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The Brauer–Clifford group and rational forms of finite groups

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ABSTRACT

Let k be a perfect field and G be a finite subgroup of $\mathrm{GL}_n(\bar{k})$. The aim of this paper is to study the following question: Is it possible to find a subgroup of $\mathrm{GL}_n(\bar{k})$, conjugate to G, with a set of fundamental polynomial invariants whose coefficients lie in the field k? Using the Brauer–Clifford group introduced by Turull this question is split into two natural parts. The first part is to recognize elements in the Brauer–Clifford group and the second part is to decide a field theoretic question involving Galois cohomology. For the first part a subgroup of the Brauer–Clifford group, which can be identified with a subdirect product of the Brauer group and a second cohomology group, is introduced. The main question is answered completely if G is an absolutely irreducible subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}})$ and isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$. Furthermore, if k is the real field or a finite field then a satisfactory treatment is possible for any finite group.

1. Introduction

Nineteenth century mathematicians like Klein, Fricke, Maschke and Valentiner constructed representations of finite groups like $\mathrm{PSL}_2(7)$, $\mathrm{SL}_2(5)$ or $3.A_6$ and calculated fundamental polynomial invariants. It often turned out that these fundamental invariants are polynomials with rational coefficients whereas the matrices of the representations have irrational entries. More recently J. Michel and I. Marin observed this phenomenon when studying automorphisms of complex reflection groups in [10].

Let k be a perfect field, K a finite extension of k and G a finite subgroup of $\mathrm{GL}_n(\bar{k})$. This paper is centered around the question: Is it possible to find a subgroup of $\mathrm{GL}_n(K)$, conjugate to G in $\mathrm{GL}_n(\bar{k})$, with a set of fundamental polynomial invariants whose coefficients lie in the field k? More precisely, find necessary and sufficient conditions on the field K to decide this question.

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The important observation, discussed in section two, is that if G is a finite subgroup of $\mathrm{GL}_n(K)$, then there exists a set of fundamental invariants whose coefficients lie in the field k if and only if the GALOIS group of the GALOIS closure of K acts on G as group automorphisms. This defines a special action of a subgroup U of the automorphism group of G on the enveloping algebra of the natural representation of G as k-algebra automorphisms. In general, algebras with a finite group U acting on them appear in CLIFFORD theory and can be studied via the BRAUER–CLIFFORD group, a group introduced by Alexandre Turull in [18].

Some natural operations on the BRAUER–CLIFFORD group are introduced in the beginning of Section 3. Those operations lead to the notion of an equivariant splitting field. Assuming that G is an absolutely irreducible matrix group, the main Theorem 3.11 says: The answer to the main question is affirmative if and only if K is an equivariant splitting field of the enveloping algebra of the natural representation of G, considered in a certain way as an element of the BRAUER–CLIFFORD group. This element admits an intrinsic definition using the group algebra. Hence the main question is split into two parts. The first part is to recognize elements in BRAUER–CLIFFORD group and the second part is to decide if a field K is an equivariant splitting field. One result is that both problems can be reduced to the case that the finite group U is a p-group.

Section 4 treats a subgroup of the BRAUER–CLIFFORD group which can be identified with a subdirect product of the BRAUER group and a second cohomology group. Given an element of this group, it is possible to calculate necessary and sufficient conditions on a field K to be an equivariant splitting field. If k is a number field, those conditions are given in terms of HASSE invariants of a certain crossed product algebra. So the main question can be answered if k is a number field and the element of the BRAUER–CLIFFORD group, associated to the finite matrix group G, lies in this particular subgroup.

The case that G is a subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}})$ and isomorphic to $\mathrm{PSL}_2(q)$ where q is a power of an odd prime is treated completely in the last section. It is shown that G is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{Q}(\chi))$, where χ is the natural character of G, with a set of fundamental invariants whose coefficients lie in the minimal possible subfield of $\mathbb{Q}(\chi)$. After that, the cases of finite fields and the real field $\mathbb{R}$ are considered for any finite group. In those cases the main question admits a satisfactory treatment. Furthermore, this provides another proof of a result of [8] concerning twisted FROBENIUS–SCHUR indicators.

2. Invariant theory and forms of finite groups

Assume that k is a perfect field and $\bar{k}$ its algebraic closure. Let G be a finite subgroup of $\mathrm{GL}_n(\bar{k})$. Then there is a natural action of G on $V := \bar{k}^{n\times 1}$ and also on the dual space V^* by $g\omega := \omega \circ g^{-1}$. Choosing a basis $x_1, \ldots, x_n$ of V^*, one can identify the ring of polynomial functions on V with $\bar{k}[x_1, \ldots, x_n]$ and gets an action of G on $\bar{k}[x_1, \ldots, x_n]$. The *ring of invariants* is denoted by $\bar{k}[x_1, \ldots, x_n]^G$.

Viewing $\mathrm{GL}_n(\bar{k})$ as an algebraic group turns G into a (totally disconnected) algebraic group. If G is defined over k then G is called a *k-form*. Let K be a finite extension of k and a subfield of $\bar{k}$. If G is a subgroup of $\mathrm{GL}_n(K)$ and defined over k then G is called a *(K/k)-form*. A finite matrix group G' is said to be a *(K/k)-form of G* if G' is a (K/k)-form and conjugate to G in $\mathrm{GL}_n(\bar{k})$.

It is well known from invariant theory that a set of fundamental invariants uniquely determines the corresponding matrix group (as an algebraic group). Hence, given a set of fundamental invariants whose coefficients lie in k the corresponding matrix group is a k-form.

Assume that G is a (K/k)-form and let Γ be the GALOIS group of the GALOIS closure of K. Then Γ acts on G as group automorphisms by applying the elements of Γ to the entries of the matrices $g \in G$.

The natural action of Γ on K induces an action on $K[x_1, \ldots, x_n]$. Combine this action with the action of Γ on G to see that Γ acts on the ring of polynomial invariants as semilinear K-algebra automorphisms. This leads to the converse of the statement that k-forms can be constructed from fundamental invariants whose coefficients lie in k.

Theorem 2.1. *Let $G \leqslant \mathrm{GL}_n(K)$ be a finite matrix group. Then G is a (K/k)-form if and only if there exists a set of fundamental polynomial invariants with coefficients in k.*

Proof. Replacing K by its GALOIS closure, it is possible to assume that (K/k) is a GALOIS extension with GALOIS group Γ. Define $R := K[x_1, \dots, x_n]$ and denote by $R^G_{\leqslant d}$ the finite dimensional vector space over K of polynomial invariants of degree at most d. By the reasoning made above the Γ-action on $R^G_{\leqslant d}$ is semilinear, and so we can apply GALOIS descent for finite dimensional vector spaces. Using the SPEISER lemma [4, Lemma 2.3.8] it follows that $R^G_{\leqslant d} = (R^G_{\leqslant d})^\Gamma \otimes_k K$ and it is clear that the elements of $(R^G_{\leqslant d})^\Gamma$ are polynomials with coefficients in k. If d is big enough this shows that it is possible find a set of fundamental invariants in $(R^G_{\leqslant d})^\Gamma$. $\square$

Example 2.2. Let $k = \mathbb{Q}$, $G = Q_8$ and $\langle a, b \mid a^4, a^2b^2, abab^{-1}\rangle$ a presentation of G. A faithful matrix representation of G is given by:

$$\Delta(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \Delta(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A set of fundamental invariants can be chosen as $\{x_1^2x_2^2, x_1^4 + x_2^4, x_1x_2^5 - x_2x_1^5\}$. Hence the matrix group $\Delta(G)$ is a $(\mathbb{Q}[i], \mathbb{Q})$-form. Applying complex conjugation to the entries of the matrices induces an injective homomorphism $\mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q}) \to \mathrm{Aut}(G)$. The image of this homomorphism is the group generated by the inner automorphism given by conjugation with $\Delta(b)$.

It is possible to construct a series of k-forms geometrically.

Example 2.3. Let $X \subset \mathbb{P}^n$ be a projective algebraic hypersurface defined over k and assume that the group G of projective motions of X is finite. Identify G with a subgroup of $\mathrm{PGL}_{n+1}(\bar{k})$ and let $p \in k[x_1, \dots, x_{n+1}]$ be a homogeneous polynomial defining X. Then it is easy to see that the stabilizer S of p in $\mathrm{GL}_{n+1}(\bar{k})$ is finite and defined over k. Note that S is a central extension of G.

Remark 2.4. If X is a plane projective algebraic curve defined over k and of genus greater than 1 then by HURWITZ theorem [6] the automorphism group of X is finite. Let k be the field of rational numbers, then the KLEIN QUARTIC [9] and the VALENTINER SEXTIC [1] are examples of such curves. Applying 2.3 one obtains the three dimensional complex representations of $\mathrm{PSL}_2(7)$ and $3.\mathrm{A}_6$ with a set of fundamental invariants whose coefficients are integers.

Recall that if G is a (K/k)-form, then the GALOIS group of the GALOIS closure of K acts on G as group automorphisms. Replacing K, if necessary, by a subfield, it can be assumed that this action induces an injective group homomorphism from $\mathrm{Gal}(K/k)$ to $\mathrm{Aut}(G)$. Let U be the image of this homomorphism, then G is called a *$(K/k, U)$-form*. The problem with this definition is that it is hard to say what is meant by a $(K/k, U)$-form of a given finite subgroup G. This is because of U depending on the (K/k)-form of G and not on G. To solve this problem representations are used in the next definition.

