QUIVER GRASSMANNIANS AND DEGENERATE FLAG VARIETIES

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Abstract. Quiver Grassmannians are varieties parametrizing subrepresentations of a quiver representation. It is observed that certain quiver Grassmannians for type A quivers are isomorphic to the degenerate flag varieties investigated earlier by the second named author. This leads to the consideration of a class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It is proven that these are (typically singular) irreducible normal local complete intersection varieties, which admit a group action with finitely many orbits, and a cellular decomposition. For type A quivers explicit formulas for the Euler characteristic (the median Genocchi numbers) and the Poincaré polynomials are derived.

1. Introduction

1.1. **Motivation.** Quiver Grassmannians, which are varieties parametrizing subrepresentations of a quiver representation, first appeared in [10, 25] in relation to questions on generic properties of quiver representations. It was observed in [4] that these varieties play an important role in cluster algebra theory [19]; namely, the cluster variables can be described in terms of the Euler characteristic of quiver Grassmannians. Subsequently, specific classes of quiver Grassmannians (for example, varieties of subrepresentations of exceptional quiver representations) were studied by several authors, with the principal aim of computing their Euler characteristic explicitly; see for example [5, 6, 7, 8]. In recent papers authors noticed that also the Poincaré polynomials of quiver Grassmannians play an important role in the study of quantum cluster algebras [22, 1]

This paper originated in the observation that a certain quiver Grassmannians can be identified with the $\mathfrak{sl}_n$-degenerate flag variety of [16, 17, 18]. This leads to the consideration of a class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It turns out that this class of varieties enjoys many of the favourable properties of quiver Grassmannians for exceptional representations. More precisely, they turn out to be (typically singular) irreducible normal local complete intersection varieties which admit a group action with finitely many orbits and a cellular decomposition. The proofs of the basic geometric properties are based on generalizations of the techniques of [24], where the case of Grassmannians of subrepresentations of injective quiver representations is treated.

1.2. **Main results.** Let Q be a quiver with set of vertices Q_0 of cardinality n and finite set of arrows Q_1. For a representation M of Q we denote by M_i the space in M attached to the i-th vertex, and by $M_\alpha : M_i \to M_j$ the linear map attached to an arrow $\alpha : i \to j$. We also denote by $\langle \_, \_ \rangle$ the Euler form on $\mathbb{Z}Q_0$. Given a dimension vector $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}_{\geq 0}Q_0$ and a representation M of Q, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M) \subset \prod_{i=1}^n \mathrm{Gr}_{e_i}(M_i)$ is the subvariety of collections of subspaces $V_i \subset M_i$ subject to the conditions $M_\alpha V_i \subset V_j$ for all $\alpha : i \to j \in Q_1$. In this paper we study a certain class of quiver Grassmannians for Dynkin quivers Q. Before describing this class, we first consider the following example.

Let Q be an equioriented quiver of type A_n with vertices $i = 1, \dots, n$ and arrows $i \to i+1$, and let $\mathbb{C}Q$ be the path algebra of Q. Then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\,\mathbb{C}Q}(\mathbb{C}Q \oplus \mathbb{C}Q^*)$ is isomorphic to the complete degenerate flag variety $\mathcal{F}^a_{n+1}$ for $G = SL_{n+1}$. Let us recall the definition (see [16],[17]). Let W be an $(n+1)$-dimensional vector space with basis $w_1, \dots, w_{n+1}$. Let $pr_k : W \to W$ for $k = 1, \dots, n+1$ be the projection operators $pr_k(\sum_{i=1}^{n+1} c_i w_i) = \sum_{i \neq k} c_i w_i$. Then the degenerate flag variety consists of collections $(V_1, \dots, V_n)$ with $V_i \subset W$ and $\dim V_i = i$, subject to the conditions $pr_{k+1} V_k \subset V_{k+1}$, $k = 1, \dots, n-1$. These varieties are irreducible singular algebraic varieties enjoying many nice properties. In particular, they are flat degenerations of classical flag varieties SL_{n+1}/B. Now let us consider the representation M of Q such that $M_i = W$ and the maps $M_i \to M_{i+1}$ are given by pr_{i+1}. For example, for $n = 3$, M has the following coefficient quiver

$$\begin{array}{ccccc} \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & & \bullet \\ \bullet & & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet \end{array}$$

where each dot represents basis vectors w_1, w_2, w_3, w_4 from bottom to top and arrows represent maps. Note that M is isomorphic to $\mathbb{C}Q \oplus \mathbb{C}Q^*$ as a representation of Q and moreover, we have

$$\mathcal{F}^a_{n+1} \simeq \mathrm{Gr}_{\mathbf{dim}\,\mathbb{C}Q}(\mathbb{C}Q \oplus \mathbb{C}Q^*). \tag{1.1}$$

Now let Q be a Dynkin quiver. Recall that the path algebra $\mathbb{C}Q$ (resp. its linear dual $\mathbb{C}Q^*$) is isomorphic as a representation of Q to the direct sum of all indecomposable projective representations (resp. all indecomposable injective representations). Motivated by the isomorphism (1.1) we consider the quiver Grassmannians $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$, where P and I are projective resp. injective representations of Q. (We note that some of our results are valid for more general Grassmannians and we discuss it in the main body of the paper. However, in the introduction we restrict ourselves to the above mentioned class of varieties). We use the isomorphism (1.1) in two different ways. On the one hand, we generalize and expand the results about $\mathcal{F}^a_{n+1}$ to the case of the above quiver Grassmannians. On the other hand, we use general results and constructions from the theory of quiver representations to understand better the structure of $\mathcal{F}^a_{n+1}$. Our first theorem is as follows:

Theorem 1.1. *The variety $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$ is of dimension $\langle \mathbf{dim}\,P, \mathbf{dim}\,I \rangle$ and irreducible. It is a normal local complete intersection variety.*

Our next goal is to construct cellular decompositions of the quiver Grassmannians and to compute their Poincaré polynomials. Let us consider the following stratification of $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$. For a point N we set $N_I = N \cap I$, $N_P = \pi N$, where $\pi : P \oplus I \to P$ is the projection. Then for a dimension vector $\mathbf{f} \in \mathbb{Z}_{\geq 0} Q_0$ we set

$$\mathcal{S}_{\mathbf{f}} = \{N \in \mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I) : \ \mathbf{dim}\,N_I = \mathbf{f}, \mathbf{dim}\,N_P = \mathbf{dim}\,P - \mathbf{f}\}.$$

We have natural surjective maps $\zeta_{\mathbf{f}} : \mathcal{S}_{\mathbf{f}} \to \mathrm{Gr}_{\mathbf{f}}(I) \times \mathrm{Gr}_{\mathbf{dim}\,P - \mathbf{f}}(P)$.

Theorem 1.2. *The map $\zeta_{\mathbf{f}}$ is a vector bundle. The fiber over a point (N_P, N_I) is isomorphic to $Hom_Q(N_P, I/N_I)$ which has dimension $\langle \mathrm{dim} P - \mathbf{f}, \mathrm{dim} I - \mathbf{f} \rangle$.*

Using this theorem, we construct a cellular decomposition for each stratum $X_{\mathbf{f}}$ and thus for the whole variety X as well. Moreover, since the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{f}}(I)$ and of $\mathrm{Gr}_{\mathbf{dim}\,P - \mathbf{f}}(P)$ can be easily computed, we arrive at a formula for the Poincaré polynomial (and thus for the Euler characteristic) of X. Recall (see [16]) that the Euler characteristic of the variety $\mathcal{F}^a_{n+1}$ is given by the normalized median Genocchi number h_{n+1} (see [11, 13, 14, 15, 26]). Using Theorem 1.2 we obtain an explicit formula for h_{n+1} in terms of binomial coefficients. Moreover, we give a formula for the Poincaré polynomial of $\mathcal{F}^a_{n+1}$, providing a natural q-version of h_{n+1}.

Finally we study the action of the group of automorphisms $Aut(P \oplus I)$ on the quiver Grassmannians. Let $G \subset Aut(P \oplus I)$ be the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P,I) & \mathrm{Aut}_Q(I) \end{bmatrix}$$

(we note that G coincides with the whole group of automorphisms unless Q is of type A_n). We prove the following theorem:

Theorem 1.3. *The group G acts on $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ with finitely many orbits, parametrized by pairs of isomorphism classes $([Q_P],[N_I])$ such that Q_P is a quotient of P, N_I is a subrepresentation of I, and $\mathbf{dim}\, Q_P = \mathbf{dim}\, N_I$. Moreover, if Q is equioriented of type A_n, then the orbits are cells parametrized by torus fixed points.*

1.3. **Outline of the paper.** Our paper is organized as follows:
In Section 2 we recall general facts about quiver Grassmannians and degenerate flag varieties.
In Section 3 we prove that the quiver Grassmannians $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$ are locally complete intersections and that they are flat degenerations of the Grassmannians in exceptional representations.
In Section 4 we study the action of the automorphism group on $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$, describe the orbits and prove the normality of $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$.
In Section 5 we construct a one-dimensional torus action on our quiver Grassmannians such that the attracting sets form a cellular decomposition.
Sections 6 and 7 are devoted to the case of the equioriented quiver of type A. In Section 6 we compute the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$ and derive several new formulas for the Euler characteristics – the normalized median Genocchi numbers.
In Section 7 we prove that the orbits studied in Section 4 are cells coinciding with the attracting cells constructed in Section 5. We also describe the connection with the degenerate group SL^a_{n+1}.

2. General facts on quiver Grassmannians and degenerate flag varieties

2.1. **General facts on quivers.** Let Q be a finite quiver with finite set of vertices Q_0 and finite set of arrows Q_1; arrows will be written as $(\alpha : i \to j) \in Q_1$ for $i, j \in Q_0$. We assume Q to be without oriented cycles. Denote by $\mathbf{Z}Q_0$ the free abelian group generated by Q_0, and by $\mathbf{N}Q_0$ the subsemigroup of dimension vectors $\mathbf{d} = (d_i)_{i \in Q_0}$ for Q. Let $\langle \_, \_ \rangle$ be the Euler form on $\mathbf{Z}Q_0$, defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{(\alpha:i \to j) \in Q_1} d_i e_j.$$

We consider finite dimensional representations M of Q over the complex numbers, viewed either as finite dimensional left modules over the path algebra $\mathbf{C}Q$ of Q, or as tuples $M = ((M_i)_{i \in Q_0}, (M_\alpha : M_i \to M_j)_{(\alpha:i\to j)\in Q_1})$ consisting of finite dimensional complex vector spaces M_i and linear maps M_α. The category $\mathrm{rep}(Q)$ of all such representations is hereditary (that is, $\mathrm{Ext}^{\geq 2}_Q(\_,\_) = 0$). Its Grothendieck group $K(\mathrm{rep}(Q))$ is isomorphic to $\mathbf{Z}Q_0$ by identifying the class of a representation M

with its dimension vector $\mathbf{dim}\, M = (\dim M_i)_{i\in Q_0} \in \mathbf{Z}Q_0$. The Euler form defined above then identifies with the homological Euler form, that is,

$$\dim \mathrm{Hom}_Q(M,N) - \dim \mathrm{Ext}^1_Q(M,N) = \langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle$$

for all representations M and N.

Associated to a vertex $i \in Q_0$, we have the simple representation S_i of Q with $(\mathbf{dim}\, S_i)_j = \delta_{i,j}$ (the Kronecker delta), the projective indecomposable P_i, and the indecomposable injective I_i. The latter are determined as the projective cover (resp. injective envelope) of S_i; more explicitly, $(P_i)_j$ is the space generated by all paths from i to j, and the linear dual of $(I_i)_j$ is the space generated by all paths from j to i.

