is an isomorphism of rings

$$\mathcal{F}(\widehat{\mathfrak{gl}}(n))_k \cong qH^\bullet(\mathrm{Gr}_{n,k+n})_{q=1}$$

between the level k fusion ring of $\widehat{\mathfrak{gl}}(n)$ (resp. $\widehat{\mathfrak{u}}(n)$) and the specialisation at $q = 1$ of the quantum cohomology ring. (A mathematical proof seems to be contained in the unpublished, unfortunately not anymore available, work [1].)

The main goal of this paper is to give a (mathematically rigorous) realisation of $\mathcal{F}(\widehat{\mathfrak{sl}}(n))_k$ as a quotient of $qH^\bullet(\mathrm{Gr}_{k,k+n})$ with the defining relations explained via Bethe Ansatz equations of a quantum integrable system.

Either of the two rings will be also be described combinatorially in terms of symmetric polynomials (Schur polynomials) in pairwise non-commuting variables. These variables will be interpreted in either case in two ways: as particle hopping operators of a certain quantum integrable system and as generators of some algebra naturally appearing in representation theory. A crucial observation here is that the generating function of the elementary symmetric polynomials turns out to be the transfer matrix of the quantum integrable system. This is the starting point of our analysis of the two rings.

As a consequence we obtain a simple particle formulation of both rings leading to novel recursion formulae for the structure constants of the fusion ring as well as for the Gromov-Witten invariants. Certain symmetries in these constants arise also naturally from our quite elementary description of both rings.

The Verlinde formula naturally emerges from our approach and in particular equips the combinatorial fusion algebra with a natural $PSL(2,\mathbb{Z})$-action (see Remark 6.14), a structural data of the abstract Verlinde algebras as introduced in [12, 0.4.1].

An abstract isomorphism of the two rings as mentioned above could be of course also obtained much more directly, since both rings are (after complexification) semisimple, and hence it is enough to compare their spectra (see for instance [3] for a description). The interesting new aspect here is our connection with Bethe Ansatz techniques of quantum integrable systems 'in the crystal limit'.

Outline of the results and methods of the paper. Let $\mathcal{H}_k$ denote the underlying vector space of the fusion ring $\mathcal{F}$. Any basis vector of $\mathcal{H}_k$ can be realised as a partition, or alternatively as an n-tuple $(m_0, m_1, \dots, m_{n-1})$ of non-negative integers which sum up to k. We therefore can consider it as a particle configuration on the circular lattice with n sites and study this underlying integrable model by viewing $\mathcal{H}_k$ as the space of k-particle states. (Details are explained in Section 2.)

We have the obvious operations of particle creation and annihilation φ_i^*, φ_i at site i and *particle hopping* $a_i = \varphi_{i+1}^*\varphi_i$; see Figure 1.1. The endomorphisms a_i of $\mathcal{H}$ form an interesting algebra which we call the *local affine plactic algebra* or the *generic affine Hall algebra*, since it generalises the plactic algebra introduced originally by Lascoux and Schützenberger [31] and Fomin and Greene [15], with its geometric incarnation as generic Hall algebra introduced by Reineke [36]. These algebras can be viewed as the $q = 0$ *crystal limit* of the positive part $U_q(\mathfrak{n}^+)$ of the quantised universal enveloping algebra of $\widehat{\mathfrak{sl}}(n)$. The action of the a_i's sends a basis vector to a basis vector or zero and defines a crystal isomorphic to the crystal of the k-symmetric power of the natural representation of $U_q(\widehat{\mathfrak{sl}}(n))$. (Details can be found in Section 3 and 4).

We then define elementary symmetric functions $e_r(a_0, a_1, \dots a_{n-1})$ in these non-commuting generators and get the following first result:

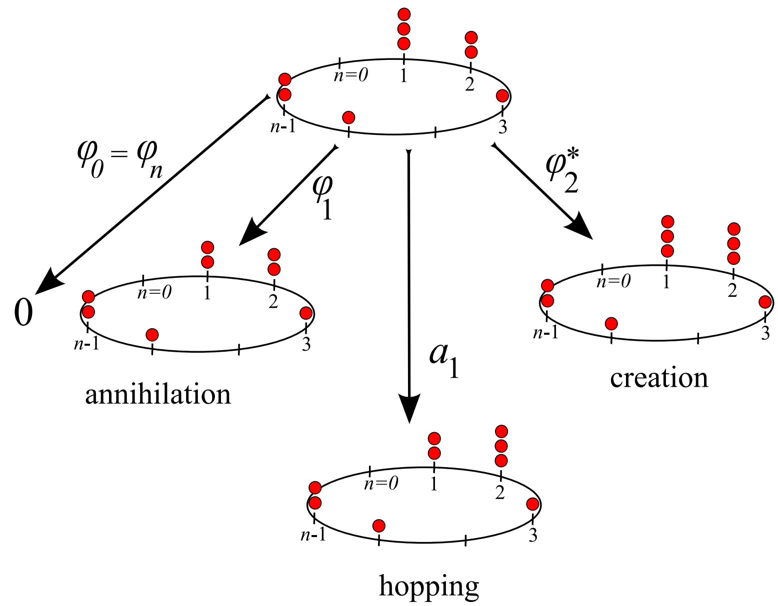

FIGURE 1.1. The dominant integral weight $(m_0, m_1, \ldots, m_{n-1}) = (0, 3, 2, 1, 0, 1, 2)$ of level $k = 9$ as particle configuration on a circle; the processes of creating and annihilating, and hopping applied to it.

Theorem 1.1. *The space $\mathcal{H}_k$ of states can be equipped with the structure of an associative unital algebra $\mathcal{F}_{\mathrm{comb}}$ by defining the (combinatorial) fusion product*

$$\lambda \circledast \mu = s_\lambda(a_0, a_1, \ldots a_{n-1})\mu.$$

This combinatorially defined fusion product coincides with the fusion product in the Verlinde algebra, hence $\mathcal{F}_{\mathrm{comb}} \cong \mathcal{F}(\widehat{\mathfrak{sl}}(n))_k$.

Here $s_\lambda(a_0, a_1, \ldots, a_{n-1})$ denotes the (noncommutative) Schur polynomials defined via the usual *Jacobi-Trudi formula* $s_\lambda = \det\left(e_{\lambda_i^t - i + j}\right)$. To ensure that this statement makes sense, we have to show that the e's pairwise commute. Instead of giving a purely combinatorial proof of this fact [39] we construct a simultaneous eigenbasis for the action of the $s_\lambda(a_0, a_1, \ldots, a_{n-1})$'s using the algebraic Bethe Ansatz and prove:

Theorem 1.2.
1. *The generating function for the noncommutative elementary symmetric functions $e_r(a_0, a_1, \ldots a_{n-1})$ is given by the transfer matrix $T(u)$ associated with a known quantum integrable system, known as* phase model.
2. *This system is integrable in the sense that $[T(u), T(v)] = 0$ for any $u, v \in \mathbb{C}$, in particular, the $e_r(a_0, a_1, \ldots a_{n-1})$'s commute, and the $s_\lambda(a_0, a_1, \ldots a_{n-1})$'s are well-defined.*
3. *The Bethe vectors* (6.1) *form an orthogonal eigenbasis for the action of the noncommutative Schur-functions $s_\lambda(a_0, a_1, \ldots a_{n-1})$.*
4. *The transformation matrix between the standard basis and the basis of Bethe vectors can be expressed in terms of Weyl characters evaluated at certain $(n+k)^{th}$*

roots of unity. The combinatorial fusion product satisfies the Verlinde formula (6.40) *expressed in the entries of the modular S-matrix.*

Let $\Lambda^{(k)} = \mathbb{Z}[e_1, \dots, e_k]$ be the ring of symmetric polynomials in k variables. We get the following main theorem, connecting the fusion algebra with the quantum cohomology (note that the algebra structure on the left hand side is defined in terms of noncommutative Schur polynomials whereas the right hand side is in terms of ordinary commutative Schur polynomials):

Theorem 1.3. *The assignment $P_k^+ \ni \hat{\lambda} \mapsto s_{\lambda^t}$ defines an isomorphism of rings*

$$\mathcal{F}_{\mathrm{comb}} = \mathcal{F}(\widehat{\mathfrak{sl}}(n), \mathbb{Z}) \cong \mathbb{Z}[e_1, \dots, e_k]/\langle h_n - 1, h_{n+1}, \dots, h_{n+k-1}, h_{n+k} + (-1)^k e_k \rangle,$$

hence, the fusion ring is isomorphic to the quotient of $qH^\bullet(\mathrm{Gr}_{k,n+k})$ obtained by imposing the additional relations $q = e_k$ and $h_n = 1$.

Note that by Lemma 6.3, the defining relations of the ring coincide exactly with the Bethe Ansatz equations (6.2). The case $n = 3$, $k = 1$ is worked out in detail in Example 6.22.

In the second part of the paper we develop the integrable system for the quantum cohomology with the resulting combinatorial description of the ring using now noncommutative Schur functions in the generators of the affine nil-Temperley-Lieb algebra, reproving a result of Postnikov [35] with our methods. The defining relations are then the Bethe Ansatz equations (10.5) of a free Fermion system. The resulting basis of Bethe vectors agrees with a basis introduced by Rietsch [37] in her comparison of the quantum cohomology with the coordinate ring of Peterson varieties. By applying our method of deducing the Verlinde formula on the fusion ring side once more on the quantum cohomology side, gives the celebrated Bertram-Vafa Intrilligator formula (10.16) expressed in terms of Schur polynomials evaluated at roots of unity (as first developed by Rietsch). Employing our free fermion formulation we present new identities relating Gromov-Witten invariants at different dimensions k leading to an inductive algorithm which starting from $k = 0$ allows one to compute and relate the entire hierarchy of structure constants for all $k > 0$; see Remark 11.5. We will provide an explicit example. This algorithm differs from the known rim-hook algorithms in [6] and [41] (see also [9] and references therein) which compute Gromov-Witten invariants for fixed k only.

The paper finishes with Part III, where we summarize the parallel construction in what we shall call *Boson-Fermion correspondence* in the present context because of its close analogy with the well-known case (see e.g. [33] and [28]). A detailed study of this correspondence from a representation theory point of view will appear in a forthcoming paper.

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General conventions. In the following vector spaces are always defined over the complex numbers $\mathbb{C}$, by an *algebra* we mean an associative unital algebra over $\mathbb{C}$, and we will abbreviate $\otimes = \otimes_{\mathbb{C}}$, $\mathrm{End} = \mathrm{End}_{\mathbb{C}}$. To avoid confusion with indices, we denote the imaginary complex number $(0,1)$ by ι.

PART I: THE FUSION RING: BOSONS ON A CIRCLE

2. THE AFFINE DYNKIN DIAGRAM AND THE VECTOR SPACE OF STATES

2.1. **The Lie algebra $\widehat{\mathfrak{sl}}(n)$, weights and partitions.** We start by setting up the notation and basics needed from the (untwisted) affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ (see e.g. [26], [46] for the general theory).

Let $\mathfrak{g} = \mathfrak{sl}(n)$ be the Lie algebra of complex traceless $n \times n$ matrices with its standard Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices. If $\mathbb{C}[t, t^{-1}]$ denotes the ring of formal Laurent series in t we have the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with Lie bracket $[x \otimes P, y \otimes Q] = [x, y] \otimes PQ$ for $x, y \in \mathfrak{g}, P, Q \in \mathbb{C}[t, t^{-1}]$. This Lie algebra has a unique central extension $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}\mathbf{C}$, and $\hat{\mathfrak{g}} := \widehat{\mathfrak{sl}}(n)$ is then obtained by adding an extra exterior derivative $\mathbf{D} = t\frac{d}{dt}$, in formulae

$$\widehat{\mathfrak{sl}}(n) = \mathfrak{sl}(n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\boldsymbol{C} \oplus \mathbb{C}\boldsymbol{D}\ .$$

with Lie bracket $[\mathbf{D}, \mathbf{C}] = 0$, $[\mathbf{D}, x \otimes P] = x \otimes \frac{d}{dt}P$ where $x \in \mathfrak{g}, P \in \mathbb{C}[t, t^{-1}]$. Consider the extended Cartan subalgebra $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}\mathbf{C} \oplus \mathbb{C}\mathbf{D}$ and let $\delta, \hat{\omega}_0 \in \hat{\mathfrak{h}}^*$ such that $\delta(\mathbf{D}) = 1$, $\delta(\mathbf{C}) = \delta(h) = 0$ and $\hat{\omega}_0(\mathbf{C}) = 1$, $\hat{\omega}_0(\mathbf{D}) = \hat{\omega}_0(h) = 0$, for $h \in \mathfrak{h}$.

Let $\{\varepsilon_i\}_{i=1}^n \in \mathfrak{h}^* \subset \hat{\mathfrak{h}}^*$ be the linear function which picks out the i^{th} diagonal entry of an element $h \in \mathfrak{h}$. The elements $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, n-1$ form a basis of $\mathfrak{h}^*$ and can be identified with the simple roots of $\mathfrak{sl}(\mathfrak{n})$. Viewed as elements of $\hat{\mathfrak{h}}^*$ (by extending trivially by zero), together with $\alpha_0 = \varepsilon_n - \varepsilon_1 + \delta \in \widehat{\mathfrak{h}}^*$ they form exactly the simple roots of $\widehat{\mathfrak{g}}$. The $\mathbb{Z}$-span of $\{\alpha_i\}_{i=0}^{n-1}$ is the affine root lattice Q. Let $\alpha_i^\vee$, $0 \leq i \leq n$, be the dual roots $\alpha_i^\vee \in \mathfrak{h}$ defined by $\langle \alpha_i, \alpha_j^\vee \rangle = a_{i,j}$, the $(i,j)^{\text{th}}$ entry in the Cartan matrix A.

We denote by Γ the Dynkin diagram of $\widehat{\mathfrak{sl}}(n)$, which we view as a circle with n equidistant marked points. We name these points $0, 1, \ldots, n-1$ in clockwise direction (see Figure 1.1).

There is a unique non-degenerate bilinear pairing on $\mathfrak{h}^*$ given by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$ and we have the fundamental weights

$$\omega_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i - \frac{i}{n}\sum_{j=1}^{n} \varepsilon_j, \quad i = 1, \ldots, n-1 \tag{2.1}$$

for $\mathfrak{g}$ characterized by $\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \langle \omega_i, \alpha_i^\vee \rangle = \delta_{i,j}$ for $i = 1, \ldots, n-1$. Let P_{fin} denote the corresponding integral weight lattice, that is the $\mathbb{Z}$-span of the fundamental weights, with its positive cone P_{fin}^+ given by the $\mathbb{Z}_{\geq 0}$-span. Set $\hat{\omega}_i = \omega_i + \hat{\omega}_0$ for $1 \leq i \leq n-1$.

Then we have the (affine) integral weight lattice $P = P_{\text{fin}} \oplus \mathbb{Z}\hat{\omega}_0 \oplus \mathbb{C}\delta$. Now fix $k \in \mathbb{Z}_{>0}$ and $\gamma \in \mathbb{C}$ and consider

$$P_k^+ = \left\{ \hat{\lambda} = \sum_{i=0}^{n-1} m_i \hat{\omega}_i + \gamma\delta \;\middle|\; \sum_{i=1}^{n-1} m_i = k\,,\ m_i \in \mathbb{Z}_{\geq 0} \right\}, \tag{2.2}$$

the set of *integral dominant weights of level* k. Since (up to a grading induced by the action of δ) the integrable highest weight modules are independent of the choice of γ, the particular choice will not be important for us and we therefore assume from now on $\gamma = 0$. The m_i appearing here are often called Dynkin labels and will also be denoted $m_i(\hat{\lambda})$ in the following. For $\hat{\lambda} = \lambda + k\hat{\omega}_0 \in P_k^+$ we call λ the *finite part* of $\hat{\lambda}$. Note that $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$.

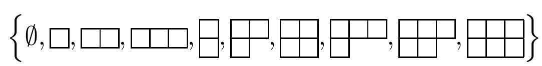
It is convenient to encode affine and non-affine weights in terms of partitions. For any two integers $r, c \in \mathbb{Z}_{\geq 0}$ let

$$\mathfrak{P}_{\leq r,c} = \{\mu = (\mu_1, \dots, \mu_r, 0, 0, \dots) \mid \mu_i \in \mathbb{Z}_{\geq 0},\ \mu_i \geq \mu_{i+1},\ \mu_1 \leq c\}$$

which is the set of all partitions whose associated Young diagram has at most r rows and c columns, i.e. they fit into a boundary box of height r and width c. Hereby we use the English notation to denote Young diagrams, often just write $(\mu_1, \dots, \mu_r)$ instead of $(\mu_1, \dots, \mu_r, 0, 0, \dots)$ and use the symbol $\emptyset$ to denote the (unique) partition of zero; so for instance

$$\mathfrak{P}_{\leq 2,3} = \{\emptyset, (1), (2), (3), (1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\} \tag{2.3}$$

corresponds to

(2.4) 

The following Lemma passes between weights and partitions. For instance, it identifies the Young diagrams in (2.4) via the map P^{-1} in (2.5) with the weights

$$3\hat{\omega}_0, 2\hat{\omega}_0+\hat{\omega}_1, \hat{\omega}_0+2\hat{\omega}_1, 3\hat{\omega}_1, 2\hat{\omega}_0+\hat{\omega}_2, \hat{\omega}_0+\hat{\omega}_1+\hat{\omega}_2, \hat{\omega}_0+2\hat{\omega}_2, 2\hat{\omega}_1+\hat{\omega}_2, \hat{\omega}_1+2\hat{\omega}_2, 3\hat{\omega}_2.$$

Lemma 2.1. *(1) With the notation from* (2.2)*, there is a bijection of sets*

$$\begin{array}{rccl} \mathrm{P}: & P_k^+ & \longrightarrow & \mathfrak{P}_{\leq n-1,k} \\ & \hat{\lambda} & \longmapsto & (\mu_1, \dots, \mu_{n-1}, 0, \dots) \text{ with } \mu_i - \mu_{i+1} = m_i. \end{array} \tag{2.5}$$

(The associated Young diagram has then exactly m_i columns of length i.)

(2) There is an injective map

$$\begin{array}{rccl} \hat{\mathrm{P}}: & P_k^+ & \longrightarrow & \mathfrak{P}_{\leq n,k} \\ & \hat{\lambda} & \longmapsto & (\mu_1 + m_0, \dots, \mu_{n-1} + m_0, m_0, 0, \dots), \text{ with } \mu_i - \mu_{i+1} = m_i. \end{array} \tag{2.6}$$

(The Young diagram $\hat{\mathrm{P}}(\hat{\lambda})$ is obtained from $\mathrm{P}(\hat{\lambda})$ by adding $m_0 = k - \sum_{i>0} m_i$ columns with n boxes.)

Given $\hat{\lambda} \in P_k^+$ we also denote by $\hat{\lambda}$ the associated Young diagram as in (2.6) and with λ the Young diagram associated to the finite part λ of $\hat{\lambda}$ as in (2.5). Given $\mu \in \mathfrak{P}_{\leq n,k}$ one can remove all columns of length n and obtain a partition $\tilde{\mu} \in \mathfrak{P}_{\leq n-1,k}$. We denote by $\mu' = \mathrm{P}^{-1}(\tilde{\mu})$, the preimage under P. In the following we identify the elements from $\{0, 1, \dots, n-1\}$ with the elements from $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$.

2.2. **Diagram automorphisms and the vector space of states.** The group of Dynkin diagram automorphisms of Γ is generated by the rotation rot of order n which sends vertex i to vertex $i+1$ modulo n and the diagram automorphism flip coming from the non-affine Dynkin diagram which fixes the 0^{th} vertex and otherwise maps vertex i to vertex $n-i$. There are corresponding automorphisms $P_k^+ \to P_k^+$:

$$\hat{\lambda} = \sum_{i\in\mathbb{Z}_n} m_i\hat{\omega}_i \quad \longmapsto \quad \mathrm{rot}(\hat{\lambda}) := \sum_{i\in\mathbb{Z}_n} m_i\hat{\omega}_{i-1} \tag{2.7}$$

$$\hat{\lambda} = k\hat{\omega}_0 + \sum_{i=1}^{n-1} m_i\omega_i \quad \longmapsto \quad \mathrm{flip}(\hat{\lambda}) := k\hat{\omega}_0 + \sum_{i=1}^{n-1} m_i\omega_{n-i} \tag{2.8}$$

In terms of partitions (via (2.5)) the rotation corresponds to adding a row of k boxes and then removing all columns which contain n boxes. The action of flip is best described in terms of taking complements of partitions: for a partition μ fitting into a box of height n and width k we denote by $\mu^\vee$ the complementary partition of μ in this box. Then $\mathrm{flip}(\hat{\lambda})$ gets mapped to the elements

$$\hat{\lambda}^* := \hat{\mathrm{P}}(\mathrm{flip}(\hat{\lambda})) = (\mathrm{P}(\hat{\lambda}))^\vee \quad \text{and} \quad \lambda^* := \mathrm{P}(\mathrm{flip}(\hat{\lambda})) = (\hat{\mathrm{P}}(\hat{\lambda}))^\vee\ . \tag{2.9}$$

For a partition λ we denote by λ^t its transpose partition (in terms of Young diagrams it just means we reflect the diagram in the diagonal).

2.3. **The vector space of states and the phase algebra.** We now introduce what we call the vector space of states. Instead of considering only a fixed level k it turns out to be more convenient to allow k ranging over the positive integers. That is, we consider the infinite-dimensional vector space

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathcal{H}_k, \qquad \mathcal{H}_k = \mathbb{C}P_k^+\ , \tag{2.10}$$

with $\mathcal{H}_0 = \mathbb{C}\{\emptyset\} = \mathbb{C}$. For each $0 \leq i \leq n-1$ we define linear endomorphisms of $\mathcal{H}$ by the following assignment on basis vectors

$$\varphi_i^* : \quad P_k^+ \to P_{k+1}^+ \qquad \varphi_i^*\hat{\lambda} = \hat{\lambda} + \hat{\omega}_i \tag{2.11}$$

$$\varphi_i : \quad P_k^+ \to P_{k-1}^+ \cup \{0\} \qquad \varphi_i\hat{\lambda} = \begin{cases} \hat{\lambda} - \hat{\omega}_i, & \text{if } \hat{\lambda} - \hat{\omega}_i \in P_{k-1}^+, \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

In terms of partitions the map φ_i^* acts on the Young diagram associated with λ by adding a column with i boxes in the appropriate place. In contrast, φ_i is the map which deletes from λ a column with i boxes or, if it has none, sends λ to zero. For $i = n$ the corresponding maps simply increase or decrease the width of the bounding box by one provided this is allowed by the shape of the diagram.

With the notation from (2.2) and $0 \leq i \leq n-1$ let N_i be the linear endomorphism of $\mathcal{H}$ which multiplies every basis vector by m_i, in formulae: $N_i\hat{\lambda} = m_i(\hat{\lambda})\hat{\lambda}$. The subalgebra of $\mathrm{End}\,\mathcal{H}$ generated by $\{\varphi_i, \varphi_i^*, N_i\}$ has been introduced previously in the physics literature and is called the *phase algebra*; compare with [7]. The operators φ_i and φ_i^* can be interpreted as respectively particle *annihilation* and *creation operators* at site i of a circular lattice which coincides with the Dynkin diagram of $\widehat{\mathfrak{sl}}(n)$. The Dynkin label $m_i(\hat{\lambda})$ is the occupation number at site i and thus $N = \sum_i N_i$ is the *total particle number operator*, and the subspace $\mathcal{H}_k$ corresponds to the physical states which contain k particles. The map $\pi_i = 1 - \varphi_i^*\varphi_i$ projects onto the subspace where no particle is sitting at site i. In the following we will denote

φ_0 also by φ_n, and similarly φ_0^* also by φ_n^*. We refer to Section 4 for the so-called phase model and to Figure 1.1 for an illustration.

3. The Phase algebra and the quantum Yang-Baxter-algebra and their connection to integrable systems

3.1. **The phase algebra (creation and annihilation of particles).**

Proposition 3.1 (Phase algebra)**.** *The φ_i generate a subalgebra $\hat{\Phi}$ of $\mathrm{End}(\mathcal{H})$ which can be realized as the algebra Φ with the following generators and relations for $0 \leq i \neq j \leq n-1$:*

$$\varphi_i\varphi_j = \varphi_j\varphi_i, \quad \varphi_i^*\varphi_j^* = \varphi_j^*\varphi_i^*, \quad N_iN_j = N_jN_i \tag{3.1}$$

$$N_i\varphi_j - \varphi_jN_i = -\delta_{ij}\varphi_i, \quad N_i\varphi_j^* - \varphi_j^*N_i = \delta_{ij}\varphi_i^*, \tag{3.2}$$

$$\varphi_i\varphi_i^* = 1, \quad \varphi_i\varphi_j^* = \varphi_j^*\varphi_i, \tag{3.3}$$

$$N_i(1-\varphi_i^*\varphi_i) = 0 = (1-\varphi_i^*\varphi_i)N_i. \tag{3.4}$$

The algebra Φ has a basis B of the form

$$\{B_{\mathbf{b},\mathbf{a},\mathbf{c}} := {\varphi_0^*}^{b_0}{\varphi_1^*}^{b_1}\cdots{\varphi_{n-1}^*}^{b_{n-1}}\varphi_0^{a_0}\varphi_1^{a_1}\cdots\varphi_{n-1}^{a_{n-1}}N_0^{c_0}N_1^{c_1}\cdots N_{n-1}^{c_{n-1}}\}, \tag{3.5}$$

where $a_i, b_i, c_i \in \mathbb{Z}_{\geq 0}, a_ib_ic_i = 0$ for $0 \leq i \leq n-1$. If we introduce the scalar product on the vector space $\mathcal{H}$ by

$$\langle \alpha\hat{\lambda}, \beta\hat{\mu}\rangle = \overline{\alpha}\beta\prod_{i=0}^{n-1}\delta_{m_i(\hat{\lambda}),m_i(\hat{\mu})},$$

for $\alpha, \beta \in \mathbb{C}$, then $\langle \varphi_i^*\hat{\lambda}, \hat{\mu}\rangle = \langle\hat{\lambda}, \varphi_i\hat{\mu}\rangle$.

Remark 3.2. One can easily check that with $\pi_i = 1 - \varphi_i^*\varphi_i$ the following (probably better known) relations hold: $\pi_i\varphi_i^* = \varphi_i\pi_i = 0, \pi_i^2 = \pi_i$.

Proof. Straightforward calculations show that the asserted relations hold in the algebra $\hat{\Phi}$. Hence, Φ surjects canonically onto $\hat{\Phi}$. The commutator relations, the relation $\varphi_i\varphi_i^* = 1$ and (3.4) imply that the elements in the proposed basis B at least span Φ. To see that they are linearly independent, we look at their action on $\mathcal{H}$. For the following argument we write a basis vector $\hat{\lambda}$ of $\mathcal{H}$ (see (2.2)) as a tuple $\hat{\lambda} = (m_0, m_1, \ldots, m_{n-1}) = \mathbf{m}$. Assume we have a finite linear combination

$$Z := \sum_{\mathbf{b},\mathbf{a},\mathbf{c}}\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}B_{\mathbf{b},\mathbf{a},\mathbf{c}} = 0 \tag{3.6}$$

with all $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}} \in \mathbb{C}$ non-zero. We have to produce a contradiction. Obviously, $Z\hat{\lambda} = 0$ for any $\hat{\lambda} = \mathbf{m}$. If we choose the m_i's big enough then

$$Z\hat{\lambda} = \sum_{\mathbf{b},\mathbf{a},\mathbf{c}}\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}m_0^{c_0}m_1^{c_1}\ldots m_{n-1}^{c_{n-1}}(m_0-a_0+b_0, m_1-a_1+b_1, \ldots, m_{n-1}-a_{n-1}+b_{n-1}).$$

The coefficient of $(m_0-x_0, m_1-x_1, \ldots, m_{n-1}-x_{n-1})$ for fixed $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ is equal to

$$\sum_{(\mathbf{b},\mathbf{a},\mathbf{c})\in X}\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}m_0^{c_0}m_1^{c_1}\ldots m_{n-1}^{c_{n-1}} \tag{3.7}$$

where the sum runs over the set X of all triples with $\mathbf{a}-\mathbf{b}=\mathbf{x}$. All these coefficients have to be zero. In other words, the polynomial

$$P(t_0,t_1,\dots,t_{n-1}) \quad = \sum_{(\mathbf{b},\mathbf{a},\mathbf{c})\in X} \gamma_{\mathbf{b},\mathbf{a},\mathbf{c}} t_0^{c_0} t_1^{c_1}\dots t_{n-1}^{c_{n-1}} \tag{3.8}$$

satisfies $P(m_0,m_1,\dots,m_{n-1})=0$ whenever the m_i's are big enough. Now consider P as a polynomial $\tilde{P}$ in t_0 by putting fixed values $t_i=m_i$ for $i>0$. Then $\tilde{P}$ has infinitely many zeroes (all the big enough m_0), so it is constant zero, and each coefficient of $t_0^{c_0}$ for fixed c_0 has to be zero. Repeating this argument finally implies that P is the zero polynomial. (Alternatively, P vanishes on the set of all the $\mathbf{m}$'s for big enough m_i's. They form a Zariski dense set in the affine space $\mathbb{A}^n$, hence P has to be the zero polynomial). In particular we can fix $\mathbf{c}$, consider the set $X(\mathbf{c})$ of all triples $(\mathbf{b},\mathbf{a},\mathbf{c})$ in X with $\mathbf{c}$ our fixed choice and get

$$\sum_{(\mathbf{b},\mathbf{a},\mathbf{c})\in X(\mathbf{c})} \gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}=0. \tag{3.9}$$

In case the components c_i of $\mathbf{c}$ are all non-zero, then the condition $a_ib_ic_i=0$ implies that for any i, $a_i=0$ or $b_i=0$, and so there is a unique element in $X(\mathbf{c})$ (namely $x_i=b_i$, $a_i=0$ if $x_i\geq 0$ and $x_i=a_i$, $b_i=0$ if $x_i\leq 0$), because $a_i,b_i\geq 0$. Therefore, $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}=0$. Hence all the in Z occurring $\mathbf{c}$'s contain at least one zero. For $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}$ occurring in Z we let $l(\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}})$ be the number of zeroes in $\mathbf{c}$. Let l be the minimum of all these $l(\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}})$. From above we know $l>0$.