Definition 2.5. Let G be a finite subgroup of $\mathrm{GL}_n(\bar{k})$, (K/k) a GALOIS extension with GALOIS group Γ and U a subgroup of the automorphism group of G. A representation $\Delta : G \to \mathrm{GL}_n(K)$ is a $(K/k, U)$-*representation of G* if

1. the representation Δ is equivalent to the natural representation of G,
2. there exists an isomorphism $^- : U \to \Gamma$ such that $\Delta(u(g)) = \bar{u}(\Delta(g))$ for all $g \in G$.

Remark 2.6. The main question can now be rephrased as: Given a finite subgroup G of $\mathrm{GL}_n(\bar{k})$ is it possible to find a subgroup U of $\mathrm{Aut}(G)$ and a GALOIS extension K/k such that there exists a $(K/k, U)$-representation of G?

Assume that the answer of this question is affirmative for a Galois extension (K/k) and a subgroup U of $\mathrm{Aut}(G)$. Then it also affirmative for a certain set of subgroups related to U.

Remark 2.7. Let G be a finite subgroup of $\mathrm{GL}_n(\bar{k})$, U a subgroup of $\mathrm{Aut}(G)$ and $\Delta : G \to \mathrm{GL}_n(K)$ a $(K/k, U)$-representation of G. Assume that φ is an automorphism of G, with the property that the representation $\Delta \circ \varphi^{-1}$ is equivalent to Δ. Then the representation

$$\Theta : G \to \mathrm{GL}_n(K) : g \mapsto \Delta\bigl(\varphi^{-1}(g)\bigr)$$

is a $(K/k, {}^{\varphi}U)$-representation of G.

From the definition of a $(K/k, U)$-representation of G one obtains an obvious necessary condition on the natural character.

Remark 2.8. For the existence of a $(K/k, U)$-representation of G it is necessary that there exists an isomorphism $^{-} : U \to \mathrm{Gal}(K/k)$ with the property that $\bar{u} \circ \chi = \chi \circ u$ for all $u \in U$.

Let Δ be the natural representation G and χ the natural character. Assume that Δ is a $(K/k, U)$-representation, then this has the following consequence on the enveloping algebra $\Delta(kG)$.

Remark 2.9. The natural U-action on G can be uniquely extended to an action of U on the enveloping algebra $\Delta(kG)$ as k-algebra automorphisms.

From now on we will restrict ourselves to the case that G is an absolutely irreducible finite matrix group. An important observation is that absolutely irreducible representations, of a given degree, of a finite group are, up to equivalence, uniquely determined by their characters (cf. [7]).

Proposition 2.10. *Let G be a finite group, K a field and $\Delta, \Theta : G \to \mathrm{GL}_n(K)$ absolutely irreducible representations. Then Δ is equivalent to Θ if and only if $\mathrm{Tr}(\Delta(g)) = \mathrm{Tr}(\Theta(g))$ for all $g \in G$.*

The next lemma shows that if the condition of Remark 2.8 is satisfied, then the natural U-action on G can be uniquely extended to an action of U on $\Delta(kG)$ as k-algebra automorphisms.

Lemma 2.11. *Let (K/k) be a finite* Galois *extension, G a finite absolutely irreducible subgroup of $\mathrm{GL}_n(K)$, $\Delta : G \to \mathrm{GL}_n(K)$ the natural representation of G, χ the natural character and U a subgroup of $\mathrm{Aut}(G)$. Assume that there exists an isomorphism $^{-} : U \to \mathrm{Gal}(K/k)$ with the property that $\bar{u} \circ \chi = \chi \circ u$ for all $u \in U$. Then the natural action of U on G can be uniquely extended to an action on the enveloping algebra $\Delta(kG)$ as k-algebra automorphisms. Furthermore, let U act on K via the isomorphism $^{-}$, then the inclusion $\mathrm{Z}(\Delta(kG)) \to K$ is U-equivariant.*

Proof. Proposition 2.10 implies that for every $u \in U$ there exists a matrix $X_u \in \mathrm{GL}_n(K)$ such that

$$u(g) = X_u \bar{u}(g) X_u^{-1} \quad \text{for all } g \in G \tag{1}$$

Then it is easy to see that the natural U-action on G can be extended uniquely to a U-action on $\Delta(kG)$ as k-algebra automorphisms.

Since G is absolutely irreducible the elements of $\mathrm{Z}(\Delta(kG))$ are scalar matrices with values in K. Hence there is a natural inclusion $\mathrm{Z}(\Delta(kG)) \to K$. Use (1) to obtain the U equivariance of this map. $\square$

3. The Brauer–Clifford group, equivariant splitting fields and forms of finite groups

Let U be a finite group. The following reviews shortly the theory of central simple U-algebras. More details can be found in the work of Turull (cf. [16]). A *U-algebra* A is a finite dimensional associative k-algebra together with a U-action on A as k-algebra automorphisms. The U-algebra A is a *simple* U-algebra if it has only the trivial two sided U-invariant ideals, and A is called *central* if k is the fixed field of the U-action restricted to the center of A. Two U-algebras are U-isomorphic if there exist a U-equivariant k-algebra isomorphism and this is denoted $A \cong_U B$. A U-algebra A is *trivial* if there exists a kU-module M such that $A \cong_U \mathrm{End}_k(M)$ with U acting on $\mathrm{End}_k(M)$ by conjugation. Two central simple U-algebras A and B are *equivalent* if there exist trivial U-algebras E_1 and E_2 such that $A \otimes_k E_1 \cong_U B \otimes_k E_2$. Let L be a central simple commutative U-algebra, then A is a central simple *U-algebra over L* if $\mathrm{Z}(A)$ is isomorphic to L as a central simple U-algebra. The notion of equivalence of central simple U-algebras defines an equivalence relation on the set of all central simple U-algebras over L. If A is a central simple U-algebra over L, denote by $[A]$ its equivalence class. The Brauer–Clifford group is defined as follows.

Definition 3.1. Let L be a commutative central simple U-algebra. As a set, the Brauer–Clifford group $\mathrm{BrCliff}(U, L)$ consists of all equivalence classes of central simple U-algebras over L. The group structure is given by:

$$\mathrm{BrCliff}(U, L) \times \mathrm{BrCliff}(U, L) \to \mathrm{BrCliff}(U, L) : \big([A], [B]\big) \mapsto [A \otimes_L B]$$

For more details, especially proofs of the statements implicit in the definition, see [18].

Now through, the commutative central simple U-algebra L is a field and hence there exists an epimorphism $^- : U \to \mathrm{Gal}(L/k)$. In particular, L is a *U-field* meaning that U acts on L as field automorphisms.

Some natural operations on the Brauer–Clifford group need to be introduced. The first operation is "forgetting the U-action" (cf. [18, Theorem 8.2])

Remark 3.2. There exists a natural homomorphism $\kappa : \mathrm{BrCliff}(U, L) \to \mathrm{Br}(L)^{\mathrm{Gal}(L/k)}$. Denote by $\mathrm{FMBrCliff}(U, L)$ the kernel of κ, then there is the following exact sequence:

$$1 \longrightarrow \mathrm{FMBrCliff}(U, L) \longrightarrow \mathrm{BrCliff}(U, L) \longrightarrow \mathrm{Br}(L)^{\mathrm{Gal}(L/k)}$$

Note that $\mathrm{FMBrCliff}(U, L)$ consists of the equivalence classes of those central simple U-algebras which are isomorphic, as central simple L-algebras, to matrix algebras over L.

The next operation is scalar extension by a U-field. A U-field K is a *U-field extension of L* if it is a field extension of L and the inclusion $L \to K$ is U-equivariant. Assume that A is a central simple U-algebra over L and K is a U-field extension of L, then *scalar extension of A by a U-field K* is the U-algebra $A \otimes_L K$ with U acting diagonally. It is clear that this is a simple U-algebra over K.

Since the fixed field of the U-action on K might be a proper extension of k, scalar extension of A by a U-field K might not be central over k. To avoid this problem, one views scalar extension of A by a U-field K as an algebra over the fixed field of K under U.

To carry the concept of scalar extension with a U-field over to the Brauer–Clifford group, one has to look at the trivial U-algebras. The next proposition deals with those U-algebras.

Proposition 3.3. *Let F be a U-field extension of k, M a kU-module and consider $M \otimes_k F$ as an FU-module. There exists a natural U-equivariant F-algebra isomorphism between the scalar extension of the trivial U-algebra $\mathrm{End}_k(M)$ by F and the U-algebra $\mathrm{End}_F(M \otimes_k F)$ where U acts by conjugation.*

Proof. Consider

$$\mathrm{End}_k(M) \otimes_k F \to \mathrm{End}_F(M \otimes_k F) : \psi \otimes c \mapsto c(\psi \otimes \mathrm{id})$$

with $\psi \in \mathrm{End}_k(M)$ and $c \in F$. $\square$

The next lemma defines the scalar extension map on the Brauer–Clifford group.

Lemma 3.4. *Let K be a U-field extension of L. Then scalar extension by K induces a group homomorphism*:

$$\mathrm{ext}_{K/L} : \mathrm{BrCliff}(U, L) \to \mathrm{BrCliff}(U, K) : [A] \mapsto [A \otimes_L K]$$

Proof. We have to show that this map is well defined. Let F be the fixed field of K under U. Assume that A and B are equivalent central simple U-algebras over L and let M_1, M_2 be kU-modules such that $A \otimes_k \mathrm{End}_k(M_1) \cong_U B \otimes_k \mathrm{End}_k(M_2)$. Consider $\mathrm{End}_F(M_i \otimes F)$ for $i = 1, 2$ as trivial U-algebras. Use Proposition 3.3 to calculate:

$$\begin{aligned}(A \otimes_L K) \otimes_F \mathrm{End}_F(M_1 \otimes_k F) &\cong_U A \otimes_L K \otimes_F \big(\mathrm{End}_k(M_1) \otimes_k F\big)\\ &\cong_U \big(A \otimes_k \mathrm{End}_k(M_1)\big) \otimes_F K\\ &\cong_U \big(B \otimes_k \mathrm{End}_k(M_2)\big) \otimes_F K\\ &\cong_U (B \otimes_L K) \otimes_F \mathrm{End}_F(M_2 \otimes_k F)\end{aligned}$$

Hence $A \otimes_L K$ is equivalent to $B \otimes_L K$ as central simple U-algebras over K and $\mathrm{ext}_{K/L}$ is well defined. The homomorphism property is obvious. $\square$

The following remark considers a sequence of U-field extensions.