Given a dimension vector $\mathbf{d} \in \mathbf{N}Q_0$, we fix complex vector spaces M_i of dimension d_i for all $i \in Q_0$. We consider the affine space

$$R_{\mathbf{d}}(Q) = \bigoplus_{(\alpha:i\to j)} \mathrm{Hom}_{\mathbf{C}}(M_i, M_j);$$

its points canonically parametrize representations of Q of dimension vector $\mathbf{d}$. The reductive algebraic group $G_{\mathbf{d}} = \prod_{i\in Q_0} \mathrm{GL}(M_i)$ acts naturally on $R_{\mathbf{d}}(Q)$ via base change

$$(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{(\alpha:i\to j)},$$

such that the orbits $\mathcal{O}_M$ for this action naturally correspond to the isomorphism classes $[M]$ of representations of Q of dimension vector $\mathbf{d}$. Note that $\dim G_{\mathbf{d}} - \dim R_{\mathbf{d}}(Q) = \langle \mathbf{d}, \mathbf{d}\rangle$. The stabilizer under $G_{\mathbf{d}}$ of a point $M \in R_{\mathbf{d}}(Q)$ is isomorphic to the automorphism group $\mathrm{Aut}_Q(M)$ of the corresponding representation, which (being open in the endomorphism space $\mathrm{End}_Q(M)$) is a connected algebraic group of dimension $\dim \mathrm{End}_Q(M)$. In particular, we get the following formulas:

$$\dim \mathcal{O}_M = \dim G_d - \dim \mathrm{End}_Q(M), \quad \mathrm{codim}_{R_{\mathbf{d}}} \mathcal{O}_M = \dim \mathrm{Ext}^1_Q(M,M). \tag{2.1}$$

2.2. **Basic facts on quiver Grassmannians.** The constructions and results in this section follow [5],[25]. Additionally to the above, fix another dimension vector $\mathbf{e}$ such that $\mathbf{e} \leq \mathbf{d}$ componentwise, and define the Q_0-graded Grassmannian $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{Gr}_{e_i}(M_i)$, which is a projective homogeneous space for $G_{\mathbf{d}}$ of dimension $\sum_{i\in Q_0} e_i(d_i - e_i)$, namely $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \simeq G_{\mathbf{d}}/P_{\mathbf{e}}$ for a maximal parabolic $P_{\mathbf{e}} \subset G_{\mathbf{d}}$. We define $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$, the universal Grassmannian of $\mathbf{e}$-dimensional subrepresentations of $\mathbf{d}$-dimensional representations of Q as the closed subvariety of $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \times R_{\mathbf{d}}(Q)$ consisting of tuples $((U_i \subset M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$ such that $M_\alpha(U_i) \subset U_j$ for all arrows $(\alpha : i \to j) \in Q_1$. The group $G_{\mathbf{d}}$ acts on $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$ diagonally, so that the projections $p_1 : \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) \to \mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ and $p_2 : \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) \to R_{\mathbf{d}}(Q)$ are $G_{\mathbf{d}}$-equivariant. In fact, the projection p_1 identifies $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$ as the total space of a homogeneous vector bundle over $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ of rank

$$\sum_{(\alpha:i\to j)\in Q_1} (d_i d_j + e_i e_j - e_i d_j).$$

Indeed, for a point $(U_i)_{i=1}^{\#Q_0}$ in $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$, we can choose complements $M_i = U_i \oplus V_i$ and identify the fiber of p_1 over $(U_i)_{i=1}^{\#Q_0}$ with

$$\left(\begin{bmatrix} \mathrm{Hom}_Q(U_i, U_j) & \mathrm{Hom}_Q(V_i, U_j) \\ 0 & \mathrm{Hom}_Q(V_i, V_j) \end{bmatrix} \subset \mathrm{Hom}_Q(M_i, M_j) \right)_{(\alpha:i\to j)} \subset R_{\mathbf{d}}(Q).$$

In particular, the universal Grassmannian $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$ is smooth and irreducible of dimension

$$\dim \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e}\rangle + \dim R_{\mathbf{d}}(Q).$$

The projection p_2 is proper, thus its image is a closed $G_{\mathbf{d}}$-stable subvariety of $R_{\mathbf{d}}$, consisting of representations admitting a subrepresentation of dimension vector $\mathbf{e}$.

We define the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M) = p_2^{-1}(M)$ as the fibre of p_2 over a point $M \in R_{\mathbf{d}}(Q)$; by definition, it parametrizes (more precisely, its closed points parametrize) $\mathbf{e}$-dimensional subrepresentations of the representation M.

Remark 2.1. Note that we have to view $\mathrm{Gr}_{\mathbf{e}}(M)$ as a scheme; in particular, it might be non-reduced. For example, if Q is the Kronecker quiver, $\mathbf{e}$ the isotropic root, and M is a regular indecomposable representation of dimension vector $2\mathbf{e}$, the quiver Grassmannian is Spec of the ring of dual numbers.

Recall that a representation M is called exceptional if $\mathrm{Ext}^1_Q(M, M) = 0$; thus, in view of (2.1), its orbit in $R_{\mathbf{d}}(Q)$ is open and dense.

Proposition 2.2. *Let M be an exceptional $\mathbf{d}$-dimensional representation of Q. Then $\mathrm{Gr}_{\mathbf{e}}(M)$ is non-empty if $\mathrm{Ext}^1_Q(N, L)$ vanishes for generic N of dimension vector $\mathbf{e}$ and generic L of dimension vector $\mathbf{d}-\mathbf{e}$. In this case, $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth of dimension $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$, and for all $\mathbf{d}$-dimensional representations N, every irreducible component of $\mathrm{Gr}_{\mathbf{e}}(N)$ has at least dimension $\langle \mathbf{e}, \mathbf{d} - \mathbf{e}\rangle$.*

Proof. The criterion for non-emptyness follows from [25, Theorem 3.3]. If $\mathrm{Gr}_{\mathbf{e}}(M)$ is non-empty, p_2 is surjective with $\mathrm{Gr}_{\mathbf{e}}(M)$ as its generic fibre. In particular, $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth of dimension $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$. For all other fibres, we obtain at least the desired estimate on dimensions of their irreducible components [20, Ch. II, Exercise 3.22 (b)]. □

We conclude this section by pointing out a useful isomorphism: let U be a point of $\mathrm{Gr}_{\mathbf{e}}(M)$ and let $T_U(\mathrm{Gr}_{\mathbf{e}}(M))$ denote the Zariski tangent space of $\mathrm{Gr}_{\mathbf{e}}(M)$ at U. As shown in [25, 5] we have the following scheme–theoretic description of the tangent space:

Lemma 2.3. *For $U \in \mathrm{Gr}_{\mathbf{e}}(M)$, we have $T_U(\mathrm{Gr}_{\mathbf{e}}(M)) \simeq \mathrm{Hom}_Q(U, M/U)$.*

2.3. **Quotient construction of (universal) quiver Grassmannians and a stratification.** We follow [24, Section 3.2]. Additionally to the choices before, fix vector spaces N_i of dimension e_i for $i \in Q_0$. We consider the universal variety $\mathrm{Hom}_Q(\mathbf{e}, \mathbf{d})$ of homomorphisms from an $\mathbf{e}$-dimensional to a $\mathbf{d}$-dimensional representation; explicitly, $\mathrm{Hom}_Q(\mathbf{e}, \mathbf{d})$ is the set of triples

$$((N_\alpha)_{\alpha\in Q_1}, (f_i : N_i \to M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1}) \in R_{\mathbf{e}} \times \prod_{i\in Q_0} \mathrm{Hom}(N_i, M_i) \times R_{\mathbf{d}}(Q)$$

such that $f_j N_\alpha = M_\alpha f_i$ for all $(\alpha : i \to j) \in Q_1$. This is an affine variety defined by quadratic relations, namely by the vanishing of the individual entries of the matrices $f_j N_\alpha - M_\alpha f_i$, on which $G_{\mathbf{e}} \times G_{\mathbf{d}}$ acts naturally. On the open subset $\mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d})$ where all $f_i : N_i \to M_i$ are injective maps, the action of $G_{\mathbf{e}}$ is free. By construction, we have an isomorphism

$$\mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d})/G_e \simeq \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$$

which associates to the orbit of a triple $((N_\alpha), (f_i), (M_\alpha))$ the pair given by $(f_i(N_i) \subset M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$. Indeed, the maps N_α are uniquely determined in this situation, and they can be reconstructed algebraically from (f_i) and (M_α) (see [24, Lemma 3.5]).

Similarly to $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$, we have a projection $\tilde{p}_2 : \mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d}) \to R_{\mathbf{d}}(Q)$ with fibres $\tilde{p}_2^{-1}(M) = \mathrm{Hom}^0_Q(\mathbf{e}, M)$, and we have a local version of the previous isomorphism

$$\mathrm{Hom}^0_Q(\mathbf{e}, M)/G_{\mathbf{e}} = \tilde{p}_2^{-1}(M)/G_{\mathbf{e}} \simeq \mathrm{Gr}_{\mathbf{e}}(M).$$

Note that the quotient map $\mathrm{Hom}^0_Q(\mathbf{e}, M) \to \mathrm{Gr}_{\mathbf{e}}(M)$ is locally trivial, since it is induced by the quotient map

$$\mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d}) = \prod_{i \in Q_0} \mathrm{Hom}^0(N_i, M_i) \to \mathrm{Gr}_{\mathbf{e}}(\mathbf{d}),$$

which can be trivialized over the standard open affine coverings of Grassmannians.

Let p be the projection from $\mathrm{Hom}^0_Q(\mathbf{e}, M)$ to $R_{\mathbf{e}}(Q)$; its fiber over N is the space $\mathrm{Hom}^0_Q(N, M)$ of injective maps. For each isomorphism class $[N]$ of representations of dimension vector $\mathbf{e}$, we can consider the subset $\mathcal{S}_{[N]}$ of $\mathrm{Gr}_{\mathbf{e}}(M)$ corresponding under the previous isomorphism to $(p^{-1}(\mathcal{O}_N))/G_{\mathbf{e}}$. It therefore consists of all subrepresentations $U \in \mathrm{Gr}_{\mathbf{e}}(M)$ which are isomorphic to N.

Lemma 2.4. *Each $\mathcal{S}_{[N]}$ is an irreducible locally closed subset of $\mathrm{Gr}_{\mathbf{e}}(M)$ of dimension $\dim \mathrm{Hom}_Q(N, M) - \dim \mathrm{End}_Q(N)$.*

Proof. Irreducibility of $\mathcal{S}_{[N]}$ follows from irreducibility of $\mathcal{O}_N$ by G_e-equivariance of p. Using the fact that the geometric quotient is closed and separating on G_e-stable subsets, an induction over $\dim \mathcal{O}_N$ proves that all $\mathcal{S}_{[N]}$ are locally closed. The dimension is calculated as

$$\dim \mathcal{S}_{[N]} = \dim \mathcal{O}_N + \dim \mathrm{Hom}^0_Q(N, M) - \dim G_e.$$

□

2.4. **Degenerate flag varieties.** In this subsection we recall the definition of the degenerate flag varieties following [16], [17], [18]. Let W be an n-dimensional vector space with a basis $w_1, \dots, w_n$. We denote by $pr_k : W \to W$ the projections along w_k to the linear span of the remaining basis vectors, that is, $pr_k \sum_{i=1}^n c_i w_i = \sum_{i \neq k} c_i w_i$.

Definition 2.5. The variety $\mathcal{F}^a_n$ is the set of collections of subspaces $(V_i \in \mathrm{Gr}_i(W))_{i=1}^{n-1}$ subject to the conditions $pr_{i+1} V_i \subset V_{i+1}$ for all $i = 1, \dots, n-2$.