Now we choose some $\mathbf{c}$ occurring in Z with exactly l zeroes. Call it $\tilde{\mathbf{c}}$. Without restriction we may assume $\tilde{c}_0=\tilde{c}_1=\ldots=\tilde{c}_{l-1}=0$. Consider the $\gamma_{\mathbf{b},\mathbf{a},\tilde{\mathbf{c}}}$'s appearing in (3.6). Amongst these pick the ones with a_0, call it $\tilde{a}_0$, minimal. Amongst these $\gamma_{\mathbf{b},\mathbf{a},\tilde{\mathbf{c}}}$ with $a_0=\tilde{a}_0$ choose a_1 minimal, call it $\tilde{a}_1$, etc. Carrying on like this defines $\tilde{a}_i$ for $0\leq i\leq l-1$. Set $\mathbf{m}=(\tilde{a}_0,\tilde{a}_1,\dots,\tilde{a}_{l-1},m_l,\dots,m_{n-1})$ and abbreviate

$$M(\mathbf{a},\mathbf{b},\mathbf{c})=(\tilde{a}_0-a_0+b_0,\dots\tilde{a}_{l-1}-a_{l-1}+b_{l-1},m_l-a_l+b_l,\dots,m_{n-1}-a_{n-1}+b_{n-1}).$$

If we choose the m_i's big enough then

$$0=Z\mathbf{m}=\sum_{\mathbf{b},\mathbf{a},\mathbf{c}} \gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}\tilde{a}_0^{c_0}\dots\tilde{a}_{l-1}^{c_{l-1}}m_l^{c_l}\dots m_{n-1}^{c_{n-1}}M(\mathbf{a},\mathbf{b},\mathbf{c}),$$

where the sum runs over all triples with the extra condition $a_i\leq\tilde{a}_i$ for $0\leq i\leq l-1$. As in (3.7) we consider the coefficient of a fixed basis vector $\hat{\lambda}$ and use the polynomial argument from (3.8) to deduce that

$$\sum_{\mathbf{b},\mathbf{a},\mathbf{c}\in X(\tilde{c},a_i\leq\tilde{a}_i)} \tilde{a}_0^{c_0}\tilde{a}_1^{c_1}\dots\tilde{a}_{l-1}^{c_{l-1}}\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}=0, \tag{3.10}$$

where the set $X(\tilde{c},a_i\leq\tilde{a}_i)$ consists of all triples with $a_i\leq\tilde{a}_i$ for $0\leq i\leq l-1$ and $c_i=\tilde{c}_i$ for $l\leq i\leq n-1$ as well as $\mathbf{a}-\mathbf{b}=\mathbf{x}$ fixed. The minimality of l implies however that $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}\neq 0$ and $c_i=\tilde{c}_i$ for $l\leq i\leq n-1$ forces $\mathbf{c}=\tilde{\mathbf{c}}$, and so we must have $\mathbf{c}=\tilde{\mathbf{c}}$. Now (3.10) refines (3.9) in the following sense

$$\sum_{(\mathbf{b},\mathbf{a},\tilde{\mathbf{c}})\in X(\tilde{\mathbf{c}})} \gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}=0, \tag{3.11}$$

with $a_i\leq\tilde{a}_i$ for $0\leq i\leq l-1$. The minimality property of the $\tilde{a}_i$ and the choice of $\mathbf{c}$ allows to replace the above condition $a_i\leq\tilde{a}_i$ by $a_i=\tilde{a}_i$ for $0\leq i\leq l-1$. The conditions $c_i\neq 0$ for $l\leq i\leq n-1$ and $a_ib_ic_i=0$ imply that for any such i,

$a_i = 0$ or $b_i = 0$. Therefore, the sum (3.11) has only one summand $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}}$ and we get $\gamma_{\mathbf{b},\mathbf{a},\mathbf{c}} = 0$. This is a contradiction. Therefore the elements of B are linearly independent and (3.5) is in fact a basis of Φ. The action of Φ on $\mathcal{H}$ is therefore faithful and factors through $\hat{\Phi}$. Hence the composition $\Phi \overset{\mathrm{can}}{\twoheadrightarrow} \hat{\Phi} \hookrightarrow \mathrm{End}(\mathcal{H})$ is injective, hence can is an isomorphism. Finally the last claim of the proposition follows directly from the definitions. □

3.2. **The quantum Yang-Baxter algebra and Pieri rules.** Using the phase algebra we want to describe a solution to the quantum Yang-Baxter equation arising in [7]. To calculate with endomorphisms of $\mathbb{C}^2 \otimes \mathcal{H}$ we use the abbreviation

$$\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes f_1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes f_2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes f_3 + u \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes f_4,$$

for endomorphisms of $\mathbb{C}^2 \otimes \mathcal{H}$, where the left hand side is a 2×2 matrix with entries in $\mathrm{End}(\mathcal{H})$. It is straightforward to check that the composition of endomorphisms corresponds to the usual matrix multiplication.

For $i \in \{1, 2, \ldots, n\}$, the i^{th} *Lax matrix* or L-operator $L_i = L_i(u)$ is the following one-parameter family of endomorphisms of $\mathbb{C}^2 \otimes \mathcal{H}$

$$L_i(u) = \begin{pmatrix} 1 & u\varphi_i^* \\ \varphi_i & u1 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathcal{H}) \ . \tag{3.12}$$

The complex variable $u \in \mathbb{C}$ is called the *spectral parameter*. The *monodromy matrix* is defined as

$$M(u) = L_n(u) \cdots L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathcal{H}). \tag{3.13}$$

If we identify $\mathbb{C}^4$ with $\mathbb{C}^2 \otimes \mathbb{C}^2$ by mapping the standard basis e_1, e_2, e_3, e_4 to $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_1$, $e_2 \otimes e_2$, then we get

Lemma 3.3 (cf. [7])**.** *The monodromy matrix is a solution to the following RTT-relation*

$$R_{12}(u/v) M_1(u) M_2(v) = M_2(v) M_1(u) R_{12}(u/v), \qquad u, v, x \in \mathbb{C}, u \neq v \neq 0, \tag{3.14}$$

with

$$R(x) = \begin{pmatrix} \frac{x}{x-1} & 0 & 0 & 0 \\ 0 & 0 & \frac{x}{x-1} & 0 \\ 0 & \frac{1}{x-1} & 1 & 0 \\ 0 & 0 & 0 & \frac{x}{x-1} \end{pmatrix} \in \mathrm{End}\, \mathbb{C}^4 \cong \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \ . \tag{3.15}$$

Proof. It is easy to check that the equation holds when we replace M by any L_i. The definition of M and the fact that the L_i pairwise commute imply the claim. □

For F from A, B, C, D as in (3.13) we introduce the power series decomposition $F(u) = \sum_{r \geq 0} F_r u^r$ with respect to the spectral parameter u. Note that it follows from (3.12) and (3.13) that $A_r, B_r, C_r, D_r = 0$ for $r > n$.

Lemma 3.4. *The monodromy matrix elements are explicitly given by*

$$\begin{array}{rclrcl} A(u) & = & \sum_{0 \leq r \leq n/2} \sum u^{|j_1 - i_1| + \cdots + |j_r - i_r|} a_{i_1, j_1} \cdots a_{i_r, j_r}, & C(u) & = & \varphi_n A(u), \\ D(u) & = & \sum_{0 \leq r \leq n/2} \sum u^{n - |j_1 - i_1| - \cdots - |j_r - i_r|} a_{j_1, i_1} \cdots a_{j_r, i_r}, & B(u) & = & D(u) \varphi_n^*, \end{array}$$

where $a_{i,j} = \varphi_i \varphi_j^*$, and the not specified sums run through the set of tuples $0 \leq i_1 < j_1 < \ldots < i_r < j_r \leq n - 1$.

Proof. For $n=2$ (or $n=1$) this formula is clear by (3.13). In general it follows from an easy induction on n. $\square$

Definition 3.5. The *quantum Yang-Baxter algebra* is the algebra $\mathfrak{B}$ generated by the $\{A_r, B_r, C_r, D_r\}_{r\geq 0}$ subject to the commutation relations (3.14) via the decompositions $F(u)=\sum_{r\geq 0} F_r u^r$.

Remark 3.6. The Yang-Baxter algebra can be equipped with a bialgebra structure with comultiplication $\Delta(A)=A\otimes A+C\otimes B$, $\Delta(B)=B\otimes A+D\otimes B$, $\Delta(C)=A\otimes C+C\otimes D$, $\Delta(D)=B\otimes C+D\otimes D$ and co-unit $\varepsilon(A)=\varepsilon(D)=1$, $\varepsilon(C)=\varepsilon(B)=0$. The above construction resembles the RTT-construction of Yangians. However, in our case the $L_i(0)$ are not invertible, so that for instance the usual construction [34, (1.27)] of the antipode is not applicable. One can in fact show that the bialgebra structure does not extend to a Hopf algebra structure.

Remark 3.7. The term 'quantum' Yang-Baxter algebra has its origin in the physical interpretation of the integrable model which is underlying our construction. By forming the state space $\mathcal{H}$ of particle configurations on a circle we allow for their complex linear superpositions which is the hallmark of a quantum mechanical system in physics. The adjective 'quantum' sets our construction apart from algebraic constructions connected with the so-called classical Yang-Baxter equation which differs from relation (3.14).

Expanding the quantum Yang–Baxter equation (or RTT-equation) leads for instance to the identities

$$\begin{aligned}(u-v)A(u)B(v) &= vB(u)A(v)-vB(v)A(u), &\quad (3.16)\\ (u-v)\,D(u)B(v) &= uB(v)D(u)-vB(u)D(v)\\ C(u)B(v) &= \frac{v}{u-v}[A(v)D(u)-A(u)D(v)]\,. &\quad (3.17)\end{aligned}$$

The action of the phase algebra on the state space $\mathcal{H}$ induces an action of the Yang-Baxter algebra. We describe this action again combinatorially. Let λ and μ be partitions which we identify with their Young diagrams. Assume that the diagram λ contains the diagram μ. Then the *skew diagram* λ/μ is obtained by removing μ from λ. It is a *vertical strip* if it contains at most 1 box in each row, or equivalently if $0\leq\lambda_i-\mu_i\leq 1$. We call it a *vertical r-strip*, denoted (1^r), if it is a vertical strip containing exactly r boxes. Horizontal strips are defined analogously.

Proposition 3.8 (Pieri-type formulae)**.** *The space of states $\mathcal{H}$ can be be turned into a $\mathfrak{B}$-module such that the action of the generators on the basis is given as follows:*

$$A_r\hat{\mu}=\sum_{\substack{\hat{\lambda}-\hat{\mu}=(1^r)\\ \hat{\lambda}\in\mathcal{H}_k}}\hat{\lambda},\quad B_r\hat{\mu}=\sum_{\substack{\hat{\lambda}-\hat{\mu}=(1^r)\\ \hat{\lambda}\in\mathcal{H}_{k+1}}}\hat{\lambda}\,,\quad C_r\hat{\mu}=\sum_{\substack{\hat{\mu}-\hat{\lambda}=(1^r)\\ \hat{\lambda}\in\mathcal{H}_{k-1}}}\hat{\lambda},\quad D_r\hat{\mu}=\sum_{\substack{\hat{\mu}-\hat{\lambda}=(1^r)\\ \hat{\lambda}\in\mathcal{H}_k}}\hat{\lambda}\,.$$

In particular, B increases, whereas C decreases the level.

Proof. The claim can be checked easily by induction using the comultiplication in Remark 3.6. Alternatively one may use the explicit formulae from Lemma 3.4. $\square$

The above action of the Yang-Baxter algebra on $\mathcal{H}$ can be described in terms of skew Schur functions. We first recall the necessary notions to explain the result and refer to [17] or [32] for more details. Given a Young diagram λ a *semi-standard standard tableau* (or just *tableau*) is a filling of the boxes of λ with the numbers from $\{1, 2, \ldots, n\}$ such that the entries are strictly increasing downwards along the columns and weakly increasing to the right along the rows. Given a tableau T we have the associated monomial $x^T := x_1^{q_1} x_2^{q_2} x_3^{q_3} \ldots x_n^{q_n}$ where q_i denotes the number of boxes filled with i. The Schur polynomial s_λ is then the sum $\sum x^T$, where T runs through all semi-standard tableaux of shape λ.

Corollary 3.9. *Let $\hat{\mu} \in \mathfrak{P}_{\leq n,k}$ be the partition associated with an affine weight of level k via (2.6). For any $x_1, \ldots, x_\ell$ complex numbers or invertible formal variables we have*

$$
\begin{aligned}
A(x_1) \cdots A(x_\ell)\hat{\mu} &= \sum_{\hat{\lambda} \in \mathrm{P}_k^+} s_{\hat{\lambda}^t/\hat{\mu}^t}(x_1, \ldots, x_\ell)\hat{\lambda}, \\
B(x_1) \cdots B(x_\ell)\hat{\mu} &= \sum_{\hat{\lambda} \in \mathrm{P}_{k+\ell}^+} s_{\hat{\lambda}^t/\hat{\mu}^t}(x_1, \ldots, x_\ell)\hat{\lambda}, \\
C(x_1) \cdots C(x_\ell)\hat{\mu} &= (x_1 \cdots x_\ell)^n \sum_{\hat{\lambda} \in \mathrm{P}_{k-\ell}^+}^{n} s_{\hat{\mu}^t/\hat{\lambda}^t}(x_1^{-1}, \ldots, x_\ell^{-1})\hat{\lambda}, \\
D(x_1) \cdots D(x_\ell)\hat{\mu} &= (x_1 \cdots x_\ell)^n \sum_{\hat{\lambda} \in \mathrm{P}_k^+} s_{\hat{\mu}^t/\hat{\lambda}^t}(x_1^{-1}, \ldots, x_\ell^{-1})\hat{\lambda} \,.
\end{aligned}
$$

Note that for $\hat{\mu} = \emptyset$ we get the usual Schur polynomials

$$
B(x_1) \cdots B(x_\ell)\hat{\mu} = \sum_{\hat{\lambda} \in \mathrm{P}_\ell^+} s_{\hat{\lambda}^t}(x_1, \ldots, x_\ell)\hat{\lambda} \,. \tag{3.18}
$$

Proof. Each semi-standard tableau T determines uniquely a sequence of partitions $\mu = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)} = \lambda$ where μ_i is obtained from λ by removing all boxes in T filled with a number greater than i. Semi-standardness implies that $\lambda^{(r+1)}/\lambda^{(r)}$ is a horizontal strip. Conversely, every sequence of partitions differing by horizontal strips arise from a semi-standard tableau in this way. Applying Proposition 3.8 yields the desired result. □

4. Transfer matrices and the phase model

We now employ the quantum Yang-Baxter algebra to define a discrete quantum integrable system, called the *phase model* in [7] due to its similarity to constructions in quantum optics. Our approach is motivated by this physical model with the so-called transfer matrix playing a central role in our combinatorial construction: as we will see below it is the generating function of the cyclic noncommutative elementary symmetric polynomials (Proposition 5.13).

First we extend scalars of the vector space of states from $\mathbb{C}$ to $\mathbb{C}[z]$, the ring of polynomials in one variable and denote it $\mathcal{H}[z] := \mathbb{C}[z] \otimes_{\mathbb{C}} \mathcal{H}$. In Section 8, when the quantum cohomology comes into the picture, z will play a role analogous to the deformation parameter q in the small quantum cohomology ring. In physical applications it is a magnetic flux parameter (or number) related to quasi-periodic boundary conditions. It enters the following definition of the one-parameter family of row-to-row transfer matrices,

$$
T(u) = \sum_{r \geq 0} T_r u^r = A(u) + zD(u) \,. \tag{4.1}
$$

The term 'transfer matrix' has its origin in yet another physical interpretation of the phase model. Namely, consider an $m \times n$ square lattice with periodic boundary conditions in both directions, i.e. a toroidal lattice. Fix a particle configuration of our circular lattice with n sites by choosing a partition. Then the operator T maps, 'transfers', this configuration into a linear combination of other configurations weighed with its matrix elements which for a special value of the spectral parameter are interpreted as Boltzmann weights, i.e. statistical probabilities that a particular configuration occurs. Taking the m^{th} power of the transfer matrix we end up with a cylinder of height m and by taking its trace we compute the so-called partition function,

$$Z(u) = \mathrm{Tr}_{\mathcal{H}[z]} T(u)^m \ ,$$

which is the sum over all allowed configuration on the torus and which is the central physical object when interpreting the phase model as a system in statistical mechanics rather than quantum mechanics.

Employing the Yang-Baxter equation one easily shows (the original idea of the proof goes back to Baxter, but the argument can also be found in e.g. [16, §3, Lemma 2]) that

$$[T(u), T(v)] = 0 \tag{4.2}$$

for any pair of spectral parameters $u, v \in \mathbb{C}$. Hence the model is *integrable*, as it possesses an infinite number of conserved quantities.

Alternatively, one can define a quantum system in terms of the Hamiltonian

$$H = -\frac{T_1 + T_{n-1}}{2} = -\frac{1}{2}\sum_{i=1}^{n} \left(\varphi_i \varphi^*_{i+1} + z\varphi^*_i \varphi_{i+1}\right) , \tag{4.3}$$

where $\varphi_{n+1} = z^{-1}\varphi_1$ and $\varphi^*_{n+1} = z\varphi^*_1$, i.e. one considers particles moving on a circle of n sites. More generally, we can define higher Hamiltonians, conserved charges by setting,

$$H^{\pm}_r = -\frac{T_r \pm T_{n-r}}{2} \ . \tag{4.4}$$

The latter are in involution, $[H^{\pm}_r, H^{\pm}_{r'}] = 0$ for any $1 \le r, r' \le (n-1)/2$, and again we have an integrable model. However, it is not the existence of these integrals of motions, but rather the quantum Yang-Baxter algebra which allows one to solve the model explicitly, i.e. to compute the eigenstates and eigenvalues of the Hamiltonian and the conserved charges.

5. Plactic algebras, universal enveloping algebras and crystal graphs

In the previous section we have largely reviewed well-known algebraic structures from the physics literature on quantum integrable systems. In this section we introduce tools from algebraic combinatorics and Lie theory and show their relations with the phase model. These connections are novel results and are crucial for our setup.

5.1. **The (local affine) plactic algebra.** We start by defining an algebra $\mathrm{Pl}(\mathcal{A})$, motivated by the plactic algebra introduced by Lascoux and Schützenberger in [31] with the following natural quotient studied for instance in [15]:

Definition 5.1. *The* local plactic algebra $\mathrm{Pl}_{\mathit{fin}}$ *is the free algebra generated by the elements of* $\{a_1, a_2, \ldots, a_{n-1}\}$ *modulo the relations*

$$a_i a_j = a_j a_i, \qquad \text{if } |i-j| > 1, \tag{5.1}$$

$$a_{i+1} a_i^2 = a_i a_{i+1} a_i, \qquad a_{i+1}^2 a_i = a_{i+1} a_i a_{i+1}, \tag{5.2}$$

whenever the expressions are defined.

Recall from [31] the plactic monoid M defined by the Knuth relations. The Robinson-Schenstedt algorithm gives a bijection between the equivalence classes of the monoid M and the set $\mathcal{T}$ of tableaux with filling from $\{1, 2, \ldots, n-1\}$ [17]. Given a tableau $\mathcal{T}$ we obtain the corresponding word by reading the columns from left to right and bottom to top, replacing each number i by the generator a_i. These words form then in particular a basis of the monoid algebra $\mathbb{C}[M]$. The *local* plactic algebra is the quotient of $\mathbb{C}[M]$ with the additional local relations (5.1). In the following we summarize a few properties of the local plactic algebra:

Proposition 5.2 (Local plactic algebra)**.**

1. *Let* $N = \frac{(n-1)n}{2}$*, then the words of the form* $a_1^{\alpha_1}(a_2a_1)^{\alpha_2}(a_2)^{\alpha_3}(a_3a_2a_1)^{\alpha_4}(a_3a_2)^{\alpha_5}(a_3)^{\alpha_6}\ldots(a_{n-1}a_{n-2})^{\alpha_{N-1}}(a_{n-1})^{\alpha_N}$ *where the* α*'s run through the nonnegative integers form a basis* B *of* $\mathrm{Pl}_{\mathit{fin}}$*.*
2. *The Robinson-Schenstedt correspondence defines a bijection between* B *and the set* $\mathcal{T}_{str}$ *of tableaux in* $\mathcal{T}$*, where the number of* j*'s appearing in row* i *is smaller or equal the number of* $j-1$*'s appearing in row* $i-1$ *for any* $i > 1$, $j \in \{1, 2, \ldots, n-1\}$*.*
3. *There is a 2-parameter deformation* $U^+_{r,s}$ *(with generic* r *and* s*) of the universal enveloping algebra* U^+ *of the strictly upper triangular matrices of* $\mathfrak{sl}(n)$ *such that the specialisation* $U^+_{q,q^{-1}}$ *is isomorphic to the usual Drinfeld-Jimbo quantisation* U_q^+ *of* U^+ *(with generic* q*) and* $U_{0,1}$ *is isomorphic to* $\mathrm{Pl}_{\mathit{fin}}$*. The basis above is a specialisation of both, a canonical basis and a PBW-basis.*

Proof. For (2) we indicate the bijection for the case $n = 4$. The general case is completely analogous. Note that the Robinson-Schenstedt algorithm transfers a word $a_i a_{i-1} a_{i_3} \ldots a_j$ with $i \geq j$ into a tableau consisting of a single column with entries $j, j+1, \ldots, i-1, i$ (starting from the top). Then more generally, the word $a_1^{\alpha_1}(a_2a_1)^{\alpha_2}(a_2)^{\alpha_3}(a_3a_2a_1)^{\alpha_4}(a_3a_2)^{\alpha_5}(a_3)^{\alpha_6}$ is mapped under the Robinson-Schensted algorithm to the tableau where the first row contains $\alpha_1 + \alpha_2 + \alpha_4$ ones, $\alpha_3 + \alpha_5$ twos and α_6 threes, the second row contains $\alpha_2 + \alpha_4$ twos and α_5 threes, the last row contains α_4 threes. In particular, the tableau is in $\mathcal{T}_{str}$ and the map is injective on B. On the other hand varying the $\alpha's$ provides all possible tableaux in $\mathcal{T}_{str}$. Hence (2) follows. Part (1) follows from part (3) or from the explicit algorithm in the next subsection. Part (3) goes back to [42] where Takeuchi defined a $\mathbb{C}[r,s]$-algebra $U^+_{r,s}$ with generators E_i, $1 \leq i \leq n$ and relations E_iE_j if $|i-j| > 1$ and

$$E_{i+1}^2E_i - (r+s)E_{i+1}E_iE_{i+1} + rsE_iE_{i+1}^2, \qquad E_{i+1}E_i^2 - (p+q)E_iE_{i+1}E_i + pqE_i^2E_{i+1}.$$

Setting $r = 0$ and $s = 1$ gives exactly the relations (5.1) and (5.2). Setting $r = q$ and $s = q^{-1}$ gives the well-known quantum Serre relations (see e.g. [24, 4.3,4.6,4.12b]).

Using standard arguments (see e.g. [24, §8]) one can show that this algebra is a free $\mathbb{C}[r,s]$-module, with a basis given by the elements

$$E_1^{\alpha_1}(E_2E_1)^{\alpha_2}(E_2)^{\alpha_3}(E_3E_2E_1)^{\alpha_4}(E_3E_2)^{\alpha_5}\dots(E_{n-1}E_{n-2})^{\alpha_{\frac{(n-1)n-2}{2}}}(E_n)^{\alpha_{\frac{(n-1)n}{2}}},$$

that is the PBW basis associated with the special reduced expression

$$w_0 = s_1(s_2s_1)s_2(s_3s_2s_1)(s_3s_2)\dots$$

of the longest element of Weyl group (i.e. the symmetric group) S_n of $\mathfrak{sl}(n)$ (here s_i denotes the elementary transposition $(i, i+1)$). The existence of a canonical basis for $U_{r,s}^+$ is proved in [36] with the property that it specialises to the basis in (1). □

Remark 5.3.
- The quantised universal enveloping algebra U_q^+ is isomorphic to (a twisted version of) *Ringel's Hall algebra* [38]. In this context, the plactic algebra appears as the Hall algebra with multiplication defined using generic extensions, hence a *generic Hall algebra* [36]. Reineke also defines the Hall algebra version of $U_{r,s}^+$.
- In Section 11 we give an explicit algorithm which turns an arbitrary tableau into a tableau in $\mathcal{T}_{str}$ using the relations of the local plactic algebra.

Definition 5.4. Let $\mathcal{A} = \{a_0, a_1, a_2, \dots a_{n-1}\}$. The *affine local plactic algebra* $\mathrm{Pl} = \mathrm{Pl}(\mathcal{A})$ is the free algebra generated by the elements of $\mathcal{A}$ modulo the relations

$$a_ia_j - a_ja_i = 0, \qquad \text{if } |i-j| > 1 \mod n, \tag{5.3}$$

$$a_{i+1}a_i^2 = a_ia_{i+1}a_i, \qquad a_{i+1}^2a_i = a_{i+1}a_ia_{i+1}, \tag{5.4}$$

where in (5.4) all variables are understood as elements in $\mathcal{A}$ by taking indices modulo n.

Example 5.5. If $n = 3$ then the defining relations are $a_2a_1^2 = a_1a_2a_1$, $a_0a_2^2 = a_2a_0a_2$, $a_2^2a_1 = a_2a_1a_2$, $a_0^2a_2 = a_0a_2a_0$. (Note that a_2 and a_0 do not commute.)

5.2. **The generalised Robinson-Schensted algorithm.** An *n-multi-partition* is an n-tuple $\pi = (\pi^{(0)}, \dots, \pi^{(n-1)})$ of partitions (resp. Young diagrams). A multi-partition is *aperiodic* if for any positive number l there is at least one Young diagram which does not have a column of height l. An *aperiodic multi-tableau* or just a multi-tableau is given by taking an aperiodic multi-partition π and putting the number $i + r - 1$ (modulo n) into the boxes in the r^{th} row of $\pi^{(i)}$.

Given a word in the local affine monoid, Deng and Du [13] assign (following ideas of Lusztig and Ringel) a multi-partition by the following algorithm: given a multi-partition $\pi = (\pi^{(0)}, \dots, \pi^{(n-1)})$ and $i \in \{0, \dots, n-1\}$ then $a_i\pi$ is the multi-partition obtained from π by adding to $\pi^{(i)}$ an extra box in the first row if $\pi^{(i+1)} = \emptyset$, and otherwise remove the first column, c, of $\pi^{(i+1)}$ and place a column of length one box longer than c in the partition $\pi^{(i)}$. (The indices are again taken modulo n). The multi-partition π associated to a word $w \in \mathrm{Pl}$ is then defined as $\pi = w(\emptyset, \dots, \emptyset)$. One can easily see that this multi-partition is aperiodic. There is also an explicit way to read off a word from a multi-partition as follows: first we convert the multi-partition into a multi-tableau (there is a unique way to do this), then we draw the multi-partitions as a diagram as indicated in Figure 5.1 by pushing down all the columns to a common baseline. Considering only the numbers on top of a column, remove all those 0's appearing higher than all 1's. Considering again only the numbers on top of a column, remove all those 1's appearing higher

FIGURE 5.1. An example of an aperiodic multipartition, $((4,2,2,2),(3,1,1,0),(4,4,2,2))$. Displayed are its multi-tableau and the multi-tableau with all columns pushed down which then corresponds to the word $a_0^2a_2^4a_1^5a_0^6a_2^3a_1^2a_0^3a_2^2$.

$$\begin{ytableau} 0 & 0 & 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{ytableau}\;,\quad \begin{ytableau} \none \\ 1 & 1 & 1 \\ 2 \\ 0 \end{ytableau}\;,\quad \begin{ytableau} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{ytableau} \qquad \rightsquigarrow \qquad \begin{ytableau} 0 & 0 & \none & \none & \none & \none & \none & 2 & 2 \\ 1 & 1 & \none & \none & 1 & \none & \none & 0 & 0 \\ 2 & 2 & \none & \none & 2 & \none & \none & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 \end{ytableau}$$

than all 2's etc. Repeat this procedure (with the cyclic ordering) as long as possible. The aperiodicity guaranties that this process can be carried on until there are no boxes left. The order in which the numbers got removed defines a word $w(\pi)$ in the local affine plactic algebra. We call these words *standard words*.

Proposition 5.6. [13, Theorem 4.1 and its proof]

(1) *The two algorithms are inverse to each other, i.e.* $\pi(w(\pi)) = \pi$ *and* $w(\pi(w))$ *is equivalent to* w *in the local plactic monoid.*
(2) *The standard words associated to aperiodic multi-partitions (resp. multi-tableau) form a basis of* Pl.
(3) *If we replace the letters in the standard words by the usual Chevalley generators* E_i, $0 \le i \le n-1$ *of the positive part* U^+ *of the universal enveloping algebra* $U(\widehat{\mathfrak{sl}}(n))$ *then we obtain a monomial basis of* U^+.

Remark 5.7. For a 2-parameter deformation of U^+ and a monomial basis in terms of Lyndon words we refer to [22].

Proposition 5.8 (Faithfulness)**.** *There is a homomorphism of algebras* $\Psi_{\mathit{fin}} : \mathrm{Pl}_{\mathrm{fin}} \to \Phi$ *such that*

$$a_j \mapsto \varphi_{j+1}^*\varphi_j, \quad j = 1, ..., n-1$$

In particular, the representation (2.12) *of the phase algebra* Φ *lifts to a representation of the local plactic algebra* $\mathrm{Pl}_{\mathit{fin}}$. *This representation is faithful. Moreover, it lifts to a representation of* Pl *on* $\mathcal{H}[z]$ *by mapping* $a_0 = a_n$ *to* $z\varphi_1^*\varphi_0$ *and the* a_j *as above. This representation is again faithful.*

Explicitly, the action on $\mathcal{H}_k$ in terms of the basis vectors given by affine weights reads for $i > 0$

$$a_i : P_k^+ \to P_k^+ \cup \{0\}, \qquad \hat\lambda \mapsto \begin{cases} \hat\lambda + \hat\omega_{i+1} - \hat\omega_i & \text{if } \hat\lambda + \hat\omega_{i+1} - \hat\omega_i \in P_k^+ \\ \\ 0 & \text{otherwise.} \end{cases}$$

In terms of Young diagrams, the endomorphism a_i adds a box in the $(i+1)^{\text{th}}$ row of the Young diagram $\hat{\mathrm{P}}(\hat\lambda)$ (see Lemma 2.1) provided the result is again a Young diagram. For $i = n$ it removes an n-column (if there is none, the result is zero), adds a box in the first row and multiplies with z. Note that these actions physically correspond to moving single particles by one site in clockwise direction on the affine Dynkin diagram (see Figure 1.1).

Remark 5.9. The set P_k^+ with the action of the local (affine) plactic generators a_i (and the obvious weight function) has the structure of an abstract crystal. This crystal is known (see [25]) to be isomorphic to the *crystal of the k^{th} symmetric tensor representation of the vector representation* of the corresponding quantized universal enveloping algebra of adjoint type (in the sense of [24, 4.5]).