Remark 3.5. Let K be a U-field extension of L and L be a U-field extension of k, then $\mathrm{ext}_{K/k} = \mathrm{ext}_{K/L} \circ \mathrm{ext}_{L/k}$.

Another natural map of the Brauer–Clifford group can be obtained via restriction to subgroups of U.

Lemma 3.6. *Let N be a subgroup of U, then the map* $\mathrm{res}_N : \mathrm{BrCliff}(U, L) \to \mathrm{BrCliff}(N, K^N L) : [A] \mapsto [A \otimes_L K^N L]$ *is a group homomorphism.*

Proof. Analogous to the proof of 3.4. $\square$

Using the operations introduced so far, it is possible to define the notion of an equivariant splitting field of a central simple U-algebra.

Definition 3.7. Let A be a central simple U-algebra over L. The U-field extension K of L is an *equivariant splitting field* of A, if $[A]$ is in the kernel of the map $\mathrm{ext}_{K/L} : \mathrm{BrCliff}(U, L) \to \mathrm{BrCliff}(U, K)$.

It is obvious that if K is an equivariant splitting field of a central simple U-algebra A, then $\mathrm{ext}_{K/L}([A])$ lies in $\mathrm{FMBrCliff}(U, K)$. For this reason it is important to identify $\mathrm{FMBrCliff}(U, L)$ with a more tractable group. In [17] Turull shows that $\mathrm{FMBrCliff}(U, L)$ is isomorphic to a second cohomology group. The next theorem is a special case of [17, Theorem 3.10].

Theorem 3.8. *Let A be a central simple U-algebra over L such that $[A] \in \mathrm{FMBrCliff}(U, L)$ and $\iota : A \to k^{l\times l}$ a k-algebra embedding with $\mathrm{C}_{k^{l\times l}}(\iota(A)) = \iota(L)$. Then there exists a map $F : U \to k^{l\times l} : u \mapsto F(u)$ such that $F(u)\iota(a)F(u)^{-1} = \iota(u(a))$ for all $a \in A$. Fix such maps ι, F then*

$$h(A) : U \times U \to L : (s, t) \mapsto F(s)F(t)F(st)^{-1}$$

is a 2-cocycle with values in L^*. This construction induces a group isomorphism $h : \mathrm{FMBrCliff}(U, L) \to \mathrm{H}^2(U, L^*) : [A] \mapsto h(A)$.

There exists an alternative way to calculate the cocycle $h(A)$. Recall that with a central simple U-algebra L comes an epimorphism $^- : U \to \mathrm{Gal}(L/k)$.

Proposition 3.9. *Let A be a central simple U-algebra over L such that $[A] \in \mathrm{FMBrCliff}(U, L)$. Identify A with $L^{n\times n}$, then there exists a collection of matrices $\{X_u\}_{u\in U}$ in $\mathrm{GL}_n(L)$ with the property $u(a) = X_u \bar{u}(a) X_u^{-1}$ for all $u \in U$ and $a \in L^{n\times n}$. Define the map*

$$\lambda : U \times U \to L : (s, t) \mapsto X_s \bar{s}(X_t) X_{st}^{-1}$$

Then λ is a 2-cocycle with values in L^*. Furthermore λ and $h(A)$ (cf. 3.8) define the same cohomology class in $\mathrm{H}^2(U, L^*)$.

Proof. Identify A with $L^{n\times n}$. Because U acts on A as semilinear L-algebra automorphisms, the existence of the matrices $\{X_u\}_{u\in U}$ is clear. We show that it is possible to choose ι, F as in 3.8 with the property that the resulting cocycle $h(A)$ is given by $h(A)(s,t) = \lambda(s,t)$ for all $s, t \in U$.

Restriction of scalars provides a k-algebra embedding $\iota : A \to k^{l\times l}$ with $\mathrm{C}_{k^{l\times l}}(\iota(A)) = \iota(L)$. Define $u(a_1, \ldots, a_n)^t := (\bar{u}(a_1), \ldots, \bar{u}(a_n))^t$, then this turns $L^{n\times 1}$ into a kU-module. Hence there exists a corresponding matrix representation $U \to k^{l\times l} : u \mapsto H_u$. Use this representation to see that $H_u \iota(a) H_u^{-1} = \iota(\bar{u}(a))$ for all $a \in A$. For $u \in U$ define $F(u) := \iota(X_u)H_u$ then a short calculation shows $F(u)\iota(a)F(u)^{-1} = \iota(u(a))$ for all $a \in A$. So ι and F fulfill the requirements of 3.8. Furthermore it is easy to see that $h(A)(s,t) = \lambda(s,t)$ for all $s, t \in U$. Hence λ is a 2-cocycle and it defines the same cohomology class as $h(A)$. □

Corollary 3.10. *Let A be a central simple U-algebra over L such that $[A] \in \mathrm{FMBrCliff}(U, L)$ and $\dim_L(A) = n^2$. Then the order of $[A]$, as an element of $\mathrm{BrCliff}(U, L)$, divides n and $|U|$.*

Proof. Since $h : \mathrm{FMBrCliff}(U, L) \to \mathrm{H}^2(U, L^*)$ is an isomophism, the order of $[A]$ is the same as the order of $h(A)$. Then it is a well-known fact in group cohomology, that the order of $h(A)$ divides $|U|$. Consider the equation $\lambda(s,t) = X_s \bar{s}(X_t) X_{st}^{-1}$ of 3.9. Taking determinants on both sides shows that λ^n is a coboundary. This implies that the order of λ as an element of $\mathrm{H}^2(U, L^*)$ divides n. Hence the order of $h(A)$ divides n. □

Recall the reformulation of the main question in 2.6. Given a finite and absolutely irreducible subgroup G of $\mathrm{GL}_n(K)$ and a subgroup U of $\mathrm{Aut}(G)$, decide if there exists a $(K/k, U)$-representation of G. The main theorem states that this can be decided by considering the enveloping algebra of G as an element of the Brauer–Clifford group.

Theorem 3.11. *Let (K/k) be a finite Galois extension, $G \leqslant \mathrm{GL}_n(K)$ a finite and absolutely irreducible matrix group, Δ the natural representation of G, $\Delta(kG)$ the enveloping algebra of the natural representation, L the center of $\Delta(kG)$ and U a subgroup of $\mathrm{Aut}(G)$. Assume that the natural action of U on G can be uniquely extended to a U-action on $\Delta(kG)$ as k-algebra automorphisms. This turns $\Delta(kG)$ into a central simple U-algebra over L. Assume further, there exists an isomorphism $^- : U \to \mathrm{Gal}(K/k)$ such that, considering K as a U-field via this isomorphism, the natural embedding $L \to K$ is U-equivariant. Then there exists a $(K/k, U)$-representation of G if and only if K is a U-equivariant splitting field of $\Delta(kG)$.*

Proof. Assume that there exists a $(K/k, U)$-representation of G. By replacing G with the image of this representation it can be assumed that G is a $(K/k, U)$-form. Scalar extension of $\Delta(kG)$ by the U-field K is the central simple U-algebra $K^{n\times n}$, where U acts by applying $\bar{u}$ to the entries of the matrices. It is easy to see that this central simple U-algebra is equivalent to the central simple U-algebra K. Since K represents the trivial element of $\mathrm{FMBrCliff}(U, K)$ it follows that $[\Delta(kG)]$ is in the kernel of the scalar extension map $\mathrm{ext}_{K/L}$. Hence K is an equivariant splitting field of $\Delta(kG)$.

For the "if" part assume that K is an equivariant splitting field of $\Delta(kG)$. Via the natural map $\Delta(kG) \otimes_L K \to K^{n\times n}$ identify the scalar extension of $\Delta(kG)$ by the U-field K with $K^{n\times n}$. By 3.9 there exists a collection of matrices $\{X_u\}_{u\in U}$ in $\mathrm{GL}_n(K)$ with

$$u(a) = X_u \bar{u}(a) X_u^{-1} \quad \text{for all } a \in K^{n\times n} \text{ and } u \in U \tag{2}$$

We prove that it is possible to choose such a collection with the additional property $X_{st} = X_s \bar{s}(X_t)$ for all $s, t \in U$.

Let $\{X_u\}_{u\in U}$ be a collection of matrices in $\mathrm{GL}_n(K)$ such that (2) is valid. Define the 2-cocycle $\lambda : U \times U \to K^* : (s,t) \mapsto X_s \bar{s}(X_t) X_{st}^{-1}$. Proposition 3.9 implies that cohomology classes of λ and $h(\mathrm{ext}_{K/L}([\Delta(kG)]))$ are the same in $\mathrm{H}^2(U, K^*)$. Because K is an equivariant splitting field of $\Delta(kG)$, the latter is trivial. Therefore λ is a coboundary and there exists a map $b : U \to K^*$ with $\lambda(s,t) = b_s \bar{s}(b_t) b_{st}^{-1}$ for $s, t \in U$. For u in U define $\tilde{X}_u := \frac{1}{b_u} X_u$, then the collection $\{\tilde{X}_u\}_{u\in U}$ clearly fulfills (2) and an easy calculation shows $\tilde{X}_{st} = \tilde{X}_s \bar{s}(\tilde{X}_t)$ for all $s, t \in U$.