The variety $\mathcal{F}^a_n$ is called the complete degenerate flag variety. It enjoys the following properties:

- $\mathcal{F}^a_n$ is a singular irreducible projective algebraic variety of dimension $\binom{n}{2}$.
- $\mathcal{F}^a_n$ is a flat degeneration of the classical complete flag variety SL_n/B.
- $\mathcal{F}^a_n$ is a normal local complete intersection variety.
- $\mathcal{F}^a_n$ can be decomposed into a disjoint union of complex cells.

We add some comments on the last property. The number of cells (which is equal to the Euler characteristic of $\mathcal{F}^a_n$) is given by the n-th normalized median Genocchi number h_n (see e.g. [17], section 3). These numbers have several definitions; here we will use the following one: h_n is the number of collections $(S_1, \dots, S_{n-1})$, where $S_i \subset \{1, \dots, n\}$ subject to the conditions

$$\# S_i = i, \ 1 \le i \le n-1; \quad S_i \subset S_{i+1} \cup \{i+1\}, \ 1 \le i \le n-2.$$

For $n = 1, 2, 3, 4, 5$ the numbers h_n are equal to $1, 2, 7, 38, 295$.

There exists a degeneration SL^a_n of the group SL_n acting on $\mathcal{F}^a_n$. Namely, the degenerate group SL^a_n is the semi-direct product of the Borel subgroup B of SL_n and a normal abelian subgroup $\mathbb{G}_a^{n(n-1)/2}$, where $\mathbb{G}_a$ is the additive group of the field. The simplest way to describe the structure of the semi-direct product is via the Lie algebra $\mathfrak{sl}^a_n$ of SL^a_n. Namely, let $\mathfrak{b} \in \mathfrak{sl}_n$ be the Borel subalgebra of upper-triangular matrices and $\mathfrak{n}^-$ be the nilpotent subalgebra of strictly lower-triangular matrices. Let $(\mathfrak{n}^-)^a$ be the abelian Lie algebra with underlying vector space $\mathfrak{n}^-$. Then $\mathfrak{n}^-$ carries a natural structure of $\mathfrak{b}$-module induced by the adjoint action on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{sl}_n/\mathfrak{b}$. Then $\mathfrak{sl}^a_n = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is abelian ideal and $\mathfrak{b}$ acts on $(\mathfrak{n}^-)^a$ as described above. The group SL_n^a (the Lie group of $\mathfrak{sl}_n^a$) acts on the variety $\mathcal{F}_n^a$ with an open $\mathbb{G}_a^{n(n-1)/2}$-orbit. We note that in contrast with the classical situation, the group SL_n^a acts on $\mathcal{F}_n^a$ with an infinite number of orbits.

For partial (parabolic) flag varieties of SL_n there exists a natural generalization of $\mathcal{F}_n^a$. Namely, consider an increasing collection $1 \le d_1 < \dots < d_s < n$. In what follows we denote such a collection by $\mathbf{d}$. Let $\mathcal{F}_{\mathbf{d}}$ be the classical partial flag variety consisting of the collections $(V_i)_{i=1}^s$, $V_i \in \mathrm{Gr}_{d_i}(W)$ such that $V_i \subset V_{i+1}$.

Definition 2.6. The degenerate partial variety $\mathcal{F}_{\mathbf{d}}^a$ is the set of collections of subspaces $V_i \in \mathrm{Gr}_{d_i}(W)$ subject to the conditions $pr_{d_i+1} \dots pr_{d_{i+1}} V_i \subset V_{i+1}$ for all $i = 1, \dots, s-1$.

We still have the following properties

- $\mathcal{F}_{\mathbf{d}}^a$ is a singular irreducible projective algebraic variety.
- $\mathcal{F}_{\mathbf{d}}^a$ is a flat degeneration of $\mathcal{F}_{\mathbf{d}}$.
- $\mathcal{F}_{\mathbf{d}}^a$ is a normal local complete intersection variety.
- $\mathcal{F}_{\mathbf{d}}^a$ is acted upon by the group SL_n^a with an open $\mathbb{G}_a^{n(n-1)/2}$-orbit.

2.5. **Comparison between quiver Grassmannians and degenerate flag varieties.** Let Q be an equioriented quiver of type A_n. We order the vertices of Q from 1 to n in such a way that the arrows of Q are of the form $i \to i+1$. Let P_i, I_i, $i = 1, \dots, n$ be the projective and injective representations attached to the i-th vertex, respectively. In particular, $\mathbf{dim}\, P_i = (0, \dots, 0, 1, \dots, 1)$ with $i-1$ zeros and $\mathbf{dim}\, I_i = (1, \dots, 1, 0, \dots, 0)$ with $n-i$ zeros.

In what follows we will use the following basis of P_i and I_i. Namely, for each $j = i, \dots, n$ we fix non-zero elements $w_{i,j} \in (P_j)_i$ in such a way that $w_{i,j} \mapsto w_{i+1,j}$. Also, for $j = 1, \dots, i$, we fix non-zero elements $w_{i,j+1} \in (I_j)_i$ in such a way that $w_{i,j} \mapsto w_{i+1,j}$ unless $j = i+1$ and $w_{i,i+1} \mapsto 0$.

Let A be the path algebra $\mathbf{C}Q$. Viewed as a representation of Q, A is isomorphic to the direct sum $\bigoplus_{i=1}^n P_i$. In particular, $\mathbf{dim}\, A = (1, 2, \dots, n)$. The linear dual A^* is isomorphic to the direct sum of injective representations $\bigoplus_{i=1}^n I_i$.

Proposition 2.7. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{dim}\, A}(A \oplus A^*)$ *is isomorphic to the degenerate flag variety* $\mathcal{F}_{n+1}^a$ *of* $\mathfrak{sl}_{n+1}$.

Proof. Consider $A \oplus A^* = \bigoplus_{i=1}^n (P_i \oplus I_i)$ as a representation of Q. Let W_j the the space attached to the j-th vertex, that is, $A \oplus A^* = (W_1, \dots, W_n)$. First, we note that $\dim W_j = n+1$ for all j. Second, we fix an $(n+1)$-dimensional vector space W with a basis $w_1, \dots, w_{n+1}$. Let us identify all W_j with W by sending $w_{i,j}$ to w_j. Then the maps $W_j \to W_{j+1}$ coincide with pr_{j+1}. Now our proposition follows from the equality $\mathbf{dim}\, A = (1, 2, \dots, n)$. $\square$

The coefficient quiver of the representation $A \oplus A^*$ is given by ($n = 4$):

$$
\begin{array}{cccccccc}
w_{1,5} & \to & w_{2,5} & \to & w_{3,5} & \to & w_{4,5} \\
w_{1,4} & \to & w_{2,4} & \to & w_{3,4} & & w_{4,4} \\
w_{1,3} & \to & w_{2,3} & & w_{3,3} & \to & w_{4,3} \\
w_{1,2} & & w_{2,2} & \to & w_{3,2} & \to & w_{4,2} \\
w_{1,1} & \to & w_{2,1} & \to & w_{3,1} & \to & w_{4,1}
\end{array}
\tag{2.2}
$$

Remark 2.8. We note that the classical SL_{n+1} flag variety has a similar realization. Namely, let $\tilde{M}$ be the representation of Q isomorphic to the direct sum of $n+1$ copies of P_1 (so, $\mathbf{dim}\, \tilde{M} = \mathbf{dim}\,(A \oplus A^*)$). Then the classical flag variety SL_{n+1}/B

is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}A}\tilde{M}$. The Q-representation $\tilde{M}$ can be visualized as ($n = 4$)

$$\begin{array}{ccccccc} \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{array} \tag{2.3}$$

We can easily generalize Proposition 2.7 to degenerate partial flag varieties:

Suppose we are given a sequence $\mathbf{d} = (0 = d_0 < d_1 < d_2 < \ldots < d_s < d_{s+1} = n+1)$. Then we define

$$P = \bigoplus_{i=1}^{s} P_i^{d_i - d_{i-1}}, \quad I = \bigoplus_{i=1}^{s} I_i^{d_{i+1} - d_i}$$

as representations of an equioriented quiver of type A_s.

Proposition 2.9. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$ *is isomorphic to the degenerate partial flag variety* $\mathcal{F}^a_{\mathbf{d}}$ *of* $\mathfrak{sl}_{n+1}$.

Proof. We note that the dimension vector of $P \oplus I$ is given by $(n+1, \ldots, n+1)$ and the dimension vector of P equals $(d_1, \ldots, d_s)$. Now let us identify the spaces $(P \oplus I)_j$ with W as in the proof of Proposition 2.7. Then the map $(P \oplus I)_j \to (P \oplus I)_{j+1}$ corresponding to the arrow $j \to j+1$ coincides with $pr_{d_j+1} \ldots pr_{d_{j+1}}$, which proves the proposition. □

3. A class of well-behaved quiver Grassmannians

3.1. **Geometric properties.** From now on, let Q be a Dynkin quiver. Then $G_{\mathbf{d}}$ acts with finitely many orbits on $R_{\mathbf{d}}(Q)$ for every $\mathbf{d}$; in particular, for every $\mathbf{d} \in \mathbf{N}Q_0$, there exists a unique (up to isomorphism) exceptional representation of this dimension vector.

The subsets $\mathcal{S}_{[N]}$ of Section 2.3 then define a finite stratification of each quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ according to isomorphism type of the subrepresentation $N \subset M$.

Proposition 3.1. *Assume that X and Y are exceptional representations of Q such that* $\mathrm{Ext}^1_Q(X, Y) = 0$. *Define* $M = X \oplus Y$ *and* $\mathbf{e} = \mathbf{dim}\, X$, $\mathbf{d} = \mathbf{dim}\,(X \oplus Y)$. *Then the following holds:*

(i) $\dim \mathrm{Gr}_{\mathbf{e}}(M) = \langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$.
(ii) *The variety* $\mathrm{Gr}_{\mathbf{e}}(M)$ *is reduced, irreducible and rational.*
(iii) $\mathrm{Gr}_{\mathbf{e}}(M)$ *is a locally complete intersection scheme.*

Proof. The representation X obviously embeds into M, thus

$$\dim \mathrm{Gr}_{\mathbf{e}}(M) \geq \dim \mathcal{S}_{[X]} = \dim \mathrm{Hom}_Q(X, M) - \dim \mathrm{End}_Q(X) = \dim \mathrm{Hom}_Q(X, Y).$$

The tangent space to any point $U \in \mathcal{S}_{[X]}$ has dimension $\dim \mathrm{Hom}_Q(X, Y)$, too, thus $\overline{\mathcal{S}_{[X]}}$ is reduced. Moreover, a generic embedding of X into $X \oplus Y$ is of the form $[\mathrm{id}_X, f]$ for a map $f \in \mathrm{Hom}_Q(X, Y)$, and this identifies an open subset isomorphic to $\mathrm{Hom}_Q(X, Y)$ of $\mathcal{S}_{[X]}$, proving rationality of $\overline{\mathcal{S}_{[X]}}$. Now suppose N embeds into $M = X \oplus Y$ and $\mathbf{dim}\, N = \mathbf{e}$. Then $\mathrm{Ext}^1_Q(N, Y) = 0$ since $\mathrm{Ext}^1_Q(X \oplus Y, Y) = 0$ by assumption, and thus $\dim \mathrm{Hom}_Q(N,Y) = \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle = \dim \mathrm{Hom}_Q(X,Y)$. Therefore,

$$\dim \mathcal{S}_{[N]} = \\ \dim \mathrm{Hom}_Q(N,X) - \dim \mathrm{Hom}_Q(N,N) + \dim \mathrm{Hom}_Q(X,Y) \leq \\ \dim \mathrm{Hom}_Q(X,Y),$$

which proves that $\dim \mathrm{Gr}_{\mathbf{e}}(M) = \dim \mathrm{Hom}_Q(X,Y) = \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$, and that the closure of $\mathcal{S}_{[X]}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{e}}(M)$. Conversely, suppose that an irreducible component C of $\mathrm{Gr}_{\mathbf{e}}(M)$ is given, then necessarily C is the closure of some stratum $\mathcal{S}_{[N]}$, and the dimension of C equals $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle = \dim \mathrm{Hom}_Q(X,Y)$ by Proposition 2.2. By the above dimension estimate, we conclude $\dim \mathrm{Hom}_Q(N,X) = \dim \mathrm{Hom}_Q(N,N)$. By [3, Theorem 2.4], this yields an embedding $N \subset X$, and thus $N = X$ by equality of dimensions. Therefore, $\mathrm{Gr}_{\mathbf{e}}(M)$ equals the closure of the stratum $\mathcal{S}_{[X]}$, thus it is irreducible, reduced and rational. The dimension of $\mathrm{Hom}^0_Q(\mathbf{e}, M)$ equals $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle + \dim G_{\mathbf{e}}$, thus its codimension in $R_{\mathbf{e}}(Q) \times \mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d})$ equals

$$\dim R_{\mathbf{e}}(Q) + \sum_i e_i d_i - \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle - \dim G_{\mathbf{e}} = \sum_{(\alpha: i \to j) \in Q_1} e_i d_j.$$