Proof of Proposition 5.8. The existence of this morphism follows directly from the definitions. That the representation $\mathcal{H}$ is faithful follows from Proposition 5.2 and Remark 5.9, but we give an explicit argument here. Let $X = \sum c_b b$ be a finite linear combination of basis elements $b \in B$ (see Proposition 5.2) in the plactic algebra P_{fin}. Assume X acts by zero on $\mathcal{H}$. Applying this to the partition containing only one box implies that $c_b = 0$ for all basis vectors b which only consist of a single monomial of the form $(a_r a_{r-1} \dots a_1)$ for arbitrary r. More generally, applying X to a column of height i, the partition (1^i), then implies that $c_b = 0$ for all basis vectors consisting of one monomial of the form $(a_r a_{r-1} \dots a_{i+1} a_i)$. In order to single out basis words consisting of two monomials consider next the partition $(2,1)$. Because of the special structure of the basis vectors application of X then implies that $c_b = 0$ for all summands b which now consist of two monomials, one of the form $(a_r a_{r-1} \dots a_1)$ for arbitrary r and the other of the form $(a_s a_{s-1} \dots a_2)$ for arbitrary s. Continuing with the partition $(3,1)$ implies that $c_b = 0$ for all basis vectors b consisting of two monomials of the form $(a_r a_{r-1} \dots a_1)$ for arbitrary r and one factor of the form $(a_s a_{s-1} \dots a_2)$ etc. Carrying on like this gives the faithfulness. The faithfulness of the second representation follows then from Proposition 5.6 and the definition of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$.

□

Henceforth, we shall always identify the local affine plactic algebra Pl with its image in $\operatorname{End} \mathcal{H}[z]$.

5.3. **Noncommutative polynomials.** Mimicking the case of the ordinary local plactic algebra we now introduce noncommutative polynomials in the generators $\{a_i\}$ of Pl which are (noncommutative, affine) analogs of the ordinary elementary and complete symmetric functions. (This approach is similar to [15], but unfortunately does not satisfy their assumption, so that we cannot use their results directly.) We need the notion of cyclically ordered products $\prod_{i\in I}^{\circlearrowright} a_i$ and $\prod_{i\in I}^{\circlearrowleft} a_j$. A monomial $a_{i_1} a_{i_2} \cdots a_{i_r}$ in the variables $\mathcal{A}$ is *clockwise cyclically ordered* (respectively *anticlockwise cyclically ordered*) if for any two indices i_j, i_k with $i_k = i_j + 1$ modulo n, the variable a_{i_j} occurs to the left (to the right) of a_{i_k}. The origin of the name becomes obvious if we identify a_i with the corresponding point on the Dynkin diagram Γ: there are two circle segments connecting the two points. If they are not of the same length we choose the shorter one and the (anti-)clockwise order is the same as the intuitively defined anti-clockwise order with respect to this segment. For any monomial $a_{i_1} a_{i_2} \cdots a_{i_r}$ not containing all the generators a_i, there is a unique clockwise (resp. anti-clockwise) cyclically monomial which differs only by a permutation of the variables. We denote it by $\prod\limits_{1\le j\le r}^{\circlearrowright} a_{i_j}$, respectively $\prod\limits_{1\le j\le r}^{\circlearrowleft} a_{i_j}$.

Definition 5.10. For $1 \leq r \leq n-1$ we define *cyclic noncommutative elementary symmetric polynomials* as the following elements of Pl

$$e_r(\mathcal{A}) = \sum_{|I|=r} \prod_{i\in I}^{\circlearrowleft} a_i \tag{5.5}$$

where the sum runs over all sets $I = \{i_1, \ldots, i_r\}$ with $i_s \neq i_t$ for $s \neq t$.

Example 5.11. *If $n = 4$ then*

$$e_2(\mathcal{A}) = a_2a_1 + a_3a_1 + a_1a_4 + a_3a_2 + a_4a_2 + a_4a_3 \ .$$

Remark 5.12. To define the noncommutative elementary symmetric polynomials we had to make a choice for the cyclic order. The other choice would give just the adjoint operators with respect to scalar product $\langle\ ,\ \rangle$ on $\mathcal{H}$ from Proposition 3.1.

We will denote by $e_r(\mathcal{A})_k$ the images of the cyclic noncommutative polynomials (5.5) in $\operatorname{End}\mathcal{H}_k[z]$. Similarly, $A(u)_k, B(u)_k, C(u)_k, D(u)_k$ will denote the restriction of the generators of the quantum Yang-Baxter algebra to $\mathcal{H}_k[z]$. The following result realizes the transfer matrix as the generating function of the elementary symmetric polynomials:

Proposition 5.13 (Generating function)**.** *Let $T(u)$ denote the transfer matrix of the phase model as before. Then*

$$T(u) = A(u) + zD(u) = \sum_{r=0}^{n} e_r(\mathcal{A})u^r, \tag{5.6}$$

where we define $e_0(\mathcal{A}) = 1$ and $e_n(\mathcal{A}) = z\,1$.

Proof. The proof uses the explicit form of the transfer matrix in terms of the phase algebra stated in Lemma 3.4, $T_r = A_r + zD_r$ with

$$A_r = \sum_{0\leq s\leq \min\{r,n/2\}} \ \sum_{\substack{i_1<j_1<\cdots<i_s<j_s\\ |j_1-i_1|+\cdots+|j_s-i_s|=r}} a_{i_1,j_1}\cdots a_{i_s,j_s}$$

and

$$D_r = \sum_{0\leq s\leq \min\{n/2,n-r\}} \ \sum_{\substack{i_1<j_1<\cdots<i_s<j_s\\ |j_1-i_1|+\cdots+|j_s-i_s|=n-r}} a_{j_1,i_1}\cdots a_{j_s,i_s} \ .$$

For $j > i$, the identities $a_{i,j} = a_{j-1}a_{j-2}\cdots a_i$ and $a_{j,i} = z^{-1}a_{i-1}\cdots a_1a_0a_{n-1}\cdots a_j$ are easily verified. The asserted equality between T_r and $e_r(\mathcal{A})$ follows. □

Corollary 5.14. *The cyclic noncommutative elementary symmetric functions pairwise commute,*

$$[e_r(\mathcal{A}), e_{r'}(\mathcal{A})] = 0, \qquad 0 \leq r, r' \leq n \ . \tag{5.7}$$

and generate a commutative subalgebra $\mathcal{A}^{\mathrm{sym}}$ of $\mathcal{A}$.

Proof. The first part is a direct consequence of (5.6) and the integrability (4.2). The second part then follows from Proposition 5.8. □

Thanks to Corollary 5.14, the following definition makes sense[1]:

[1]Note also the stronger (independent) result of Theorem 6.7, namely the construction of a simultaneous eigenbasis.

Definition 5.15. Given a partition λ, the *Jacobi-Trudy formula*

$$s_\lambda(\mathcal{A}) = \det\left(e_{\lambda_i^t - i + j}(\mathcal{A})\right). \tag{5.8}$$

gives well-defined polynomials (in the generators of the affine plactic algebra), which we call the *cyclic noncommutative Schur polynomials*.

In further analogy with the commutative case we also introduce noncommutative versions of the complete symmetric polynomials.

Definition 5.16. Define the set of *cyclic noncommutative complete symmetric polynomials* via

$$h_r(\mathcal{A}) = \det(e_{1-i+j}(\mathcal{A}))_{1\le i,j\le r}\ . \tag{5.9}$$

In particular, we have for $\lambda = (r)$, a horizontal r-strip, that $s_{\lambda=(r)}(\mathcal{A}) = h_r(\mathcal{A})$ analogous to the commutative case.

6. Bethe Ansatz and the isomorphism of rings

Employing the algebraic Bethe Ansatz (or Quantum Inverse Scattering Method) one can compute the eigenvectors of the transfer matrix (4.1), which according to (5.6) generates the cyclic noncommutative elementary symmetric functions. Algebraic Bethe Ansatz calculations for the phase model were first undertaken by Bogoliubov et al. [7]. We shall give here a far more detailed analysis of the solutions to the Bethe Ansatz equations and generalize their discussion from periodic (z=1) to quasi-periodic (z generic) boundary conditions. The important novel aspect of our work with regard to the discussion of the phase model in [7] is the connection with representation theory and finally the identification of the abelian algebra generated by the commuting transfer matrices of this integrable system with the Verlinde algebra (by showing that the so-called Bethe vectors are given in terms of Weyl characters).

6.1. **Bethe vectors.** We introduce the following notation for the special vector already discussed in (3.18)

$$b(x) := B(x_1^{-1})\cdots B(x_k^{-1})\emptyset = \sum_{\hat\lambda\in P_k^+} s_{\hat\lambda^t}(x_1^{-1},...,x_k^{-1})\hat\lambda\ . \tag{6.1}$$

The vectors $b(x)$ are by definition in $\mathcal{H}_k$. We first like to find precise conditions on x, ensuring that $b(x)$ is an eigenvector of the transfer matrix $T(u)$, and hence a simultaneous eigenvector of the noncommutative elementary symmetric functions e_r and then also for the whole algebra of noncommutative symmetric functions. We will show that the Bethe vectors form an orthogonal basis of $\mathcal{H}_k$. The Ansatz for the algebraic form of the eigenvectors (the first identity in (3.18)) and the derivation of the resulting conditions on x, usually called *Bethe Ansatz equations*, exploiting the quantum Yang-Baxter algebra, is standard in the physics literature, (see for example [8]). The following proposition states the result of the algebraic Bethe Ansatz. For convenience we sketch its proof:

Proposition 6.1. *(1) For the vector $b(x)$ depending on invertible indeterminates $x = (x_1,\dots,x_k)$ to be an eigenvector for the transfer matrix $T(u)$, x must obey the Bethe Ansatz equations*

$$x_1^{n+k} = \cdots = x_k^{n+k} = (-1)^{k-1} z e_k(x_1,\dots,x_k), \tag{6.2}$$

where $e_k(x) = \prod_{i=1}^k x_i$ *is the (ordinary)* k^{th} *elementary symmetric polynomial.*

(2) *Given a solution* $x = (x_1, \ldots, x_k)$ *of the Bethe Ansatz equations (6.2),* $b(x)$ *is an eigenvector of the transfer matrix (4.1) and of the integrals of motion* $H_r^{\pm}$ *(4.4) for the phase model. The eigenvalues are given by*

$$T(u)b(x) = \left[1 + (-1)^k z e_k(x) u^{n+k}\right] \prod_{i=1}^k \frac{1}{1 - ux_i}\, b(x) \tag{6.3}$$

and

$$H_r^{\pm} b(x) = -\frac{h_r(x_1, \ldots, x_k) \pm z h_r(x_1^{-1}, \ldots, x_k^{-1})}{2}\, b(x)\ , \tag{6.4}$$

where e_r, h_r *is the* r^{th} *ordinary (commutative) elementary and complete symmetric polynomials, respectively.*

Sketch of proof: It is easy to verify $A(u)\emptyset = \emptyset$ and $D(u)\emptyset = u^n\emptyset$. One can show via induction on k that

$$\begin{aligned} T(u)B(x_1^{-1})...B(x_k^{-1})\emptyset &= \left(\prod_{i=1}^k \frac{1}{1-ux_i} + zu^n \prod_{i=1}^k \frac{ux_i}{ux_i - 1}\right) B(x_1^{-1})...B(x_k^{-1})\emptyset \\ &\quad + \sum_{i=1}^k F_i(u)\ B(x_1^{-1})...B(x_{i-1}^{-1})B(x_{i+1}^{-1})...B(x_k^{-1})\emptyset, \end{aligned}$$

where F_i is given by

$$\begin{aligned} F_i(u) &= \frac{1}{1-ux_i}\left(\prod_{j\neq i} \frac{x_i}{x_i - x_j} - z x_i^{-n} \prod_{j\neq i} \frac{x_j}{x_j - x_i}\right) \\ &= \frac{x_i^{k-1}}{1-ux_i}\left(1 + z(-1)^k x_i^{-n-k} e_k(x)\right) \prod_{j\neq i} \frac{1}{x_i - x_j} \end{aligned}$$

(The case $k = 1$ is immediate from the commutation relation (3.16)). The second sum is referred to as the *unwanted terms* which have to vanish in order to turn the Bethe vector into an eigenstate of the transfer matrix. This leads to the condition

$$1 + z(-1)^k x_i^{-n-k} e_k(x) = 0\ , \qquad i = 1, 2, \ldots, k,$$

which are the Bethe Ansatz equations. Hence $b(x)$ is an eigenvector if and only if x satisfies the Bethe Ansatz equations. The eigenvalues can now be simply deduced by a series expansion in the spectral parameter u and using Lemma 6.2 below. □

Lemma 6.2. *Assume* $x := (x_1, x_2, \ldots x_n)$ *satisfies the Bethe equations* (6.2). *Then*

$$h_{n-r}(x_1, x_2, \ldots x_n) = z h_r(x_1^{-1}, \ldots, x_k^{-1}) \tag{6.5}$$

for any $0 \leq r \leq n$.

Proof. We use $ze_k(x) = (-1)^{k-1} x_i^{n+k}$ and calculate

$h_{n-r}(x_1, \ldots, x_k) = \sum_{i=1}^k x_i^{n-r} \prod_{j,j\neq i} \frac{x_i}{x_i - x_j} = \sum_{i=1}^k x_i^{n-r+(k-1)} \prod_{j,j\neq i} \frac{1}{x_i - x_j}$
$= z \sum_{i=1}^k x_i^{-r-1} e_k(x) \prod_{j,j\neq i} \frac{1}{x_j - x_i} = z \sum_{i=1}^k x_i^{-r-1} e_k(x) \prod_{j,j\neq i} \frac{x_i^{-1}x_j^{-1}}{x_i^{-1} - x_j^{-1}}$
$= z \sum_{i=1}^k x_i^{-r} \prod_{j,j\neq i} \frac{x_i^{-1}}{x_i^{-1} - x_j^{-1}} = z h_r(x_1^{-1}, \ldots, x_k^{-1}),$

where only the first and last equality need some explanation. Following [32, page 209] we define $g_s(x_1, \dots, x_k; q)$ through the formula

$$\prod_i \frac{1 - uq\, x_i}{1 - u\, x_i} = \sum_{s \geq 0} g_s(x_1, \dots, x_k; q) u^s \tag{6.6}$$

In particular, $g_0 = 1$. In case $s > 0$, $g_s(x_1, \dots, x_k; q) = (1-q) P_{(s)}(x_1, \dots, x_k; q)$ where $P_{(s)}(x_1, \dots, x_k; q)$ is the so-called *Hall-Littelwood polynomial*, explicitly

$$g_s(x_1, \dots, x_k; q) = (1-q) \sum_{i=1}^{k} x_i^s \prod_{j \neq i} \frac{x_i - q x_j}{x_i - x_j}.$$

Our two formulae in questions follow then from the specialisation $q \mapsto 0$. $\square$

The Bethe Ansatz equations (6.2) can be reformulated as follows

Lemma 6.3. *The equations (6.2) are equivalent to the conditions*

$$h_{n+1}(x) = \cdots = h_{n+k-1}(x) = h_{n+k}(x) + (-1)^k z e_k(x) = 0, \tag{6.7}$$

where the h_r are again the ordinary complete symmetric polynomials.

Proof. Assume the equations (6.2) hold. Integrability, property (4.2), implies that the transfer matrix $T(u)$ has a simultaneous (i.e. independent of u) eigenspace decomposition. The definition of the transfer matrix (4.1) implies that the eigenvalues in (6.3) must be polynomial in the spectral parameter u and at most of degree n. The eigenvalues in (6.3) are of the form $(1 + (-1)^k z e_k(x) u^{n+k}) H(u)$, where $H(u)$ is the generating function of the complete symmetric polynomials. The coefficients of u^j for $j > n$ have to vanish, hence $0 = h_{n+j}(x)$ for $1 \leq j \leq k-1$ and $h_{n+k+j}(x) + (-1)^k z e_k(x) h_j(x) = 0$ for $j \geq 0$ which of course implies (6.7) (it is in fact equivalent, because $h_{n+k+j}(x) + (-1)^k z e_k(x) h_j(x) = 0$ implies $0 = (\sum_{i=1}^k x_i)(h_{n+k+j}(x) + (-1)^k z e_k(x) h_j(x)) = h_{n+k+j+1}(x) + (-1)^k z e_k(x) h_{j+1}(x))$. Conversely if we assume (6.7) holds, then (6.3) is a polynomial in u (of degree n). If we choose a generic point x, fix x_i and formally replace u by $u - x_i^{-1}$, then (6.3) turns into $p(u) q(u) u x_i^{-1}$, where $p(u)$ is a polynomial, and $q(u)$ is a formal power series in u. Since this has to be a polynomial in u again, its residue, which is a multiple of $1 + (-1)^k z e_k(x) h_{j+1}(x) x_i^{-(n+k)}$, must be zero and the equations (6.2) hold. $\square$

To state the explicit form of the Bethe roots, i.e. solutions to (6.2), we formally want to allow n^{th} roots of z and also inverses, hence work in the ring $R = \mathbb{C}[z^{\mp 1}, z^{\mp \frac{1}{n}}] = \mathbb{C}[z^{\mp \frac{1}{n}}]$. We extend the complex conjugation to R by setting $\overline{z} := z^{-1}$. The expansion (6.1) into Schur polynomials shows that the Bethe vectors $b(x)$ do not really depend on the tuple $x = (x_1, x_2, \dots, x_k)$, but only on the set $\{x_1, x_2, \dots, x_k\}$ or its S_k-orbit. Furthermore, in case $x_i = x_j$ for some $i \neq j$ we get $b(x) = 0$. Therefore we denote

$$\mathrm{Sol}(n,k) := \{x \in R^k \mid x \text{ solves (6.2) and the } x_i \text{ are pairwise distinct}\}$$

and want to describe the orbit space $\widehat{\mathrm{Sol}(n,k)} := \mathrm{Sol}(n,k)/S_k$.

Theorem 6.4 (Solutions of the Bethe Ansatz)**.** *There is a bijection*

$$\begin{aligned} \mathfrak{P}_{\leq n-1,k} &\cong \widehat{\mathrm{Sol}(n,k)}, \\ \sigma &\mapsto x_\sigma := z^{\frac{1}{n}}\zeta^{\frac{|\sigma|}{n}}\left(\zeta^{I_1},\dots,\zeta^{I_k}\right), \end{aligned} \tag{6.8}$$

where $\zeta = \exp\frac{2\pi i}{k+n}$ *and*

$$I = I(\sigma^t) := \left(\tfrac{k+1}{2}+\sigma^t_k - k,\dots,\tfrac{k+1}{2}+\sigma^t_1-1\right). \tag{6.9}$$

Moreover, $\widehat{\mathrm{Sol}(n,k)}$ *decomposes into the disjoint union of orbits under the* $\mathbb{Z}_n$*-action given by* $\ell.x := e^{\frac{2\pi\iota\ell}{n}}x$ *with* $\ell\in\mathbb{Z}$, $x\in\widehat{\mathrm{Sol}(n,k)}$.

Proof. It is easy to check that $\sigma\mapsto I(\sigma^t)$ defines a bijection between $\mathfrak{P}_{\leq n-1,k}$ and the following set of tuples of (half)-integers

$$\mathcal{I}_{n-1,k} := \left\{(I_1,...,I_k)\mid -\tfrac{k-1}{2}\leq I_1<\dots<I_k\leq n-1+\tfrac{k-1}{2}, I_j\in\tfrac{k+1}{2}+\mathbb{Z},\forall j\right\}.$$

For instance, $\mathfrak{P}_{\leq 2,3}$ from (2.3) gets identified with the tuples

$$\{(-1,0),(-1,1),(-1,2),(-1,3),(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}.$$

Hence it is enough to show that the x_σ as in (6.8) with $I\in\mathcal{I}_{n-1,k}$ are a full list of representatives for $\widehat{\mathrm{Sol}(n,k)}$.

Assume $x\in R^k$ solves (6.2). First we describe the dependence on z. Let s_i be the highest exponent of z in x_i. Then the highest exponent in x_i^{n+k} is $s_i(n+k)$. In particular, $s_i=s_j$ for any $1\leq i,j\leq n$ and then also $s_i(n+k)=1+ks_i$, hence $s_in=1$. The same argument also applies to the lowest exponent, hence any x_i is a monomial in $z^{\frac{1}{n}}$. In the following it will be enough to consider therefore the special case $z=1$.

First note that (6.2) implies that all x_j are of norm 1, hence of the form $x_j = e^{2\pi\iota\alpha_j}$ for some $\alpha_j\in\mathbb{R}$. Then the Bethe Ansatz equations (6.2) for the phase model can be rewritten (by taking the logarithm) as

$$\alpha_j = \frac{J_j}{k+n-1}+\frac{1}{k+n-1}\sum_{l\neq j}\alpha_l \tag{6.10}$$

with the J_j being half-integers in case k is even, and integers in case k is odd. For each fixed generic configuration $J=(J_1,\dots,J_k)$ one easily checks (using our general assumption $n+k>2$) that there is precisely one solution to this system of linear equations, namely $\alpha_j = \frac{1}{(k+n)}(\frac{n+1}{n}\ J_j+\frac{1}{n}\sum_{l\neq j}J_l) = \frac{1}{(k+n)}(J_j+\frac{1}{n}||J||)$ with $||J||=\sum_{l=1}^k J_l$). Let $x(J)=z^{\frac{1}{n}}\zeta^{\frac{||J||}{n}}(\zeta^{J_1},\dots,\zeta^{J_k})$ with $\zeta=e^{\frac{2\pi\iota}{k+n}}$ be the corresponding element in $\mathrm{Sol}(n,k)$. The choice $J_j:=I_j(\sigma^t)$ gives $x(J)=x(I(\sigma^t))=x_\sigma$. Hence we get all the solutions x_σ predicted in the theorem, but have to show there are not more (up to permutations).

The assumption $x_j\neq x_i$ for $j\neq i$ implies $\alpha_i\notin\alpha_j+\mathbb{Z}$ and then $J_i\not\equiv J_j \mod (n+k)$. Therefore, modulo $n+k$, the (half)-integers J_i are contained in the interval $[-\frac{k-1}{2},n+\frac{k-1}{2}]$ and are distinct. Since $k<n+k$ there is at least one (half)-integer m in the interval which does not occur amongst the J_i's. Hence we can find some $l\in\mathbb{Z}$ such that $J'=J+(l,l,\dots,l)$ and all the (half)-integer J'_i are contained in the interval $[-\frac{k-1}{2},n-1+\frac{k-1}{2}]$ modulo $n+k$ (in other words we shift

the 'gap' m to the end of the interval). The two operations (taking J modulo $n+k$ and adding some $(l,l,\dots,l)$) however only rescale the solutions, because

$$\begin{aligned} x(J_1,\dots,J_j+k+n,\dots,J_k) &= z^{\frac{1}{n}}\zeta^{\frac{||J||}{n}+\frac{k+n}{n}}(\zeta^{J_1},\dots,\zeta^{J_k}) = e^{\frac{2\pi\iota}{n}}x(J)\\ x(J_1+1,\dots,J_k+1) &= z^{\frac{1}{n}}\zeta^{\frac{||J||}{n}+\frac{k}{n}+1}(\zeta^{J_1},\dots,\zeta^{J_k}) = e^{\frac{2\pi\iota}{n}}x(J)\end{aligned}$$

Therefore, it is enough to show that multiplication with $\eta := e^{\frac{2\pi\iota}{n}}$ preserves the set $X := \{x(I(\sigma^t))\}$, in other words we have to show that

$$\eta x = z^{\frac{1}{n}}\zeta^{\frac{||I||+k}{n}}(\zeta^{I_1+1},\dots,\zeta^{I_k+1}) = z^{\frac{1}{n}}\zeta^{\frac{||I'||}{n}}(\zeta^{I'_1},\dots,\zeta^{I'_k}) \in X$$

with $I'_j = I_j+1$ for $1\leq j\leq k$. In case $I_k < n-1+\frac{k-1}{2}$ there is nothing to do, hence assume $I_k = n-1+\frac{k-1}{2}$. Now note that $x(I') = \eta x(I'_1,\dots,I'_{k-1},I'_k-(n+k)) = \eta x(I'_k-(n+k),I'_1,\dots,I'_{k-1}) = \eta^{-1}\eta x(I'_k-(n+k)+1,I'_1+1,\dots,I'_{k-1}+1)$. Since $n+k-1>k$, applying these two equations repeatedly, we finally obtain an element $x = x(I') \in X$. Hence the elements in $\widehat{\mathrm{Sol}(n,k)}$ are all of the form as claimed. This proves that we have a bijection of sets $\mathfrak{P}_{\leq n-1,k} \cong \widehat{\mathrm{Sol}(n,k)}$ and that $\widehat{\mathrm{Sol}(n,k)}$ decomposes into disjoint sets whose elements are obtained by rescaling with an n^{th} root of unity. □

Before deducing interesting consequences we collect well-known formulae (see e.g. [37], [11, Proposition 38.2]):

Lemma 6.5. *Let $\mu = \hat{\lambda} \in \mathfrak{P}_{\leq n,k}, \sigma \in \mathfrak{P}_{\leq n-1,k}$. With the notation from Theorem 6.4 and Section 2.1 we have the following identities*

(1) $s_\mu(\zeta^{I(\sigma)}) = s_{\mu^t}(\zeta^{-I(\sigma^t)})$
(2) $s_\mu(\zeta^{I(\sigma)}) = e_n(\zeta^{I(\sigma)})^{m_n(\mu)}s_{\tilde{\mu}}(\zeta^{I(\sigma)}),\qquad where \quad m_n(\mu) = \frac{|\mu|-|\tilde{\mu}|}{n}$.
(3) $s_\mu(\zeta^{I(\sigma)}) = e_n(\zeta^{I(\sigma)})^k s_{\mu^\vee}(\zeta^{-I(\sigma)})$.
(4) Suppose $m_n(\mu)=0$ then $s_{\mathrm{rot}(\mu)}(\zeta^{I(\sigma)}) = \zeta^{|\sigma|}s_\mu(\zeta^{I(\sigma)})$.

Proof. Statement (1) follows directly from the formulae

$$e_r(\zeta^{I(\sigma)}) = h_r(\zeta^{-I(\sigma^t)}) \qquad \text{and} \qquad h_r(\zeta^{I(\sigma)}) = e_r(\zeta^{-I(\sigma^t)})\ . \tag{6.11}$$

of [37]. From (6.8) and (6.7) it follows

$$h_{n+1}(\zeta^{I(\sigma^t)}) = \cdots = h_{n+k-1}(\zeta^{I(\sigma^t)}) = h_{n+k}(\zeta^{I(\sigma^t)}) + (-1)^k = 0$$

If $\mu = \tilde{\mu}$ then there is nothing to do in (2). So assume that $\mu_1^t = n$. By expanding the determinant of the Jacobi-Trudy formula $s_\mu(y) = \det(h_{\mu_i-i+j}(y))$ for $y = \zeta^{I(\sigma^t)}$ with respect to the first row together with (6.7) implies

$$s_\mu(\zeta^{I(\sigma^t)}) = h_n(\zeta^{I(\sigma^t)})s_{(\mu_2^t,\dots,\mu_k^t)}(\zeta^{I(\sigma^t)})$$

Repeating the same step until all parts of size n are removed, we obtain the asserted identity (2). Now (3) follows from the general formula ([37, (4.3)])

$$s_{(\mu^\vee)^t}(y_1^{-1},\dots,y_k^{-1}) = s_{(\mu^t)^\vee}(y_1^{-1},\dots,y_k^{-1}) = \frac{s_{\mu^t}(y_1,\dots,y_k)}{s_{(n^k)}(y_1,\dots,y_k)}\ . \tag{6.12}$$

To obtain (4) we employ the Pieri formula to find

$$s_{\mathrm{rot}(\mu)}(\zeta^{I(\sigma^t)}) = e_k(\zeta^{I(\sigma^t)})s_\mu(\zeta^{I(\sigma^t)}) = \zeta^{\frac{k+n}{n}|\sigma|}s_\mu(\zeta^{I(\sigma^t)})\ .$$

and then apply (1) once more. □

We need another Schur function identity which we will frequently use in what follows.

Lemma 6.6. *Adopting the notation from Theorem 6.4 we set $\tilde{x} = \zeta^{|\sigma|/n}\zeta^{I(\sigma^t)}$ and $\tilde{y} = \zeta^{|\sigma|/n}\zeta^{-I(\sigma)}$, where $\sigma \in \mathfrak{P}_{\leq n-1,k}$. Then we have for any $\lambda \in \mathfrak{P}_{\leq n-1,k}$ the identities*

$$s_{\lambda^t}(\tilde{x}) = s_\lambda(\tilde{y}) \quad \text{and} \quad \overline{s_\lambda(\tilde{y})} = s_\lambda(\tilde{y}^{-1}) = s_{\hat{\lambda}^*}(\tilde{y}) \tag{6.13}$$

Proof. The first equality is immediate from Lemma 6.5 (1). Employing Lemma 6.5 (3) and the following identities $e_n(\zeta^{I(\sigma)})^k = \zeta^{|\sigma|k}$ from (6.9) and $|\lambda^\vee| = |\hat{\lambda}^*| = kn - |\lambda|$ (see Section 2.2) we calculate

$$\begin{aligned} s_\lambda(\zeta^{-\frac{|\sigma|}{n}}\zeta^{I(\sigma)}) &= \zeta^{-\frac{|\sigma||\lambda|}{n}} e_n(\zeta^{I(\sigma)})^k s_{\lambda^\vee}(\zeta^{-I(\sigma)}) \\ &= s_{\hat{\lambda}^*}(\zeta^{\frac{|\sigma|}{n}}\zeta^{-I(\sigma)}) = \overline{s_\lambda(\zeta^{\frac{|\sigma|}{n}}\zeta^{-I(\sigma)})} \end{aligned} \tag{6.14}$$

which proves the assertion. □

6.2. **Eigenbasis and Weyl characters.** In this section we show that the Bethe vectors form a complete set of pairwise orthogonal eigenvectors, determine their eigenvalues in terms of Weyl characters.

Given $g \in \mathrm{GL}(k, \mathbb{C})$ diagonalisable with eigenvalues $t_1, \dots, t_n$ and χ_λ the character of the finite dimensional irreducible module corresponding to the partition λ, we have the equality $\chi_\lambda(g) = s_\lambda(t_1, \dots, t_n)$. We consider for $\xi \in \mathbb{Z}^n$ the values

$$\chi_\lambda(\xi) := s_\lambda(e^{\frac{2\pi\iota\xi_1}{k+n}}, \dots, e^{\frac{2\pi\iota\xi_k}{k+n}})$$

(where e denotes here the usual exponential function). Recall the scalar product $\langle\ ,\rangle$ on $\mathcal{H}$ from Proposition 3.1.