Given such a collection of matrices the generalized Hilbert 90 theorem [13, p. 151] says there exists a matrix $Y \in \mathrm{GL}_n(K)$ such that $X_u = Y \bar{u}(Y^{-1})$, for all $u \in U$. Define the representation $\Theta : G \to \mathrm{GL}_n(K) : g \mapsto Y^{-1} g Y$ and use (2) to calculate:

$$\begin{aligned}
\bar{u}\big(\Theta(g)\big) &= \bar{u}\big(Y^{-1}\big)\bar{u}(g)\bar{u}(Y) \\
&= Y^{-1}\big(X_u \bar{u}(g) X_u^{-1}\big)Y \\
&= Y^{-1} u(g) Y \\
&= \Theta\big(u(g)\big)
\end{aligned}$$

Hence, Θ is a $(K/k, U)$-representation of G. $\square$

Some remarks should be made about the main theorem.

Remark 3.12.

1. The proof of Theorem 3.11 is constructive in the sense that it provides an explicit construction of a $(K/k, U)$-representation of G. Nevertheless the actual computations are non-trivial and involve computations in the second cohomology group and computational class field theory.
2. Lemma 2.11 implies that the assumptions of 3.11 can be checked using the natural character χ. Specifically they are fulfilled if and only if the natural character fulfills the condition of 2.8.
3. Consider the group algebra kG, then the natural action of U on G can be uniquely extended to an action of U on kG as k-algebra automorphisms. Furthermore, this induces a U-action on $kG/\mathrm{rad}(kG)$ because $\mathrm{rad}(kG)$ is U-invariant. If the assumptions of the main theorem are fulfilled, then the map Δ induces a U-equivariant k-algebra homomorphism $kG/\mathrm{rad}(kG) \to \Delta(kG)$. Since $kG/\mathrm{rad}(kG)$ is semisimple, it is easy to see that $\Delta(kG)$ is U-isomorphic to a simple component of $kG/\mathrm{rad}(kG)$. Therefore, the central simple U-algebra $\Delta(kG)$ can be defined intrinsically using the group algebra.

To recognize elements of the Brauer–Clifford group and to decide if a U-field K is an equivariant splitting field come up as natural problems. A first reduction will be a restriction to certain subgroups of U. For this reason it is important to study the interplay of the scalar extension map $\mathrm{ext}_{K/L}$ and the subgroup restriction map of 3.6.

Let $\mathrm{BrCliff}(U, L, K)$ be the set of equivalence classes of central simple U-algebras over L for which K is a splitting field (as a central simple L-algebra). Clearly, this is a subgroup of $\mathrm{BrCliff}(U, L)$.

Given a U-field K and a subgroup N of U, the inclusion map $N \to U$ induces a natural map $\mathrm{res} : \mathrm{H}^n(U, K^*) \to \mathrm{H}^n(N, K^*)$ on the cohomology. This map is called the restriction homomorphism. For further details see [13].

Lemma 3.13. *Let N be a subgroup of U and K a U-field extensions of L. Denote by $\mathrm{ext}_{K/L}$, res_N and h the maps of* 3.4, 3.6 *and* 3.8. *Then the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{BrCliff}(U, L, K) \arrow[r, "\mathrm{ext}_{K/L}"'] \arrow[d, "\mathrm{res}_N"'] & \mathrm{FMBrCliff}(U, K) \arrow[r, "h"'] \arrow[d, "\mathrm{res}_N"'] & \mathrm{H}^2(U, K^*) \arrow[d, "\mathrm{res}"'] \\
\mathrm{BrCliff}(N, K^N L, K) \arrow[r, "\mathrm{ext}_{K/K^N L}"'] & \mathrm{FMBrCliff}(N, K) \arrow[r, "h"'] & \mathrm{H}^2(N, K^*)
\end{tikzcd}$$

Proof. A verification. □

The idea is to use the following fact from group cohomology: An element λ of $\mathrm{H}^2(U, K^*)$ is trivial if and only if λ is in the kernel of $\mathrm{res} : \mathrm{H}^2(U, K^*) \to \mathrm{H}^2(U_p, K^*)$ for all Sylow p-subgroups of U. Combining this fact with Lemma 3.13 leads to the following theorem, which shows that it is possible to restrict oneself to the case that U is a p-group.

Theorem 3.14. *Let K be a U-field extension of L and A be a central simple U-algebra such that $[A]$ is an element of* $\mathrm{BrCliff}(U, L, K)$. *Then $[A]$ is in the kernel of the map $\mathrm{ext}_{K/L}$ if and only if it is in the kernel of $\mathrm{ext}_{K/K^{U_p}L} \circ \mathrm{res}_{U_p}$ for all* Sylow *p-subgroups of U.*

4. A tractable subgroup

Assume that K and L are Galois extensions of k and L is a subfield of K. Let U be the abstract finite group corresponding to $\mathrm{Gal}(K/k)$ and $^- : U \to \mathrm{Gal}(K/k)$ the natural isomorphism. Now through, view K and L as U-fields via the natural action of U.

Consider a central simple k-algebra as a central simple U-algebra over k with trivial action of U. This provides an injective homomorphism $\mathrm{Br}(k) \to \mathrm{BrCliff}(U, k)$. Use this homomorphism to identify the Brauer group $\mathrm{Br}(k)$ with a subgroup of $\mathrm{BrCliff}(U, k)$. Then the scalar extensions map $\mathrm{ext}_{L/k}$ (cf. 3.4) is a homomorphism $\mathrm{Br}(k) \to \mathrm{BrCliff}(U, L)$. We will show that this map is injective, hence it is possible to identify the Brauer group with a subgroup of the Brauer–Clifford group.

Denote by $\mathrm{Br}(K|k)$ the set of equivalence classes of central simple k-algebras, for which K is a splitting field. This is a subgroup of the Brauer group $\mathrm{Br}(k)$. In [13, Chapter X] Serre obtains an isomorphism $\delta : \mathrm{Br}(K|k) \to \mathrm{H}^2(U, K^*)$ via descent theory.

Let B be a central simple k-algebra of dimension n^2, for which K is a splitting field. View B as a central simple U-algebra with trivial action of U on B. Since K is a splitting field of B, the scalar extension of B by the U-field K can be identified with $K^{n\times n}$. Via this identification view $K^{n\times n}$ as a central simple U-algebra over K. Applying Proposition 3.9 there exists a collection of matrices $\{X_u\}_{u\in U}$ with the property

$$X_u \bar{u}(x) X_u^{-1} = u(x) \quad \text{for all } x \in K^{n\times n} \tag{3}$$

Note that B is isomorphic, as k-algebras, to the fixed algebra of $K^{n\times n}$ under U. Then use (3) to see that this fixed algebra is given by $\{x \in K^{n\times n} \mid X_u \bar{u}(x) X_u^{-1} = x\}$. A 2-cocycle, representing the cohomology class of $\delta(B)$, is $U \times U \to K^* : (s,t) \mapsto X_s \bar{s}(X_t) X_{st}^{-1}$ [13, Example 2, p. 159]. This cocycle also represents the cohomology class of $h(\mathrm{ext}_{K/k}([B]))$ by 3.9. Going over to the Brauer–Clifford group yields the following lemma.

Lemma 4.1. *The diagram*

$$\begin{tikzcd} \mathrm{Br}(K|k) \arrow[rr, "\delta"] \arrow[dr, "\mathrm{ext}_{K/k}"'] & & \mathrm{H}^2(U, K^*) \\ & \mathrm{FMBrCliff}(U, K) \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative.

Note that h and δ in Lemma 4.1 are isomorphisms. Since the kernel of $\mathrm{ext}_{K/k} : \mathrm{Br}(k) \to \mathrm{BrCliff}(U, K)$ is contained in $\mathrm{Br}(K|k)$ the map $\mathrm{ext}_{K/k}$ is injective. Furthermore, the factorization $\mathrm{ext}_{K/k} = \mathrm{ext}_{K/L} \circ \mathrm{ext}_{L/k}$ implies that $\mathrm{ext}_{L/k}$ is injective.

Now look at the subgroup of $\mathrm{BrCliff}(U, L)$ generated by $\mathrm{FMBrCliff}(U, L)$ and $\mathrm{ext}_{L/k}(\mathrm{Br}(k))$. Denote this group by $\mathrm{ExFMBrCliff}(U, L)$. It is easy to see that the intersection of $\mathrm{ext}_{L/k}(\mathrm{Br}(k))$ and $\mathrm{FMBrCliff}(U, L)$ is isomorphic to $\mathrm{Br}(L|k)$. Identify the Brauer group $\mathrm{Br}(k)$ with $\mathrm{ext}_{L/k}(\mathrm{Br}(k))$, then the next remark describes the structure of $\mathrm{ExFMBrCliff}(U, L)$.

Remark 4.2. The group $\mathrm{ExFMBrCliff}(U, L)$ is a subdirect product of $\mathrm{Br}(k)$ with $\mathrm{FMBrCliff}(U, L)$ over $\mathrm{Br}(L|k)$. Note that $\mathrm{FMBrCliff}(U, L)$ is isomorphic to a second cohomology group.

The group $\mathrm{ExFMBrCliff}(U, L)$ is interesting for two reasons. Firstly, its elements can be recognized via cohomological methods and hence it is a tractable group. Secondly, in a number of cases it coincides with Brauer–Clifford group. For example this happens if (L/k) is a cyclic extension.

Theorem 4.3. *If (L/k) is cyclic, then* $\mathrm{BrCliff}(U, L) = \mathrm{ExFMBrCliff}(U, L)$.