But this is exactly the number of equations defining $\mathrm{Hom}_Q(\mathbf{e}, M)$. Thus $\mathrm{Hom}^0_Q(\mathbf{e}, M)$ is locally a complete intersection. The map $\mathrm{Hom}^0_Q(\mathbf{e}, M) \to \mathrm{Gr}_{\mathbf{e}}(M)$ is locally trivial with smooth fiber $G_{\mathbf{e}}$, hence the last statement follows. □

On a quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$, the automorphism group $\mathrm{Aut}_Q(M)$ acts algebraically. In the present situation, this implies that the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(X) & 0 \\ \mathrm{Hom}_Q(X,Y) & \mathrm{Aut}_Q(Y) \end{bmatrix}$$

acts on $\mathrm{Gr}_{\mathbf{e}}(X \oplus Y)$.

3.2. **Flat degeneration.** Now let $\tilde{M}$ be the unique (up to isomorphism) exceptional representation of the same dimension vector as M. By Proposition 2.2, we also have $\dim \mathrm{Gr}_{\mathbf{e}}(\tilde{M}) = \langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$. It is thus reasonable to ask for good properties of the degeneration from $\mathrm{Gr}_{\mathbf{e}}(\tilde{M})$ to $\mathrm{Gr}_{\mathbf{e}}(M)$.

Theorem 3.2. *Under the previous hypotheses, the quiver Grassmannian* $\mathrm{Gr}_{\mathbf{e}}(M)$ *is a flat degeneration of* $\mathrm{Gr}_{\mathbf{e}}(\tilde{M})$.

Proof. Let Y be the open subset of $R_{\mathbf{d}}(Q)$ consisting of all representations Z whose orbit closure $\overline{\mathcal{O}_Z}$ contains the orbit $\mathcal{O}_M$; in particular, Y contains $\mathcal{O}_{\tilde{M}}$. We consider the diagram

$$\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \stackrel{p_1}{\leftarrow} \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) \stackrel{p_2}{\rightarrow} R_{\mathbf{d}}(Q)$$

of the previous section. In particular, we consider the restriction $q : \tilde{Y} \to Y$ of p_2 to $\tilde{Y} = p_2^{-1}(Y)$. This is a proper morphism (since p_2 is so) between two smooth and irreducible varieties (since they are open subsets of the smooth varieties $R_{\mathbf{d}}(Q)$ and $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d})$, respectively). The general fibre of q is $\mathrm{Gr}_{\mathbf{e}}(\tilde{M})$, since the orbit of $\tilde{M}$, being exceptional, is open in Y, and the special fibre of q is $\mathrm{Gr}_{\mathbf{e}}(M)$, since the orbit of M is closed in Y by definition. By semicontinuity, all fibres of q have the same dimension $\langle \mathbf{e}, \mathbf{d}-\mathbf{e}\rangle$. By [21, Corollary to Theorem 23.1], a proper morphism between smooth and irreducible varieties with constant fibre dimension is already flat. □

Remark 3.3. Theorem 3.2 generalizes Proposition 3.15 of [16] (see also subsections 2.4, 2.5), where the flatness of the degeneration $\mathcal{F}_n \to \mathcal{F}^a_n$ was proved using complicated combinatorial tools.

Note that the degeneration from $\tilde{M}$ to M in $R_{\mathbf{d}}(Q)$ can be realized along a one-parameter subgroup of $G_{\mathbf{d}}$ in the following way:

Lemma 3.4. *Under the above hypothesis, there exists a short exact sequence* $0 \to X \to \tilde{M} \to Y \to 0$.

Proof. By [25, Theorem 3.3], a generic representation Z of dimension vector $\mathbf{d}$ admits a subrepresentation of dimension vector $\mathbf{e}$ if $\mathrm{Ext}^1_Q(N, L)$ vanishes for generic N of dimension vector $\mathbf{e}$ and generic L of dimension vector $\mathbf{d} - \mathbf{e}$. In the present case, these generic representations are $Z = \tilde{M}$, $N = X$ and $L = Y$, and the lemma follows. □

This lemma implies that $\tilde{M}$ can be written, up to isomorphism, in the following form

$$\tilde{M}_\alpha = \begin{bmatrix} X_\alpha & \zeta_\alpha \\ 0 & Y_\alpha \end{bmatrix}$$

for all $\alpha \in Q_1$. Conjugating with the one-parameter subgroup

$$\left(\begin{bmatrix} t \cdot \mathrm{id}X_i & 0 \\ 0 & \mathrm{id}Y_i \end{bmatrix} \right)_{i \in Q_0}$$

of $G_{\mathbf{d}}$ and passing to the limit $t = 0$, we arrive at the desired degeneration.

Since Q is a Dynkin quiver, the isomorphism classes of indecomposable representations of Q are parametrized by the positive roots Φ^+ of the corresponding root system. We view Φ^+ as a subset of $\mathbf{N}Q_0$ by identifying the simple root α_i with with the vector having 1 at the i-th place and zeros everywhere else. Denote by V_α the indecomposable representation corresponding to $\alpha \in \Phi^+$; more precisely, $\mathbf{dim}V_\alpha = \alpha$. Using this parametrization of the indecomposables and the Auslander-Reiten quiver of Q, we can actually construct $\tilde{M}$ explicitly from X and Y (or, more precisely, from their decompositions into indecomposables), using the algorithm of [23, Section 3].

4. The group action and normality

In this section we put $X = P$ and $Y = I$, where P and I are projective and injective representations of a Dynkin quiver Q. We consider the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P, I) & \mathrm{Aut}_Q(I) \end{bmatrix}.$$

Theorem 4.1. *The group G acts on $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ with finitely many orbits, parametrized by pairs of isomorphism classes $([Q_P], [N_I])$ such that Q_P is a quotient of P, N_I is a subrepresentation of I, and Q_P and N_I have the same dimension vector.*

Proof. Suppose N is a subrepresentation of $P \oplus I$ of dimension vector $\mathbf{dim}\, N = \mathbf{dim}\, P$, and denote by $\iota : N \to P \oplus I$ the embedding. Define $N_I = N \cap I$ and $N_P = N/(N \cap I)$. Then $N_P \simeq (N + I)/I$ embeds into $(P \oplus I)/I \simeq P$, thus N_P is projective since $\mathrm{rep}(Q)$ is hereditary. Therefore, the short exact sequence

$$0 \to N_I \to N \to N_P \to 0$$

splits. We thus have a retraction $r : N_P \to N$ such that N is the direct sum of N_I and $r(N_P)$, and such that N_I embeds into the component I of $P \oplus I$ under ι. Without loss of generality, we can thus write the embedding of N into $P \oplus I$ as

$$\iota = \begin{bmatrix} \iota_P & 0 \\ f & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I$$

for ι_P (resp. ι_I) an embedding of N_P (resp. N_I) into P (resp. I), and $f : N_P \to I$. Since I is injective, the map f factors through ι_P, yielding a map $x : P \to I$ such that $x\iota_P = f$. We can then conjugate ι with

$$\begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} \in G.$$

We have thus proved that each G-orbit in $\mathrm{Gr}_{\mathbf{dim}_P}(P \oplus I)$ contains an embedding of the form

$$\begin{bmatrix} \iota_P & 0 \\ 0 & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I,$$

such that N_I is a subrepresentation of I, the representation $Q_P = P/N_P$ is a quotient of P, and their dimension vectors obviously add up to $\mathbf{dim}\, P$. We now have to show that the isomorphism classes of such Q_P and N_I already characterize the corresponding G-orbit in $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$. To do this, suppose we are given two such embeddings

$$\begin{bmatrix} \iota_P & 0 \\ 0 & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I, \quad \begin{bmatrix} \iota'_P & 0 \\ 0 & \iota'_I \end{bmatrix} : N'_P \oplus N'_I \to P \oplus I$$

such that the cokernels Q_P and Q'_P of ι_P and ι'_P, respectively, are isomorphic, and such that N_I and N'_I are isomorphic. By [24, Lemma 6.3], an arbitrary isomorphism $\psi_I : N_I \to N'_I$ lifts to an automorphism φ_I of I, such that $\varphi_I \iota_I = \iota'_I \psi_I$. By the obvious dual version of the same lemma, an arbitrary isomorphism $\xi_P : Q_P \to Q'_P$ lifts to an automorphism φ_P of P, which in turn induces an isomorphism $\psi_P : N_P \to N'_P$ such that $\varphi_P \iota_P = \iota'_P \psi_P$. This proves that the two embeddings above are conjugate under G. Finally, given representations Q_P and N_I as above, we can define N_P as the kernel of the quotient map and get an embedding as above. $\square$

Remark 4.2. We can obtain an explicit parametrization of the orbits as follows: we write

$$P = \bigoplus_{i \in Q_0} P_i^{a_i}, \quad I = \bigoplus_{i \in Q_0} I_i^{b_i}.$$

By [24, Lemma 4.1] and its obvious dual version, we have:

A representation N_I embeds into I if and only if $\dim \mathrm{Hom}_Q(S_i, N_I) \leq b_i$ for all $i \in Q_0$,

a representation Q_P is a quotient of P if and only if $\dim \mathrm{Hom}_Q(Q_P, S_i) \leq a_i$ for all $i \in Q_0$.

The previous result establishes a finite decomposition of the quiver Grassmannians into orbits. In particular the tangent space is equidimensional along every such orbit. The following examples shows that in general such orbits are not cells.

Example 4.3. Let

$$Q := \qquad \begin{array}{ccccc} & & & 3 & \\ & & \nearrow & & \\ 1 \longrightarrow & 2 & \longrightarrow & \longrightarrow & 4 \end{array}$$

be a Dynkin quiver of type D_4. The quiver Grassmannian $\mathrm{Gr}_{(1211)}(I_3 \oplus I_4)$ is isomorphic to $\mathbf{P}^1$, with the points 0 and ∞ corresponding to two decomposable representations, whereas all points in $\mathbf{P}^1 \setminus \{0, \infty\}$, which is obviously not a cell, correspond to subrepresentations which are isomorphic to the indecomposable representation of dimension vector (1211).