Theorem 6.7. (1) *For each $k \in \mathbb{Z}_{\geq 0}$ the Bethe vectors $b(x_\sigma)$, $\sigma \in \mathfrak{P}_{\leq n-1,k}$, form a complete set of pairwise orthogonal eigenvectors for the action of the cyclic noncommutative symmetric functions on $\mathcal{H}_k[z]$. The eigenvalues are given by the following formulae*

$$e_r(\mathcal{A})b(x_\sigma) = \begin{cases} h_r(x_\sigma)b(x_\sigma) & \text{if } r \neq n, \\ h_n(x_\sigma)b(x_\sigma) = zb(x_\sigma) & \text{if } r = n. \end{cases} \tag{6.15}$$

$$s_\lambda(\mathcal{A})b(x_\sigma) = s_{\lambda^t}(x_\sigma)b(x_\sigma) = z^{\frac{|\lambda|}{n}}\chi_\lambda(\xi_\sigma)b(x_\sigma). \tag{6.16}$$

(2) *The norm of the Bethe vectors is given by the following formula*

$$\langle b_\sigma, b_\sigma\rangle = \frac{n(n+k)^{n-1}}{|\operatorname{Van}_\sigma|^2}, \tag{6.17}$$

where $\operatorname{Van}_\sigma$ denotes the Vandermonde determinant $\prod_{1\leq i<j\leq n}(\zeta^{I_i(\sigma)} - \zeta^{I_j(\sigma)})$.

Proof. Thanks to (6.3) and (5.6) we have for any solution x of (6.2) the explicit formula for the eigenvalues $e_r(\mathcal{A})b(x) = h_r(x)b(x)$ for $r \neq n$ and $e_n(\mathcal{A})b(x) = zb(x)$ and $zb(x) = h_n(x)$ by (6.5).

The (noncommutative versus the commutative) Jacobi-Trudy formula implies then $s_\lambda(\mathcal{A})b(x) = s_{\lambda^t}(x)b(x)$. Using (6.8) and Lemma 6.5(1) we get $s_{\lambda^t}(x_\sigma) = z^{\frac{|\lambda|}{n}} s_\lambda(\zeta^{\frac{|\sigma|}{n}}\zeta^{I_1(\sigma)}, \dots, \zeta^{\frac{|\sigma|}{n}}\zeta^{I_n(\sigma)})$ and (6.16) follows then from the definition of χ_λ.

Now consider the action of the commutative algebra $\mathcal{A}^{\mathrm{sym}}$ (from Corollary (5.14)) on $\mathcal{H}_k$. It decomposes the $\mathcal{H}_k$ into simultaneous generalised eigenspaces, and the

Bethe vectors are simultaneous eigenvectors with the eigenvalues as computed above. To show that they form a complete eigenbasis it is enough to see that they are in different eigenspaces, because we have previously established that there are $\dim \mathcal{H}_k$ solutions to the Bethe Ansatz equations. Assume $e_r(\mathcal{A})b(x_\sigma) = e_r(\mathcal{A})b(x_\tau)$ for all r.

Then $h_r(x_\sigma) = h_r(x_\tau)$ for all r. Now the h_r generate the ring of symmetric functions in k variables which in turn can be identified with the (polynomial) ring of regular functions on the orbit space $\mathbb{C}^k/S_k$ of the symmetric group S_k acting on $\mathbb{C}^k$ by permuting the variables. In particular, $h_r(x_\sigma) = h_r(x_\tau)$ for all r implies that x_σ lies in the same S_k orbit as x_τ. Because of definition (6.9) this is only possible if $\sigma = \tau$.

Next we claim that $\langle b_\sigma, b_\tau\rangle = 0$ if $\sigma \neq \tau$. The adjointness of φ_i with φ_i^* from Proposition 3.1 and the definition of the elementary symmetric functions implies that $e_{n-r}(\mathcal{A})$ has adjoint $z^{-1}e_r(\mathcal{A})$. In particular,

$$\begin{aligned} 0 &= \langle e_{n-r}(\mathcal{A})b(x_\sigma), b(x_\tau)\rangle - \langle b(x_\sigma), z^{-1}e_r(\mathcal{A})b(x_\tau)\rangle \\ &= (\overline{h_{n-r}(x_\sigma)} - z^{-1}h_r(x_\tau)\langle b(x_\sigma), b(x_\tau)\rangle \\ &= (z^{-1}h_r(x_\sigma) - z^{-1}h_r(x_\tau))\langle b(x_\sigma), b(x_\tau)\rangle. \end{aligned}$$

where we used Lemma 6.2 for the last equality. The claim follows then as above.

To compute the norm, note that the ordinary Schur polynomials can be expressed (see e.g. [11, Theorem 38.1]) as a ratio

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{D_\lambda(x_1, x_2, \ldots, x_n)}{\mathrm{Van}(x)},$$

where Van denotes the Vandermond determinant and

$$D_\lambda(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_{n-1}^{\lambda_1+n-1} & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_{n-1}^{\lambda_2+n-2} & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ x_1^{\lambda_{n-1}+1} & x_2^{\lambda_{n-1}+1} & \ldots & x_{n-1}^{\lambda_{n-1}+1} & x_n^{\lambda_{n-1}+1} \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \ldots & x_{n-1}^{\lambda_n} & x_n^{\lambda_n} \end{pmatrix}$$

Let $\lambda \in \mathfrak{P}_{\leq n,k}$. Then $\lambda \in \mathfrak{P}_{\leq n-1,k}$ if and only if $\lambda_n = 0$ which is if and only if the last row of $D_\lambda = D_\lambda(x_1, x_2, \ldots, x_n)$ consists of 1's only. Let us assume this is the case and expand D_λ with respect to the last row to obtain $D_\lambda = \sum_{r=1}^n (-1)^r D_\lambda^{(r)}$ where the $D_\lambda^{(r)}$ denotes the determinant of the matrix obtained by removing the r^{th} column and the n^{th} row. On the other hand, for fixed r, $1 \leq r \leq n$, we get

$$D_\lambda(x_1, x_2, \ldots, x_n)_{x_r=0} = \begin{cases} 0 & \text{if } \lambda_n \neq 0, \\ (-1)^r D_\lambda^{(r)} & \text{if } \lambda_n = 0. \end{cases} \tag{6.18}$$

Thanks to (6.1) and Proposition 6.5(1) we have

$$\langle b_\sigma, b_\sigma\rangle = \sum_{\lambda \in \mathfrak{P}_{\leq n-1,k}} \zeta^{\frac{|\sigma|-|\sigma|}{n}|\lambda|} s_{\lambda^t}(\zeta^{-I(\sigma^t)}) s_{\lambda^t}(\zeta^{I(\sigma^t)}) = \sum_{\lambda \in \mathfrak{P}_{\leq n-1,k}} s_\lambda(\zeta^{I(\sigma)}) s_{\lambda^t}(\zeta^{I(\sigma^t)})$$

Recall the Cauchy identity $\sum_{\lambda \in \mathfrak{P}_{\leq n,k}} s_\lambda(z_1, z_2, \ldots z_n) s_{\lambda^t}(w_1, w_2, \ldots w_k) = \prod_{i=1}^n \prod_{j=1}^k (1+ z_i w_j)$. Rietsch [37, (4.5)] showed that for $(z_1, z_2, \ldots z_n) = \zeta^{I(\sigma)}$ and $(w_1, w_2, \ldots w_k) =$

$\zeta^{I(\sigma^t)}$ it is $|\operatorname{Van}(\zeta^{I(\sigma)})|^2\prod_{j=1}^k(1+z_iw_j)=(n+k)$ for any $1\leq i\leq n$. Putting everything together we get

$$\begin{aligned}
&|\operatorname{Van}_\sigma|^2\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}s_\lambda(\zeta^{I(\sigma)})s_{\lambda^t}(\zeta^{I(\sigma^t)})\\
=\;&\overline{\operatorname{Van}_\sigma}\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}D_\lambda(\zeta^{I(\sigma)})s_{\lambda^t}(\zeta^{I(\sigma^t)})\\
=\;&\overline{\operatorname{Van}_\sigma}\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}\sum_{r=1}^n(-1)^rD_\lambda^{(r)}(\zeta^{I(\sigma)})s_{\lambda^t}(\zeta^{I(\sigma^t)})\\
=\;&\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}\sum_{r=1}^n(-1)^rD_\lambda^{(r)}(\zeta^{I(\sigma)})D_{\lambda^t}(\zeta^{I(\sigma^t)})\\
\overset{(6.18)}{=}\;&\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}\sum_{r=1}^n(-1)^rD_\lambda(y_1^{(r)},y_2^{(r)},\ldots y_n^{(r)})D_{\lambda^t}(\zeta^{I(\sigma^t)})\\
\overset{(6.18)}{=}\;&\sum_{\lambda\in\mathfrak{P}_{\leq n,k}}\sum_{r=1}^n(-1)^rD_\lambda(y_1^{(r)},y_2^{(r)},\ldots y_n^{(r)})D_{\lambda^t}(\zeta^{I(\sigma^t)})\\
=\;&\sum_{r=1}^n\sum_{\lambda\in\mathfrak{P}_{\leq n,k}}(-1)^rD_\lambda(y_1^{(r)},y_2^{(r)},\ldots y_n^{(r)})D_{\lambda^t}(\zeta^{I(\sigma^t)})
\end{aligned}$$

where $y_i^{(r)}=\zeta_i^{I(\sigma)}$ if $i\neq r$ and $y_r^{(r)}=0$, and with Rietsch's formula gives

$$\begin{aligned}
\langle b_\sigma,b_\sigma\rangle=\sum_{\lambda\in\mathfrak{P}_{\leq n-1,k}}s_\lambda(\zeta^{I(\sigma)})s_{\lambda^t}(\zeta^{I(\sigma^t)})\;&=\;|\operatorname{Van}_\sigma|^{-2}\sum_{r=1}^n(n+k)^{n-1}\\
&=\;|\operatorname{Van}_\sigma|^{-2}n(n+k)^{n-1}.
\end{aligned}$$

Hence the formula follows. □

Corollary 6.8. *The known determinant formulae from the ring of commutative functions are also true for the cyclic noncommutative functions, in particular*

$$s_\lambda(\mathcal{A})=\det(h_{\lambda_i-i+j}(\mathcal{A}))_{1\leq i,j\leq n}\;.$$

The cyclic noncommutative Schur polynomials pairwise commute.

Proof. Specialising in (6.16) to a horizontal r-strip it follows from Definition 5.9 that

$$\begin{aligned}
h_r(\mathcal{A})b(x)\;&=\;\det(e_{1-i+j}(\mathcal{A}))_{1\leq i,j\leq r}b(x)\\
&=\;\det(h_{1-i+j}(x))_{1\leq i,j\leq r}b(x)=e_r(x)b(x)\;.
\end{aligned}\tag{6.19}$$

Employing the familiar relations form the ring of commutative symmetric functions we therefore have

$$\begin{aligned}
&s_\lambda(\mathcal{A})b(x)=s_{\lambda^t}(x)b(x)=\\
&\qquad\det(e_{\lambda_i-i+j}(x))_{1\leq i,j\leq n}b(x)=\det(h_{\lambda_i-i+j}(\mathcal{A}))_{1\leq i,j\leq n}b(x)\;.
\end{aligned}$$

According to Theorem 6.7 the Bethe vectors (6.1) form a basis in each subspace $\mathcal{H}_k[z]$ for any $k\in\mathbb{Z}_{\geq0}$ and, hence, the last identity implies the assertion. □

Corollary 6.9. *For $0 \leq r \leq n-1$ the complete symmetric polynomials $h_r(\mathcal{A})$ defined in* (5.9) *have the following explicit, simpler form*

$$h_r(\mathcal{A}) = \sum_{|J|=r} \prod_{j\in J}^{\circlearrowright} a_j, \tag{6.20}$$

where the sum runs over all multisets $J = \{j_1, \ldots, j_r\}$ (i.e. $s \neq t$ does not necessarily imply that $j_s \neq j_t$).

Example 6.10. *If $n = 4$ then*

$$h_3(\mathcal{A}) = (\sum_{i=1}^{4} a_i^3) + a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + a_1 a_3^2 + a_4^2 a_1 + a_4 a_1^2 + a_2^2 a_3 + a_2 a_3^2$$
$$+ a_2^2 a_4 + a_2 a_4^2 + a_3^2 a_4 + a_3 a_4^2 + a_1 a_2 a_3 + a_4 a_1 a_2 + a_2 a_3 a_4\ .$$

Proof. Because of Theorem 6.7 it suffices to show that for any $k \in \mathbb{Z}_{\geq 0}$ the Bethe vectors are eigenvectors of the polynomials defined on the right hand side of equation (6.20) and have eigenvalues $e_r(x)$ for $0 \leq r \leq n-1$. We will make repeated use of the maps $\hat{\mathrm{P}}$ and P defined in Section 2.1 and employ the expansion of the Bethe vector $b(x)$ into Schur functions; see the second equation in (6.1). Let us start by describing the action of the polynomial in (6.20) on a weight $\hat{\lambda}$. Given a composition $p = (p_1, ..., p_n)$ of $r < n$, there must be at least one $p_i = 0$ and, hence, we may rewrite the polynomial in (6.20) as

$$\sum_{|J|=r} \prod_{j\in J}^{\circlearrowright} a_j = \sum_{p\vdash r} \prod_{1\leq j\leq n}^{\circlearrowright} a_j^{p_j}, \qquad \prod_{1\leq j\leq n}^{\circlearrowright} a_j^{p_j} \overset{p_i=0}{=} a_{i+1}^{p_{i+1}} \cdots a_n^{p_n} a_1^{p_1} \cdots a_{i-1}^{p_{i-1}}\ . \tag{6.21}$$

For simplicity let us first assume that $p_n = 0$ and set $\hat{\mu} = a_1^{p_1} \cdots a_{n-1}^{p_{n-1}} \hat{\lambda}$. Since each a_j acts on $\hat{\lambda}$ by adding a box in the $(j+1)^{\text{th}}$ row of the associated Young diagram $\hat{\mathrm{P}}(\hat{\lambda})$, it is not difficult to see that either $\hat{\mu}$ is the null vector or $\hat{\mathrm{P}}(\hat{\mu})/\hat{\mathrm{P}}(\hat{\lambda}) = (r)$ with p_{j-1} boxes in the j^{th} row.

Now assume that $p_n > 0$ and some other $p_i = 0$. Then $a_n^{p_n}$ acts by removing p_n-columns of height n (if possible), adding p_n boxes in the first row *and* multiplying with z^{p_n}. Thus, the weights $\hat{\mu} = a_{i+1}^{p_{i+1}} \cdots a_n^{p_n} a_1^{p_1} \cdots a_{i-1}^{p_{i-1}} \hat{\lambda}$ and $\hat{\lambda}$ are now related in terms of Young diagrams as follows (we assume $a_{i+1}^{p_{i+1}} \cdots a_n^{p_n} a_1^{p_1} \cdots a_{i-1}^{p_{i-1}} \hat{\lambda} \neq 0$): consider the Young diagram $\mathrm{P}(\hat{\lambda})$ under the bijection $\mathrm{P} : P_k^+ \to \mathfrak{P}_{\leq n-1,k}$, then add the unique horizontal strip with the prescribed number of p_j boxes in the $(j+1)^{\text{th}}$ row and finally add $p_n = \mu_1 - \lambda_1$ columns of height n to obtain a Young diagram in $\hat{\mathrm{P}}(P_k^+) \subset \mathfrak{P}_{\leq n,k}$ which has width k. Putting both results, for $p_n = 0$ and $p_n > 0$, together we arrive at the formula

$$\sum_{|J|=r} \big(\prod_{j\in J}^{\circlearrowright} a_j\big) b(x) = \sum_{\hat{\lambda}\in P_k^+} \sum_{\substack{\mu\in\mathfrak{P}_{\leq n,k} \\ \mu/\lambda=(r)}} z^{\mu_1-\lambda_1} s_{\hat{\lambda}^t}(x^{-1})\hat{\mu}, \tag{6.22}$$

where $\hat{\mu}$ is the weight whose Young diagram $\hat{\mathrm{P}}(\hat{\mu})$ is obtained by adding $k - \mu_1$ columns of height n to μ.

Let us now compare this result with multiplying the Bethe vector $b(x)$ in (6.1) by $e_r(x)$. Using the well-known Pieri formula for *commutative* Schur functions and the identity (6.13) we compute

$$e_r(x)b(x) = \sum_{\hat\lambda\in P_k^+} e_r(x)s_{\hat\lambda^t}(x^{-1})\hat\lambda = \sum_{\hat\lambda\in P_k^+} z^{-\frac{|\hat\lambda|-r}{n}} e_r(\tilde x)s_{(\lambda^*)^t}(\tilde x)\hat\lambda \tag{6.23}$$

$$= \sum_{\hat\lambda\in P_k^+} z^{-\frac{|\hat\lambda|-r}{n}} \sum_{\mu/\lambda^*=(r)} s_{\mu^t}(\tilde x)\hat\lambda, \tag{6.24}$$

where $\tilde x(\sigma) = \zeta^{\frac{|\sigma|}{n}}\zeta^{I(\sigma^t)}$ are the Bethe roots (6.8) with the z-dependence removed. Note that the complement $\mu^\vee \in \mathfrak{P}_{\le n,k}$ is well-defined, since $s_{\mu^t}(\tilde x) = 0$ if $\mu_1 > k$ and $\lambda^* \in \mathfrak{P}_{\le n-1,k}$. Furthermore, because $\hat\lambda = (\lambda^*)^\vee$, it easy to see that $\mu/\lambda^* = (r)$ is equivalent to $\hat\lambda/\mu^\vee = (r)$. Thus, we can rewrite the last expression as

$$\begin{aligned}\sum_{\hat\lambda\in P_k^+} z^{-\frac{|\hat\lambda|-r}{n}} \sum_{\mu/\lambda^*=(r)} s_{\mu^t}(\tilde x)\hat\lambda &= \sum_{\hat\lambda\in P_k^+}\sum_{\hat{\mathrm{P}}(\hat\lambda)/\mu^\vee=(r)} z^{-\frac{|\mu^\vee|}{n}} s_{(\mu^\vee)^t}(\tilde x^{-1})\hat\lambda \\ &= \sum_{\hat\lambda\in P_k^+}\sum_{\hat\lambda/\mu^\vee=(r)} s_{(\mu^\vee)^t}(x^{-1})\hat\lambda\ .\end{aligned} \tag{6.25}$$

We wish to compare with the above result (6.22). To this end we now construct for each $\mu^\vee \in \mathfrak{P}_{\le n,k}$ an associated weight $\hat\mu^\vee \in P_k^+$ and show that there exists a partition $\tilde\lambda$ such that $\tilde\lambda/\tilde\mu = (r)$ with $\tilde\mu = \mathrm{P}(\hat\mu^\vee)$. Recall that the image of P_k^+ under the map $\hat{\mathrm{P}} : P_k^+ \to \mathfrak{P}_{\le n,k}$ consists of all partitions $\nu \in \mathfrak{P}_{\le n,k}$ with $\nu_1 = k$. Noting that $\mu_1^\vee = k - \mu_n$, we can conclude that by adding $\mu_n = k - \mu_1^\vee$ columns of height n to $\mu^\vee$ we have $\mu^\vee \in \hat{\mathrm{P}}(P_k^+)$. Denote by $\hat\mu^\vee \in P_k^+$ the corresponding weight and set $\tilde\mu = P(\hat\mu^\vee)$, i.e. $\tilde\mu$ is obtained by removing all n-columns in the Young diagram of $\mu^\vee$. Let $m_n(\mu^\vee)$ be the number of n-columns in the Young diagram of $\mu^\vee$. Then set $\tilde\lambda$ to be the partition obtained by removing $m_n(\mu^\vee)$ columns of height n in the Young diagram of $\hat\lambda$. The partition $\tilde\lambda$ is well-defined, since $\mu^\vee \subset \hat{\mathrm{P}}(\hat\lambda)$ and hence $m_n(\hat\lambda) \ge m_n(\mu^\vee)$. Furthermore, since we subtract the same number of n-columns from $\mathrm{P}(\hat\lambda)$ and $\mu^\vee$ it is obvious that $\mathrm{P}(\hat\lambda)/\mu^\vee = (r)$ is equivalent to $\tilde\lambda/\tilde\mu = \tilde\lambda/\mathrm{P}(\hat\mu^\vee) = (r)$. With these definitions of $\tilde\lambda, \tilde\mu$ and $\hat\mu^\vee$ in place, we may write

$$\begin{aligned} e_r(x)b(x) &= \sum_{\hat\lambda\in P_k^+}\sum_{\hat\lambda/\mu^\vee=(r)} s_{(\mu^\vee)^t}(x^{-1})\hat\lambda \\ &= \sum_{\hat\mu^\vee\in P_k^+}\sum_{\substack{\tilde\lambda\in\mathfrak{P}_{\le n,k}\\ \tilde\lambda/\tilde\mu=(r)}} z^{\tilde\lambda_1-\tilde\mu_1} s_{(\hat\mu^\vee)^t}(x^{-1})\hat\lambda,\end{aligned} \tag{6.26}$$

where we have used that the Bethe ansatz equations imply $s_{(\mu^\vee)^t}(\tilde x^{-1}) = s_{(\hat\mu^\vee)^t}(\tilde x^{-1})$ (as both partitions, $\mu^\vee$ and $\hat\mu^\vee$, only differ by columns of length n); compare with (6.5) implying $h_n(\tilde x^{-1}) = \overline{h_n(\tilde x)} = 1$ and Lemma 6.5, (1) and (2). The z-dependence then follows by observing that the Schur polynomial s_ν is homogeneous of degree $|\nu|$ and that

$$\begin{aligned} |\hat\mu^\vee| &= |\mu^\vee| + n(k-\mu_1^\vee) \\ &= |\mu^\vee| + n(\hat\lambda_1 - m_n(\mu^\vee) - \mu_1^\vee + m_n(\mu^\vee)) \\ &= |\mu^\vee| + n(\tilde\lambda_1 - \tilde\mu_1)\ .\end{aligned}$$

Thus, upon renaming the summation variables in (6.26) we obtain (6.22) and the assertion now follows from Theorem 6.7 and (6.19). $\square$

The last corollary prompts us to introduce the following counterpart of the transfer matrix. Namely, we set

$$Q(u) = \sum_{r \geq 0} h_r(\mathcal{A}) u^r \tag{6.27}$$

as the generating operator for the noncommutative complete symmetric functions where, similar as before, we choose the convention $h_0(\mathcal{A}) = 1$. Note that this operator is well-defined since upon restricting to a subspace $\mathcal{H}_k[z]$ with fixed particle number k only the first $(k+1)$ terms in the sum (6.27) do not vanish, i.e. $h_r(\mathcal{A})_k = 0$ for $r > k$. Moreover, the operator $h_{r=k}(\mathcal{A})_k$ becomes the cyclic translation operator corresponding to the Dynkin diagram automorphism rot if we specialise $z = 1$. Thus, with $Q(u)_k$ being a polynomial in u of degree k the result (6.3) can be rewritten as the following functional equation,

$$T(u)_k Q(-u)_k = \left[Q(-uq)_k + z u^n q^k Q(-uq^{-1})_k \right]_{q=0} \ . \tag{6.28}$$

This equation is the 'crystal limit' of what is known as Baxter's TQ-equation in the literature on quantum integrable systems. In particular, it is immediate from our results that for any pair $u, v \in \mathbb{C}$ we have

$$[Q(u), Q(v)] = [Q(u), T(v)] = 0 \ . \tag{6.29}$$

The operator $Q(u)$ can therefore be identified as the analogue of Baxter's *Q-operator* for the XYZ and XXZ models [2]; further details will be presented in a forthcoming paper [30].

6.3. **The combinatorial fusion ring and Verlinde algebra.** Employing the cyclic noncommutative symmetric functions generated from the transfer matrix of the phase model, we now show that the k-particle space $\mathcal{H}_k[z]$ can be turned into a commutative, associative and unital algebra. To explicitly compute its structure constants we first need the following result:

Proposition 6.11 (Transformation matrix)**.** *Let $\hat{\sigma} \in P_k^+$ and $\sigma = \mathrm{P}(\hat{\sigma})$ the corresponding partition. Let $\xi_\sigma = \sigma + \rho$, where $\rho = (\frac{n+1}{2} - 1, \ldots, \frac{n+1}{2} - n)$. Then we have the identity*

$$\langle \hat{\lambda}, b(x_\sigma) \rangle = z^{-\frac{|\hat{\lambda}|}{n}} s_\lambda(\zeta^{-\frac{|\sigma|}{n}} \zeta^{I(\sigma)}) = z^{-\frac{|\hat{\lambda}|}{n}} \overline{\chi_\lambda(\xi_\sigma)} = z^{-\frac{|\hat{\lambda}|}{n}} \chi_{\lambda^*}(\xi_\sigma). \tag{6.30}$$

Proof. From the definition (6.1) we have $\langle \hat{\lambda}, b(x_\sigma) \rangle = s_{\hat{\lambda}^t}(x_\sigma^{-1})$. We claim that

$$s_{\hat{\lambda}^t}(x_\sigma^{-1}) = h_n(x_\sigma^{-1})^{m_n(\hat{\lambda})} s_{\lambda^t}(x_\sigma^{-1}) = z^{-m_n(\hat{\lambda})} s_{\lambda^t}(x_\sigma^{-1}) = z^{-\frac{|\hat{\lambda}| - |\lambda|}{n}} s_{\lambda^t}(x_\sigma^{-1}). \tag{6.31}$$

The first equality here is just Lemma 6.5 (2) and (6.11). The identity $h_n(x_\sigma^{-1}) = z^{-1} h_0(x_\sigma)^{-1} = z^{-1} = h_n(x_\sigma)^{-1}$ holds by Lemma 6.2 and (6.15). Since the Schur function s_λ is a homogeneous polynomial of degree $|\lambda|$, Lemma 6.5 (1) and the explicit form of the Bethe roots (6.8) imply $s_{\lambda^t}(x_\sigma^{-1}) = z^{-\frac{|\lambda|}{n}} s_\lambda(\zeta^{-\frac{|\sigma|}{n}} \zeta^{I(\sigma)})$. Hence,

$$\langle \hat{\lambda}, b(x_\sigma) \rangle = z^{-|\hat{\lambda}|/n} s_\lambda(\zeta^{-\frac{|\sigma|}{n}} \zeta^{I(\sigma)}). \tag{6.32}$$

Finally we use the identity (6.13). $\square$

The main theorem of this section is the following ring structure on $\mathcal{H}_k[z]$:

Theorem 6.12 (Combinatorial fusion ring)**.** *Fix $k \in \mathbb{Z}_{\geq 0}$ and consider the k-particle subspace $\mathcal{H}_k[z] \subset \mathcal{H}[z]$. The assignment*

$$(\hat{\lambda}, \hat{\mu}) \mapsto \hat{\lambda} \circledast \hat{\mu} := s_{\hat{\lambda}}(\mathcal{A})\hat{\mu} \tag{6.33}$$

for basis elements $\hat{\lambda}, \hat{\mu} \in P_k^+$ turns $\mathcal{H}_k[z]$ into a commutative, associative and unital $\mathbb{C}[z]$-algebra $\mathcal{F}_{\mathrm{comb}}$.

Proof. The unit element is obviously given by the weight corresponding to the empty partition as $s_\emptyset(\mathcal{A}) = 1_{\mathcal{H}_k[z]}$. We have to check the commutativity and associativity of the product. To achieve this we compute the matrix elements

$$s_{\hat{\lambda}}(\mathcal{A})\hat{\mu} = \sum_{\hat{\nu}\in P_k^+} \langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \hat{\nu} \tag{6.34}$$

in the eigenbasis of Bethe vectors $\{b(x_\sigma)\}_{\sigma\in\mathfrak{P}_{\leq k,n-1}}$. Note that (6.31) together with Theorem 6.7 implies the identity

$$s_{\hat{\lambda}}(\mathcal{A}) = z^{\frac{|\hat{\lambda}|-|\lambda|}{n}} s_\lambda(\mathcal{A}). \tag{6.35}$$

Moreover, thanks to Theorem 6.7 and Proposition 6.11 we have

$$\begin{aligned}
\langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle &= \sum_{\sigma\in\mathfrak{P}_{\leq n-1,k}} \frac{\langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})b(x_\sigma)\rangle\langle b(x_\sigma),\hat{\mu}\rangle}{\langle b(x_\sigma), b(x_\sigma)\rangle} \\
&= \frac{z^{\frac{|\hat{\lambda}|+|\hat{\mu}|-|\hat{\nu}|}{n}}}{n(k+n)^{n-1}} \sum_{\sigma\in\mathfrak{P}_{\leq k,n-1}} \zeta^{|\sigma|\frac{|\lambda|+|\mu|-|\nu|}{n}} \frac{s_\lambda(\zeta^{-I(\sigma)})s_\mu(\zeta^{-I(\sigma)})s_\nu(\zeta^{I(\sigma)})}{\prod_{i<j}^n |\zeta^{I_i(\sigma)}-\zeta^{I_j(\sigma)}|^{-2}}
\end{aligned} \tag{6.36}$$

from which it is obvious that the product is commutative as the matrix element is invariant under exchanging $\hat{\lambda}$ and $\hat{\mu}$. Associativity is also clear:

$$\begin{aligned}
\hat{\lambda} \circledast (\hat{\mu} \circledast \hat{\nu}) &= \sum_{\hat{\rho}\in P_k^+} \langle \hat{\rho}, s_{\hat{\mu}}(\mathcal{A})\hat{\nu}\rangle \hat{\lambda} \circledast \hat{\rho} = \sum_{\hat{\rho},\hat{\rho}'\in P_k^+} \langle \hat{\rho}', s_{\hat{\lambda}}(\mathcal{A})\hat{\rho}\rangle\langle \hat{\rho}, s_{\hat{\nu}}(\mathcal{A})\hat{\mu}\rangle \hat{\rho}' \\
&= \sum_{\hat{\rho}'\in P_k^+} \langle \hat{\rho}', s_{\hat{\lambda}}(\mathcal{A})s_{\hat{\nu}}(\mathcal{A})\hat{\mu}\rangle \hat{\rho}' = \sum_{\hat{\rho}'\in P_k^+} \langle \hat{\rho}', s_{\hat{\nu}}(\mathcal{A})s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \hat{\rho}' \\
&= \sum_{\hat{\rho},\hat{\rho}'\in P_k^+} \langle \hat{\rho}', s_{\hat{\nu}}(\mathcal{A})\hat{\rho}\rangle\langle \hat{\rho}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \hat{\rho}' = \sum_{\hat{\rho}\in P_k^+} \langle \hat{\rho}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \hat{\nu} \circledast \hat{\rho} \\
&= \sum_{\hat{\rho}\in P_k^+} \langle \hat{\rho}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \hat{\rho} \circledast \hat{\nu} = (\hat{\lambda} \circledast \hat{\mu}) \circledast \hat{\nu}\ ,
\end{aligned}$$

where in the first and last line we have exploited commutativity of the product and in the second line we used the non-degeneracy of the bilinear form and the commutativity of the polynomials $s_{\hat{\lambda}}(\mathcal{A})$, $s_{\hat{\nu}}(\mathcal{A})$ (see Corollary 6.8). □

Definition 6.13. For $\hat{\lambda}, \hat{\mu}, \hat{\sigma} \in P_k^+$ we abbreviate $\mathcal{S}_{0,\hat{\sigma}} = \frac{1}{\sqrt{n(n+k)^{n-1}}}|\operatorname{Van}_\sigma|$ and define $\mathcal{S}_{\hat{\lambda}\hat{\sigma}} = \mathcal{S}_{0\hat{\sigma}}\chi_\lambda(\xi_\sigma)$ with $\xi_\sigma = -(\mathrm{P}(\sigma)+\rho)$. We call the matrix $\mathcal{S} := \mathcal{S}_{\hat{\lambda},\hat{\mu}}$ with columns and rows indexed by P_k^+(or by $\mathfrak{P}_{\leq n-1,k}$ via (2.5)) the *modular S-matrix*.