Proof. Let $[A] \in \mathrm{BrCliff}(U, L)$, $H := \mathrm{Gal}(L/k)$ and $\kappa : \mathrm{BrCliff}(U, L) \to \mathrm{Br}(L)^H$ as in 3.2. Denote by res the map $\mathrm{Br}(k) \to \mathrm{Br}(L) : C \mapsto C \otimes_k L$. The following exact sequence is due to Teichmüller [15]

$$\mathrm{H}^2(H, L^*) \longrightarrow \mathrm{Br}(k) \xrightarrow[\mathrm{res}]{} \mathrm{Br}(L)^H \longrightarrow \mathrm{H}^3(H, L^*)$$

Cohomology of cyclic groups yields that $\mathrm{H}^3(H, L^*)$ is trivial. Hence there exists a $B \in \mathrm{Br}(k)$ such that $\mathrm{res}(B) = \kappa([A])$. Therefore $\mathrm{ext}_{L/k}(B)^{-1}[A]$ is an element of $\mathrm{FMBrCliff}(U, L)$. This proves the theorem. $\square$

Choosing $L = k$ yields [17, Corollary 3.13].

Corollary 4.4. *If $L = k$, then* $\mathrm{BrCliff}(U, k) \cong \mathrm{Br}(k) \times \mathrm{FMBrCliff}(U, k)$.

Remark 4.5. For the last theorem and the corollary the assumption on K can be dropped. The only assumption needed, implicit in the definition of the Brauer–Clifford group, is that U acts on L as field automorphisms and k is the fixed field of this action.

Let $\mathrm{ExFMBrCliff}(U, L, K)$ be the group generated by $\mathrm{FMBrCliff}(U, L)$ and $\mathrm{ext}_{L/k}(\mathrm{Br}(K|k))$. The next problem is to find necessary and sufficient conditions on K to be an equivariant splitting field of a central simple U-algebra A over L with $[A] \in \mathrm{ExFMBrCliff}(U, L)$. In particular if K is an equivariant splitting field of A, then K is a splitting field of A considered as a central simple L-algebra. This provides some necessary conditions on K and we can assume that $[A] \in \mathrm{ExFMBrCliff}(U, L, K)$.

It is clear that there exists a, not unique, factorization of $[A]$ as:

$$[A] = \mathrm{ext}_{L/k}\big([B]\big)[C] \quad \text{with } [B] \in \mathrm{Br}(K|k) \quad \text{and} \quad [C] \in \mathrm{FMBrCliff}(U, L)$$

The next step is to calculate $h(\mathrm{ext}_{K/L}([A]))$. Use Lemma 4.1 to see:

$$\begin{aligned} h\big(\mathrm{ext}_{K/L}\big([A]\big)\big) &= h\big(\mathrm{ext}_{K/k}\big([B]\big)\big)h\big(\mathrm{ext}_{K/L}\big([C]\big)\big) \\ &= \delta\big([B]\big)h\big(\mathrm{ext}_{K/L}\big([C]\big)\big) \end{aligned}$$

The next proposition shows that it is possible to calculate $h(\mathrm{ext}_{K/L}([C]))$ from $h([C])$.

Proposition 4.6. *The inclusion $L \to K$ is U-equivariant and induces a map $\tau_{L,K} : \mathrm{H}^n(U, L^*) \to \mathrm{H}^n(U, K^*)$ on the cohomology. Then $h(\mathrm{ext}_{K/k}([C])) = \tau_{L,K}(h([C]))$ for all $[C] \in \mathrm{FMBrCliff}(U, L)$.*

Proof. Directly from 3.9 and the natural embedding $L^{n\times n} \to K^{n\times n}$. □

Summing up gives:

Lemma 4.7. *Let $[A] \in \mathrm{ExFMBrCliff}(U, L, K)$, $[B] \in \mathrm{Br}(K|k)$ and $[C] \in \mathrm{FMBrCliff}(U, L)$ such that $[A] = [\mathrm{ext}_{L/k}([B])][C]$, then*

$$h\big(\mathrm{ext}_{K/k}\big([A]\big)\big) = \delta\big([B]\big)\tau_{L,K}\big(h\big([C]\big)\big) \quad \text{in } \mathrm{H}^2\big(U, K^*\big)$$

Since $[B]$ and $h([C])$ do not depend on K, Lemma 4.7 can be used to obtain necessary and sufficient conditions on K to be an equivariant splitting field.

If k is a number field, this can be concretized. First of all, recall some constructions from the theory of central simple algebras. Given a 2-cocycle in $\mathrm{Z}^2(U, K^*)$, it is possible to construct a crossed product algebra. This construction induces an isomorphism between $\mathrm{H}^2(U, K^*)$ and $\mathrm{Br}(K|k)$.

Let B be a central simple k-algebra B, for which K is a splitting field. The crossed product algebra defined by the 2-cocycle $\delta(B)$ is equivalent to the opposite algebra B^o of B (cf. [13, p. 159]).

Recall the theorem of Brauer–Hasse–Noether, which says that the sequence

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus\nolimits_{\mathfrak{p}} \mathrm{Br}(k_{\mathfrak{p}}) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z} \tag{4}$$

is exact, where $\mathfrak{p}$ runs through all primes of the field k. The map inv is called the Hasse invariant map, computed locally on each component: $\mathrm{inv} = \sum \mathrm{inv}_{k_{\mathfrak{p}}}$. The maps $\mathrm{inv}_{k_{\mathfrak{p}}}$ are called Hasse invariants. For more details on Hasse invariants and central simple algebras see [12].

We saw that there exists a factorization of $[A]$ as

$$[A] = \big[\mathrm{ext}_{L/k}\big([B]\big)\big][C] \quad \text{with } [B] \in \mathrm{Br}(k) \text{ and } [C] \in \mathrm{FMBrCliff}(U, L)$$

Let $\tau_{L,K} : \mathrm{H}^2(U, L^*) \to \mathrm{H}^2(U, K^*)$ be the map of 4.6 and $D_{L,K}$ the crossed product algebra corresponding to the cocycle $\tau_{L,K}(h([C]))$. Lemma 4.7 implies that K is an equivariant splitting field of A if and only if the 2-cocycle $\delta([B])\tau_{L,K}(h([C]))$ is a coboundary. This is the case if and only if $[B^o][D_{L,K}]$ is trivial in the Brauer group. Using sequence (4), this is equivalent to

$$\mathrm{inv}_{\mathfrak{p}}\big(\big[B^o\big]\big) + \mathrm{inv}_{\mathfrak{p}}\big([D_{L,K}]\big) = 0 \in \mathbb{Q}/\mathbb{Z} \quad \text{for all primes } \mathfrak{p} \text{ of } k$$

Hence K is an equivariant splitting field of A if and only if the crossed product algebra $D_{L,K}$ is equivalent to B. Note that B is independent of K. The next example will illustrate this strategy.

Example 4.8. Let $k=\mathbb{Q}$, $G=Q_8$, $\langle a,b \mid a^4, a^2b^2, abab^{-1}\rangle$ a presentation of G. Then

$$\Delta(a)=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \Delta(b)=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

defines a faithful representation of G. Let u be the inner automorphism of G induced by conjugation with $\Delta(b)$ and U the group generated by u. Extend the natural action of U on $\Delta(G)$ to an action on the enveloping algebra $\Delta(\mathbb{Q}G)$ as $\mathbb{Q}$-algebra automorphisms. View $\Delta(\mathbb{Q}G)$ as a central simple U-algebra over $\mathbb{Q}$. Since $\mathrm{BrCliff}(U,\mathbb{Q})$ is a direct product of $\mathrm{FMBrCliff}(U,\mathbb{Q})$ and $\mathrm{Br}(\mathbb{Q})$ by 4.4, there exists a unique factorization

$$\bigl[\Delta(\mathbb{Q}G)\bigr]=[B][C] \quad \text{with } [B]\in \mathrm{Br}(k) \text{ and } [C]\in \mathrm{FMBrCliff}(U,\mathbb{Q})$$

More precisely B is the quaternion algebra $Q\,(\frac{(-1,-1)}{\mathbb{Q}})$ with trivial U-action and C is the matrix algebra $\mathbb{Q}^{2\times 2}$, where the U-action is given by $u(X)=\Delta(b)X\Delta(b)^{-1}$ for $X\in\mathbb{Q}^{2\times 2}$.

Let K be a quadratic extension of $\mathbb{Q}$ and view K as a U-field via the isomorphism $U\to \mathrm{Gal}(K/\mathbb{Q})$. We want to find necessary and sufficient conditions on K to be an equivariant splitting field of $\Delta(\mathbb{Q}G)$.