We note the following generalization of the tautological bundles

$$\mathcal{U}_i = \{(U,x) \in \mathrm{Gr}_{\mathbf{e}}(X) \times X_i \,:\, x \in U_i\}$$

on $\mathrm{Gr}_{\mathbf{e}}(X)$:

Given a projective representation P, the trivial vector bundle $\mathrm{Hom}_Q(P,X)$ on $\mathrm{Gr}_{\mathbf{e}}(X)$ admits the subbundle

$$\mathcal{V}_P = \{(U,\alpha) \in \mathrm{Gr}_{\mathbf{e}}(X) \times \mathrm{Hom}_Q(P,X) \,:\, \alpha(P) \subset U\}.$$

We then have $\mathcal{V}_P \simeq \bigoplus_{i\in Q_0} \mathcal{V}_i^{m_i}$ if $P \simeq \bigoplus_{i\in Q_0} P_i^{m_i}$. Dually, given an injective representation I, the trivial vector bundle $\mathrm{Hom}_Q(X,I)$ admits the subbundle

$$\mathcal{V}_I = \{(U,\beta) \in \mathrm{Gr}_e(X) \times \mathrm{Hom}_Q(X,I) \,:\, \beta(U) = 0\}.$$

We then have $\mathcal{V}_I \simeq \bigoplus_{i\in Q_0} (\mathcal{V}_i^*)^{m_i}$ if $I \simeq \bigoplus_{i\in Q_0} I_i^{m_i}$.

Given a decomposition of the dimension vector $\mathbf{dim}\, P = \mathbf{e} = \mathbf{f} + \mathbf{g}$, recall the subvariety $\mathcal{S}_{\mathbf{f}}(P \oplus I) \subset \mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ consisting of all representations N such that $\mathbf{dim}\, N \cap I = \mathbf{f}$ and $\mathbf{dim}\, \pi(N) = \mathbf{g}$, where $\pi : P \oplus I \to P$ is the natural projection. We have a natural surjective map $\zeta : \mathrm{Gr}_{\mathbf{f},\mathbf{g}}(P\oplus I) \to \mathrm{Gr}_{\mathbf{g}}(P) \times \mathrm{Gr}_{\mathbf{f}}(I)$. We note that since P is projective, all the points of $\mathrm{Gr}_{\mathbf{g}}(P)$ are isomorphic as representations of Q. Also, since I is injective, for any two points $M_1, M_2 \in \mathrm{Gr}_{\mathbf{f}}(I)$ the representations I/M_1 and I/M_2 of Q are isomorphic. Therefore, the dimension of the vector space $\mathrm{Hom}_Q(N_P, I/N_I)$ is independent of the points $N_P \in \mathrm{Gr}_{\mathbf{g}}(P)$ and $N_I \in \mathrm{Gr}_{\mathbf{f}}(I)$. We denote this dimension by D.

Proposition 4.4. *The map ζ is a D-dimensional vector bundle (in the Zariski topology).*

Proof. Associated to N_P and N_I, we have exact sequences

$$0 \to N_P \to P \to Q_P \to 0 \quad \text{and} \quad 0 \to N_I \to I \to Q_I \to 0.$$

These induce the following commutative diagram with exact rows and columns (the final zeroes arising from projectivity of N_P and injectivity of Q_I; we abbreviate $\mathrm{Hom}_Q(\_,\_)$ by $(\_,\_)$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (Q_P, N_I) & \to & (Q_P, I) & \to & (Q_P, Q_I) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (P, N_I) & \to & (P, I) & \to & (P, Q_I) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (N_P, N_I) & \to & (N_P, I) & \to & (N_P, Q_I) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram yields an isomorphism

$$\mathrm{Hom}_Q(N_P, Q_I) \simeq \mathrm{Hom}_Q(P,I)/(\mathrm{Hom}_Q(P,N_I) + \mathrm{Hom}_Q(Q_P,I)).$$

Pulling back the tautological bundles constructed above via the projections

$$\mathrm{Gr}_{\mathbf{g}}(P) \overset{\mathrm{pr}_1}{\leftarrow} \mathrm{Gr}_{\mathbf{g}}(P) \times \mathrm{Gr}_{\mathbf{f}}(I) \overset{\mathrm{pr}_2}{\rightarrow} \mathrm{Gr}_{\mathbf{f}}(I),$$

we get subbundles $\mathrm{pr}_2^*\mathcal{V}_P$ and $\mathrm{pr}_1^*\mathcal{V}_I$ of the trivial bundle $\mathrm{Hom}_Q(P,I)$ on $\mathrm{Gr}_{\mathbf{g}}(P) \times \mathrm{Gr}_{\mathbf{f}}(I)$. By the above isomorphism, the quotient bundle

$$\mathrm{Hom}_Q(P,I)/(\mathrm{pr}_1^*\mathcal{V}_P + \mathrm{pr}_2^*\mathcal{V}_I)$$

identifies with the fibration $\zeta : \mathcal{S}_{\mathbf{f}}(P \oplus I) \to \mathrm{Gr}_{\mathbf{g}}(P) \times \mathrm{Gr}_{\mathbf{f}}(I)$, proving Zariski local triviality of the latter. $\square$

The methods established in the two previous proofs now allow us to prove normality of the quiver Grassmannians.

Theorem 4.5. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ *is a normal variety.*

Proof. We already know that $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ is locally a complete intersection, thus normality is proved once we know that $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ is regular in codimension 1. By the proof of Theorem 4.1, we know that a subrepresentation N of $P \oplus I$ of dimension vector $\mathbf{dim}P$ is of the form $N = N_P \oplus N_I$, with exact sequences

$$0 \to N_P \to P \to Q_P \to 0, \quad 0 \to N_I \to I \to Q_I \to 0,$$

such that N_I and Q_P are of the same dimension vector $\mathbf{f}$. By the tangent space formula, N defines a singular point of $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ if and only if

$$\mathrm{Ext}^1_Q(N_P \oplus N_I, Q_P \oplus Q_I) = \mathrm{Ext}^1_Q(N_I, Q_P)$$

is non-zero. In particular, singularity of the point N only depends on the isomorphism types of $N_I = N \cap I$ and $Q_P = (P \oplus I)/(N + I)$. Consider the locally closed subset Z of $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$ consisting of subrepresentations N' such that $N' \cap I \simeq N_I$ and $(P \oplus I)/(N' + I) \simeq Q_P$; thus $Z \subset \mathcal{S}_{\mathbf{f}}$. The vector bundle $\zeta : \mathcal{S}_{\mathbf{f}} \to \mathrm{Gr}_{\mathbf{f}}(I) \times \mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P)$ of the previous proposition restricts to a vector bundle $\zeta : Z \to Z_I \times Z_P$, where $Z_I = \mathcal{S}_{[N_I]} \subset \mathrm{Gr}_{\mathbf{f}}(I)$ consists of subrepresentations isomorphic to N_I, and $Z_P \subset \mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P)$ consists of subrepresentations with quotient isomorphic to Q_P. By the dimension formula for the strata $\mathcal{S}_{[N_I]}$, the codimension of Z_I in $\mathrm{Gr}_{\mathbf{f}}(I)$ equals $\dim \mathrm{Ext}^1_Q(N_I, N_I)$; dually, the codimension of Z_P in $\mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P)$ equals $\dim \mathrm{Ext}^1_Q(Q_P, Q_P)$. Since the rank of the bundle ζ is $\dim \mathrm{Hom}_Q(N_P, Q_I)$, we have

$$\begin{gathered}
\dim \mathrm{Gr}_{\mathbf{e}}(P \oplus I) - \dim \zeta^{-1}(Z_I \times Z_P) = \\
= \dim \mathrm{Gr}_{\mathbf{e}}(P \oplus I) - \dim \mathrm{Hom}_Q(N_P, Q_I) - (\dim \mathrm{Gr}_{\mathbf{f}}(I) - \dim \mathrm{Ext}^1_Q(N_I, N_I)) - \\
-(\dim \mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P) - \dim \mathrm{Ext}^1_Q(Q_P, Q_P)) = \\
= \langle \mathbf{e}, \mathbf{d} \rangle - \langle \mathbf{e}-\mathbf{f}, \mathbf{d}-\mathbf{f} \rangle - \langle \mathbf{f}, \mathbf{d}-\mathbf{f} \rangle - \langle \mathbf{e}-\mathbf{f}, \mathbf{f} \rangle + \dim \mathrm{Ext}^1_Q(N_I, N_I) + \dim \mathrm{Ext}^1_Q(Q_P, Q_P) = \\
= \langle \mathbf{f}, \mathbf{f} \rangle + \dim \mathrm{Ext}^1_Q(N_I, N_I) + \dim \mathrm{Ext}^1_Q(Q_P, Q_P)
\end{gathered}$$

for the codimension of Z in $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$. Assume that this codimension equals 1. Since the Euler form (Q being Dynkin) is positive definite, the summand $\langle \mathbf{f}, \mathbf{f} \rangle$ is nonnegative. If it equals 0, then $\mathbf{f}$ equals 0, and N_I and Q_P are just the zero representations, a contradiction to the assumption $\mathrm{Ext}^1_Q(N_I, Q_P) \neq 0$. Thus $\langle \mathbf{f}, \mathbf{f} \rangle = 1$ and both other summands are zero, thus N_I and Q_P are both isomorphic to the exceptional representation of dimension vector $\mathbf{f}$. But this implies vanishing of $\mathrm{Ext}^1_Q(N_I, Q_P)$ and thus nonsingularity of N. □

5. Cell decomposition

Let Q be a Dynkin quiver, P and I respectively a projective and an injective representation of Q. Let $M := P \oplus I$ and let $\mathrm{Gr} = \mathrm{Gr}_{\mathbf{e}}(M)$ where $\mathbf{e} = \mathbf{dim}\, P$. In this section we construct a cellular decomposition of Gr.

The indecomposable direct summands of M are either injective or projective. In particular they are thin, that is, the vector space at every vertex is at most one–dimensional. The set of generators of these one–dimensional spaces form a linear basis of M which we denote by $\mathbf{B}$. To each indecomposable summand L of M we assign an integer $d(L)$, the *degree* of L, so that if $\mathrm{Hom}_Q(L, L') \neq 0$ then $d(L) < d(L')$ and so that all the degrees are different. In particular the degrees of the homogeneous vectors of I are strictly bigger than the ones of P (in case there is a projective–injective summand in both P and I we choose the degree of the copy in I to be bigger than the degree of the copy in P). To every vector of L we assign degree $d(L)$. In particular every element v of $\mathbf{B}$ has an assigned degree $d(v)$. In view of [6] the one–dimensional torus $T = \mathbf{C}^*$ acts on Gr as follows: for every $v \in \mathbf{B}$ and every $\lambda \in T$ we define

$$\lambda \cdot v := \lambda^{d(v)} v. \tag{5.1}$$

This action extends uniquely to an action on M and induces an action on Gr. The T–fixed points are precisely the points of Gr generated by a part of $\mathbf{B}$, that is, the coordinate sub–representations of $P \oplus I$ of dimension vector $\mathbf{dim}\, P$.

We denote the (finite) set of T–fixed points of Gr by Gr^T.

For every $L \in \mathrm{Gr}^T$ the torus acts on the tangent space $T_L(\mathrm{Gr}) \simeq \mathrm{Hom}_Q(L, M/L)$. More explicitly, the vector space $\mathrm{Hom_Q(L, M/L)}$ has a basis given by elements which associate to a basis vector $v \in L \cap \mathbf{B}$ a non–zero element $v' \in M/L \cap \mathbf{B}$ and such element is homogeneous of degree $d(v') - d(v)$ [10]. We denote by $\mathrm{Hom}_Q(L, M/L)^+$ the vector subspace of $\mathrm{Hom}_Q(L, M/L)$ generated by the basis elements of *positive* degree.