Remark 6.14. From Theorem (6.7) it follows directly that the matrix $\mathcal{S}$ is *unitary*, i.e. $\sum_{\hat{\lambda}\in P_k^+} \mathcal{S}_{\hat{\lambda}\hat{\sigma}}\bar{\mathcal{S}}_{\hat{\lambda}\hat{\rho}} = \delta_{\hat{\sigma},\hat{\rho}}$. From Lemma 6.5 (4) it follows that $\mathcal{S}$ is almost rotation invariant: $\mathcal{S}_{\mathrm{rot}(\hat{\lambda})\hat{\mu}} = e^{-\frac{2\pi\iota}{n}|\mu|}\mathcal{S}_{\hat{\lambda}\hat{\mu}}$. Moreover, we have the *duality* $\mathcal{S}_{\hat{\lambda}\hat{\mu}} =$

$\sqrt{k/n}e^{\frac{2\pi\iota}{kn}|\lambda||\mu|}\overline{\mathcal{S}_{\hat{\lambda}^t\hat{\mu}^t}}$ and *charge conjugation formula* $\bar{\mathcal{S}}_{\hat{\lambda}\hat{\mu}} = \mathcal{S}_{\hat{\lambda}^*\hat{\mu}}$. Let $\mathcal{T}$ be the diagonal matrix with entries $e^{2\pi\iota\,\mathrm{m}(\hat{\lambda})}$ where $\mathrm{m}(\hat{\lambda}) := \frac{|\lambda+\rho|^2}{2(k+n)} - \frac{|\rho|^2}{2n}$ is the so-called *modular anomaly*. Then $\mathrm{S} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \mathcal{S}$, $\mathrm{T} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \mapsto \mathcal{T}$ provides a representation of $\mathrm{PSL}(2,\mathbb{Z})$, while $\mathrm{S} \mapsto \bar{\mathcal{S}}$, $\mathrm{T} \mapsto \mathcal{T}$ yields a representation of $\mathrm{SL}(2,\mathbb{Z})$. Thus, together with complex conjugation this turns $\mathcal{F}_{\mathrm{comb}}$ into an abstract Verlinde algebra in the sense of [12, 0.4.1]. While defined here in a combinatorial setting these actions of the modular group correspond to the familiar modular invariance in the context of conformal field theory; see e.g. [14, Chapters 10, 14 and 16].

The following key result relates our combinatorial fusion ring to the $\widehat{\mathfrak{sl}}(n)_k$ Verlinde algebra and what is usually called the modular S-matrix in the literature. It states that our modular S- matrix satisfies the famous Kac-Peterson formula [27].

Proposition 6.15 (Kac-Peterson formula)**.** *Let $\hat{\lambda}, \hat{\sigma} \in P_k^+$. Then (in the notation of Section 2.1)*

$$\mathcal{S}_{\hat{\lambda}\hat{\sigma}} = \frac{e^{\iota\pi n(n-1)/4}}{\sqrt{n(k+n)^{n-1}}} \sum_{w\in S_n} (-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}(\sigma+\rho, w(\lambda+\rho))}$$

Here $\hat{\lambda} = k\hat{\omega}_0 + \lambda$ with finite part $\lambda = \sum_{i=1}^n (\lambda_i - |\lambda|/n)\varepsilon_i$ and $\rho = \sum_{i=1}^n \left(\frac{n+1}{2} - i\right)\varepsilon_i$.

Remark 6.16. Using Weyl's denominator formula one can also deduce an explicit expression

$$\mathcal{S}_{0\hat{\sigma}} = \frac{1}{\sqrt{n(k+n)^{n-1}}} \prod_{\alpha>0} 2\sin\frac{\pi(\rho+\sigma,\alpha)}{k+n} \tag{6.37}$$

$$= \frac{e^{\iota\pi n(n-1)/4}}{\sqrt{n(k+n)^{n-1}}} \sum_{w\in S_n} (-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}(\sigma+\rho, w\rho)} . \tag{6.38}$$

Remark 6.17 (Uniqueness theorem)**.** Goodman and Wenzl ([21, Theorem 3.2], see also [20]) established a characterisation of the fusion ring $\mathcal{F}$ by the data of its basis P_k^+, the associativity and the explicit formulae for $\hat{\lambda} \circledast \hat{\mu} = \hat{\mu} \circledast \hat{\lambda}$ where $\hat{\lambda}, \hat{\mu} \in P_k^+$ with $\hat{\mu}$ of the form (1^r). Theorem 6.12 and the Pieri-type formulae from Proposition 3.8 can then be used to verify the assumption of [21] and so to deduce an isomorphism of rings $\mathcal{F}_{\mathrm{comb}} \cong \mathcal{F}$. We will establish such an isomorphism (in Theorem 6.18) in a different way, involving the Verlinde formula and the modular S-matrix.

Proof of Proposition 6.15. Let $\delta = (n-1, n-2, \ldots, 1, 0)$ and for $j = 1, \ldots, n$ set

$$y_j = e^{-\frac{2\pi\iota}{k+n}\,(\sigma+\rho,\varepsilon_j)} = \zeta^{\frac{|\sigma|}{n} - I_j(\sigma)}, \quad y^\mu = y_1^{\mu_1} y_2^{\mu_2} \cdots y_n^{\mu_n}.$$

Using (6.32), with $z = 1$, and relating Schur polynomials with Weyl's character formula [32, p.40 in particular (3.1)] we get that the term $\langle b(x_\sigma), \hat{\lambda}\rangle$ equals

$$s_\lambda(\zeta^{\frac{|\sigma|}{n}}\zeta^{-I(\sigma)}) = \frac{\sum\limits_{w\in S_n} (-1)^{\ell(w)} w(y^{\lambda+\delta})}{\sum\limits_{w\in S_n} (-1)^{\ell(w)} w(y^\delta)} = \frac{\sum\limits_{w\in S_n} (-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}\sum_{j=1}^n (x,\varepsilon_j)(n+\lambda_j-j)}}{\sum\limits_{w\in S_n} (-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}\sum_{j=1}^n (x,\varepsilon_j)(n-j)}}$$

with $x = w(\sigma+\rho)$. On the other hand $(w(x), \epsilon_j)\,(n + \lambda_j - j)$ equals

$$\left(w(x), \left(\lambda_j - \frac{|\lambda|}{n}\right)\epsilon_j\right) - \left(w(x), \left(j - \frac{n+1}{2}\right)\epsilon_j\right) + \left(w(x), \left(\frac{n-1}{2} + \frac{|\lambda|}{n}\right)\epsilon_j\right).$$

From the definitions, and using $\sum_j I_j(\sigma) = |\sigma|$, we get

$$\langle b(x_\sigma), \hat{\lambda}\rangle \quad = \quad \frac{\sum_{w\in S_n}(-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}(\sigma+\rho, w(\rho+\lambda))}}{\sum_{w\in S_n}(-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}(\sigma+\rho, w\rho)}}$$

The assertion then follows from (6.38). □

6.4. **The Verlinde algebra and the modular S-matrix.** The Verlinde algebra is the fusion algebra of the integrable highest weight modules of level k. It plays a prominent role in conformal field theory (for details see e.g. [14], [26], [27]).

More precisely, the *Verlinde algebra* $V_k = V_k(\widehat{\mathfrak{sl}}(n), \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}$ is the $\mathbb{C}$-algebra with basis indexed by the elements from P_k^+ together with the multiplication (called *fusion product*)

$$\hat{\lambda} * \hat{\mu} := \sum_{\hat{\nu}\in P_k^+} \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} \hat{\nu}, \tag{6.39}$$

where the structure constants, known as *fusion coefficients* are given in terms of the *Verlinde formula* [45]

$$\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} = \sum_{\hat{\sigma}\in P_k^+} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}}\mathcal{S}_{\hat{\mu}\hat{\sigma}}\mathcal{S}_{\hat{\nu}^*\hat{\sigma}}}{\mathcal{S}_{0\hat{\sigma}}} \tag{6.40}$$

and the $\mathcal{S}_{\hat{\lambda}\hat{\sigma}}$ are given by the Kac-Peterson formula (precisely the formula from Proposition (6.15)). We already saw that one could alternatively define $\mathcal{S}_{\hat{\lambda}\hat{\sigma}} = \mathcal{S}_{0\hat{\sigma}}\chi_\lambda(\xi_\sigma)$, $\xi_\sigma = -(\mathrm{P}(\hat{\sigma}) + \rho)$, with $\mathcal{S}_{0\hat{\sigma}}$ as above.

Theorem 6.18 (Combinatorial description of the Verlinde algebra)**.** *The combinatorial algebra $\mathcal{F}$ specialises to the integral Verlinde (or fusion) algebra, i.e.*

$$\mathcal{F}_{\mathrm{comb}}/\langle z-1\rangle \cong \mathcal{F}. \tag{6.41}$$

In particular, we have for $z = 1$ the following equalities

$$\hat{\lambda} * \hat{\mu} = \hat{\lambda} \circledast \hat{\mu} = s_{\hat{\lambda}}(\mathcal{A})\hat{\mu} = s_\lambda(\mathcal{A})\hat{\mu}, \qquad \forall \hat{\lambda}, \hat{\mu} \in P_k^+ . \tag{6.42}$$

identifying the fusion product (6.39) with the combinatorially defined product (6.33).

Proof. We relate the matrix elements of the cyclic noncommutative Schur functions to the fusion constants:

$$\begin{aligned} \langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle &= \frac{z^{\frac{|\hat{\lambda}|+|\hat{\mu}|-|\hat{\nu}|}{n}}}{n(k+n)^{n-1}} \sum_{\sigma\in\mathfrak{P}_{\le k, n-1}} \frac{\chi_\lambda(\xi_\sigma)\chi_\mu(\xi_\sigma)\chi_{\nu^*}(\xi_\sigma)}{\prod_{i<j}^k |\zeta^{I_i(\sigma)} - \zeta^{I_j(\sigma)}|^{-2}} \\ &= z^{\frac{|\hat{\lambda}|+|\hat{\mu}|-|\hat{\nu}|}{n}} \sum_{\hat{\sigma}\in P_k^+} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}}\mathcal{S}_{\hat{\mu}\hat{\sigma}}\mathcal{S}_{\hat{\nu^*}\hat{\sigma}}}{\mathcal{S}_{0\hat{\sigma}}} = z^{\frac{|\hat{\lambda}|+|\hat{\mu}|-|\hat{\nu}|}{n}} \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} . \end{aligned} \tag{6.43}$$

The first equality here holds thanks to (6.36), (6.32) and (6.13), the second by Definition 6.13, and the last one is the Verlinde formula (6.40). Specialising $z \mapsto 1$ and employing (6.35) gives the result. □

Remark 6.19. Note that it is known that the fusion coefficients $\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}}$ are non-negative integers which coincide with the dimension of certain moduli spaces of generalized θ-functions [3]. From the combinatorial nature of the affine plactic algebra we can conclude completely combinatorially that $\langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu}\rangle \in \mathbb{Z}$. A purely combinatorial proof of non-negativity seems to be missing. First steps in this direction will appear in a forthcoming paper.

Theorem 6.20. *Let $\Lambda^{(k)} = \mathbb{Z}[e_1, \dots, e_k]$ be the ring of symmetric polynomials in k variables. The assignment $\hat{\lambda} \mapsto s_{\lambda^t}$ defines an isomorphism of rings*

$$\mathcal{F} = \mathcal{F}(\widehat{\mathfrak{sl}}(n))_k \cong \mathbb{Z}[e_1, \dots, e_k]/\langle h_n - 1, h_{n+1}, \dots, h_{n+k-1}, h_{n+k} + (-1)^k e_k \rangle, \tag{6.44}$$

where h_r denotes the r^{th} complete symmetric function in k variables.

Remark 6.21. We will see later (Corollary 8.2) that $V_k(\widehat{\mathfrak{sl}}(n), \mathbb{Z})$ is a quotient of the quantum cohomology ring of the Grassmannian by factoring out the extra relations $q = e_k$ and $h_n = 1$.

Proof. The notation $\Lambda^{(k)} = \mathbb{Z}[e_1, \dots, e_k]$ makes sense, since $\Lambda^{(k)}$ is generated by the e_i's and they are algebraically independent ([11, Prop. 35.1]). Let $\Lambda^{(k)}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda^{(k)}$ be the complexification. Denote by I be the ideal generated by f_i, $0 \le i \le k$, where $f_0 = h_n - 1$, $f_k = h_{n+k} + (-1)^k e_k$ and $f_i = h_{n+i}$ otherwise; and let $R := \Lambda^{(k)}_{\mathbb{C}}/I$ be the quotient, which is of course exactly the complexification of the right hand side of (6.44).
Claim 1: I is a radical ideal, i.e $I = \sqrt{I}$.
First note that we could also write $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k]$ [11, Prop. 35.1], in particular $e_k \in \mathbb{Z}[h_1, \dots, h_k]$. Now consider $\Lambda = \varprojlim \Lambda^{(k)} = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$, the ring of symmetric functions with its natural projection $p : \Lambda \twoheadrightarrow \Lambda^{(k)}$. Set $\tilde{h}_i = h_i \in \Lambda$ if $i \notin \{n, n+k\}$, $\tilde{h}_n = h_n - 1$ and $\tilde{h}_{n+k} = h_{n+k} + (-1)^k e_k$. For μ a partition define $\tilde{h}_\mu = \tilde{h}_{\mu_1} \tilde{h}_{\mu_2} \dots$. The h_i, $i \in \{1, 2, \dots, k, n, n+1, \dots, n+k\}$ are algebraically independent, and therefore also the corresponding $\tilde{h}_i$. Consequently, the $\tilde{h}_\mu$'s form a linearly independent subset of Λ. Assume now $f \in \Lambda^{(k)} \subset \Lambda$ is not in the ideal I' generated by the $\tilde{h}_i$, $n \le i \le n+k$. If we expand f in the $\tilde{h}_\mu$'s there must appear some $\tilde{h}_\mu$ where $\mu_j \notin \{n, n+1, \dots, n+k\}$ for all j. Let s be a positive integer. Then f^s must contain $\tilde{h}_{\mu^s}$ in its basis expansion, where μ^s is the partition which contains each part μ_j exactly s times. In particular $f^s \notin I'$ for any s. Now one can use the projection p to deduce the result.
Claim 2: Let $f, g \in \Lambda$. Then $f = g \in R$ if and only if $f(x) - g(x) = 0$ for any solution x of the Bethe Ansatz equations (6.2).
Using the alternative version (6.7) of (6.2) (with $z = 1$), together with the formula $h_n(x_\sigma) = 1$ from (6.15), the claim is a direct consequence of Claim 1 and Hilbert's Nullstellensatz, since then I equals the vanishing ideal of the zero set of I.
Claim 3: Sending $\hat{\lambda}$ to the class of s_{λ^t} in R defines a ring homomorphism

$$F : \quad V_k(\widehat{\mathfrak{sl}}(n), \mathbb{C}) \to R.$$

It is enough to show that for any $\hat{\lambda}, \hat{\mu} \in P^+_k$ we have $F(\hat{\lambda} * \hat{\mu}) = F(\hat{\lambda}) F(\hat{\mu})$ (thanks to (6.42)). Let $b(x) = b(x_\sigma)$ be a (non-zero) Bethe vector. Using (6.1) and (6.13) for $z = 1$ we expand $b(x) = \sum_{\hat{\nu} \in P^+_k} s_{\nu^{t*}}(x) \hat{\nu}$ and obtain from Theorem 6.7

$$s_\lambda(\mathcal{A}) b(x) = s_{\lambda^t}(x) b(x) = \sum_{\hat{\mu} \in P^+_k} s_{\lambda^t}(x) s_{\mu^t}(x) \hat{\mu}^* = \sum_{\hat{\mu}, \hat{\nu} \in P^+_k} \langle \hat{\mu}^*, s_\lambda(\mathcal{A}) \hat{\nu}^* \rangle s_{\nu^t}(x) \hat{\mu}^* .$$

Comparing coefficients of each basis element in the last two terms we obtain

$$s_{\lambda^t}(x)s_{\mu^t}(x) = \sum_{\hat{\nu}\in P_k^+}\langle\hat{\mu}^*, s_\lambda(\mathcal{A})\hat{\nu}^*\rangle s_{\nu^t}(x) = \sum_{\hat{\nu}\in P_k^+}\langle\hat{\nu}, s_\lambda(\mathcal{A})\hat{\mu}\rangle s_{\nu^t}(x) \tag{6.45}$$

Note that our convention $s_{(1^n)}(\mathcal{A}) = e_n(\mathcal{A}) = 1$ (with $z = 1$) precisely corresponds under F to the extra imposed relation $s_n = h_n = 1$. To see the equality of the matrix elements recall from Section 2.2 that

$$\langle\hat{\mu}^*, s_\lambda(\mathcal{A})\hat{\nu}^*\rangle = \overline{\langle\hat{\nu}, (\mathrm{flip}\circ s_\lambda(\mathcal{A})^*\circ\mathrm{flip})\hat{\mu}\rangle},$$

where $s_\lambda(\mathcal{A})^*$ denotes the right adjoint of $s_\lambda(\mathcal{A})$. One easily verifies that $\mathrm{flip}\circ a_i\circ\mathrm{flip} = a_{n-i}^*$ which implies $\mathrm{flip}\circ e_r(\mathcal{A})^*\circ\mathrm{flip} = e_r(\mathcal{A})$ and, thus, $\mathrm{flip}\circ s_\lambda(\mathcal{A})^*\circ\mathrm{flip} = s_\lambda(\mathcal{A})$. Together with the reality of the matrix elements, which is immediate from the action of the affine plactic algebra this gives the desired identity (6.45).

Now, fom the definition of F one obtains

$$A := F(\hat{\lambda}*\hat{\mu}) = F\left(s_\lambda(\mathcal{A})\hat{\mu}\right) = F\left(\sum_{\hat{\nu}\in P_k^+}\langle\hat{\nu}, s_\lambda(\mathcal{A})\hat{\mu}\rangle\hat{\nu}\right) = \sum_{\hat{\nu}\in P_k^+}\langle\hat{\nu}, s_\lambda(\mathcal{A})\hat{\mu}\rangle s_{\nu^t},$$

whereas

$$B := F(\hat{\lambda})F(\hat{\mu}) = s_{\lambda^t}s_{\mu^t}.$$

By (6.45) we have $A(x) = B(x)$ for any Bethe root $x = x_\sigma$ or any other in the S_k-orbit of x_σ. Moreover, $A(x) = B(x)$ for any Bethe root where not all the entries are different, since both sides of the equality just vanish. Hence $A(x) = B(x)$ for any solution of the Bethe Ansatz equations. The Claim follows now from Claim 2. *Claim 4: F is an isomorphism.* Since via (2.5) the $\lambda\in\mathfrak{P}_{\le n-1,k}$ form a basis of the complexification of $V_k(\widehat{\mathfrak{sl}}(n),\mathbb{Z})$ it is enough to show that the s_{λ^t}, $\lambda\in\mathfrak{P}_{\le n-1,k}$ form a basis of R/I. Recall that the the $\tilde{h}_\mu$'s form a basis of $\Lambda^{(k)}$, if μ runs through the set of partitions with at most k parts [17, p 73]. In particular, the $\tilde{h}_\mu$'s with $\mu\in\mathfrak{P}_{\le n-1,k}$ span R/I. To see the linearly independence assume that $\sum_{\mu\in\mathfrak{P}_{\le n-1,k}} a_\mu\tilde{h}_\mu = 0$ in R/I for some $a_\mu\in\mathbb{C}$, equivalently $\sum_{\mu\in\mathfrak{P}_{\le n-1,k}} a_\mu\tilde{h}_\mu = \sum_{i=n}^{n+k} r_i\tilde{h}_i$ for some $r_i\in R$. Expanding this equality in the basis of the $\tilde{h}_\mu$'s we see that the basis vectors on the left hand side are all of the form $\tilde{h}_\mu$, where $\mu\in\mathfrak{P}_{\le n-1,k}$, whereas the basis vectors on the right hand side are all of the form $\tilde{h}_\mu$ with $\mu\notin\mathfrak{P}_{\le n-1,k}$. In particular, $a_\mu = 0$ for all $\mu\in\mathfrak{P}_{\le n-1,k}$ and so the $\tilde{h}_\mu$ form a basis. Moreover, $\tilde{h}_\mu = h_\mu$ in this case. Now the transformation matrix between the h_μ's and the s_μ's is triangular with 1's on the diagonal [17, p 75 (4)], hence the linearly independence follows.

Since all the constructions are defined over the integers, F induces an isomorphism for the integral version as well. $\square$

Example 6.22. Consider the case $n = 3$ and $k = 1$. The ring $\mathbb{C}[e_1]/\langle h_4 + e_1\rangle$ is isomorphic to $\mathbb{C}[e_1]/\langle e_1^4 - e_1\rangle$ (since $h_4 = e_1^4 - 3e_2e_1^2 + 2e_3e_1 + e_2^2$ evaluated at $x = (x_1, 0, 0, 0, \ldots)$ gives $h_4(x) = e_1^4(x)$), and has basis $1, e_1, e_1^2, e_1^3 = h_3$. If we impose the additional relation $h_3 = 1$, we get a ring on basis $1, e_1, e_1^2$ and multiplication $e_1e_1 = e_1^2$, $e_1e_1^2 = e_1^3 = 1$. Under the map Φ, the three partitions $\emptyset$, (1), (1^2) are mapped to $s_\emptyset = 1$, $s_{(1)} = h_1 = e_1$, $s_{(2)} = h_2 = e_1^2$, and the multiplication in the combinatorial fusion ring is precisely the given one.

7. Applications

7.1. **Symmetries of fusion coefficients.** The following symmetry formulae for the fusion coefficients are now easily verified using (6.42):

Proposition 7.1. *For $\hat\lambda$, $\hat\mu$, $\hat\nu \in P_k^+$ set $\mathcal{N}_{\hat\lambda\hat\mu\hat\nu} := \mathcal{N}_{\hat\lambda\hat\mu}^{(k),\hat\nu^*}$ then*

(i) $\mathcal{N}_{\pi(\hat\lambda)\pi(\hat\mu)\pi(\hat\nu)} = \mathcal{N}_{\hat\lambda\hat\mu\hat\nu}$ *for any permutation $\pi \in S_3$ of $(\hat\lambda, \hat\mu, \hat\nu)$*

(ii) $\mathcal{N}_{\mathrm{rot}(\hat\lambda)\hat\mu\hat\nu} = \mathcal{N}_{\hat\lambda\,\mathrm{rot}(\hat\mu)\hat\nu} = \mathcal{N}_{\hat\lambda\hat\mu\,\mathrm{rot}(\hat\nu)}$

(iii) $\overline{\mathcal{N}_{\hat\lambda\hat\mu\hat\nu}} = \mathcal{N}_{\hat\lambda^*\hat\mu^*\hat\nu^*} = \mathcal{N}_{\hat\lambda\hat\mu\hat\nu}$

Proof. According to (6.43) and (6.36) (with $\hat\nu$ replaced by $\hat\nu^*$) we have

$$\mathcal{N}_{\hat\lambda\hat\mu\hat\nu} = \frac{1}{n(k+n)^{n-1}} \sum_{\sigma\in\mathfrak{P}_{\le n-1,k}} \zeta^{|\sigma|\frac{|\lambda|+|\mu|+|\nu|}{n}} \frac{s_\lambda(\zeta^{-I(\sigma)})s_\mu(\zeta^{-I(\sigma)})s_\nu(\zeta^{-I(\sigma)})}{\prod_{i<j}^n |\zeta^{I_i(\sigma)} - \zeta^{I_j(\sigma)}|^{-2}},$$

and the statement (i) follows. The fusion coefficients do not depend on z, therefore we may set $z = 1$ and calculate

$$\begin{aligned} \mathcal{N}_{\mathrm{rot}(\hat\lambda)\hat\mu\hat\nu} &\overset{\text{(i)}}{=} \mathcal{N}_{\hat\mu\hat\nu\,\mathrm{rot}(\lambda)} \overset{(6.43)}{=} \langle \mathrm{rot}(\hat\lambda)^*, s_\mu(\mathcal{A})\hat\nu\rangle = \langle \mathrm{rot}^{-1}(\hat\lambda^*), s_\mu(\mathcal{A})\hat\nu\rangle \\ &\overset{(\dagger)}{=} \langle \hat\lambda^*, s_\mu(\mathcal{A})\,\mathrm{rot}(\hat\nu)\rangle \overset{(6.43)}{=} \mathcal{N}_{\hat\mu\,\mathrm{rot}(\hat\nu)\lambda} \overset{\text{(i)}}{=} \mathcal{N}_{\hat\lambda\hat\mu\,\mathrm{rot}(\hat\nu)}, \end{aligned}$$

(where (†) follows from Corollary 6.8 using the fact that $\mathrm{rot} = h_k(\mathcal{A})_k$ when $z = 1$) and then apply again part (i) repeatedly to deduce $\mathcal{N}_{\mathrm{rot}(\hat\lambda)\hat\mu\hat\nu} = \mathcal{N}_{\lambda\,\mathrm{rot}(\hat\mu)\hat\nu}$ as well. To prove (iii) we make once more use of the relation $\mathrm{flip}\circ s_\lambda(\mathcal{A})^* = s_\lambda(\mathcal{A}) \circ \mathrm{flip}$ already discussed when deriving (6.45). Together with $s_\lambda(\mathcal{A})^* = s_{\lambda^*}(\mathcal{A})$, which follows from (6.16) and (6.13) and the definitions in Section 2.2, we calculate

$$\mathcal{N}_{\hat\lambda^*\hat\mu^*\hat\nu^*} = \langle \hat\nu, s_{\lambda^*}(\mathcal{A})\hat\mu^*\rangle = \langle \hat\nu^*, \mathrm{flip}\circ s_\lambda(\mathcal{A})^* \circ \mathrm{flip}\,\hat\mu\rangle = \langle \hat\nu^*, s_\lambda(\mathcal{A})\hat\mu\rangle = \mathcal{N}_{\hat\lambda\hat\mu\hat\nu}.$$

□

7.2. **Noncommutative Cauchy identities.** Exploiting the combinatorial definition (6.42) of the fusion product, we now state new identities for the fusion coefficients.

Corollary 7.2 (Noncommutative Cauchy identities)**.** *Let $\ell \in \{1, \dots, n-1\}$, $u_i \in \mathbb{C}$ for $1 \le i \le \ell$ and $k \in \mathbb{Z}_{>0}$. Recalling the analogue of Baxter's Q-operator from* (6.27) *and the transfer matrix* (4.1) *we have the following equalities of endomorphisms of $\mathcal{H}_k$:*

$$Q(u_1)_k \cdots Q(u_\ell)_k = \sum_{\lambda\in\mathfrak{P}_{\le\ell,k}} s_\lambda(u_1,\dots,u_\ell)s_\lambda(\mathcal{A})_k, \tag{7.1}$$

$$T(u_1)_k \cdots T(u_\ell)_k = \sum_{\lambda\in\mathfrak{P}_{\le n,\ell}} s_{\lambda^t}(u_1,\dots,u_\ell)s_\lambda(\mathcal{A})_k\ . \tag{7.2}$$

In particular,

$$h_{\alpha_1}(\mathcal{A})_k \cdots h_{\alpha_\ell}(\mathcal{A})_k = \sum_{\lambda\in\mathfrak{P}_{\le\ell,k}} K_{\lambda\alpha}s_\lambda(\mathcal{A})_k \tag{7.3}$$

$$e_{\alpha_1}(\mathcal{A})_k \cdots e_{\alpha_\ell}(\mathcal{A})_k = \sum_{\lambda\in\mathfrak{P}_{\le n,\ell}} K_{\lambda^t\alpha}s_\lambda(\mathcal{A})_k, \tag{7.4}$$

with $K_{\lambda\alpha}$ being the Kostka numbers, i.e. the number of semi-standard tableau of shape λ and weight $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{Z}_{>0}^\ell$.

Proof. We employ once more the eigenbasis of Bethe states; see Theorem 6.7. The formula (7.1) is a consequence of the Cauchy-identity $\sum_\lambda s_\lambda(x)s_{\lambda^t}(y)=\prod_{i,j}(1+x_iy_j)$ for commutative Schur functions. Namely, the definition of the Q-operator in (6.27) together with (6.19) gives $Q(u)b(x)=\prod_{j=1}^k(1+ux_j)b(x)$ for any Bethe state $b(x)\in\mathcal{H}_k$. Therefore, it follows that

$$\begin{aligned}Q(u_1)\cdots Q(u_\ell)b(x) &= \prod_{i=1}^{\ell}\prod_{j=1}^{k}(1+u_ix_j)b(x)=\sum_{\lambda\in\mathfrak{P}_{\le\ell,k}}s_\lambda(u)s_{\lambda^t}(x)b(x)\\ &= \sum_{\lambda\in\mathfrak{P}_{\le\ell,k}}s_\lambda(u)s_\lambda(\mathcal{A})b(x)\ .\end{aligned}$$

Since the Bethe states form a basis of $\mathcal{H}_k$ (Theorem 6.7), we have proved (7.1). To derive (7.3), recall the well-known expansion $s_\lambda(u)=\sum_\alpha K_{\lambda\alpha}m_\alpha(u)$ with m_α being the basis of symmetric monomial functions (see e.g. [32, Chapter I, Section 6, Table 1 and (6.4)]). Expanding both sides of (7.1) with respect to the basis of symmetric monomial functions, we obtain the asserted formula (7.3).
The proof of (7.2) and (7.4) is completely analogous to the known one for the ring of commutative symmetric functions, since we have shown in Corollary 6.8 that the cyclic noncommutative Schur polynomials satisfy all the familiar relations of their commutative counterparts. □

As an application of the generalised Cauchy identities we now derive novel identities for the fusion coefficients. Let ℓ and α be as in Corollary 7.2 and $\mu,\nu\in\mathfrak{P}_{\le n-1,\ell}$. Then denote by $\hat\mu,\hat\nu$ and $\widehat{(\alpha_i)}$, $\widehat{(1^{\alpha_i})}$ the affine weights in P_k^+ corresponding to the pre-images of μ,ν and a horizontal and vertical strip of length α_i under the bijection (2.5).