The $\mathbb{Q}$-algebra B is isomorphic to B^o and the Hasse invariants are

$$\mathrm{inv}_2\bigl([B]\bigr)=\frac{1}{2}, \qquad \mathrm{inv}_\infty\bigl([B]\bigr)=\frac{1}{2}, \qquad \mathrm{inv}_{\mathfrak{p}}\bigl([B]\bigr)=0 \quad \text{other primes } \mathfrak{p} \text{ of } \mathbb{Q}$$

It is well known that K is a splitting field of B if and only if

$$(K_{\mathfrak{P}}:\mathbb{Q}_2)=2 \quad \text{with } \mathfrak{P}_{|\mathbb{Q}}=2 \quad \text{and} \quad (K_\infty:\mathbb{R})=2 \tag{5}$$

Since $\Delta(b)^2=-1$,

$$h\bigl([C]\bigr):\ U\times U\to\mathbb{Q}:(\sigma,\tau)\mapsto \begin{cases} -1, & \text{if } \sigma=\tau=u \\ 1, & \text{otherwise}\end{cases}$$

is a cocycle representing the cohomology class of $h([C])$. Let $D_{\mathbb{Q},K}$ be the crossed product algebra defined by $\tau_{\mathbb{Q},K}(h([C]))$. Then the conditions on K to be an equivariant splitting field are (5) and

$$\mathrm{inv}_2\bigl([D_{\mathbb{Q},K}]\bigr)=\frac{1}{2}, \qquad \mathrm{inv}_\infty\bigl([D_{\mathbb{Q},K}]\bigr)=\frac{1}{2}, \qquad \mathrm{inv}_{\mathfrak{p}}\bigl([D_{\mathbb{Q},K}]\bigr)=0 \quad \text{other primes } \mathfrak{p} \text{ of } \mathbb{Q}$$

Because $D_{\mathbb{Q},K}$ is a cyclic algebra $(K/\mathbb{Q},\sigma,-1)$, with σ a generator of $\mathrm{Gal}(K/\mathbb{Q})$, it is possible to give the conditions on K in a more satisfactory way. Then K is an equivariant splitting field of the central simple U-algebra $\Delta(\mathbb{Q}G)$ if and only if K is the splitting field of an Eisenstein polynomial $x^2+2ax+b$ at prime 2, where $a,b\in\mathbb{Z}$ have the following properties:

$$a^2-b<0, \qquad \left(\frac{-1,a^2-b}{2}\right)=-1 \quad \text{and} \quad \left(\frac{-1,a^2-b}{\mathfrak{p}}\right)=1$$

where $\mathfrak{p}$ runs through all odd primes of $\mathbb{Q}$ dividing a^2-b and $(\frac{\circ,\circ}{\mathfrak{p}})$ is the Hilbert-symbol [11].

This classifies all quadratic extensions K of $\mathbb{Q}$ with the following property: There exist a $(K/\mathbb{Q})$-form G' of Q_8 and applying the non-trivial GALOIS automorphism to the elements of G' induces an inner automorphism of G'.

5. Applications and examples

Let k be a number field, K a finite GALOIS extension of k, G a finite and absolutely irreducible subgroup of $\mathrm{GL}_n(K)$, χ the corresponding natural character, Δ the natural representation, and U a subgroup of $\mathrm{Aut}(G)$. Assume that the SCHUR index of χ is one and the condition of 2.8 is satisfied. In this case it is possible to give sufficient conditions for the existence of a $(K/k, U)$-representation of G, depending solely on character theoretic data.

The idea to find such is to look for conditions which imply that the central simple U-algebra $\Delta(kG)$ (cf. 3.11) represents the trivial element of the BRAUER–CLIFFORD group. If this is the case, then it is easy to see that K is an equivariant splitting field. Hence there exists a $(K/k, U)$-representation of G by 3.11. Denote the SCHUR index of χ over k by $\mathrm{m}_k(\chi)$.

Lemma 5.1. *Assume there exists an irreducible complex character ζ of the semidirect product $G \rtimes U$ with*

$$k(\zeta) = k, \qquad \mathrm{m}_k(\zeta) = 1 \quad \text{and} \quad \zeta_{|G} = \sum_{\sigma \in H} \sigma \circ \chi \quad \text{where } H = \mathrm{Gal}\big(k(\chi)/k\big)$$

Then there exists a $(K/k, U)$-representation of G.

Proof. The SCHUR index of χ is one, so it can be assumed that G is a subgroup $\mathrm{GL}_n(\mathbb{Q}(\chi))$. Because the condition of 2.8 is satisfied, the natural action of U on G can be extended k-linearly to $\Delta(kG)$. Consider $\Delta(kG)$ as a central simple U-algebra as in 3.11. We claim that $[\Delta(kG)]$ represents the trivial element of $\mathrm{BrCliff}(U, k(\chi))$. It is easy to see that $[\Delta(kG)] \in \mathrm{FMBrCliff}(U, k(\chi))$, hence it is enough to show that $h(\Delta(kG))$ is a coboundary.

Let $\iota : k(\chi)^{n\times n} \to k^{l\times l}$ be the k-algebra homomorphism induced by restricting scalars. Obviously, there exists a representation $\Theta : G \rtimes U \to \mathrm{GL}_l(k)$, affording ζ, with the property $\Theta(g) = \iota(\Delta(g))$ for every $g \in G$. Define the map $F : U \to k^{l\times l} : u \mapsto \Theta(u)$, then ι and F fulfill the requirements of 3.8. Hence $h([\Delta(kG)])(s,t) = \Theta(s)\Theta(t)\Theta(st)^{-1} = 1$ for all $s, t \in U$, and this shows that $h(\Delta(kG))$ is a coboundary. Therefore K is an equivariant splitting field of $\Delta(kG)$ and the result follows from 3.11. $\square$

Using 3.10, another sufficient condition can be obtained [10, Proposition 8.1].

Corollary 5.2. *If $|U|$ is prime to n, the degree of χ, then there exists a $(K/k, U)$-representation of G.*

Assume that G is an absolutely irreducible subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}})$ and isomorphic to $\mathrm{PSL}_2(q)$, where q is a power of an odd prime p. Let k be the minimal subfield of $\mathbb{Q}(\chi)$ with the property that a subgroup of $\mathrm{Aut}(G)$ acts transitively on the set of GALOIS conjugates of χ over k. The aim is to show that there exists a $(\mathbb{Q}(\chi)/k)$-form of G.

To determine k, identify G with $\mathrm{PSL}_2(q)$ and consider an arbitrary faithful and irreducible complex character χ of $\mathrm{PSL}_2(q)$. Note that every automorphism of $\mathrm{PSL}_2(q)$ is induced by a unique automorphism of $\mathrm{SL}_2(q)$ [2]. View χ as a character of $\mathrm{SL}_2(q)$ then it is possible to work with $\mathrm{SL}_2(q)$. So let χ be an irreducible complex character of $\mathrm{SL}_2(q)$.

At first, recall some basic facts about $\mathrm{SL}_2(q)$. Let ν be a generator of $\mathbb{F}_q^*$ and define

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad z = \begin{pmatrix} -10 & \\ 0 & -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$$

$$a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \quad b \text{ element of order } q+1$$

Table 1

	1	z	c	d	a^l	b^m
1_G	1	1	1	1	1	1
ψ	q	q	0	0	1	-1
χ_i	$q+1$	$(-1)^i(q+1)$	1	1	$\rho^{il}+\rho^{-il}$	0
Θ_j	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$
ζ_1	$\frac{q+1}{2}$	$\frac{1}{2}\epsilon(q+1)$	$\frac{1}{2}(1+\sqrt{\epsilon q})$	$\frac{1}{2}(1-\sqrt{\epsilon q})$	$(-1)^l$	0
ζ_2	$\frac{q+1}{2}$	$\frac{1}{2}\epsilon(q+1)$	$\frac{1}{2}(1-\sqrt{\epsilon q})$	$\frac{1}{2}(1+\sqrt{\epsilon q})$	$(-1)^l$	0
η_1	$\frac{q-1}{2}$	$-\frac{1}{2}\epsilon(q-1)$	$\frac{1}{2}(-1+\sqrt{\epsilon q})$	$\frac{1}{2}(-1-\sqrt{\epsilon q})$	0	$(-1)^{m+1}$
η_2	$\frac{q-1}{2}$	$-\frac{1}{2}\epsilon(q-1)$	$\frac{1}{2}(-1-\sqrt{\epsilon q})$	$\frac{1}{2}(-1+\sqrt{\epsilon q})$	0	$(-1)^{m+1}$

Table 2

C	(1)	(z)	(c)	(d)	(a^l)	(b^m)
${}^uC$	(1)	(z)	(d)	(c)	(a^l)	(b^m)
${}^FC$	(1)	(z)	(c)	(d)	(a^{pl})	(b^{pm})

Then the conjugacy classes of $\mathrm{SL}_2(q)$ are:

$$(1), (z), (c), (d), (cz), (dz), \left(a^l\right), \left(b^m\right) \quad \text{with } 1 \leqslant l \leqslant \frac{q-3}{2} \text{ and } 1 \leqslant m \leqslant \frac{q-1}{2}$$

Let $\epsilon := (-1)^{\frac{q-1}{2}}$, $\rho \in \mathbb{C}$ a primitive $(q-1)$th root of unity and $\sigma \in \mathbb{C}$ a primitive $(q+1)$th root of unity. Then the complex character table of $\mathrm{SL}_2(q)$ is given by Table 1 with $1 \leqslant i \leqslant \frac{q-3}{2}$ and $1 \leqslant j \leqslant \frac{q-1}{2}$.

It is well known that $\mathrm{Aut}(\mathrm{SL}_2(q))$ is isomorphic to the projective semilinear group $\mathrm{P}\Gamma\mathrm{L}_2(q)$ [2]. This group is a semidirect product of $\mathrm{PGL}_2(q)$ with $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. The subgroup of $\mathrm{PGL}_2(q)$ generated by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if } q \equiv 3 \mod 4 \quad \text{and} \quad \begin{pmatrix} 0 & \nu \\ -1 & 0 \end{pmatrix} \quad \text{if } q \equiv 1 \mod 4 \tag{6}$$

is a complement of $\mathrm{PSL}_2(q)$ in $\mathrm{PGL}_2(q)$ and cyclic of order 2. Hence $\mathrm{PGL}_2(q)$ is a semidirect product $\mathrm{PSL}_2(q) \rtimes \mathrm{C}_2$.