Since Gr is projective, for every $N \in \mathrm{Gr}$ the limit $\lim_{\lambda\to 0} \lambda.N$ exists and moreover it is T–fixed (see e.g. [9, Lemma 2.4.3]). For every $L \in \mathrm{Gr}^T$ we consider its *attracting set*

$$\mathcal{C}(L) = \{N \in \mathrm{Gr} |\ \lim_{\lambda\to 0} \lambda \cdot N = L\}.$$

The action (5.1) on Gr induces an action on $\mathrm{Gr}_{\mathbf{f}}(I)$ and $\mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P)$ so that the map

$$\zeta : \mathcal{S}_{\mathbf{f}} \to \mathrm{Gr}_{\mathbf{f}}(I) \times \mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P) \tag{5.2}$$

is T–equivariant. Since both $\mathrm{Gr}_{\mathbf{f}}(I)$ and $\mathrm{Gr}_{\mathbf{e}-\mathbf{f}}(P)$ are smooth (P and I being rigid), we apply [2] and we get cellular decompositions into attracting sets

$$\mathrm{Gr}_{\mathbf{f}}(I) = \textstyle\coprod_{L_I \in \mathrm{Gr}_{\mathbf{f}}(I)^T} \mathcal{C}(L_I) \quad \text{and} \quad \mathrm{Gr}_{\mathbf{g}}(P) = \textstyle\coprod_{L_P \in \mathrm{Gr}_{\mathbf{g}}(P)^T} \mathcal{C}(L_P)$$

and moreover $\mathcal{C}(L_I) \simeq \mathrm{Hom}_Q(L_I, I/L_I)^+$ and $\mathcal{C}(L_P) \simeq \mathrm{Hom}_Q(L_P, P/L_P)^+$.

Theorem 5.1. *For every $L \in \mathrm{Gr}^T$ its attracting set is an affine space isomorphic to $\mathrm{Hom}_Q(L, M/L)^+$. In particular we get a cellular decomposition $\mathrm{Gr} = \coprod_{L \in \mathrm{Gr}^T} \mathcal{C}(L)$. Moreover*

$$\mathcal{C}(L) = \zeta^{-1}(\mathcal{C}(L_I) \times \mathcal{C}(L_P)) \simeq \mathcal{C}(L_I) \times \mathcal{C}(L_P) \times \mathrm{Hom}_Q(L_P, I/L_I). \tag{5.3}$$

Proof. The subvariety $\mathcal{S}_{\mathbf{f}} := \zeta^{-1}(\mathrm{Gr}_{\mathbf{f}}(I) \times \mathrm{Gr}_{\mathbf{dim}\, P - \mathbf{f}}(P))$ is smooth but not projective. Nevertheless it enjoys the following property

$$\text{for all } \ N \in \mathcal{S}_{\mathbf{f}}, \ \ \lim_{\lambda\to 0} \lambda \cdot N \in \mathcal{S}_{\mathbf{f}}. \tag{5.4}$$

Indeed let N be a point of $\mathcal{S}_{\mathbf{f}}$ and let $w_1, \cdots, w_{|\mathbf{e}|}$ be a basis of it (here $|\mathbf{e}| = \sum_{i \in Q_0} e_i$). We write every w_i in the basis $\mathbf{B}$ and we find a vector $v_i \in \mathbf{B}$ which has minimal degree in this linear combination and whose coefficient can be assumed to be 1. We call v_i the leading term of w_i. The sub–representation $N_I = N \cap I$ is generated by those w_i's which belong to I while $N_P = \pi(N) \simeq N/N_I$ is generated by the remaining ones. The torus action is chosen in such a way that the leading term of every $w_j \in N_P$ belongs to P. The limit point $L := \lim_{\lambda\to 0} \lambda \cdot N$ has $v_1, \cdots, v_{|\mathbf{e}|}$ as its basis. The sub–representation $L_I = L \cap I$ is generated precisely by the v_i's which are the leading terms of $w_i \in N_I$. In particular $\mathbf{dim}\, L_I = \mathbf{dim}\, N_I = \mathbf{f}$ and hence $L \in \mathrm{Gr}_{\mathbf{f}}$.

Since the map ζ is T–equivariant, (5.3) follows from (5.4).

It remains to prove that $\mathcal{C}(L) \simeq \mathrm{Hom}_Q(L, M/L)^+$. This is a consequence of the following

$$\mathcal{C}(L_I) \simeq \mathrm{Hom}_Q(L_I, I/L_I)^+, \qquad \mathcal{C}(L_P) \simeq \mathrm{Hom}_Q(L_P, P/L_P)^+$$

$$\mathrm{Hom}_Q(L_P, I/L_I)^+ = \mathrm{Hom}_Q(L_P, I/L_I), \qquad \mathrm{Hom}_Q(L_I, P/L_I)^+ = 0,$$

together with the isomorphism (5.3). $\square$

The following example shows that for $L \in \mathrm{Gr}^T$ and $N \in \mathcal{C}(L)$ it is not true that the tangent spaces at N and L have the same dimension.

Example 5.2. Let

$$\begin{tikzcd}[row sep=small] & & 3 & \\ Q := \quad 1 \arrow[r] & 2 \arrow[ur] \arrow[rr] & & 4 \end{tikzcd}$$

be a Dynkin quiver of type D_4. For every vertex $k \in Q_0$ let P_k and I_k be respectively the indecomposable projective and injective Q–representation at vertex k. Let $P := P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and $I := I_1 \oplus I_2 \oplus I_3 \oplus I_4$. We consider the variety $\mathrm{Gr}_{(1233)}(I \oplus P)$. We assign degree $deg(P_k) := 4 - k$ and $deg(I_k) := 4 + k$ for $k = 1, 2, 3, 4$. We notice that $I_4 \oplus I_3 \oplus I_2$ has an indecomposable sub–representation N_I of dimension vector (1211) such that $\lim_{\lambda \to 0} \lambda \cdot N_I = I_4 \oplus (0110) := L$, where (0110) denotes the indecomposable sub–representation of I_3 of dimension vector (0110). We have $I/L_I \simeq I/N_I \simeq I_1 \oplus I_1 \oplus I_2$ and $\dim \mathrm{Hom}_Q(N_I, I/N_I) = \dim \mathrm{Hom}_Q(L_I, I/L_I) = 3$. Let us choose L_P inside P of dimension vector (0022) so that $L_I \oplus L_P \in \mathrm{Gr}$. We choose $L_P \simeq P_3^2 \oplus P_4^2$ where P_3^2 is a sub–representation of $P_1 \oplus P_3$ and P_4^2 is in $P_1 \oplus P_2$. The quotient $P/L_P \simeq I_2 \oplus (0110) \oplus P_4$. Now $\dim \mathrm{Ext}^1_Q(N_I, P/L_P) = \dim \mathrm{Ext}^1_Q(N_I, P_4) = 1$ but $\dim \mathrm{Ext}^1_Q(L_I, P/L_P) = \dim \mathrm{Ext}^1_Q(I_4, (0110)) + \dim \mathrm{Ext}^1_Q((0110), P_4) = 2$.

6. Poincaré polynomials in type A and Genocchi numbers

In this section we compute the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ for equioriented quiver of type A and derive some combinatorial consequences.

6.1. **Equioriented quiver of type** A. For two non-negative integers k and l the q-binomial coefficient $\binom{k}{l}_q$ is defined by the formula

$$\binom{k}{l}_q = \frac{k_q!}{l_q!(k-l)_q!}, \text{ where } k_q! = (1-q)(1-q^2)\dots(1-q^k).$$

We also set $\binom{k}{l}_q = 0$ if $k < l$ or $k < 0$ or $l < 0$.

Recall (see Proposition 2.7) that $\mathcal{F}^a_{n+1}$ is isomorphic to $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$, where P (resp. I) is the direct sum of all projective (resp. injective) indecomposable representations of Q. According to Proposition 4.4, in order to compute the Poincaré polynomial of $\mathrm{Gr}_{\mathbf{e}}(P \oplus I)$, we only need to compute the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{g}}(P)$ and $\mathrm{Gr}_{\mathbf{f}}(I)$ for arbitrary dimension vectors $\mathbf{g} = (g_1, \dots, g_n)$ and $\mathbf{f} = (f_1, \dots, f_n)$. Let us compute these polynomials in a slightly more general setting. Namely, fix two collections of non-negative integers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ and set $P = \bigoplus_{i=1}^n P_i^{a_i}$, $I = \bigoplus_{i=1}^n I_i^{b_i}$.

Lemma 6.1.

$$P_{Gr_{\mathbf{g}}(P)}(q) = \prod_{k=1}^{n} \binom{a_1 + \dots + a_k - g_{k-1}}{g_k - g_{k-1}}_q, \tag{6.1}$$

$$P_{Gr_{\mathbf{f}}(I)}(q) = \prod_{k=1}^{n} \binom{b_{n+1-k} + f_{n+2-k}}{f_{n+1-k}}_q \tag{6.2}$$

with the convention $g_0 = 0$, $f_{n+1} = 0$.

Proof. We first prove the first formula by induction on n. For $n = 1$ the formula reduces to the well-known formula for the Poincaré polynomials of the classical Grassmannians. Let $n > 1$. Consider the map $\mathrm{Gr}_{\mathbf{g}}(P) \to \mathrm{Gr}_{g_1}((P)_1)$. We claim that this map is a fibration with the base $\mathrm{Gr}_{g_1}(\mathbb{C}^{a_1})$ and a fiber isomorphic to $\mathrm{Gr}_{(g_2-g_1,g_3-g_1,\dots,g_n-g_1)}(P_1^{a_1-g_1}\bigoplus_{i=2}^n P_i^{a_i})$. In fact, an element of $\mathrm{Gr}_{\mathbf{g}}(P)$ is a collection of spaces $(V_1,\dots,V_n)$ such that $V_i\subset(P)_i$. We note that all the maps in P corresponding to the arrows $i\to i+1$ are embeddings. Therefore, if one fixes a g_1-dimensional subspace $V_1\subset P_1$, this automatically determines the g_1-dimensional subspaces to be contained in $V_2,\dots,V_n$. This proves the claim. Now formula (6.1) follows by induction.

In order to prove (6.2), we consider the map

$$\mathrm{Gr}_{\mathbf{f}}(I)\to\mathrm{Gr}_{\mathbf{f}^*}(I^*): N\mapsto\{\varphi\in I^*\,|\,\varphi(N)=0\},$$

where $I^*=\mathrm{Hom}_{\mathbf{C}}(I,\mathbf{C})$ and $\mathbf{f}^*=(f_1^*,\cdots,f_{n-1}^*)=\mathbf{dim}\,I-\mathbf{f}$ is defined by

$$f_k^*=b_k+b_{k+1}+\cdots+b_n-f_k.$$

Now I^* can be identified with $\bigoplus_{i=1}^n P_{n+1-i}^{b_i}$ by acting on the vertices of Q with the permutation $\omega: i\mapsto n-i$ for every $i=1,2,\cdots,n-1$. We hence have an isomorphism

$$\mathrm{Gr}_{\mathbf{f}}(I)\simeq\mathrm{Gr}_{\omega\mathbf{f}^*}(\bigoplus_{i=1}^n P_i^{b_i}).$$

Substituting into (6.1), we obtain (6.2). □

Theorem 6.2. *Let* $\mathrm{Gr}=\mathrm{Gr}_{\mathbf{e}}(I\oplus P)$ *with* I *and* P *as above. Then the Poincaré polynomial of* Gr *is given by* $P_{\mathrm{Gr}}(q)=$

(6.3)

$$=\sum_{\mathbf{f}+\mathbf{g}=\mathbf{e}}q^{\sum_{i=1}^n g_i(a_i-f_i+f_{i+1})}\prod_{k=1}^n\binom{a_1+\cdots+a_k-g_{k-1}}{g_k-g_{k-1}}_q\prod_{k=1}^n\binom{b_{n+1-k}+f_{n+2-k}}{f_{n+1-k}}_q.$$

Proof. Recall the decomposition $\mathrm{Gr}_{\mathbf{e}}(P\oplus I)=\sqcup_{\mathbf{f}}\mathcal{S}_{\mathbf{f}}$. Each stratum $\mathcal{S}_{\mathbf{f}}$ is a total space of a vector bundle over $\mathrm{Gr}_{\mathbf{g}}(P)\times\mathrm{Gr}_{\mathbf{f}}(I)$ with fiber over a point $(N_P,N_I)\in\mathrm{Gr}_{\mathbf{g}}(P)\times\mathrm{Gr}_{\mathbf{f}}(I)$ isomorphic to $\mathrm{Hom}_Q(N_P,I/N_I)$. Since $\mathrm{Ext}^1_Q(N_P,I/N_I)=0$, we obtain $\dim\mathrm{Hom}_Q(N_P,I/N_I)=\langle\mathbf{g},\mathbf{dim}\,I-\mathbf{f}\rangle$. Since Q is the equioriented quiver of type A_n, we obtain

$$\langle\mathbf{g},\mathbf{dim}\,I-\mathbf{f}\rangle=\sum_{i=1}^n g_i(a_i-f_i+f_{i+1}).$$

Now our Theorem follows from formulas (6.1) and (6.2). □

Now let $a_i=b_i=1$, $i=1,\dots,n$. Then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\,P}(P\oplus I)$ is isomorphic to $\mathcal{F}^a_{n+1}$. We thus obtain the following corollary.