Corollary 7.3 (Kostka numbers and fusion coefficients)**.** *Given a sequence $S=(\nu=\mu^{(0)},\mu^{(1)},\ldots,\mu^{(\ell)}=\mu)$ of partitions in $\mathfrak{P}_{\le n-1,\ell}$ set $N_i(S)=\mathcal{N}^{(k)\widehat{\mu^{(i-1)}}}_{\widehat{(\alpha_i)}\,\widehat{\mu^{(i)}}}$ and $\mathcal{N}'_i(S)=\mathcal{N}^{(k)\widehat{\mu^{(i-1)}}}_{\widehat{(1^{\alpha_i})}\,\widehat{\mu^{(i)}}}$. Then we have the identities*

$$\sum_S\mathcal{N}_1(S)\mathcal{N}_2(S)\ldots\mathcal{N}_\ell(S) = \sum_{\lambda\in\mathfrak{P}_{\le n-1,\ell}}K_{\lambda\alpha}\mathcal{N}^{(k)\hat\nu}_{\hat\lambda\hat\mu} \tag{7.5}$$

$$\sum_S\mathcal{N}'_1(S)\mathcal{N}'_2(S)\ldots\mathcal{N}'_\ell(S) = \sum_{\lambda\in\mathfrak{P}_{\le n,\ell}}K_{\lambda^t\alpha}\mathcal{N}^{(k)\hat\nu}_{\hat\lambda\hat\mu}\ . \tag{7.6}$$

where the left sums run through the set of all sequences S as above and $\hat\lambda\in P_k^+$ is the pre-image of (2.5)*. If $\lambda\in\mathfrak{P}_{\le n,\ell}$ we first delete all columns of length n.*

Proof. Consider the matrix element $\langle\hat\nu,h_{\alpha_1}(\mathcal{A})\cdots h_{\alpha_\ell}(\mathcal{A})\hat\mu\rangle$ with $\hat\mu,\hat\nu\in P_k^+$. Employing (7.3) we obviously have

$$\langle\hat\nu,h_{\alpha_1}(\mathcal{A})\cdots h_{\alpha_\ell}(\mathcal{A})\hat\mu\rangle=\sum_{\lambda\in\mathfrak{P}_{\le n-1,\ell}}K_{\lambda\alpha}\langle\hat\nu,s_\lambda(\mathcal{A})\hat\mu\rangle$$

and (7.3) now follows from (6.43). The second equality is proved along the same lines by considering the matrix element $\langle\hat\nu,e_{\alpha_1}(\mathcal{A})\cdots e_{\alpha_\ell}(\mathcal{A})\hat\mu\rangle$ and applying the identity (7.4). □

Using the phase algebra generators we now derive a relation between fusion coefficients of level k and level $k+1$. Such a recursion relation is new and is a direct consequence of our combinatorial particle description of the fusion ring.

Corollary 7.4 (Recursion relation for fusion coefficients)**.** *Adopt the same conventions as in the previous corollary. Choose any integer vector* $\mathbf{j} = (j_1, \dots, j_\ell) \in \mathbb{Z}_n^\ell$ *and define for level* $k+1$ *the fusion coefficients* $\mathcal{N}'_i(S, \mathbf{j}) = \mathcal{N}^{(k+1)\ \varphi_{j_i^*} \widehat{\mu^{(i-1)}}}_{\widehat{(1^{\alpha_i})}\ \varphi_{j_i^*} \widehat{\mu^{(i)}}}$*, then we have the recursion relation*

$$\sum_S \mathcal{N}'_1(S,\mathbf{j})\mathcal{N}'_2(S,\mathbf{j}) \dots \mathcal{N}'_\ell(S,\mathbf{j}) = \sum_{\lambda \in \mathfrak{P}_{\leq n,\ell}} K_{\lambda^t \alpha} \mathcal{N}^{(k)\hat{\nu}}_{\hat{\lambda}\hat{\mu}} \ . \tag{7.7}$$

In particular, for $\alpha = (r, 0, \dots, 0)$ *we simply have*

$$\mathcal{N}^{(k+1)\ \varphi_j^* \hat{\nu}}_{\widehat{(1^r)}\ \varphi_j^* \hat{\mu}} = \mathcal{N}^{(k)\ \hat{\nu}}_{\widehat{(1^r)}\ \hat{\mu}} \tag{7.8}$$

for any $j \in \mathbb{Z}_n$.

Proof. Employing Lemma 3.4 one easily verifies the following relations between the generators of the Yang-Baxter and the phase algebra,

$$A(u)\varphi_1 = \varphi_1 A(u), \quad \varphi_1 D(u) = uC(u), \quad D(u) = uC(u)\varphi_1^* \ .$$

From the latter identities one now obtains for the transfer matrix (4.1),

$$\varphi_1 T(u) \varphi_1^* = \varphi_1 A(u) \varphi_1^* + z\varphi_1 D(u)\varphi_1^* = A(u)\varphi_1\varphi_1^* + zuC(u)\varphi_1^* = T(u).$$

But since for $z=1$ one has $\mathrm{rot} \circ T(u) = T(u) \circ \mathrm{rot}$, this is obvious as each $e_r(\mathcal{A})$ is invariant under a cyclic rotation when $z=1$, we must have

$$\varphi_i T(u) \varphi_i^* = T(u)$$

for all $i = 0, 1, \dots, n-1$. Therefore, we have $\varphi_i e_r(\mathcal{A}) \varphi_i^* = e_r(\mathcal{A})$ and the assertion follows from (7.4),

$$\begin{aligned} \langle \hat{\nu}, e_{\alpha_1}(\mathcal{A}) \cdots e_{\alpha_\ell}(\mathcal{A}) \hat{\mu} \rangle &= \langle \hat{\nu}, \varphi_{j_1} e_{\alpha_1}(\mathcal{A}) \varphi_{j_1}^* \cdots \varphi_{j_\ell} e_{\alpha_\ell}(\mathcal{A}) \varphi_{j_\ell}^* \hat{\mu} \rangle \\ &= \sum_{\lambda \in \mathfrak{P}_{\leq n,\ell}} K_{\lambda^t \alpha} \mathcal{N}^{(k)\hat{\nu}}_{\hat{\lambda}\hat{\mu}} \ . \end{aligned}$$

□

Example 7.5. Let $n=3$ and $k=2$. Choose $\ell = k = 2$ and set $\mu = (2,1)$. Then it is not difficult to compute, using the combinatorial formula (6.42), the fusion product

$$\ydiagram{2,1} * \ydiagram{2,1} = \ydiagram{2,1} + \emptyset \ .$$

Fixing $\nu = \emptyset$ and making the choices $\alpha = (1,1,1)$, $\mathbf{j} = (2,3,1)$ one finds that on the left hand side of (7.7) only two sequences S contribute, namely

$$\left(\nu = \emptyset, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{2,1} = \mu\right) \text{ and } \left(\nu = \emptyset, \ydiagram{1,1}, \ydiagram{2,2}, \ydiagram{2,1} = \mu\right) \ .$$

Again one computes combinatorially

$$\mathcal{N}^{(k+1)\ \varphi_2^* \hat{\emptyset}}_{\widehat{(1)}\ \varphi_2^* \widehat{(1,1)}} = \mathcal{N}^{(k+1)\ \varphi_3^* \widehat{(1,1)}}_{\widehat{(1)}\ \varphi_3^* \widehat{(1,0)}} = \mathcal{N}^{(k+1)\ \varphi_3^* \widehat{(1,1)}}_{\widehat{(1)}\ \varphi_3^* \widehat{(2,2)}} = \mathcal{N}^{(k+1)\ \varphi_1^* \widehat{(1,0)}}_{\widehat{(1)}\ \varphi_1^* \hat{\mu}} = \mathcal{N}^{(k+1)\ \varphi_1^* \widehat{(2,2)}}_{\widehat{(1)}\ \varphi_1^* \hat{\mu}} = 1$$

leading to the identity

$$\sum_S \mathcal{N}_1'(S,\mathbf{j})\mathcal{N}_2'(S,\mathbf{j})\mathcal{N}_3'(S,\mathbf{j}) = 2\ .$$

The only possible semi-standard tableaux of weight α lying within the bounding box of size 3×2 are $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$. Obviously, $\mathcal{N}^{(k)\hat\emptyset}_{\hat\emptyset\hat\mu}=0$ and, thus, we compute for the right hand side of (7.7)

$$\sum_{\lambda\in\mathfrak{P}_{\leq n,k}} K_{\lambda^t\alpha}\mathcal{N}^{(k)\hat\nu}_{\hat\lambda\hat\mu} = 2\mathcal{N}^{(k)\hat\emptyset}_{\widehat{(2,1)}\hat\mu} = 2$$

which yields the desired equality.

PART II: THE QUANTUM COHOMOLOGY RING: FERMIONS ON A CIRCLE

8. FREE FERMION FORMULATION OF THE SMALL QUANTUM COHOMOLOGY RING

We want to develop structures parallel to the ones discussed before for the fusion ring, but now for the quantum cohomology ring and finally connect the two sides. Building on earlier results of for instance Bertram [5], Rietsch [37], and Postnikov [35] we present a free fermion formulation of the small quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{k,k+n})$ of the complex Grassmannian. Here k and n are analogous to the level and rank in the context of the fusion ring.

More precisely, let $n,k\in\mathbb{Z}_+$ and denote $N=k+n$. Let $\mathrm{Gr}_{k,n+k}$ be the Grassmannian of k-planes inside $\mathbb{C}^N=\mathbb{C}^{n+k}$. The integral cohomology ring $H^\bullet(\mathrm{Gr}_{k,k+n})$ can explicitly be described using Schubert calculus (see for example [17, 9.4]): the Schubert classes $[\Omega_\lambda]$ form a basis, the structure constants are the intersection numbers, which coincide with the Littlewood-Richardson coefficients. (Here λ runs through $\mathfrak{P}_{\leq k,n}$.)

The small quantum cohomology $qH^\bullet(\mathrm{Gr}_{k,n+k})$ is a deformation of the usual (singular) cohomology. More precisely, it is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathbb{Z}[q]\otimes_{\mathbb{Z}} H^\bullet(\mathrm{Gr}_{k,n+k})$ as a $\mathbb{Z}[q]$-module. The Schubert classes give a $\mathbb{Z}[q]$-basis $\{1\otimes[\Omega_\lambda]\}$. This module can be equipped (in a non-obvious way) with a ring structure where the structure constants $C^\nu_{\lambda,\mu}(q)=\sum C^{\nu,d}_{\lambda,\mu}q^d$ are given by the so-called 3-point, genus 0 Gromov-Witten invariants $C^{d,\nu^\vee}_{\lambda,\mu}$ which count the number of rational curves of degree d passing through generic translates of Ω_λ, Ω_μ, Ω_ν. (In the cases $|\lambda|+|\mu|+|\nu|\neq kn+d(k+n)$, where the number of curves could be infinite, one just puts $C^{d,\nu}_{\lambda,\mu}=0$.) For a general overview we refer to [18] and [29]. Note that for $q=0$ we obtain the ordinary cohomology $H^\bullet(\mathrm{Gr}_{k,n+k})$, with the ring structure given by the cup or intersection product. For a description of the quantum cohomology ring from the Schubert calculus point of view we refer to [5], [9], [10], [43].

In the rest of the paper we will develop a combinatorial (fermionic model) for the quantum cohomology similar to our combinatorial description of the fusion algebra.

8.1. **01-words, partitions and Schubert cells.** Throughout our discussion we will use the following well-known bijections between 01-words of length N and partitions (see e.g. [35]) as well as the basis vectors of the wedge space $\bigwedge\mathbb{C}^N$.

Definition 8.1. Let $w = w_1 w_2 \cdots w_N$ be a word with N letters w_i which are either 0 or 1. The *weight* of w is the sum $\sum_i w_i = k$. Let

$$W_{k,N} = \left\{ w = w_1 w_2 \cdots w_N \;\middle|\; |w| := \sum_i w_i \;= k \right\}$$

be the set of words of weight k. Let $\mathrm{n}_i(w) := w_1 + \cdots + w_i$ the number of 1-letters w_j for j in the closed interval $[0, i]$.

For $w \in W_{k,N}$ denote by $1 \leq \ell_1 < \ldots < \ell_k \leq N$ the positions of the 1-letters in w counting from the left. Then there is obviously a bijection $\mathfrak{P}_{\leq k,n} \to W_{k,N}$ given by

$$\lambda \mapsto w(\lambda) = 0 \cdots 0 \underset{\ell_1}{1} 0 \cdots 0 \underset{\ell_k}{1} 0 \cdots 0,$$

where the positions ℓ_i of the letters 1 are determined according to the formula

$$\ell_i = \lambda_{k+1-i} + i \; . \tag{8.1}$$

We shall denote the inverse image under this bijection $w \mapsto \lambda(w)$. Graphically this correspondence is simply assigning to each λ the path which is traced out by its Young diagram in the $k \times n$ rectangle. Namely, starting from the left bottom corner in the $k \times n$ rectangle each move by a box to the right corresponds to a letter 0 and each move by a box up to a letter 1. For instance, the partitions displayed by the Young diagrams from (2.3) correspond under this bijection to the set of words

$$\{11000, 10100, 10010, 10001, 01100, 01010, 00110, 010001, 00101, 00011\}.$$

Recall the well-known correspondence (see e.g. [17], [35]) between 01-words of weight k and Schubert classes in $qH^\bullet(\mathrm{Gr}_{k,N})$: fix a full standard flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^N$ and assign to each word w the Schubert cell

$$\Omega^\circ_w = \{ V \in \mathrm{Gr}_{k,N} \mid \dim(V \cap \mathbb{C}^k) = w_N + w_{N-1} \cdots + w_{N+1-k} \} \; . \tag{8.2}$$

The associated Schubert class $[\Omega_w]$ is the fundamental cohomology class of the Schubert variety Ω_w (i.e. the closure of the Schubert cell.)

Siebert and Tian [40] gave an explicit presentation of $qH^\bullet(\mathrm{Gr}_{k,N})$ in terms of the ring of symmetric polynomials:

$$qH^\bullet(\mathrm{Gr}_{k,N}) \cong \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathbb{Z}[e_1, e_2, \ldots e_k] / \left\langle h_{n+1}, \ldots, h_{N-1}, h_N + (-1)^k q \right\rangle \tag{8.3}$$

where the e_i's are the (commutative) elementary symmetric functions in k variables and the h_i's are the corresponding complete symmetric functions. A $\mathbb{Z}[q]$-basis is given by the images of the Schur polynomials s_λ for $\lambda \in \mathfrak{P}_{\leq k,n}$. The following is then a direct consequence of Theorem 6.20:

Corollary 8.2 (Verlinde algebra as a quotient of $qH^\bullet$)**.**
The Verlinde algebra $V_k(\widehat{\mathfrak{sl}}(n), \mathbb{Z})$ is isomorphic to the quotient of the quantum cohomology ring $qH^\bullet(\mathrm{Gr}_{k,k+n})$ by imposing the extra relations $q = e_k$ and $h_n = 1$.

Finally the words of weight k (or the partitions from $\mathfrak{P}_{\leq k,n}$) are also in bijection to basis vectors of $\bigwedge \mathbb{C}^N$ by sending a partition λ to the vector $\lambda_i = k - 1 - i$. (This will turn out to be a small 'shadow' of the so-called Fock space which appears in the Fermion-Boson correspondence [26], [28]; see the table in Part III.)

8.2. **State space and fermions.** Paralleling our previous discussion for the fusion ring we now introduce a vector space of states for $qH^\bullet(\mathrm{Gr}_{k,N})$ which will again lead to a description of the ring structure in terms of quantum particles hopping on circle but now with N sites. Choose $N = n+k \in \mathbb{Z}_{>0}$ to be fixed and define the vector space

$$\mathcal{F} = \bigoplus_{k=0}^{N} \mathcal{F}_k, \qquad \mathcal{F}_k = \mathbb{C}\mathfrak{P}_{\leq k, N-k} = \mathbb{C}W_{k,N} \tag{8.4}$$

with $\mathcal{F}_0 = \mathbb{C}\{\emptyset\}$ as before. If we denote the basis vector in $\mathcal{F}_k$ given by the empty partition by $\emptyset = \emptyset_k$, then $\emptyset_0$ has the physical interpretation of the vacuum, i.e. no particles in the system. Note that in contrast to the fusion ring we are now considering a finite-dimensional vector space.

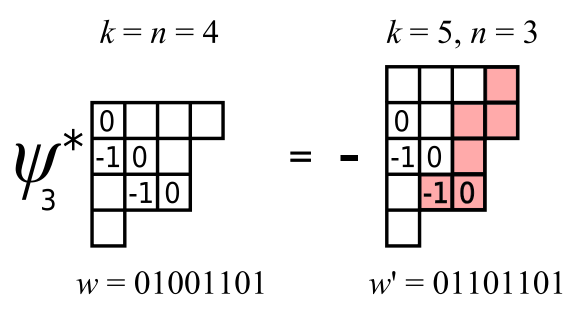
We want to construct an analogue of the phase algebra for our vector space $\mathcal{F}$. For $1 \leq i \leq N$ define the following linear endomorphisms ψ_i^*, ψ_i of $\mathcal{F}$:

$$\psi_i^*(w) \;:=\; \begin{cases} (-1)^{\mathrm{n}_{i-1}(w)} w', & \text{if } w_i = 0. \text{ Here } w' \text{ differs from } w \text{ only by } w_i' = 1 \\ 0 & \text{if } w_i \neq 0 \end{cases}$$

$$\psi_i(w) \;:=\; \begin{cases} (-1)^{\mathrm{n}_{i-1}(w)} w', & \text{if } w_i = 1. \text{ Here } w' \text{ differs from } w \text{ only by } w_i' = 0 \\ 0 & \text{if } w_i \neq 1. \end{cases}$$

The associated Young diagram $\psi_i^*\lambda$ is obtained by adding the top row of the Young diagram λ to itself (thereby increasing its height) and then subtracting a boundary ribbon starting in the $(i-k)$-diagonal and ending in the top row. In contrast $\psi_i\lambda$ is the Young diagram which is obtained from λ by subtracting the top row (thereby decreasing the height) and then adding a boundary ribbon with same start and end points as in the previous case.

Example 8.3. To visualize the action of ψ_i^* consider the special case $n = k = 4$ and $\mu = (4,3,3,1)$: $\psi_3^*\mu$ is depicted in the figure below, where the entries in the diagram label the diagonals. The $(3-k) = -1$-diagonal determines the start of the boundary ribbon (the shaded boxes) which has to be subtracted:



Proposition 8.4 (Fermion/Clifford algebra). *(1) The endomorphisms $\psi_i, \psi_i^* : \mathcal{F}_k \to \mathcal{F}_{k\pm1}$ create a subalgebra $\mathfrak{C} \subset \mathrm{End}\,\mathcal{F}$ isomorphic to a 2^{2N}-dimensional Clifford algebra with the following defining relations:*

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \quad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}\ .$$

(2) If we introduce the following scalar product on $\mathcal{F}$, $\langle \alpha w, \beta w' \rangle = \overline{\alpha}\beta\delta_{w,w'} = \prod_{i=1}^{N} \delta_{w_i,w_i'}$, then $\langle \psi_i\lambda, \lambda' \rangle = \langle \lambda, \psi_i^*\lambda' \rangle$.

(3) *If we introduce a grading on C by putting ψ_i in degree -1 and ψ_i^* in degree 1, we get a triangular decomposition $C = C_- \oplus C_0 \oplus C_+$. Then C_- annihilates the vacuum vector $\emptyset_0$, and $\mathcal{F}$ is a free C_+-module of rank 1, with graded decomposition $\mathcal{F} = \oplus_k \mathcal{F}_k$. In particular, $\mathcal{F}$ becomes an irreducible lowest weight Verma module module for the Clifford algebra C.*

Proof. All assertions can be easily verified by straightforward calculations. □

If we consider a word w as an N-tuple of integers, similar to the phase algebra the maps ψ_i^*, ψ_i have the interpretation of particle creation and annihilation operators at site i of a circular lattice but this time of size $N = n + k$. Moreover, they now describe free fermions which obey the Pauli exclusion principle, i.e. only one particle per site is allowed. The dimension (level) k is again the total number of particles which is given by the operator $K = \sum_{i=1}^N K_i, K_i = \psi_i^* \psi_i$. The codimension $n = N - k$ is the number of unoccupied sites, i.e. holes in the system. Again we scalar extend our phase space (8.4)) to

$$\mathcal{F}[q] = \mathbb{C}[q] \otimes_{\mathbb{C}} \mathcal{F}\,, \tag{8.5}$$

because later on we introduce quasi-periodic boundary conditions on the circle $\psi_{N+1}^* = q\psi_1^*$ and $\psi_{N+1} = q^{-1}\psi_1$.

Note that our interpretation of $\mathcal{F}[q]$ as state space of a *quantum mechanical* particle system is also valid when setting $q = 0$. In particular setting $q = 0$ does not correspond to the classical limit of our particle system (in the physical sense) but simply alters the boundary conditions. The adjective *quantum* in *quantum cohomology* is in our context simply synonymous with *q-deformed* and has nothing to do with its physical meaning.

8.3. **Discrete symmetries: parity and time reversal, particle-hole duality, shift.** We now introduce a number of mappings which arise naturally in the context of the physical particle picture. Later on we relate them to symmetries of Gromov-Witten invariants.

Parity Reversal/Poincare duality: We introduce the parity reversal operator $\mathcal{P}$ (an involution) which reverses the order of letters in a word by setting,

$$\mathcal{P}w = w^\vee, \qquad w^\vee := w_N w_{N-1} \cdots w_1\ .$$

This operator corresponds to Poincaré duality, since $w(\lambda^\vee) = w(\lambda)^\vee$. The transformation property of the fermion operators under parity reversal is (with the notation $e^{\iota\pi K}(?) = (-1)^{K(?)}$)

$$\mathcal{P}\psi_i\mathcal{P} = e^{\iota\pi K}\psi_{N+1-i}, \qquad \mathcal{P}\psi_i^*\mathcal{P} = e^{\iota\pi K}\psi_{N+1-i}^*\ . \tag{8.6}$$

Time reversal/complex conjugation: In continuum quantum physics time reversal is related to complex conjugation of the wave function through the Schrödinger equation. In the present context of a discrete system we generalize this notion in terms of the antilinear involution

$$\mathcal{T}\sum_w \alpha_w w = \sum_w \bar{\alpha}_w w$$

with $\overline{q} = q^{-1}$. In particular,

$$\mathcal{T}\psi_i\mathcal{T} = \psi_i, \qquad \mathcal{T}\psi_i^*\mathcal{T} = \psi_i^*. \tag{8.7}$$

Particle-Hole Duality/transpose: This involution, which we shall denote by the symbol $\mathcal{C}$, is associated with the bijection $\mathfrak{P}_{\leq k,n} \to \mathfrak{P}_{\leq n,k}$ or to the swapping of the letters in each word w: 0-letters become 1-letters and vice versa. In terms of partitions we define

$$\mathcal{C}\lambda = (\lambda^\vee)^t = (\lambda^t)^\vee, \tag{8.8}$$

i.e. we take the transpose and the Poincaré dual. The transformation properties for the Clifford algebra generators are

$$\mathcal{C}\psi_i\mathcal{C} = (-1)^{i-1}\psi_i^*, \qquad \mathcal{C}\psi_i^*\mathcal{C} = (-1)^{i-1}\psi_i\ . \tag{8.9}$$

Together with parity reversal this gives the known isomorphism of rings

$$\omega : qH^\bullet(\mathrm{Gr}_{k,N}) \to qH^\bullet(\mathrm{Gr}_{n,N}), \qquad 1\otimes[\Omega_\lambda] \mapsto 1\otimes[\Omega_{\lambda^t}]\ , \tag{8.10}$$

which in the language of symmetric functions just amounts to swapping the bases of elementary and complete symmetric functions, $\omega : e_r \mapsto h_r$.

Shift operator: There is an analogue Rot of the cyclic shift operator rot for the fusion ring and defined on words as follows

$$\mathrm{Rot}(w) = w_2\cdots w_N w_1\ . \tag{8.11}$$

This operator appears already in the context of Gromov-Witten invariants in [35]. In terms of Young diagrams its action corresponds to shifting the entire diagram within the $k\times n$ bounding box by one column to the left if $w_1 = 0$ and by one row down if $w_1 = 1$. Expressed in formulae this means,

$$\mathrm{Rot}(\lambda) = \begin{cases} (\lambda_1 - 1, \dots, \lambda_k - 1), & \text{if } w_1(\lambda) = 0 \\ (n, \lambda_1, \lambda_2, \dots, \lambda_{k-1}), & \text{if } w_1(\lambda) = 1 \end{cases}\ . \tag{8.12}$$

Under the identification of partitions with particle configurations on a circle via (2.5) it simply rotates the particle positions on the N-circle, alias the Dynkin diagram, but this time of $\widehat{\mathfrak{sl}}(N)$.

9. THE AFFINE NIL-TEMPERLEY-LIEB ALGEBRA AND NONCOMMUTATIVE POLYNOMIALS

Mimicking the passing from the phase algebra to the local plactic algebra we define the following weight preserving endomorphism of $\mathcal{F}$:

$$u_i = \psi_{i+1}^*\psi_i\ , \quad \text{if } 1 \leq i \leq N-1, \qquad u_N \mapsto -qe^{\iota\pi\,\mathrm{n}_N}\psi_1^*\psi_N\ . \tag{9.1}$$

Proposition 9.1. *Let $N \geq 2$.*

(1) The subalgebra E_{fin} of $\mathrm{End}(\mathcal{F})$ generated by the u_i's $1 \leq i \leq N-1$ is isomorphic to the nil-Temperley-Lieb algebra 0-TL_N.

(2) The subalgebra E_{aff} of $\mathrm{End}(\mathcal{F}[q])$ generated by the u_i's $1 \leq i \leq N$ is isomorphic to the affine nil-Temperley-Lieb algebra 0-$\widehat{TL}_N$.

Proof. Recall (see e.g. [15], [35]) that 0-$\widehat{TL}_N$ is the unital associative algebra with generators $\{\hat{u}_1, \dots, \hat{u}_N\}$ and relations

$$\hat{u}_i^2 = \hat{u}_i\hat{u}_{i+1}\hat{u}_i = \hat{u}_{i+1}\hat{u}_i\hat{u}_{i+1} = 0, \qquad \hat{u}_i\hat{u}_j = \hat{u}_j\hat{u}_i \quad \text{if } i-j \neq \pm 1 \bmod N \tag{9.2}$$

where all indices are understood modulo N. The subalgebra generated by the $\hat{u}_i$, $i > 0$ is the ordinary nil-Temperley-Lieb algebra 0-TL_N. Using the Clifford relations it is straight-forward to check that the u_i's satisfy the nil-Temperley-Lieb relations, hence $\hat{u}_i \mapsto u_i$ defines surjective algebra morphisms 0-$TL_N \twoheadrightarrow E_{\mathrm{fin}}$ and

$0\text{-}\widehat{TL}_N \twoheadrightarrow E_{\mathrm{aff}}$. By [4, Proposition 2.4.1], $\mathcal{F}$ is a faithful representation of $0\text{-}TL_N$ from which one easily deduces that $\mathcal{F}[q]$ is a faithful representation of $0\text{-}\widehat{TL}_N$. $\square$

In the following we will consider the nil-Temperley algebras as algebras of endomorphisms and will not distinguish in the notation between u_i and $\hat{u}_i$.

Remark 9.2 (Fermion hopping)**.** In terms of our particle description the operators u_i simply correspond to one fermion hopping in clockwise direction on the circle. The action of the affine nil-Temperley-Lieb algebra leaves the k-particle space invariant, i.e. $u_i\mathcal{F}_k[q] \subset \mathcal{F}_k[q]$. In terms of Young diagrams u_i adds a box in the $(i-k)^{\text{th}}$-diagonal for $i = 1, \ldots, N-1$. (The p^{th}-diagonal of a Young diagram is formed by all squares (i,j) with content $p = j - i$). The action of the affine generator u_N corresponds to removing (adding) a rim hook of size $N-1$. If either of these actions is not allowed the image is 0; see [35] for details.

Lemma 9.3. *We have the following transformation properties of the affine nil-Temperley-Lieb generators, $1 \le i \le N-1$, where $u_i^* = \psi_i^*\psi_{i+1}$ denotes the right adjoint of u_i:*

$$\begin{array}{lllllll} \mathcal{P}u_N\mathcal{P} & = & -qe^{\iota\pi\,\mathrm{n}_N}\psi_N^*\psi_1, & \qquad & \mathcal{P}u_i\mathcal{P} & = & u_{N-i}^* \\ \mathcal{T}u_N\mathcal{T} & = & -q^{-1}e^{\iota\pi\,\mathrm{n}_N}\psi_1^*\psi_N, & & \mathcal{T}u_i\mathcal{T} & = & u_i, \\ \mathcal{C}u_N\mathcal{C} & = & -qe^{\iota\pi\,\mathrm{n}_N}\psi_N^*\psi_1, & & \mathcal{C}u_i\mathcal{C} & = & u_i^*. \end{array} \tag{9.3}$$

Analogously to Section 5.3, we again consider noncommutative symmetric polynomials:

Definition 9.4. *For $r = 1, 2, \ldots, N-1$ let*

$$e_r(\mathcal{U}) = \sum_{|I|=r} \prod_{i\in I}^{\circlearrowright} u_i, \quad h_r(\mathcal{U}) = \sum_{|I|=r} \prod_{i\in I}^{\circlearrowleft} u_i. \tag{9.4}$$

We also set $e_N(\mathcal{U}) = -qe^{\iota\pi\,\mathrm{n}_N}$ and $h_N(\mathcal{U})_{|\mathcal{F}_N[q]} = q$. Note that we have $h_N(\mathcal{U})_{|\mathcal{F}_k[q]} = 0$ for $k < N$ and $h_r(\mathcal{U}) = 0$ for $r > N$. For $\lambda \in \mathfrak{P}_{\le N,N}$ define the noncommutative Schur polynomial

$$s_\lambda(\mathcal{U}) \overset{(1)}{=} \det(e_{\lambda_i^t - i + j}(\mathcal{U}))_{1\le i,j\le N} \overset{(2)}{=} \det(h_{\lambda_i - i + j}(\mathcal{U}))_{1\le i,j\le N}\ . \tag{9.5}$$

Remark 9.5. The definition of $s_\lambda(\mathcal{U})$ only makes sense after we have proved that the $e_r(\mathcal{U})$ resp. $h_r(\mathcal{U})$ pairwise commute (see Corollary 10.7 below). We also have to show that the two definitions from (9.5) agree (Theorem 10.6). However, we would like to accept the definitions and draw a few consequences first. We also want to point out that both, the definitions (9.4) as well as Corollary 10.7 are due to Postnikov [35]. Our approach provides a new point of view of these results.