Let u be the automorphism of $\mathrm{SL}_2(q)$ induced by conjugation with the matrix of (6) and F the automorphism induced by the Frobenius automorphism of $\mathbb{F}_q$. Note that the images of u and F under the natural epimorphism $\mathrm{Aut}(\mathrm{SL}_2(q)) \to \mathrm{Out}(\mathrm{SL}_2(q))$ generate the outer automorphism group.

To study the action of $\mathrm{Aut}(\mathrm{SL}_2(q))$ on the irreducible complex characters, we have to analyze the action on the conjugacy classes. It is clear that the automorphism group can be replaced by the group of outer automorphisms. So it is enough to consider u and F. Since u and F are given explicitly, Table 2 can be calculated easily.

Table 2 determines the action of $\mathrm{Out}(\mathrm{SL}_2(q))$ on the conjugacy classes completely. A close look at the character table reveals the following fact and determines the field k.

Remark 5.3. Assume that the character field of χ is a proper extension of $\mathbb{Q}$. The natural action of $\mathrm{Out}(\mathrm{SL}_2(q))$ on the set of irreducible complex characters induces an action on $\mathbb{Q}(\chi)$ as field automorphisms. Let k be the fixed field of this action or, if χ has values in $\mathbb{Q}$, let $k = \mathbb{Q}$. In all cases k is a minimal subfield of the character field with a subgroup of $\mathrm{Aut}(\mathrm{SL}_2(q))$ acting transitively on the set of Galois conjugates of χ over k. Furthermore, $\mathrm{Gal}(\mathbb{Q}(\chi)/k)$ is a cyclic group.

In case that the character field is a proper extension of $\mathbb{Q}$, there exists a natural epimorphism $\pi : \mathrm{Aut}(\mathrm{SL}_2(q)) \to \mathrm{Gal}(\mathbb{Q}(\chi)/k)$ with the property $\pi(\varphi) \circ \chi = \chi \circ \varphi$ for all $\varphi \in \mathrm{Aut}(\mathrm{SL}_2(q))$.

Lemma 5.4. *Assume that the character field of χ is a proper extension of $\mathbb{Q}$. Then the natural epimorphism $\pi : \mathrm{Aut}(\mathrm{SL}_2(q)) \to \mathrm{Gal}(\mathbb{Q}(\chi)/k)$ admits a section.*

Proof. Recall that u is the automorphism of $\mathrm{SL}_2(q)$ induced by conjugation with the matrix of (6) and F the automorphism induced by the FROBENIUS. It is easy to see that u has order 2 and F has order ν where $q = p^\nu$. Let λ be a generator of the cyclic group $\mathrm{Gal}(\mathbb{Q}(\chi)/k)$. If $\chi \in \{\zeta_{1,2}, \eta_{1,2}\}$, then $k = \mathbb{Q}$ and the map $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \to \mathrm{Aut}(\mathrm{SL}_2(q)) : \lambda \mapsto u$ defines a section of π. In case that $\chi \in \{\chi_i, \Theta_j\}$, then it is easy to see that there exists an $l \in \mathbb{N}$ such that $\mathrm{Gal}(\mathbb{Q}(\chi)/k) \to \mathrm{Aut}(\mathrm{SL}_2(q)) : \lambda \mapsto F^l$ defines a section. $\square$

Note that Remark 5.3 and Lemma 5.4 are, mutatis mutandis, true if χ is a complex character of $\mathrm{PSL}_2(q)$.

Theorem 5.5. *Let G be an absolutely irreducible subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}})$ isomorphic to $\mathrm{PSL}_2(q)$ and χ the natural character of G. Then there exists a $(\mathbb{Q}(\chi)/k)$-form of G with k given in Remark* 5.3.

Proof. Recall that every character of $\mathrm{PSL}_2(q)$ has SCHUR index one [14]. Assume that χ has values in $\mathbb{Q}$, then there exists a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ conjugate to G. This subgroup is clearly a $\mathbb{Q}$-form of G.

Assume that the character field of χ is a proper extension of $\mathbb{Q}$ and identify G with $\mathrm{PSL}_2(q)$. Take a section $\alpha : \mathrm{Gal}(\mathbb{Q}(\chi)/k) \to \mathrm{Aut}(G)$ of π (cf. 5.4) and let U be the image of α. Then π induces an isomorphism $^- : U \to \mathrm{Gal}(\mathbb{Q}(\chi)/k)$ such that $\bar{u} \circ \chi = \chi \circ u$ for all $u \in U$. This shows that the condition of 2.8 is fulfilled.

Assume that $\chi \in \{\eta_1, \eta_2, \zeta_1, \zeta_2\}$. Recall that in those cases $k = \mathbb{Q}$, U is cyclic of order 2 and n, the degree of χ, is odd. Then by 5.2 there exists a $(\mathbb{Q}(\chi)/\mathbb{Q}, U)$-representation of G.

Let $1 \leqslant i \leqslant \frac{q-3}{2}$, i even and $\chi = \chi_i$. Choose the section of π given in the proof of Lemma 5.4 and let U be the image of this section. Note that $U = \langle F^l \rangle$ for an $l \in \mathbb{N}$. Use CLIFFORD theory to see that $\chi^{G \rtimes U}$ is an irreducible complex character of $G \rtimes U$ with the following properties:

$$k\big(\chi^{G \rtimes U}\big) = k \quad \text{and} \quad \big(\chi^{G \rtimes U}\big)_{|G} = \sum_{\sigma \in H} \sigma \circ \chi \quad \text{where } H = \mathrm{Gal}\big(k(\chi)/k\big)$$

To apply 5.1, it remains to check that the SCHUR index over k of $\chi^{G \rtimes U}$ is one. View χ as a character of $\mathcal{G} = \mathrm{SL}_2(q)$ and U as a subgroup of $\mathrm{Aut}(\mathcal{G})$, then it is enough to show that the SCHUR index over k of $\chi^{\mathcal{G} \rtimes U}$ is one.

Let $\mathcal{N} := \langle a, x \rangle$ and $\mathcal{A} := \langle a \rangle$ with

$$x := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad a := \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

and for $1 \leqslant \mathbf{k} \leqslant \frac{(q-3)}{2}$ define the characters $\lambda_{\mathbf{k}} : \mathcal{A} \to \mathbb{C}^* : a \mapsto \rho^{\mathbf{k}}$. The proof of [14, Lemma 2.1] shows that:

$$\langle \chi_{\mathbf{k}}, \lambda_{\mathbf{k}} \rangle_{\mathcal{A}} = 3 \quad \text{and} \quad \langle \chi_{\mathbf{k}}, \lambda_{\mathbf{l}} \rangle_{\mathcal{A}} = 2 \quad \text{if } \mathbf{l} \neq \mathbf{k}$$

In particular, $\langle \gamma \circ \chi, \lambda_i \rangle = 2$ for any non-trivial GALOIS automorphism $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/k)$. Use this, CLIFFORD theory and FROBENIUS reciprocity to see that $\langle \chi_i^{\mathcal{G} \rtimes U}, \lambda_i^{\mathcal{G} \rtimes U} \rangle$ is odd. Assume for the moment that the character $\lambda_i^{\mathcal{G} \rtimes U}$ is afforded by a representation over k. Since the SCHUR index of $\chi^{\mathcal{G} \rtimes U}$ over

k divides $\langle \chi_i^{\mathcal{G} \rtimes U}, \lambda_i^{\mathcal{G} \rtimes U} \rangle$, it has to be odd. Applying the Brauer–Speiser theorem one sees that the Schur index of $\chi_i^{\mathcal{G} \rtimes U}$ over k has to be one.

It remains to show that there exists a representation over k affording $\lambda_i^{\mathcal{G} \rtimes U}$. The group $\mathcal{N}$ is obviously U-invariant, hence it is sufficient to show that $\lambda_i^{\mathcal{N} \rtimes U}$ is afforded by a representation over k. Let $a \in \mathcal{N}$ act on $\mathbb{Q}(\rho_i)$ as left multiplication with ρ^i, $x \in \mathcal{N}$ by applying complex conjugation and F^l by applying the Galois-automorphism $\rho^i \mapsto \rho^{ip^l}$. This turns $\mathbb{Q}(\rho^i)$ into a $k(\mathcal{N} \rtimes U)$-module and it is easy to see that the corresponding representation over k affords $\lambda_i^{\mathcal{N} \rtimes U}$.

Let $1 \leqslant j \leqslant \frac{(q-1)}{2}$, j even and $\chi = \Theta_j$. This case can be treated analogously to the last case. Note that $\mathcal{N}$ has to be replaced by $\mathrm{N}_{\mathcal{G}}(\langle b \rangle)$ and Lemma (2.2) of [14] by Lemma (2.1). $\square$

Let G be a (K/k)-form and Γ the Galois group of (K/k). The fixed group G^Γ is a subgroup of $\mathrm{GL}_n(k)$ and is called the *subgroup of k-rational points*.

Remark 5.6. Denote by $\mathbb{F}_{p^r} \leqslant \mathbb{F}_q = \mathbb{F}_{p^f}$ the fixed field of the map F^l used in the proof of 5.5. The $(\mathbb{Q}(\chi)/k)$-forms constructed in this proof have the following subgroups of k-rational points:

- $\mathrm{PSL}_2(p^f)$ if χ has values in $\mathbb{Q}$,
- $\mathbb{D}_{q-1}$ if $\chi \in \{\eta_1, \eta_2\}$,
- $\mathbb{D}_{q+1}$ if $\chi \in \{\zeta_1, \zeta_2\}$,
- $\mathrm{PGL}_2(p^r)$ if $2r \mid f$ and χ is χ_i or Θ_j,
- $\mathrm{PSL}_2(p^r)$ if $2r \nmid f$ and χ is χ_i or Θ_j.