Corollary 6.3. *The Poincaré polynomial of the complete degenerate flag variety* $\mathcal{F}^a_{n+1}$ *is equal to*

$$\sum_{f_1,\dots,f_n\geq0}q^{\sum_{k=1}^n(k-f_k)(1-f_k+f_{k+1})}\prod_{k=1}^n\binom{1+f_{k-1}}{f_k}_q\prod_{k=1}^n\binom{1+f_{k+1}}{f_k}_q,\tag{6.4}$$

(we assume $f_0=f_{n+1}=0$*).*

Now fix a collection $\mathbf{d}=(d_1,\dots,d_s)$ with $0=d_0<d_1<\cdots<d_s<d_{s+1}=n+1$. We obtain the following corollary:

Corollary 6.4. *Define* $a_i=d_i-d_{i-1}$, $b_i=d_{i+1}-d_i$. *Then formula* (6.3) *gives the Poincaré polynomial of the partial degenerate flag variety* $\mathcal{F}^a_{\mathbf{d}}$.

Proof. Follows from Proposition 2.9. □

6.2. **The normalized median Genocchi numbers.** Recall that the Euler characteristic of $\mathcal{F}^a_{n+1}$ is equal to the $(n+1)$-st normalized median Genocchi number h_{n+1} (see [17], Proposition 3.1 and Corollary 3.7). In particular, the Poincaré polynomial (6.4) provides natural q-deformation $h_{n+1}(q)$. We also arrive at the following formula.

Corollary 6.5.

$$h_{n+1} = \sum_{f_1,\dots,f_n\geq 0} \prod_{k=1}^{n} \binom{1+f_{k-1}}{f_k} \prod_{k=1}^{n} \binom{1+f_{k+1}}{f_k} \tag{6.5}$$

with $f_0 = f_{n+1} = 0$.

We note that formula (6.5) can be seen as a sum over the set M_{n+1} of Motzkin paths starting at $(0,0)$ and ending at $(n+1,0)$. Namely, we note that a term in (6.5) is zero unless $f_{i+1} = f_i$ or $f_{i+1} = f_i + 1$ or $f_{i+1} = f_i - 1$ for $i = 1,\dots,n$ (recall that $f_i \geq 0$ and $f_0 = f_{n+1} = 0$). Therefore the terms in (6.5) are labeled by Motzkin paths (see e.g. [12]). We can simplify the expression for h_{n+1}. Namely, for a Motzkin path $\mathbf{f} \in M_{n+1}$ let $l(\mathbf{f})$ be the number of "rises" $(f_{i+1} = f_i + 1)$ plus the number of "falls" $(f_{i+1} = f_i - 1)$. Then we obtain

Corollary 6.6.

$$h_{n+1} = \sum_{\mathbf{f}\in M_{n+1}} \frac{\prod_{k=1}^n (1+f_k)^2}{2^{l(\mathbf{f})}}.$$

We note also that Remark 4.2 produces one more combinatorial definition of the numbers h_{n+1}. Namely, for $1 \leq i \leq j \leq n$ we denote by $S_{i,j}$ the indecomposable representation of Q such that

$$\mathbf{dim}\, S_{i,j} = (\underbrace{0,\dots,0}_{i-1}, \underbrace{1,\dots,1}_{j-i+1}, 0,\dots,0).$$

In particular, the simple indecomposable representation S_i coincides with $S_{i,i}$. Then we have

$$\dim \mathrm{Hom}_Q(S_k, S_{i,j}) = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{otherwise;} \end{cases} \qquad \dim \mathrm{Hom}_Q(S_{i,j}, S_k) = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Recall (see Theorem 4.1) that the Euler characteristic of $\mathcal{F}^a_{n+1}$ is equal to the number of isomorphism classes of pairs $[Q_P], [N_I]$ such that N_I is embedded into $I = \bigoplus_{k=1}^n I_k$, Q_P is a quotient of $P = \bigoplus_{k=1}^n P_k$ and $\mathbf{dim}\, N_I = \mathbf{dim}\, Q_P$. Let

$$N_I = \bigoplus_{1\leq i\leq j\leq n} S_{i,j}^{r_{i,j}}, \quad Q_P = \bigoplus_{1\leq i\leq j\leq n} S_{i,j}^{m_{i,j}}.$$

Then from Remark 4.2 we obtain the following Proposition.

Proposition 6.7. *The normalized median Genocchi number h_{n+1} is equal to the number of pairs of collections of non-negative integers $(r_{i,j})$, $(m_{i,j})$, $1 \leq i \leq j \leq n$ subject to the following conditions for all $k = 1,\dots,n$:*

$$\sum_{k=i}^{n} r_{i,k} \leq 1, \quad \sum_{k=1}^{j} m_{k,j} \leq 1, \quad \sum_{i\leq k\leq j} r_{i,j} = \sum_{i\leq k\leq j} m_{i,j}.$$

7. Cells and the group action in type A

In this section we fix Q to be the equioriented quiver of type A_n.

7.1. **Description of the group.** Let $P = \bigoplus_{i=1}^n P_i$ and $I = \bigoplus_{i=1}^n I_i$. As in the general case, we consider the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P,I) & \mathrm{Aut}_Q(I) \end{bmatrix},$$

which is a subgroup of $\mathrm{Aut}_Q(P \oplus I)$.

Remark 7.1. The whole group of automorphisms $\mathrm{Aut}_Q(P \oplus I)$ is generated by G and $\exp(\mathrm{Hom}_Q(I,P))$. We note that $\mathrm{Hom}_Q(I,P)$ is a one-dimensional space. In fact, $\mathrm{Hom}_Q(I_k,P_l) = 0$ unless $k = n$, $i = 1$ and $I_n \simeq P_1$. Thus G "almost" coincides with $\mathrm{Aut}(P \oplus I)$.

We now describe G explicitly.

Lemma 7.2. *The groups $\mathrm{Aut}_Q(P)$ and $\mathrm{Aut}_Q(I)$ are isomorphic to the Borel subgroup B_n of the Lie group GL_n, that is, to the group of non-degenerate upper-triangular matrices.*

Proof. For $g \in \mathrm{Aut}(P \oplus I)$ let g_i be the component acting on $(P \oplus I)_i$ (the vector space corresponding to the i-th vertex). Then the map $g \mapsto g_n$ gives a group isomorphism $\mathrm{Aut}_Q(P) \simeq B_n$. In fact, $\mathrm{Hom}_Q(P_k,P_l) = 0$ if $k > l$; otherwise $(k \le l)$ $\mathrm{Hom}_Q(P_k,P_l)$ is one-dimensional and is completely determined by the n-th component. Similarly, the map $g \mapsto g_1$ gives a group isomorphism $\mathrm{Aut}_Q(I) \simeq B_n$. □

In what follows, we denote $\mathrm{Aut}_Q(P)$ by B_P and $\mathrm{Aut}_Q(I)$ by B_I.

Proposition 7.3. *The group G is isomorphic to the semi-direct product $\mathbb{G}_a^{n(n+1)/2} \rtimes (B_P \times B_I)$.*

Proof. First, the groups B_P and B_I commute inside G. Second, the group G is generated by B_P, B_I and $\exp(\mathrm{Hom}_Q(P,I))$. The group $\exp(\mathrm{Hom}_Q(P,I))$ is abelian and isomorphic to $\mathbb{G}_a^{n(n+1)/2}$ (the abelian version of the unipotent subgroup of the lower-triangular matrices in SL_{n+1}). In fact, $\mathrm{Hom}_Q(P_i,I_j)$ is trivial if $i > j$ and otherwise $(i \le j)$ it is one-dimensional. Also, $\exp(\mathrm{Hom}_Q(P,I))$ is normal in G. □

We now describe explicitly the structure of the semi-direct product. For this we pass to the level of the Lie algebras. So let $\mathfrak{b}_P$ and $\mathfrak{b}_I$ be the Lie algebras of B_P and B_I, respectively ($\mathfrak{b}_P$ and $\mathfrak{b}_I$ are isomorphic to the Borel subalgebra of $\mathfrak{sl}_n$). Let $(\mathfrak{n}^-)^a$ be the abelian $n(n+1)/2$-dimensional Lie algebra, that is, the Lie algebra of the group $\mathbb{G}_a^{n(n+1)/2}$. Also, let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_{n+1}$. Recall that the degenerate Lie algebra $\mathfrak{sl}_{n+1}^a$ is defined as $(\mathfrak{n}^-)^a \oplus \mathfrak{b}$, where $(\mathfrak{n}^-)^a$ is an abelian ideal and the action of $\mathfrak{b}$ on $(\mathfrak{n}^-)^a$ is induced by the adjoint action of $\mathfrak{b}$ on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{sl}_{n+1}/\mathfrak{b}$. Consider the embedding $\imath_P : \mathfrak{b}_P \to \mathfrak{b}$, $E_{i,j} \mapsto E_{i,j}$ and the embedding $\imath_I : \mathfrak{b}_I \to \mathfrak{b}$, $E_{i,j} \mapsto E_{i+1,j+1}$. These embeddings define the structures of $\mathfrak{b}_P$- and $\mathfrak{b}_I$-modules on $(\mathfrak{n}^-)^a$.

Proposition 7.4. *The Lie algebra of the group G is isomorphic to $(\mathfrak{n}^-)^a \oplus \mathfrak{b}_P \oplus \mathfrak{b}_I$, where $(\mathfrak{n}^-)^a$ is an abelian ideal and the structure of $\mathfrak{b}_P \oplus \mathfrak{b}_I$-module on $(\mathfrak{n}^-)^a$ is defined by the embeddings $\imath_P$ and $\imath_I$.*

Proof. The Lie algebra of G is isomorphic to the direct sum $\mathrm{End}_Q(P) \oplus \mathrm{End}_Q(I) \oplus \mathrm{Hom}_Q(P,I)$. Recall that the identification $\mathrm{Hom}_Q(P,P) \simeq \mathfrak{b}_P$ is given by $a \mapsto a_n$ and the identification $\mathrm{Hom}_Q(I,I) \simeq \mathfrak{b}_I$ is given by $a \mapsto a_1$, where a_i denotes its i-th component for $a \in \mathrm{Hom}_Q(P \oplus I, P \oplus I)$. Recall (see subsection 2.5) that $(P \oplus I)_1$ is spanned by the vectors $w_{1,j}$, $j = 1, \dots, n+1$ and $w_{1,1} \in (P_1)_1$, $w_{1,j} \in (I_{j-1})_1$ for $j > 1$. Therefore, we have a natural embedding $\mathfrak{b}_I \subset \mathfrak{b}$ mapping the matrix unit $E_{i,j}$ to $E_{i+1,j+1}$. Similarly, $(P \oplus I)_n$ is spanned by the vectors $w_{n,j}$, $j = 1, \dots, n+1$ and

$w_{n,n+1} \in (I_n)_n$, $w_{n,j} \in (P_j)_n$ for $j < n+1$, giving the natural embedding $\mathfrak{b}_I \subset \mathfrak{b}$, $E_{i,j} \mapsto E_{i,j}$. With such a description it is easy to compute the commutator of an element of $\mathfrak{b}_P \oplus \mathfrak{b}_I$ with an element of $\mathrm{Hom}_Q(P,I) \simeq (\mathfrak{n}^-)^a$. □

We now compare G with SL^a_{n+1}. We note that the Lie algebra $\mathfrak{sl}^a_{n+1}$ and the Lie group SL^a_{n+1} have one-dimensional centers. Namely, let θ be the highest root of $\mathfrak{sl}_{n+1}$ and let $e_\theta = E_{1,n} \in \mathfrak{b} \subset \mathfrak{sl}_{n+1}$ be the corresponding element. Then e_θ commutes with everything in $\mathfrak{sl}^a_{n+1}$ and thus the exponents $\exp(te_\theta) \in SL^a_{n+1}$ form the center Z. From Proposition 7.4 we obtain the following corollary.