Lemma 9.6. *We have the following transformation properties*

$$\mathcal{PC}e_r(\mathcal{U}) = h_r(\mathcal{U})\mathcal{PC}, \qquad \mathcal{PC}s_\lambda(\mathcal{U}) = s_{\lambda^t}(\mathcal{U})\mathcal{PC} \tag{9.6}$$

Proof. Using (9.3) and $\mathcal{PC}u_N\mathcal{PC} = u_N$ one verifies that each clockwise cyclically ordered monomial is transformed into an anti-clockwise cyclically ordered one and vice versa, $\mathcal{PC}\prod_{i\in I}^{\circlearrowright} u_i\mathcal{PC} = \prod_{i\in I}^{\circlearrowleft} u_i$. The first assertion follows then from (9.4), the second from the first with (9.5) $\square$

10. Construction of a common eigenbasis

We now construct a common eigenbasis for the polynomials $e_r(\mathcal{U}), h_r(\mathcal{U})$, then define the algebra $\mathcal{U}^{\rm sym}$ of noncommutative symmetric functions and establish in this way Postnikov's result that the noncommutative Schur polynomial $s_\lambda(\mathcal{U})$ are well-defined. Our construction and discussion of the eigenbasis will naturally lead to the Bertram-Vafa-Intrilligator formula for Gromov-Witten invariants allowing us to make contact with the results in [37] and also set it in parallel to the Verlinde formula (6.43).

Analogous to the definition of the Bethe vectors in the context of the fusion ring we now introduce (free) k-Fermion states $\tilde{b}(x)$ through the following identity which resembles equation (6.1).

Lemma 10.1. *Let $y=(y_1,\ldots,y_k)\in\left(\mathbb{C}[q^{\pm\frac{1}{N}}]\right)^k$ and set $\hat{\psi}^*(y_i)=\sum_{p=1}^N y_i^{-p}\psi_p^*$. Then we have the identity*

$$\tilde{b}(y):=\eta(y)\;\hat{\psi}^*(y_1)\cdots\hat{\psi}^*(y_k)\emptyset_0=\sum_{\lambda\in\mathfrak{P}_{\leq k,n}}s_\lambda(y_1^{-1},\ldots,y_k^{-1})\;\lambda, \tag{10.1}$$

where the normalisation factor $\eta(y)$ is given by

$$\eta(y)=\frac{(-1)^{\frac{k(k-1)}{2}}e_k(y)}{\prod_{i<j}(y_i^{-1}-y_j^{-1})}\,.$$

Proof. Inserting the definition of the creation operators $\hat{\psi}^*(y_i)$ one derives the expansion

$$\begin{aligned}\tilde{b}(y)&=\eta(y)\,\hat{\psi}^*(y_1)\cdots\hat{\psi}^*(y_k)\emptyset_0\\&=\eta(y)\sum_{\pi\in S_k}(-1)^{\ell(\pi)}\sum_{\ell_1<\ldots<\ell_k}y_{\pi(1)}^{-\ell_1}\cdots y_{\pi(k)}^{-\ell_k}\psi_{\ell_1}^*\cdots\psi_{\ell_k}^*\emptyset_0\\&=\sum_{\pi\in S_k}(-1)^{\ell(\pi)+\frac{k(k-1)}{2}}\sum_{\ell_1<\ldots<\ell_k}\frac{y_{\pi(1)}^{-\ell_1+1}\cdots y_{\pi(k)}^{-\ell_k+1}}{\prod_{i<j}(y_i^{-1}-y_j^{-1})}\;\psi_{\ell_1}^*\cdots\psi_{\ell_k}^*\emptyset_0\;.\end{aligned} \tag{10.2}$$

Recalling the relationship (8.1) between the particle positions and the associated partition, $\ell_{k+1-j}-1=k+\lambda_j-j$ the statement follows after applying the permutation $w_0=(k\ldots 21)$ from the following determinant formula for Schur functions (see e.g. [32]),

$$s_\lambda(y^{-1})=\frac{\det(y_i^{-k-\lambda_j+j})_{1\leq i,j\leq k}}{\prod_{i<j}(y_i^{-1}-y_j^{-1})}\;.$$

□

In the context of the fusion ring we used the relations of the quantum Yang-Baxter algebra to derive the Bethe Ansatz equations; see Proposition 6.1 and the sketched proof. For the quantum cohomology ring we will employ the Clifford algebra relations from Proposition 8.4 (1) and (10.2) to derive a set of conditions on $y=(y_1,\ldots,y_k)$ such that the vector $\tilde{b}(y)$ is an eigenvector of the noncommutative symmetric functions.

Lemma 10.2. *Provided $y=(y_1,\ldots,y_k)$ is a solution to the equations*

$$y_1^N=\cdots=y_k^N=(-1)^{k-1}q \tag{10.3}$$

the associated vector $\tilde{b}(y)$ from (10.1) *is a common eigenvector of the noncommutative elementary and complete symmetric polynomials* (9.4). *In particular, the eigenvalues are of the following form,*

$$e_r(\mathcal{U})\tilde{b}(y) = e_r(y)\tilde{b}(y) \qquad \text{and} \qquad h_r(\mathcal{U})\tilde{b}(y) = h_r(y)\tilde{b}(y)\,. \tag{10.4}$$

Proof. To prove our claim, we first compute the action of $e_r(\mathcal{U})$ on $\tilde{b}(y)$ and then show that the result matches with multiplying $\tilde{b}(y)$ by $e_r(y)$ if we impose the conditions (10.3). We employ once more the expansion (10.2) of the vector $\tilde{b}(y)$. Define $v_{\ell_1,\ldots,\ell_k} = \psi^*_{\ell_1}\cdots\psi^*_{\ell_k}\emptyset_0$ with $v_{\ell_1,\ldots,\ell_k} = 0$ if and only if $\ell_i = \ell_j$ for $i \neq j$ because of the Clifford algebra relations. From equation (9.1) it follows that

$$u_N v_{\ell_1,\ldots,\ell_{k-1},N} = (-1)^{k+1}\psi^*_1\psi_N\; v_{\ell_1,\ldots,\ell_{k-1},N} = \\ -\,q\psi^*_1\psi^*_{\ell_1}\cdots\psi^*_{\ell_{k-1}}\psi_N\psi^*_N\emptyset_0 = -qv_{1,\ell_1,\ldots,\ell_{k-1}},$$

whence we have the quasi-periodic boundary conditions

$$v_{\ell_1,\ldots,\ell_{k-1},N+1} = -qv_{1,\ell_1,\ldots,\ell_{k-1}}.$$

For convenience, we define in addition $v_{0,\ell_2,\ldots,\ell_k} := -q^{-1}v_{\ell_2,\ldots,\ell_k,N}$. The computation is further simplified by introducing the idempotent $\mathrm{A}_k = \sum_{\pi\in S_k}(-1)^{\ell(\pi)}\,\pi$ which projects onto the totally antisymmetric subspace; it allows us to rewrite (10.2) as follows,

$$\tilde{b}(y) = \eta(y)\,\mathrm{A}_k \sum_{0\leq\ell_1<\cdots<\ell_k\leq N} y_1^{-\ell_1}\cdots y_k^{-\ell_k}v_{\ell_1,\ldots,\ell_k}\,.$$

Without loss of generality we may restrict to $r \leq k$, since $e_r(\mathcal{U})_k = 0$ for $r > k$. Via induction one then shows that for $i_1 < \cdots < i_r$ (in the cyclic sense) one has

$$u_{i_1}\cdots u_{i_r}v_{\ell_1,\ldots,\ell_k} = \sum_{1\leq j_1<\cdots<j_r\leq k}\delta_{i_1\ell_{j_1}}\cdots\delta_{i_r\ell_{j_r}}v_{\ell_1,\ldots,\ell_{j_1}+1,\ldots,\ell_{j_r}+1,\ldots,\ell_k},$$

Hence, we arrive at the following expression

$$e_r(\mathcal{U})\hat{\psi}^*(y_1)\cdots\hat{\psi}^*(y_k)\emptyset_0 = \tag{10.5}$$
$$\mathrm{A}_k\sum_{0\leq\ell_1<\cdots<\ell_k\leq N} y_1^{-\ell_1}\cdots y_k^{-\ell_k}\sum^{\circlearrowleft}_{i_1<\cdots<i_r}u_{i_1}\cdots u_{i_r}v_{\ell_1,\ldots,\ell_k} = \\ \mathrm{A}_k\sum_{0\leq\ell_1<\cdots<\ell_k\leq N} y_1^{-\ell_1}\cdots y_k^{-\ell_k}\sum_{1\leq j_1<\cdots<j_r\leq k}v_{\ell_1,\ldots,\ell_{j_1}+1,\ldots,\ell_{j_r}+1,\ldots,\ell_k}\;.$$

In comparison, multiplication of $\tilde{b}(y)$ by $e_r(y)$ yields the formula

$$e_r(y)\hat{\psi}^*(y_1)\cdots\hat{\psi}^*(y_k)\emptyset_0 = \tag{10.6}$$
$$\left(\sum_{1\leq j_1<\cdots<j_r\leq k}y_{j_1}\cdots y_{j_r}\right)\left(\mathrm{A}_k\sum_{0\leq\ell_1<\cdots<\ell_k\leq N}y_1^{-\ell_1}\cdots y_k^{-\ell_k}v_{\ell_1,\ldots,\ell_k}\right) = \\ \mathrm{A}_k\sum_{\substack{0\leq\ell_1<\cdots<\ell_k\leq N\\ 1\leq j_1<\cdots<j_r\leq k}}y_1^{-\ell_1}\cdots y^{-\ell_{j_1}+1}\cdots y^{-\ell_{j_r}+1}\cdots y_k^{-\ell_k}v_{\ell_1,\ldots,\ell_k}\;.$$

Notice that the summands for which $\ell_{j_a}+1 = \ell_{j_a+1}$ vanish under multiplication with y_{j_a} due to the antisymmetrizer. This ensures compatibility with the Clifford algebra relations which imply $v_{\ell_1,\dots,\ell_k} = 0$ if two indices are equal. Comparing coefficients in both expressions, (10.5) and (10.6), we are led to the equations, $y_i^N = (-1)^{k-1}q$.

The proof for $h_r(\mathcal{U})$ is slightly more complicated, since now shifts in the particle positions larger than one can occur, but follows along similar lines. We therefore omit it. $\square$

Remark 10.3. The vectors (10.1) are well-known in the physics literature: via a *Jordan-Wigner transformation* they diagonalize the Hamiltonian $H_{XX} = e_1(\mathcal{U}) + e_1(\mathcal{U})^*$ of the so-called *XX spin-chain*, mapping it to a free fermion system. Our computation shows that the eigenvectors of H_{XX} are also eigenvectors of the algebra of noncommutative functions $\mathcal{U}^{\mathrm{sym}}$.

We use analogous conventions and notation as in Theorem 6.4 and denote by $\widehat{\mathrm{Sol}(n,k)}_{\mathrm{fermi}}$ the solutions $y \in \left(\mathbb{C}[q^{\pm\frac{1}{N}}]\right)^k$ of (10.3) up to permutation and giving rise to non-trivial $\tilde{b}(y)$. In particular, we assume the y_i's to be pairwise distinct as (10.1) shows that the eigenvector vanishes otherwise.

Proposition 10.4. *Let $\zeta = \exp\frac{2\pi\iota}{k+n}$. Then there is a bijection*

$$\begin{aligned} \mathfrak{P}_{\le n,k} &\cong \widehat{\mathrm{Sol}(n,k)}_{\mathrm{fermi}} \\ \sigma &\mapsto y_\sigma := q^{\frac{1}{N}}\left(\zeta^{I_1},\dots,\zeta^{I_k}\right), \qquad (10.7)\\ \textit{where } I = I(\sigma^t) &:= \left(\tfrac{k+1}{2}+\sigma_k^t-k,\dots,\tfrac{k+1}{2}+\sigma_1^t-1\right). \qquad (10.8)\end{aligned}$$

Proof. In contrast to the the Bethe Ansatz equations (6.2) we discussed in the context of the fusion ring, the equations (10.3) are decoupled and we simply have to take the N^{th} root in each of the k equations. The q dependence is obvious. Thus, for each single equation there are N solutions. Since we are only interested in solutions $y = (y_1,\dots,y_k)$ where the components are pairwise distinct and identify solutions up to permutation, we are left with $\binom{N}{k} = |\mathfrak{P}_{\le n,k}| = |\mathfrak{P}_{\le k,n}| = \dim\mathcal{F}_k$ possible solutions to (10.3). Proceeding similar to the case of the fusion ring we define a bijection, (10.8), but this time from $\mathfrak{P}_{\le n,k}$ into the set of integers

$$\mathcal{I}_{n,k} := \left\{(I_1,...,I_k) \mid -\tfrac{k-1}{2} \le I_1 < \cdots < I_k \le n+\tfrac{k-1}{2}, I_j \in \tfrac{k+1}{2}+\mathbb{Z}, \forall j\right\}.$$

Taking the N^{th} power of each component in (10.7) one verifies that these indeed solve (10.3) giving rise to $\binom{N}{k}$ distinct solutions. $\square$

Remark 10.5. The solutions to (10.3) are discussed in [37] in the context of the Bertram-Vafa-Intrilligator formula. Employing the well-known recursion relation $h_r(y_1,\dots,y_k) = h_r(y_1,\dots,y_{k-1}) + y_k h_{r-1}(y_1,\dots,y_k)$ one can show (compare with Lemma 6.3) that the system of equations (10.3) is equivalent to

$$h_{N-k+1}(y) = \cdots = h_{N-1}(y) = h_N(y) + (-1)^k q = 0. \qquad (10.9)$$

This provides a new perspective: the set of equations (10.3) coincide with the Bethe Ansatz equations of free fermions on a circular lattice with quasi-periodic boundary conditions. The integers I correspond to the momenta p_r of the individual particles via $p_r = 2\pi I_r/N$ and the total momentum $P = \sum_{r=1}^k p_r$ is simply given by

$$P=\frac{2\pi}{N}\sum_{r=1}^{k}I_r(\sigma)=\frac{2\pi|\sigma|}{N}. \tag{10.10}$$

Theorem 10.6. *(1) The Bethe vectors $\tilde{b}(y_\sigma)$, $\sigma\in\mathfrak{P}_{\leq k,n}$, form a complete set of pairwise orthogonal eigenvectors for the action of the cyclic noncommutative symmetric functions on $\mathcal{F}_k[q]$. The eigenvalues are given by the following formulae*

$$e_r(\mathcal{U})\tilde{b}(y_\sigma) \quad = \quad \begin{cases} e_r(y_\sigma)\tilde{b}(y_\sigma) & \text{if } r\neq N,\\ 0 & \text{if } r>k.\end{cases} \tag{10.11}$$

$$h_r(\mathcal{U})\tilde{b}(y_\sigma) \quad = \quad \begin{cases} h_r(y_\sigma)\tilde{b}(y_\sigma) & \text{if } r\neq N,\\ -qe^{\iota\pi k}\tilde{b}(y_\sigma) & \text{if } r=N.\end{cases} \tag{10.12}$$

(2) The norm of the Bethe vectors is given by the following formula

$$\langle\tilde{b}_\sigma,\tilde{b}_\sigma\rangle \quad = \quad \frac{n(n+k)^n}{|\operatorname{Van}_\sigma|^2}, \tag{10.13}$$

where $\operatorname{Van}_\sigma$ denotes the Vandermonde determinant $\prod_{1\leq i<j\leq n}(\zeta^{I_i(\sigma)}-\zeta^{I_j(\sigma)})$.

Proof. The eigenvalue expressions for $e_r(\mathcal{U}),\ h_r(\mathcal{U})$ follow from Lemma 10.2. The special cases $e_r(\mathcal{U}),\ r>k$ and $h_N(\mathcal{U})$ are immediate from the fact that there are only k-particles in the system and (10.9), respectively.

Orthogonality and the norm of the Bethe vectors follows from (10.1) and the summation identity for Schur functions in [37]; see Proposition 4.3 (3) therein. □

We arrive at the following statement, compare with [35] where the commutativity of $\{e_r(\mathcal{U}),h_s(\mathcal{U})\}$ was shown by different methods.

Corollary 10.7. *The elements in the set $\{e_r(\mathcal{U}),h_s(\mathcal{U})\}$ pairwise commute. Thus, the noncommutative Schur polynomials $s_\lambda(\mathcal{U})$ are well defined and one has the eigenvalue equation*

$$s_\lambda(\mathcal{U})\tilde{b}(y_\sigma)=s_\lambda(y_\sigma)\tilde{b}(y_\sigma)=q^{\frac{|\lambda|}{N}}s_\lambda(\zeta^{I(\sigma)})\tilde{b}(y_\sigma). \tag{10.14}$$

Hence, both determinant formulae in (9.5) *for $s_\lambda(\mathcal{U})$ are valid.*

Proof. Theorem 10.6 and Proposition 10.4 imply that if we formally set $q\cdot\bar{q}=q\cdot q^{-1}=1$ then the non-zero vectors (10.1) form a complete set of pairwise orthogonal eigenvectors, hence an eigenbasis of the noncommutative polynomials $e_r(\mathcal{U}),h_s(\mathcal{U})$. The eigenvalue equation for the noncommutative Schur polynomial is then immediate from the definition (1) in (9.5). Moreover, the equality (2) in (9.5) is then deduced from the corresponding equality in the commutative case. □

Theorem 10.8 (Combinatorial quantum cohomology ring)**.** *Fix $k\in\mathbb{Z}_{\geq 0}$ and consider the k-particle subspace $\mathcal{F}_k[q]\subset\mathcal{F}[q]$. The assignment*

$$(\lambda,\mu)\mapsto\lambda\star\mu:=s_\lambda(\mathcal{U})\mu \tag{10.15}$$

for basis elements $\lambda,\mu\in\mathfrak{P}_{\leq k,n}$ turns $\mathcal{F}_k[q]$ into a commutative, associative and unital $\mathbb{C}[q]$-algebra $H^\bullet_{\text{comb}}$.

Proof. The proof is completely analogous, with the obvious replacements, to the one of Theorem 6.12. □

Define the structure constants $C^{\nu,d}_{\lambda,\mu} \in \mathbb{Z}$ of $H^{\bullet}_{\text{comb}}$ through

$$\lambda \star \mu = \sum_{\nu \in \mathfrak{P}_{\leq k,n}, d \in \mathbb{Z}} C^{\nu,d}_{\lambda,\mu} q^d \nu.$$

Corollary 10.9. *The following formula describes the structure constants $C^{\nu,d}_{\lambda,\mu}$ for $\lambda, \mu, \nu \in \mathfrak{P}_{\leq k,n}$:*

$$C^{\nu,d}_{\lambda,\mu} q^d = \frac{q^{\frac{|\lambda|+|\mu|-|\nu|}{N}}}{N^k} \sum_{\sigma \in \mathfrak{P}_{\leq k,n}} \frac{s_\lambda(\zeta^{I(\sigma)}) s_\mu(\zeta^{I(\sigma)}) s_{\nu^\vee}(\zeta^{-I(\sigma)})}{\zeta^{k|\sigma|} \prod_{i<j}^{k} |\zeta^{I_i(\sigma)} - \zeta^{I_j(\sigma)}|^{-2}} \tag{10.16}$$

In particular, $C^{\nu,d}_{\lambda,\mu} = 0$ unless $|\lambda| + |\mu| - |\nu| = dN$ and $d \geq 0$.

Proof. From (9.1) and (10.15) we can conclude that d is non-negative as the non-commutative Schur polynomial $s_\lambda(\mathcal{U})$ is polynomial in the generators of the affine nil-Temperley-Lieb algebra; see (9.5). According to (10.15) the coefficients in the product expansion $\lambda \star \mu$ are given by the matrix elements of $s_\lambda(\mathcal{U})$ which we compute in the eigenbasis (10.1), completely analogous to the derivation of (6.36). $\square$

We illustrate the last formula by an example:

Example 10.10. Consider the case $k = 4$ and $n = 3$, hence $N = 7$. Set $\lambda = (3, 3, 2, 1)$ and $\mu = (2, 2, 1, 0)$. Then we have $|\lambda| + |\mu| = 9 + 5 = 14$ and therefore $C^{\nu,d}_{\lambda,\mu} = 0$ unless $7d + |\nu| = 14$. Therefore $d \in \{0, 1, 2\}$. The case $d = 0$ gives a partition ν with 14 boxes which is impossible, $d = 2$ implies $\nu = \emptyset$ and $d = 1$ implies that ν has 7 boxes, hence we get a formula of the form

$$\ydiagram{3,3,2,1} \star \ydiagram{2,2,1} = a_1 q \ydiagram{2,2,2,1} + a_2 q \ydiagram{3,2,1,1} + a_3 q \ydiagram{3,2,2} + a_4 q \ydiagram{3,3,1} + a_5 q^2 \emptyset \ .$$

for some $a_i \in \mathbb{Z}$. We used a computer to evaluate the formula (10.9) and obtained $a_i = 1$ except for $a_2 = 2$.

The following reproves a result of Postnikov [35], and at the same time connects his approach with the results of Rietsch [37]:

Theorem 10.11. *The map $s_\lambda(\mathcal{U}) \mapsto s_\lambda$ defines an isomorphism of rings $H^{\bullet}_{\text{comb}} \cong qH^{\bullet}(\text{Gr}_{k,N})$. In particular,*

$$H^{\bullet}_{\text{comb}} \cong \mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathbb{Z}[e_1, e_2, \ldots e_k] / \left\langle h_{n+1}, \ldots, h_{n+k-1}, h_{n+k} + (-1)^k q \right\rangle \tag{10.17}$$

and the Gromov-Witten invariants are given by the matrix elements of the noncommutative Schur function,

$$q^d C^{\nu,d}_{\lambda,\mu} = \langle \nu, s_\lambda(\mathcal{U}) \mu \rangle \ . \tag{10.18}$$

Proof. The expansion (10.16) is the celebrated Bertam-Vafa-Intrilligator formula for Gromow-Witten invariants. In the stated form it has appeared in [37]. The isomorphism (10.17) follows then from Remark 10.5 by the same arguments as explained in detail in the proof of Theorem 6.20. The remaining statements are then direct consequences from the presentation (8.3) of the quantum cohomology ring. $\square$

Remark 10.12. We want to point out here that the definitions and basic properties of the combinatorial ring $H^\bullet_{\mathrm{comb}}$ only rely on (10.15). From there one derives in an elementary way the associativity, the Bertram-Vafa-Intrilligator formula (10.16) and the presentation (10.17). All of these are non-trivial results in the quantum cohomology of the Grassmannian.

11. Symmetries, Recursion Formulae and an Algorithm for Gromov-Witten Invariants

The combinatorial description of the quantum cohomology ring can be exploited to derive in a very simple manner non-trivial identities, referred to as 'symmetries', between Gromov-Witten invariants. The main idea is to employ the combinatorial description (10.15) of the product and then exploit certain invariance or transformation properties of the noncommutative Schur polynomials.

To explain our approach in more detail, assume we have a linear endomorphism F of $\mathcal{F}[q]$. For notational convenience we shall abbreviate throughout this section $S_\lambda = s_\lambda(\mathcal{U})$. In general we cannot expect the endomorphism F to be compatible with the product structure in the sense that $S_\lambda \circ F = F \circ S_\lambda$. However, we might be able to choose F such that $S_\lambda \circ F = \sum_\mu F_\mu \circ S'_\mu$ for certain (well-understood) endomorphisms F_μ and $S'_\mu = S_\mu$ at least after replacing q by $q' = -q$ or q^{-1}. Applying this equality to a basis vector in $\mathcal{F}[q]$ we obtain identities of Gromov-Witten invariants by comparing coefficients in the respective product expansions,

$$\sum_\nu q^d C^{\nu,d}_{\lambda,F(\sigma)}\nu = S_\lambda(F(\sigma)) = \sum_\mu F_\mu(S'_\mu(\sigma)) = \sum_\nu\sum_\mu (q')^d C^{\nu,d}_{\mu,\sigma} F_\mu(\nu)\,.$$

In case that the endomorphisms F, F_μ preserve the dimension k and $q' = q$, we obtain relations between structure constants of the same quantum cohomology ring. Our standard examples are the discrete symmetry operators from Section 8.3. In this way we deduce in particular (mostly well-known) symmetry properties of Gromov-Witten invariants, albeit providing simpler and shorter proofs (Proposition 11.1 below). By choosing for F particle generation and annihilation operators ψ_i and ψ_i^* we obtain, however, more interesting (new) results, namely formulae relating Gromov-Witten invariants from *different* quantum cohomology rings. In particular, one obtains an explicit recursion formula and an inductive algorithm to compute all the structure constants.

We start with the following (apart from (3) probably) well-known symmetry formulae:

Proposition 11.1. *Let* $C_{\lambda,\mu,\nu}(q) = q^d C^{\nu^\vee,d}_{\lambda,\mu}$ *with* $d = \frac{|\lambda|+|\mu|+|\nu|-kn}{N}$ *, then we have the following identities:*

1. S_3 symmetry: *for any* $p \in S_3$ *we have* $C_{\lambda,\mu\,\nu}(q) = C_{p(\lambda),p(\mu),p(\nu)}(q)\,.$
2. Level-rank duality: $C_{\lambda,\mu,\nu}(q) = C_{\lambda^t,\mu^t,\nu^t}(q)$.
3. Rotation invariance: *let* $\mathrm{R} = \mathrm{Rot}$ *be as in* (8.12) *and extend the definition of* n_i *from Section 8.1 to any* $i \in \mathbb{Z}$ *by setting* $\mathrm{n}_{i+N} = \mathrm{n}_i + k$*. Then*
$$C_{\mathrm{R}^a(\lambda),\mu,\nu}(q) = q^{\mathrm{n}_a(\mu)-\mathrm{n}_a(\lambda)} C_{\lambda,\mathrm{R}^a(\mu),\nu}(q) = q^{\mathrm{n}_a(\nu)-\mathrm{n}_a(\lambda)} C_{\lambda,\mu,\mathrm{R}^a(\nu)}(q),$$
and in particular for $a+b+c=0$ *we have*
$$C_{\mathrm{R}^a(\lambda),\mathrm{R}^b(\mu),\mathrm{R}^c(\nu)}(q) = q^{\mathrm{n}_a(\lambda)+\mathrm{n}_b(\mu)+\mathrm{n}_c(\nu)} C_{\lambda,\mu,\nu}(1)\,.$$

(4) Postnikov's 'curious duality':

$$C_{\mathrm{R}^a(\lambda),\mathrm{R}^b(\mu),\mathrm{R}^c(\nu)}(q)=q^{\mathrm{n}_a(\lambda)+\mathrm{n}_b(\mu)+\mathrm{n}_c(\nu)}C_{\lambda^\vee,\mu^\vee,\nu^\vee}(q^{-1})$$

with $a+b+c=N-k$.

Proof. The first identity is obvious from the Bertram-Vafa-Intrilligator formula (10.16). The second equality is easily obtained by employing parity-reversal and particle-hole duality,

$$C_{\lambda,\mu,\nu}(q)=\langle\nu^\vee,S_\lambda\mu\rangle=\langle\left(\nu^t\right)^\vee,\mathcal{PC}S_\lambda\mathcal{PC}\mu\rangle=\langle\left(\nu^t\right)^\vee,S_{\lambda^t}\mu^t\rangle=C_{\lambda^t,\mu^t,\nu^t}(q)\ .$$

To prove statement (3) we first specialise to $q=1$, denoting the corresponding noncommutative Schur polynomial by $\tilde{S}_\lambda$. Under this condition the noncommutative elementary and symmetric polynomials in (9.4) become invariant under a cyclic permutation of the letters u_i. Together with (9.5) this implies

$$\mathrm{R}\circ\tilde{S}_\lambda=\tilde{S}_\lambda\circ\mathrm{R}\ . \tag{11.1}$$

Hence, the following equalities hold true

$$\langle\nu^\vee,\mathrm{R}\circ\tilde{S}_\lambda(\mu)\rangle=\langle\nu^\vee,\tilde{S}_\lambda(\mathrm{R}\,\mu)\rangle=\langle\nu^\vee,\mathrm{R}\circ\tilde{S}_\lambda(\mu)\rangle=$$
$$\langle\nu^\vee,\mathrm{R}\circ\tilde{S}_\mu(\lambda)\rangle=\langle\nu^\vee,\tilde{S}_\mu(\mathrm{R}\,\lambda)\rangle=\langle\nu^\vee,\mathrm{R}\circ\tilde{S}_\mu(\lambda)\rangle$$

Using the S_3-symmetry from (1) statement (3) follows for $q=1$. Because of Corollary 10.9 it suffices to compute the difference in the degrees of the respective Gromov-Witten invariants in order to treat the case of generic q. The respective degrees in (3) can be deduced with help of the general formula

$$|\,\mathrm{R}^a(\mu)|=|\mu|+N\,\mathrm{n}_a(\mu)-ak\ , \tag{11.2}$$

which is easily verified using the action of R on Young diagrams as described in (8.12). For instance, the degrees for the last equality in (3) follow from the computation

$$|\,\mathrm{R}^a(\lambda)|+|\,\mathrm{R}^b(\mu)|+|\,\mathrm{R}^c(\nu)|=$$
$$|\lambda|+|\mu|+|\nu|+N(\mathrm{n}_a(\lambda)+\mathrm{n}_b(\mu)+\mathrm{n}_c(\nu))+(a+b+c)k$$

and, hence, we arrive at the asserted identity under the stated assumption that $a+b+c=0$.