A 3-dimensional representation of $\mathrm{PSL}_2(7)$ over $\mathbb{Q}(\sqrt{-7})$ which has S_3 as the subgroup of $\mathbb{Q}$-rational points, can be found in [3].

Proof. Identify G with $\mathrm{PSL}_2(q)$ and obtain $U \leqslant \mathrm{Aut}(G)$ from the proof of 5.5. Then the group of k-rational points is the fixed group of $\mathrm{PSL}_2(q)$ under U. Since the subgroups of $\mathrm{PSL}_2(q)$ are well known (cf. [5, Chapter 2, Theorem 8.5]) and due to the explicit description of U, the fixed groups can be calculated easily. $\square$

Assume that k is a finite field and K a finite Galois extension of k with Galois group Γ. Because of the fact that $\mathrm{H}^2(\Gamma, K^*)$ is trivial, the main Theorem 3.11 can be simplified.

Corollary 5.7. *Let $G \leqslant \mathrm{GL}_n(K)$ be a finite absolutely irreducible matrix group, Δ the natural representation of G, χ the natural character, and U a subgroup of* $\mathrm{Aut}(G)$ *such that there exists an isomorphism* $^- : U \to \mathrm{Gal}(K/k)$. *Then there exists a $(K/k, U)$-representation of G if and only if $\bar{u} \circ \chi = \chi \circ u$ for all $u \in U$.*

Proof. The only non-trivial part is the "if" part. Assume $\bar{u} \circ \chi = \chi \circ u$ for all $u \in U$. Lemma 2.11 implies that the assumptions of the main Theorem 3.11 are fulfilled. So, consider $\Delta(kG)$ as a central simple U-algebra over its center L (cf. 3.11). We have to show that K is an equivariant splitting field of $\Delta(kG)$, and to see this we have to calculate $\mathrm{ext}_{K/L}([\Delta(kG)])$. Note that $\mathrm{ext}_{K/L}([\Delta(kG)])$ is an element of $\mathrm{FMBrCliff}(U, K)$, and by 3.8 this group is isomorphic to $\mathrm{H}^2(U, K^*)$. Since U acts on K^* as Galois automorphisms, it is well known from Galois cohomology that $\mathrm{H}^2(U, K^*)$ is trivial. Hence K is an equivariant splitting field of $\Delta(kG)$. $\square$

Remark 5.8. In case of finite fields, the condition of 2.8 is necessary and sufficient for the existence of a $(K/k, U)$-representation.

Let k be the real field $\mathbb{R}$, G be a finite and absolutely irreducible subgroup of $\mathrm{GL}_n(\mathbb{C})$ with natural character χ and natural representation $\Delta : G \to \mathrm{GL}_n(\mathbb{C})$. Furthermore, let U be an order 2 subgroup of the automorphism group $\mathrm{Aut}(G)$. Assume that $\chi \circ u = \overline{\chi}$ for $U = \langle u \rangle$. Hence the enveloping algebra $\Delta(\mathbb{R}G)$ can be considered as a central simple U-algebra over $\mathbb{R}(\chi)$ as in 3.11. Distinguish two cases:

Case 1. $\mathbb{R}(\chi) = \mathbb{C}$.

Since $\mathbb{C}$ is algebraically closed, one can identify $\mathrm{BrCliff}(U, \mathbb{C})$ and $\mathrm{FMBrCliff}(U, \mathbb{C})$. View the map h of 3.8 as an isomorphism $h : \mathrm{BrCliff}(U, \mathbb{C}) \to \mathrm{H}^2(U, \mathbb{C}^*)$. Because U acts on $\mathbb{C}^*$ as GALOIS automorphisms, $\mathrm{H}^2(U, \mathbb{C}^*)$ is isomorphic to $\mathrm{Br}(\mathbb{R})$ and the latter group is cyclic of order 2. View h as a map $h : \mathrm{BrCliff}(U, \mathbb{C}) \to \mathbb{Z}_2$. Then

$$h\big(\Delta(\mathbb{R}G)\big) = \begin{cases} 1, & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi^{G \rtimes U}) = 1 \\ -1, & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi^{G \rtimes U}) = 2 \end{cases}$$

Case 2. $\mathbb{R}(\chi) = \mathbb{R}$.

Identify $\mathrm{BrCliff}(U, \mathbb{R})$ with $\mathrm{Br}(\mathbb{R}) \times \mathrm{FMBrCliff}(U, \mathbb{R})$ (cf. 4.4). Hence the map

$$(\delta, h) : \mathrm{BrCliff}(U, \mathbb{R}) \to \mathrm{H}^2\big(U, \mathbb{C}^*\big) \times \mathrm{H}^2\big(U, \mathbb{R}^*\big) : [B][C] \mapsto \big(\delta(B), h\big([C]\big)\big)$$

is an isomorphism. Identify $\mathrm{H}^2(U, \mathbb{C}^*)$ with $\mathbb{Z}_2$ as in Case 1. Because U acts trivially on $\mathbb{R}$, the group $\mathrm{H}^2(U, \mathbb{R}^*)$ is isomorphic to $\mathbb{R}^*/\mathbb{R}^{*2}$ and the latter is a cyclic group of order 2. View (δ, h) as a map $\mathrm{BrCliff}(U, \mathbb{R}) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. Use CLIFFORD theory to see that there exists an extension $\Theta \in \mathrm{Irr}(G \rtimes U)$ of the character χ. This extension is unique up to complex conjugation. Then

$$(\delta, h)\big(\Delta(\mathbb{R}G)\big) = \begin{cases} (1, 1), & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi) = 1,\ \mathbb{R}(\Theta) = \mathbb{R} \text{ and } \mathrm{m}_{\mathbb{R}}(\Theta) = 1 \\ (1, -1), & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi) = 1,\ \mathbb{R}(\Theta) = \mathbb{C} \text{ and } \mathrm{m}_{\mathbb{R}}(\Theta) = 1 \\ (-1, 1), & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi) = 2,\ \mathbb{R}(\Theta) = \mathbb{R} \text{ and } \mathrm{m}_{\mathbb{R}}(\Theta) = 2 \\ (-1, -1), & \text{if and only if } \mathrm{m}_{\mathbb{R}}(\chi) = 2,\ \mathbb{R}(\Theta) = \mathbb{C} \text{ and } \mathrm{m}_{\mathbb{R}}(\Theta) = 1 \end{cases}$$

The sum $\epsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$ is called the FROBENIUS–SCHUR indicator. It has the well-known property that:

$$\epsilon(\chi) = \begin{cases} 1, & \text{if and only if } \mathbb{R}(\chi) = \mathbb{R} \text{ and } \mathrm{m}_{\mathbb{R}}(\chi) = 1 \\ -1, & \text{if and only if } \mathbb{R}(\chi) = \mathbb{R} \text{ and } \mathrm{m}_{\mathbb{R}}(\chi) = 2 \\ 0, & \text{if and only if } \mathbb{R}(\chi) = \mathbb{C} \end{cases} \tag{7}$$

Observe that all possibilities of the Cases 1 and 2 can be distinguished using the FROBENIUS–SCHUR indicator of the characters χ and Θ. A twisted version of the FROBENIUS–SCHUR indicator was defined in [8] as

$$\epsilon^u(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi\big(gu(g)\big)$$

where u is an order 2 element of $\mathrm{Aut}(G)$. The next lemma shows that the twisted FROBENIUS–SCHUR indicator can be calculated from the FROBENIUS–SCHUR indicators of χ and Θ.

Lemma 5.9. *Let u be an element of order 2 in $\mathrm{Aut}(G)$. Assume that θ is an irreducible complex character of $G \rtimes U$ with $\langle \theta_{|G}, \chi \rangle \neq 0$, then:*

$$\epsilon(\theta) = \frac{1}{2}\frac{\theta(1)}{\chi(1)}\big(\epsilon(\chi) + \epsilon^u(\chi)\big)$$

Proof. Directly from Clifford theory and a calculation. □

Assume we are in Case 2 of the discussion above. Lemma 4.7 implies that the following diagram is commutative

$$\begin{tikzcd} \mathrm{BrCliff}(U,\mathbb{R}) \arrow[r, "\mathrm{ext}_{\mathbb{C}/\mathbb{R}}"] \arrow[d, "{(\delta,h)}"'] & \mathrm{FMBrCliff}(U,\mathbb{C}) \arrow[d, "h"] \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r, "{(a,b)\to ab}"] & \mathbb{Z}_2 \end{tikzcd}$$

Hence it is easy to decide if $\mathbb{C}$ is an equivariant splitting field.

Use the formula of Lemma 5.9 and the property (7) to see that the twisted Frobenius–Schur is 1 if $\mathbb{C}$ is an equivariant splitting field and -1 if $\mathbb{C}$ is not. Furthermore, the same result is obtained in Case 1. This shows that if $\chi \circ u = \overline{\chi}$, then the twisted Frobenius–Schur indicator determines whether or not there exists a $(\mathbb{C}/\mathbb{R})$-form of G.

Assume $\chi \circ u \neq \overline{\chi}$. Use Clifford theory, (7) and Lemma 5.9 to see that $\epsilon^u(\chi) = 0$. Summing up, one gets a theorem due to Kawanaka and Matsuyama.

Theorem 5.10. *Let u be an element of order 2 in $\mathrm{Aut}(G)$. Then*

$$\epsilon^u(\chi) = \begin{cases} 1, & \text{if and only if } \chi \circ u = \overline{\chi} \text{ and there exists a } (\mathbb{C}/\mathbb{R})\text{-form of } G \\ -1, & \text{if and only if } \chi \circ u = \overline{\chi} \text{ and there is no } (\mathbb{C}/\mathbb{R})\text{-form of } G \\ 0, & \text{otherwise} \end{cases}$$

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