Corollary 7.5. *The group SL^a_{n+1}/Z is embedded into G.*

7.2. **Bruhat-type decomposition.** The goal of this subsection is to study the G-orbits on the degenerate flag varieties. So let $\mathbf{d} = (d_1, \dots, d_s)$ for $0 = d_0 < d_1 < \dots < d_s < d_{s+1} = n+1$.

Lemma 7.6. *The group G acts naturally on all degenerate flag varieties $\mathcal{F}^a_{\mathbf{d}}$.*

Proof. By definition, G acts on the degenerate flag variety $\mathcal{F}^a_{n+1}$. We note that there exists a map $\mathcal{F}^a_{n+1} \to \mathcal{F}^a_{\mathbf{d}}$ defined by $(V_1, \dots, V_n) \mapsto (V_{d_1}, \dots, V_{d_s})$. Since G acts fiberwise with respect to this projection, the G-action on $\mathcal{F}^a_{n+1}$ induces a G-action on $\mathcal{F}^a_{\mathbf{d}}$. □

We first work out the case $s = 1$, that is, the G-action on the classical Grassmannian $\mathrm{Gr}_d(n+1)$. We first recall the cellular decomposition from [17]. The cells are labelled by torus fixed points, that is, by collections $L = (l_1, \dots, l_d)$ with $1 \le l_1 < \dots < l_d \le n+1$. The corresponding cell is denoted by $\mathcal{C}_L$. Explicitly, the elements of $\mathcal{C}_L$ can be described as follows. Let k be an integer such that $l_k \le d < l_{k+1}$. Recall the basis $w_1, \dots, w_{n+1}$ of $W = \mathbb{C}^{n+1}$. We denote by $x_L \in \mathrm{Gr}_{d+1}(n+1)$ the linear span of $w_{l_1}, \dots, w_{l_d}$. Then a d-dimensional subspace V belongs to $\mathcal{C}_L$ if and only if it has a basis $e_1, \dots, e_d$ such that for some constants c_p, we have

$$e_j = w_{l_j} + \sum_{p=1}^{l_j-1} c_p w_p + \sum_{p=d+1}^{n+1} c_p w_p \qquad \text{for } j = 1, \dots, k; \tag{7.1}$$

$$e_j = w_{l_j} + \sum_{p=d+1}^{l_j-1} c_p w_p \qquad \text{for } j = k+1, \dots, d. \tag{7.2}$$

For example, $x_L \in \mathcal{C}_L$.

Lemma 7.7. *Each G-orbit on the Grassmannian $\mathcal{F}^a_{(d)}$ contains exactly one torus fixed point x_L. The orbit $G \cdot x_L$ coincides with $\mathcal{C}_L$.*

Proof. Follows from the definition of G. □

We prove now that the G-orbits in $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ described in Theorem 4.1 are cells. Moreover, we prove that this cellular decomposition coincides with the one of [17]. Let

$$P = \bigoplus_{i=1}^{s} P_i^{d_i - d_{i-1}}, \quad I = \bigoplus_{i=1}^{s} I_i^{d_{i+1} - d_i}.$$

We start with the following lemma.

Lemma 7.8. *Let $N_I \subset I$ be a subrepresentation of I. Then there exists a unique torus fixed point $N_I^\circ \in \mathrm{Gr}_{\mathbf{dim}N_I}(I)$ such that $N_I \simeq N_I^\circ$. Similarly, for $N_P \subset P$ there exists a unique torus fixed point $N_P^\circ \in \mathrm{Gr}_{\mathbf{dim}N_P}(P)$ such that $P/N_P \simeq P/N_P^\circ$.*

Proof. We prove the first part, the second part can be proved similarly. Recall the vectors $w_{i,j} \in (I_{j-1})_i$, $i = 1, \ldots, n$, $j = i+1, \ldots, n+1$ such that $w_{i,j} \mapsto w_{i+1,j}$ if $j \neq i+1$ and $w_{i,j} \mapsto 0$ if $j = i+1$. For each indecomposable summand $S_{k,l}$ of N_I we construct the corresponding indecomposable summand of N_I°. Namely, we take the subrepresentation in I_l of dimension vector

$$(\underbrace{0, \ldots, 0}_{k-1}, \underbrace{1, \ldots, 1}_{l-k+1}, 0, \ldots, 0).$$

Since each I_l is torus-fixed, our lemma is proved. □

Remark 7.9. This lemma is not true for injective modules over Dynkin quivers in general. Namely, consider the quiver from Example 4.3 and let $N_I \subset I_3 \oplus I_4$ be indecomposable Q-module of dimension $(1, 2, 1, 1)$. Then for such N_I Lemma 7.8 does not hold.

Corollary 7.10. *Each G-orbit in $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ contains exactly one torus fixed point, and each such point is contained in some orbit.*

Proof. Follows from Theorem 4.1 and Lemma 7.8. □

We note that any torus fixed point in $\mathcal{F}^a_{\mathbf{d}}$ is the product of fixed points in the Grassmannians $\mathcal{F}^a_{(d_i)}$, $i = 1, \ldots, s$. Therefore any such point is of the form $\prod_{i=1}^s x_{L^i}$. We denote this point by $x_{L^1,\ldots,L^s}$.

Theorem 7.11. *The orbit $G \cdot x_{L^1,\ldots,L^s}$ is the intersection of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ with the product of cells $\mathcal{C}_{L^i}$.*

Proof. First, obviously $G \cdot x_{L^1,\ldots,L^s} \subset \mathcal{F}^a_{\mathbf{d}} \bigcap \prod_{i=1}^s \mathcal{C}_{L^i}$. Second, since each orbit contains exactly one torus fixed point and the intersection on the right hand side does not contain other fixed points but $x_{L^1,\ldots,L^s}$, the theorem is proved. □

Corollary 7.12. *The G-orbits on $\mathcal{F}^a_{\mathbf{d}}$ produce the same cellular decomposition as the one constructed in* [17].

Proof. The cells from [17] are labeled by collections $L^1, \ldots, L^s$ (whenever $x_{L^1,\ldots,L^s} \in \mathcal{F}^a_{\mathbf{d}}$) and the corresponding cell $\mathcal{C}_{L^1,\ldots,L^s}$ is given by

$$\mathcal{C}_{L^1,\ldots,L^s} = \mathcal{F}^a_{\mathbf{d}} \bigcap \prod_{i=1}^s \mathcal{C}_{L^i}.$$

□

7.3. **Cells and one-dimensional torus.** In this subsection we show that the cellular decomposition described above coincides with the one constructed in section 5. We describe the case of the complete flag varieties (in the parabolic case everything works in the same manner). Recall that the action of our torus is given by the formulas

$$\lambda \cdot w_{i,j} = \begin{cases} \lambda^{j-2} w_{i,j}, & \text{if } j > i, \\ \lambda^{j+n-1} w_{i,j}, & \text{if } j \leq i. \end{cases} \tag{7.3}$$

For $n = 4$ we have the following picture (compare with (2.2)):

$$
\begin{array}{llllllll}
\lambda^7 & & & & & & & w_{4,4} \\
\lambda^6 & & & & & w_{3,3} & \to & w_{4,3} \\
\lambda^5 & & & w_{2,2} & \to & w_{3,2} & \to & w_{4,2} \\
\lambda^4 & w_{1,1} & \to & w_{2,1} & \to & w_{3,1} & \to & w_{4,1} \\
\lambda^3 & w_{1,5} & \to & w_{2,5} & \to & w_{3,5} & \to & w_{4,5} \\
\lambda^2 & w_{1,4} & \to & w_{2,4} & \to & w_{3,4} & & \\
\lambda & w_{1,3} & \to & w_{2,3} & & & & \\
1 & w_{1,2} & & & & & &
\end{array}
\tag{7.4}
$$

Proposition 7.13. *The attracting* $(\lambda \to 0)$*-cell of a fixed point* x *of the one-dimensional torus* (7.3) *coincides with the* G*-orbit* $G \cdot x$.

Proof. First, consider the action of our torus on each Grassmannian $\mathrm{Gr}_d((P \oplus I)_d)$. Then formulas (7.1) and (7.2) imply that the attracting cells $(\lambda \to 0)$ coincide with the cells $\mathcal{C}_L$. Now Theorem 7.11 implies our proposition. □

We note that the one-dimensional torus (7.3) does not belong to SL^a_{n+1} (more precisely, to the image of SL^a_{n+1} in the group of automorphisms of the degenerate flag variety). However, it does belong to a one-dimensional extension $SL^a_{n+1} \rtimes \mathbb{C}^*_{PBW}$ of the degenerate group (see [17], Remark 1.1). Recall that the extended group is the Lie group of the extended Lie algebra $\mathfrak{sl}^a_{n+1} \oplus \mathbb{C}d_{PBW}$, where d_{PBW} commutes with the generators $E_{i,j} \in \mathfrak{sl}_{n+1}$ as follows: $[d_{PBW}, E_{i,j}] = 0$ if $i < j$ and $[d_{PBW}, E_{i,j}] = E_{i,j}$ if $i > j$. In particular, the action of the torus $\mathbb{C}^*_{PBW} = \{\exp(\lambda d_{PBW}), \lambda \in \mathbb{C}\}$ on $w_{i,j}$ is given by the formulas: $\lambda \cdot w_{i,j} = w_{i,j}$ if $i \geq j$ and $\lambda \cdot w_{i,j} = \lambda w_{i,j}$ if $i < j$. For example, for $n = 4$ one has the following picture (vectors come equipped with the weights):

$$
\begin{array}{ccccccc}
\lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet \\
\lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet & & 1\bullet \\
\lambda\bullet & \to & \lambda\bullet & & 1\bullet & \to & 1\bullet \\
\lambda\bullet & & 1\bullet & \to & 1\bullet & \to & 1\bullet \\
1\bullet & \to & 1\bullet & \to & 1\bullet & \to & 1\bullet
\end{array}
$$

Proposition 7.14. *The one-dimensional torus* (7.3) *sits inside the extended group* $SL^a_{n+1} \rtimes \mathbb{C}^*_{PBW}$.

Proof. For any collection of integers $k_1, \ldots, k_{n+1}$ there exists a one-dimensional torus $\mathbb{C}^*_{(k_1,\ldots,k_{n+1})}$ inside the Cartan subgroup of SL^a_{n+1} which acts on $w_{i,j}$ by the formula $w_{i,j} \mapsto \lambda^{k_j} w_{i,j}$. Direct check shows that the torus (7.3) acts as $\mathbb{C}^*_{(n,n+1,\ldots,2n)} \times (\mathbb{C}^*_{PBW})^{-n-1}$. □

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