To show the last identity (4) we again specialise to $q=1$. Employing the parity and time reversal operator of Section 8.3 one shows that the eigenvector (10.1) satisfies

$$\mathcal{PT}\tilde{b}(y)=\sum_{\mu\in\mathfrak{P}_{\le k,n}}s_\mu(y)\mu^\vee=e_k(y)^n\sum_{\mu\in\mathfrak{P}_{\le k,n}}s_{\mu^\vee}(y^{-1})\mu^\vee=e_k(y)^n\tilde{b}(y),$$

see Lemma 6.5 (3). Applying Theorem 10.6, Corollary 10.7 and once more Lemma 6.5 (3) one obtains the identity $\mathcal{PT}\tilde{S}_\lambda\mathcal{PT}=\mathrm{R}^n\circ\tilde{S}_{\lambda^\vee}$. Thus, we calculate

$$C_{\lambda,\mu,\nu}=\langle\nu^\vee,\tilde{S}_\lambda\mu\rangle=\langle\nu,\mathcal{PT}\tilde{S}_\lambda\mathcal{PT}\mu^\vee\rangle=\langle\nu,\mathrm{R}^n\,\tilde{S}_{\lambda^\vee}\mu^\vee\rangle=$$
$$\langle\mathrm{R}^{-c}(\nu),\tilde{S}_{\mathrm{R}^a(\lambda^\vee)}\,\mathrm{R}^{-b}(\mu)^\vee\rangle=C_{\mathrm{R}^{-a}(\lambda)^\vee,\mathrm{R}^{-b}(\mu)^\vee,\mathrm{R}^{-c}(\nu)^\vee}$$

for any a, b, c with $a + b + c = n = N - k$. Here we have used again (11.1) in the penultimate equality. This proves (4) when $q = 1$. The general case now follows by comparing degrees on both sides of (4). We find

$$C_{\lambda^\vee,\mu^\vee,\nu^\vee}(q) = q^d C_{\lambda^\vee,\mu^\vee,\nu^\vee}(1), \qquad d = -\frac{|\lambda| + |\mu| + |\nu| - 2k(N-k)}{N}$$

and

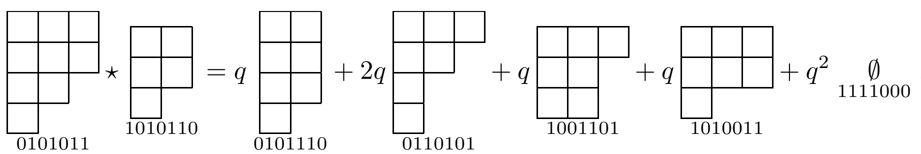
$$C_{\mathrm{R}^a(\lambda),\mathrm{R}^b(\mu),\mathrm{R}^c(\nu)}(q) = q^{d'} C_{\mathrm{R}^a(\lambda),\mathrm{R}^b(\mu),\mathrm{R}^c(\nu)}(1)$$

with

$$d' = \frac{|\lambda| + |\mu| + |\nu| - k(N-k)}{N} + \mathrm{n}_a(\lambda) + \mathrm{n}_b(\mu) + \mathrm{n}_c(\nu) - \frac{a+b+c}{N}k$$

and the proposition follows. □

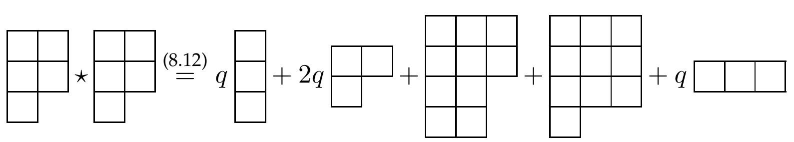
Example 11.2. Let us illustrate the second equality in the first formula of Proposition 11.1 (3). Recall from Example 10.10 the product expansion for $\lambda \star \mu$,



Exploiting that $\mathrm{Rot}(\nu^\vee) = (\mathrm{Rot}^{-1}(\nu))^\vee$, Proposition 11.1 (3) can be rewritten as

$$\begin{aligned} \mathrm{Rot}(\lambda) \star \mu &= \sum_\nu C_{\lambda,\mu,\mathrm{Rot}(\nu^\vee)}(q)\nu = \sum_\nu q^{d'} C_{\lambda\mu}^{\mathrm{Rot}^{-1}(\nu),d'} \nu \\ &= \sum_\nu q^{d-k+\mathrm{n}_1(\lambda)+\mathrm{n}_6(\mathrm{Rot}(\nu))} C_{\lambda\mu}^{\nu,d}\, \mathrm{Rot}(\nu)\ , \end{aligned}$$

where $d'(\nu) = d + k - \mathrm{n}_{N-1}(\nu) + \mathrm{n}_1(\lambda)$ according to the definition of $C_{\lambda,\mu,\nu}(q)$ in Proposition 11.1 and (11.2). Since, $\mathrm{n}_1(\lambda) = 0$ and $\mathrm{Rot}(\lambda) = \mu$ we find by applying Rot to each diagram in the product expansion



We now derive recursion formulae for Gromov-Witten invariants. As a preliminary step we need to derive the commutation relations between ψ_i^*, ψ_i and the noncommutative Schur polynomials. We start with the elementary symmetric polynomials and then, once more, make use of the Jacobi-Trudy formula. Henceforth let S'_λ and E'_r denote S_λ and E_r with q replaced by $-q$, respectively.

Lemma 11.3. *We have the following formula*

$$E_r \psi_j^* = \psi_j^* E'_r + \psi_{j+1}^* E'_{r-1},$$

where for $j = N$ we have $\psi_{N+1}^* = -qe^{i\pi K}\psi_1^*$.

Proof. We start from the relations

$$\begin{aligned} u_i \psi_l^* &= \psi_l^* u_i + \delta_{i,l}\psi_{i+1}^*, \qquad i = 1, 2, \dots, N-1 \\ u_N(q)\psi_l^* &= \psi_l^* u_N(-q) + \delta_{N,l}\, q\psi_1^* e^{i\pi K} \end{aligned}$$

which are easily verified. More generally we claim that for any $i_1 > \cdots > i_r$ (with respect to the cyclic order) the following identity is true,

$$u_{i_r}(q)\cdots u_{i_1}(q)\psi_j^* = \psi_j^* u_{i_r}(-q)\cdots u_{i_1}(-q) + \psi_{j+1}^*\sum_{s=1}^r \delta_{j,i_s} u_{i_r}(-q)\cdots \not{u}_{i_s}(-q)\cdots u_{i_1}(-q)\;,$$

where $\not{u}$ means we leave out the factor u. Assume this is true for r, then multiplication on both sides with $u_{i_{r+1}}(q)$ yields

$$u_{i_{r+1}}(q)\cdots u_{i_1}(q)\psi_j^* = \psi_j^* u_{i_{r+1}}(-q)\cdots u_{i_1}(-q) + \delta_{j,i_{r+1}}\psi_{j+1}^* u_{i_r}(-q)\cdots u_{i_1}(-q) + u_{i_{r+1}}(q)\psi_{j+1}^*\sum_{s=1}^r \delta_{j,i_s} u_{i_r}(-q)\cdots \not{u}_{i_s}(-q)\cdots u_{i_1}(-q)\;.$$

Provided that $i_{r+1} < i_r$ one has $u_{i_{r+1}}(q)\psi_{j+1}^* = \psi_{j+1}^* u_{i_{r+1}}(-q)$ which concludes the induction step and, thus, proves the claim. Noting that $\psi_{j+1}^* u_j = 0$ the assertion follows from the definition (9.4) of the cyclic noncommutative symmetric functions. □

Proposition 11.4. *The following commutation relations hold true,*

$$S_\lambda \psi_i^* = \psi_i^* S'_\lambda + \sum_{r=1}^{\lambda_1}\psi_{i+r}^* \sum_{\lambda/\mu=(r)} S'_\mu \tag{11.3}$$

$$S_\lambda \psi_i = \psi_i S'_\lambda + \sum_{r=1}^{\ell(\lambda)}(-1)^r\psi_{i-r} \sum_{\lambda/\mu=(1^r)} S'_\mu \tag{11.4}$$

where we again impose the quasi-periodic boundary conditions $\psi_{j-N} = -qe^{i\pi K}\psi_j$ *and* $\psi_{j+N}^* = -qe^{i\pi K}\psi_j^*$.

Proof. We start with the first identity and perform an induction on the length of λ with the previous lemma as starting point. Assume as induction hypothesis that (11.3) holds true for λ and let $\tilde\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots)$ be λ with an additional row added to the top. For the induction step we will exploit $S_{\tilde\lambda^t} = \det(E_{\tilde\lambda_i - i + j})$. Expanding with respect to the first column one finds $S_{\tilde\lambda^t} = \sum_{i=0}^{\ell}(-1)^i E_{\lambda_i - i} S_{(\tilde\lambda^{(i)})^t}$, where $\tilde\lambda^{(0)} = \lambda$ and for $i>0$ we set $\tilde\lambda^{(i)} = (\lambda_0+1, \lambda_1+1, \ldots, \lambda_{i-1}+1, \lambda_{i+1}, \ldots, \lambda_\ell)$. The induction hypothesis then yields

$$\begin{aligned} S_{\tilde\lambda^t}\psi_j^* &= \sum_{i=0}^{\ell}(-1)^i E_{\lambda_i - i}\sum_{r=0}^{\ell}\psi_{j+r}^* \sum_{\tilde\lambda^{(i)}/\tilde\mu=(1^r)} S'_{\tilde\mu^t} \\ &= \sum_{i,r=0}^{\ell}(-1)^i \left\{ \left(\psi_{j+r}^* E'_{\lambda_i - i} + \psi_{j+r+1}^* E'_{\lambda_i - i - 1}\right) \sum_{\tilde\lambda^{(i)}/\tilde\mu=(1^r)} S'_{\tilde\mu^t} \right\} \end{aligned}$$

We now rearrange the sum over r and find

$$S_{\tilde{\lambda}^t}\psi_j^* = \psi_j^* S'_{\tilde{\lambda}^t} + \\ + \sum_{r=1}^{\ell}\psi_{j+r}^* \sum_{i=0}^{\ell}(-1)^i \left\{ E'_{\lambda_i - i} \sum_{\tilde{\lambda}^{(i)}/\tilde{\mu}=(1^r)} S'_{\tilde{\mu}} + E'_{\lambda_i-i-1} \sum_{\tilde{\lambda}^{(i)}/\tilde{\mu}=(1^{r-1})} S'_{\tilde{\mu}^t} \right\} \\ + \sum_{i=0}^{\ell}(-1)^i \psi_{j+\ell+1}^* E'_{\lambda_i-i-1} S'_{(\lambda_0,\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1}-1,\ldots,\lambda_\ell-1)^t}$$

The two sums in the second line give the two possible contributions when one subtracts a vertical r-strip from $\tilde{\lambda}$ with $1 \leq r \leq \ell$: either a single box or no box is deleted in the i^{th} row. The last term is the removal of a vertical $(\ell+1)$-strip from $\tilde{\lambda}$. Thus, we arrive at

$$S_{\tilde{\lambda}^t}\psi_j^* = \psi_j^* S'_{\tilde{\lambda}^t} + \sum_{r=1}^{\ell}\psi_{j+r}^* \sum_{\tilde{\lambda}/\tilde{\mu}=(1^r)} S'_{\tilde{\mu}^t} + \psi_{j+\ell+1}^* S'_{(\lambda_0-1,\lambda_1-1,\ldots,\lambda_\ell-1)^t}$$

which upon taking the transpose in all partitions is the desired result.

The second formula (11.4) now follows from applying the transformations

$$\mathcal{PC}\psi_j^*\mathcal{PC} = (-1)^{j-1}e^{i\pi K}\psi_{N+1-j} \quad \text{and} \quad \mathcal{PC}S_\lambda\mathcal{PC} = S_{\lambda^t}$$

which we proved earlier; see Section 8.3 and Lemma 9.6. □

Remark 11.5 (Algorithm)**.** The above commutation relations imply the following product formulae

$$\lambda \star \psi_i^*(\mu) = S_\lambda \psi_i^*(\mu) = \sum_{r=0}^{\lambda_1} \sum_{\lambda/\nu=(r)} \psi_{i+r}^*(\nu \star' \mu), \tag{11.5}$$

$$\lambda \star \psi_i(\mu) = S_\lambda \psi_i(\mu) = \sum_{r=0}^{\lambda_1}(-1)^r \sum_{\lambda/\nu=(1^r)} \psi_{i-r}(\nu \star' \mu), \tag{11.6}$$

where $\star'$ denotes the product where we replace the deformation parameter q with $-q$. Thus, one can successively create for $k = 0, 1, \ldots, N$ the entire hierarchy $qH^\bullet(\mathrm{Gr}_{k,N})$ of quantum cohomology rings starting with either $k = 0$ or $k = N$.

Example 11.6. We illustrate the algorithm by computing the (quite trivial) example of projective space and some product in $qH^\bullet(\mathrm{Gr}_{2,5})$ starting from the quantum cohomology of a point. First of all we create the basis vectors of $qH^\bullet(\mathrm{Gr}_{1,N})$ according to the formula

$$c_j = \underset{10000}{\emptyset} \star \psi_j^*(\underset{00000}{\emptyset}) = \psi_j^*(\emptyset) = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{j-1}, \quad j = 1, \ldots, N\,.$$

Then the product of two arbitrary basis vectors is given by

$$c_i \star c_j = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{i} \star \psi_j^*(\underset{00000}{\emptyset}) = \sum_{i'=0}^{i} \psi_{i'+j}^*(h_{i-i'}(\mathcal{U}, -q)\underset{00000}{\emptyset}) = \psi_{i+j}^*(\underset{00000}{\emptyset}) \\ = q^p \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{i+j-pN},$$

where $p=0$ if $i+j<N=n+1$ and $p=1$ otherwise (by invoking the quasi-periodic boundary conditions $\psi^*_{N+r}=(-1)^{K+1}q\psi^*_r$). Now we can pass to the case $k=2$ and compute in $qH^\bullet(\mathrm{Gr}_{2,5})$ for instance

$$\underset{00101}{\ydiagram{3,2}}\star\psi^*_2\left(\underset{00010}{\ydiagram{3}}\right)=\underset{00101}{\ydiagram{3,2}}\star\underset{01010}{\ydiagram{2,1}}$$
$$=\psi^*_{2+2}\left(\ydiagram{3}\;\star'\;\ydiagram{3}\right)+\psi^*_{2+3}\left(\ydiagram{2}\;\star'\;\ydiagram{3}\right)$$
$$=-q\psi^*_{2+2}\left(\underset{01000}{\ydiagram{1}}\right)-q\psi^*_{2+3}\left(\underset{10000}{\emptyset}\right)=q\,\ydiagram{2,1}+q\,\ydiagram{3}\,.$$

Note that the cases $k=3,4,5$ and $N=5$ are obtained via level-rank duality, see Proposition 11.1 (2).

Using the identity $\psi^*_i\psi_i+\psi_i\psi^*_i=1$ one now derives the corresponding recursion formulae for the Gromov-Witten invariants.

Corollary 11.7 (Recursion formulae)**.** *Let $j=1,2,\ldots,N$. One has the following recursion formulae for Gromov-Witten invariants: if $\psi_j\mu=0$ then*

$$C^{\nu,d}_{\lambda\mu}(k,N)=\sum_{r=0}^{\ell(\lambda)}(-1)^{d+r+\mathrm{n}_{j-1}(\mu)+\mathrm{n}_{j-r}(\nu)}\sum_{\lambda/\rho=(1^r)}C^{\psi^*_{j-r}\nu,d_r}_{\rho,\psi^*_j\mu}(k+1,N)\tag{11.7}$$

else

$$C^{\nu,d}_{\lambda\mu}(k,N)=\sum_{r=0}^{\lambda_1}(-1)^{d+\mathrm{n}_{j-1}(\mu)+\mathrm{n}_{j+r-1}(\nu)}\sum_{\lambda/\rho=(r)}C^{\psi_{j+r}\nu,d'_r}_{\rho,\psi_j\mu}(k-1,N),\tag{11.8}$$

where $\mathrm{n}_{j+N}=\mathrm{n}_j+k$ and

$$d_r=\begin{cases}d-1, & j<r\\ d, & \textit{else}\end{cases}\;,\qquad d'_r=\begin{cases}d-1, & j+r>N\\ d, & \textit{else}\end{cases}\;.\tag{11.9}$$

Proof. We start from the identity

$$q^dC^{\nu,d}_{\lambda,\mu}(k,N)=\langle\nu,S_\lambda\mu\rangle=\langle\nu,S_\lambda\psi_j\psi^*_j\mu\rangle+\langle\nu,S_\lambda\psi^*_j\psi_j\mu\rangle\;.$$

Only one of the matrix elements on the right hand side of the last equation can be nonzero. Assume $\psi_j\mu=0$, then it follows from the previous proposition that

$$\begin{aligned}q^dC^{\nu,d}_{\lambda,\mu}(k,N)&=\langle\nu,S_\lambda\psi_j\psi^*_j\mu\rangle\\&=\sum_{r=0}^{\lambda_1^t}(-1)^r\sum_{\lambda^t/\rho^t=(r)}\langle\nu,\psi_{j-r}S'_\rho\psi^*_j\mu\rangle\\&=q^d\sum_{r=0}^{\ell(\lambda)}(-1)^{r+d+\mathrm{n}_{j-1}(\mu)+\mathrm{n}_{j-r-1}(\nu)}\sum_{\lambda/\rho=(1^r)}C^{\psi^*_{j-r}\nu,d_r}_{\rho,\psi^*_j\mu}(k+1,N)\;.\end{aligned}$$

Here we have used that

$$|\lambda|=\sum_{i=1}^k\ell_i-\frac{k(k+1)}{2}\qquad\text{and}\qquad|\psi^*_j\lambda|=|\lambda|+j-k-1,$$

from which one deduces that for $j-r>0$

$$|\rho|+|\psi^*_j\mu|-|\psi^*_{j-r}\nu|=|\lambda|+|\mu|-|\nu|=dN\;.$$

If $j-r<0$ it follows from the quasi-periodic boundary conditions that

$$\langle \nu, \psi_{j-r} S'_\rho \psi^*_j \mu\rangle = (-1)^{k-1} q \langle \nu, \psi_{j-r+N} S'_\rho \psi^*_j \mu\rangle$$

and the degree is decreased by one. Since we set $\mathrm{n}_{a+N} = \mathrm{n}_a + k$ the overall sign works out correctly.

The second case, $\psi^*_j \mu = 0$, leads to the second recursion formula via a similar computation. To determine the equality of the degrees one now uses the relation $|\psi_j \lambda| = |\lambda| - j + k$. □

Example 11.8. Consider once more the product expansion from Example 10.10. Choose $\nu = (3,2,1,1)$, then $C^{\nu,1}_{\lambda\mu} = 2$. Since $w(\mu) = 1010110$ and $w(\nu) = 0110101$, $\psi^*_j \mu$ is only non-vanishing for $j = 2,4,7$,

$$\psi^*_2 \mu = -(1,1,0,0,0), \quad \psi^*_4 \mu = (1,1,1,1,0), \quad \psi^*_7 \mu = (2,2,2,1,0)\;.$$

In contrast, $\psi^*_{j-r}\nu \neq 0$ only if $j-r = 1,4,6$,

$$\psi^*_6 \nu = -(2,2,2,1,1), \quad \psi^*_1 \nu = (2,1,0,0,0), \quad \psi^*_4 \nu = (2,1,1,1,1)\;,$$

whence the only allowed value for $j = 2,7$ is $r = 3$, while for $j = 4$ we have the possible values $r = 0,3$. The corresponding partitions ρ in the non-trivial skew diagrams λ/ρ are $\rho_1 = (2,2,1,1,0)$ and $\rho_2 = (2,2,2,0,0)$. We obtain the following identities between Gromov-Witten invariants, where the first equality in each line follows from (11.7), while the actual numerical values are calculated with the help of a computer using (10.16):

$$\begin{aligned}
j=2:\ C^{\nu,1}_{\lambda\mu}(4,7) &= C^{\psi^*_6\nu,0}_{\rho_1\psi^*_2\mu}(5,7) + C^{\psi^*_6\nu,0}_{\rho_2\psi^*_2\mu}(5,7) = 1+1 = 2,\\
j=4:\ C^{\nu,1}_{\lambda\mu}(4,7) &= C^{\psi^*_4\nu,1}_{\lambda\psi^*_4\mu}(5,7) + C^{\psi^*_1\nu,1}_{\rho_1\psi^*_4\mu}(5,7) + C^{\psi^*_1\nu,1}_{\rho_2\psi^*_4\mu}(5,7) = 0+1+1 = 2,\\
j=7:\ C^{\nu,1}_{\lambda\mu}(4,7) &= C^{\psi^*_4\nu,1}_{\rho_1\psi^*_7\mu}(5,7) + C^{\psi^*_4\nu,1}_{\rho_2\psi^*_7\mu}(5,7) = 1+1 = 2\;.
\end{aligned}$$

Note that for $j = 2$ we have used the quasi-periodic boundary conditions, $\psi_{-1} = q\psi_6 e^{\iota\pi K}$ to determine the overall sign factor and degree.

PART III: THE 'BOSON-FERMION-CORRESPONDENCE'

The following table summarizes the analogy of the constructions on both sides. We call it the 'Boson-Fermion-correspondence' (although a precise representation theoretic statement will appear elsewhere).

fusion ring $\mathcal{F}(\hat{\mathfrak{sl}}(n))_k$	quantum cohomology $qH^\bullet(\mathrm{Gr}_{k,k+n})$
integrable system:	
(bosonic) configuration of k particles on a circle with n sites	(fermionic) configurations of k particles on a circle with $N = n+k$ sites
creation and annihilation operators:	
phase algebra	Clifford algebra
noncommuting variables:	
Local affine plactic algebra	affine nil-Temperley-Lieb algebra
crystal structure:	
Crystal of the k^{th} symmetric power	Crystal of the k^{th} exterior power
Bethe Ansatz equations:	
$x_1^{n+k} = \cdots = x_k^{n+k} = (-1)^{k-1} z e_k(x_1, \ldots, x_k)$	$y_1^N = \cdots = y_k^N = (-1)^{k-1} q.$
Bethe roots ($\zeta = \exp \frac{2\pi i}{k+n}$)	
$x_\sigma = z^{\frac{1}{n}} \zeta^{\frac{\lvert\sigma\rvert}{n}} \left(\zeta^{I_1}, \ldots, \zeta^{I_k}\right), \sigma \in \mathfrak{P}_{\leq n-1,k}$	$y_\sigma = q^{\frac{1}{N}} \left(\zeta^{I_1}, \ldots, \zeta^{I_k}\right), \sigma \in \mathfrak{P}_{\leq k,n}$,
ring structure	
$\lambda \circledast \mu = s_\lambda(a_0, a_1, \ldots a_{n-1})\mu$	$\lambda \star \mu = s_\lambda(u_0, u_1, \ldots u_{n-1})$
Verlinde formula	Bertram-Vafa-Intrilligator formula

APPENDIX: LOCAL PLACTIC ALGEBRA AND TABLEAUX

We found the following explicit algorithm which produces a sequence of (generalised) tableaux, $\mathcal{T} \ni T \rightsquigarrow \hat{T} \rightsquigarrow T' \rightsquigarrow D(T) \in \mathcal{T}_{str}$ (using the notation from Proposition 5.2 (2)) such that the following statement is true.

Proposition 11.9. *The algorithm associates to each tableau T a tableau $D(T)$ in $\mathcal{T}_{str}$ with equivalent column word under the relations* (5.1) *and* (5.2)*.*

In particular, from $D(T)$ one can read off the corresponding basis element in B.

Definition 11.10. *Assume we are given a tableau T (or any more general diagram, where the following definition makes sense). We say a column C of T* has property (S) *if there are no two numbers $i < j$, appearing in C in two consecutive rows and satisfying $j - i > 1$. (That means the numbers in C form an increasing sequence of consecutive numbers).*

The algorithm:

Step 0. If the tableau is already in $\mathcal{T}_{str}$ then there is nothing to do.

Step 1: Creating $\hat{T}$. Read the tableau T column by column from right to left (and from top to bottom) until the first column C which has not property (S). Let i, j be the first pair of numbers destroying property (S).

Now do the following procedure *basic procedure* $P(C, j)$ (associated with the pair (C, j)): cross out the number j and consider the column, D, next to C on the right, deal with the first applicable situation from the following list, where the term 'contains' stands for 'contains an uncrossed'.

(1) D is empty. Place j in the first row of D.
(2) D contains $j-1$ and j. Do $P(D, j)$.
(3) D contains $j-1$, not j. Place j below $j-1$, (replacing any crossed out number there) and pushing down all greater numbers by one row.
(4) D contains $j+1$, not j. Then $j+1$ is necessarily on the top of D (due to the choice of i and j). Replace $j+1$ by j, and do $P(D, j+1)$.
(5) D contains $j+1$ and j. Do $P(D, j)$.
(6) D contains neither $j+1$ nor $j-1$ and is not empty.
   (a) If it contains just j then insert an extra copy of the column D to the right of D.
   (b) If all entries in D are greater than j then replace the entry, x, in the first row by j, and do $P(D, x)$.
   (c) If all entries in D are less than j do $P(D, j)$.

Since T has a finite number of rows and columns, we are done after applying finitely many basic procedures. The multi-set of numbers which are not crossed out will be the same as at the beginning, but some numbers could have been moved to the right or up. There are no numbers crossed out in the first row.

Now repeat the above rule with all (not crossed out) pairs i, j as long as possible. The whole procedure stops after finitely many steps, latest when all numbers which are not crossed out appear in the first row. We claim that this produces a tableau $\hat{T}$ with some numbers crossed out (for a proof see below). By construction, each step does not change the equivalence class of column words (under the relations (5.1) and (5.2).

Step 2: Creating T' and $D(T)$. Now remove the boxes with crossed out numbers to obtain T'. The first row is weakly increasing, the columns are connected and are increasing sequences $i(i+1)(i+2)\ldots$ of consecutive numbers. Thanks to property (S) the local plactic relations make sure that the corresponding column word is in the same equivalence class as the column word of T. Finally, the tableau $D(T)$ is obtained from T' by moving all the boxes to the left as far as possible. The specific shape of T' makes sure that $D(T)$ is in fact a tableau (since the rows are of course still weakly increasing and the columns are not necessarily sequences anymore, but definitely strictly increasing). The column word of $D(T)$ is obviously equivalent to the column word of T' via the relations (5.1). This is the end of the algorithm.

Example 11.11. We illustrate the algorithm with an example

$$T = \begin{matrix} 1&1&1&2&2&3&3&4&6&10 \\ 2&2&3&3&3&4&5&6&7 \\ 9 \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} 1&1&1&2&2&3&3&4&6&6&10 \\ 2&2&3&3&3&4&5&\cancel{6}&7 \\ 9 \end{matrix}$$

$$\begin{matrix} 1&1&1&2&2&3&3&4&6&6&10 \\ 2&2&3&3&3&4&\cancel{5}&5&7 \\ 9 \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} 1&1&1&2&2&3&3&3&4&6&6&10 \\ 2&2&\cancel{3}&3&3&4&\cancel{5}&\cancel{5}&5&7 \\ 9 \end{matrix}$$

$$\hat{T} = \begin{matrix} 1&1&1&2&2&3&3&3&4&6&6&9&10 \\ 2&2&\cancel{3}&3&3&4&\cancel{5}&\cancel{5}&5&7 \\ \cancel{9} \end{matrix}$$

In the first step we applied (5), (6b), (1), then (3), then twice (2) followed by (5) and (6a), and finally ten times (6c) followed by (4) and (1). Moreover,

$$
\begin{aligned}
T' &= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&3&4&6&6&9&10\\ \hline 2&2&&3&3&4&&&5&7&&&\\ \hline\end{array}\\
D(T) &= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&3&4&6&6&9&10\\ \hline 2&2&3&3&4&5&7&&&&&&\\ \hline\end{array}
\end{aligned}
$$

The column word for $D(T)$ is $a_2a_1a_2a_1a_3a_1a_3a_2a_4a_2a_5a_3a_7a_3a_3a_4a_6a_6a_9a_{10}$. Under the local plactic relations, (5.1) and (5.2), the corresponding element in B is then

$$a_1(a_2a_1)(a_2a_1)(a_3a_2)(a_3a_2)(a_4a_3)(a_3)(a_3)(a_5a_4)(a_6)(a_7a_6)(a_9)(a_{10}),$$

which is obviously equivalent to $D(T)$ under the Robinson-Schensted correspondence. We demonstrate the reordering for the subsequence $a_2a_1a_2a_1a_3a_1a_3a_2a_4a_2\dots$ of the column word: apply (5.1) to obtain $a_2a_1a_2a_1a_1a_3a_3a_2a_2a_4\dots$ and then repeatedly (5.2) to arrive at $a_2a_1a_1a_2a_1a_3a_2a_3a_2a_4\dots = a_1a_2a_1a_2a_1a_3a_2a_3a_2a_4\dots$ Continuing in the same manner one derives the above basis word.

Proof that $\hat{T}$ is a tableau. We have to show that the rules from Step 1 produce a tableaux. We do this by carefully examining what happens in the different cases (1) -(6)(c). So let (i, j) be as in Step 1. The cases (1) and (4) are obvious, there is nothing to check, since the top entry in C is smaller than j. In case (3) we place j below $j-1$, the last entry in the column thanks to property (S). The entry a to the left of $j-1$ is at most $j-1$, hence there must be an entry $b \leq j$ below a. The entry a' to the right of $j-1$ is at least $j-1$, hence any possible entry b' below a' is at least j. Cases (6a) and (6b) are obvious. So placing j always produces a new tableau, but (disregarding the crossed out ones) possibly one entry removed.

(i) Assume we removed j via case (2) applied to some column C' which then has to contain at least j and $j-1$. Applying then cases (1), (3), (4) or (6b) cause no problems. Case (6a) is obviously producing a tableau.

(ii) Assume we removed $j+1$ via case (4) from some column C'. Again case (1) is obvious. Then case (3) is not applicable, since C' originally contained $j+1$, hence the entry to the right of it is at least $j+1$. In case (4) we originally have $j+1$ at the top of column C' and $j+2$ to the right of it. This gets changed into j and $j+1$ which produces a tableau. Case (6a) is obviously causing no problems. In case (6b) we originally had $j+1$ on the top of C' and all entries in the column to the right of it are greater than $j+2$. In particular, we can replace the top entry by $j+1$.

(iii) Assume we applied case (5) for (C', j). Then applying (1) is fine. For case (3) we argue as in (11). For cases (4) and (6b) note that the top entry in C' is at most j. Case (6a) is clear.

(iv) Assume we applied case (6b) for (C', j) with j replacing x. Then applying (1) is fine, because $x > j$. Case (3) is not applicable. Cases (4) and (6a) are clear. In case (6b) we originally have x on top of column C'. There might be some a below it. Let z be to the right of x and possibly some b below z. Then x gets replaced by j and z by x, hence the first row is still weakly increasing. Since $j < x < a$ and $b > z \leq x$, the columns are still strictly increasing.

(v) Assume we applied case (6c). Then place j at the bottom of column C' and consider only the diagram containing the columns to the right of C', including C'. By induction on the number of columns, we are done.

□

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