August 16, 2012

CALDERO-CHAPOTON ALGEBRAS

GIOVANNI CERULLI IRELLI, DANIEL LABARDINI-FRAGOSO, AND JAN SCHRÖER

ABSTRACT. Motivated by the representation theory of quivers with potential introduced by Derksen, Weyman and Zelevinsky and by work of Caldero and Chapoton, who gave explicit formulae for the cluster variables of Dynkin quivers, we associate a *Caldero-Chapoton algebra* $\mathcal{A}_\Lambda$ to any (possibly infinite dimensional) basic algebra Λ. By definition, $\mathcal{A}_\Lambda$ is (as a vector space) generated by the *Caldero-Chapoton functions* $C_\Lambda(\mathcal{M})$ of the decorated representations $\mathcal{M}$ of Λ. If $\Lambda = \mathcal{P}(Q,W)$ is the Jacobian algebra defined by a 2-acyclic quiver Q with non-degenerate potential W, then we have $\mathcal{A}_Q \subseteq \mathcal{A}_\Lambda \subseteq \mathcal{A}_Q^{\rm up}$, where $\mathcal{A}_Q$ and $\mathcal{A}_Q^{\rm up}$ are the cluster algebra and the upper cluster algebra associated to Q. The set $\mathcal{B}_\Lambda$ of generic Caldero-Chapoton functions is parametrized by the strongly reduced components of the varieties of representations of the Jacobian algebra $\mathcal{P}(Q,W)$ and was introduced by Geiss, Leclerc and Schröer. Plamondon parametrized the strongly reduced components for finite-dimensional basic algebras. We generalize this to arbitrary basic algebras. Furthermore, we prove a decomposition theorem for strongly reduced components. We define $\mathcal{B}_\Lambda$ for arbitrary Λ, and we conjecture that $\mathcal{B}_\Lambda$ is a basis of the Caldero-Chapoton algebra $\mathcal{A}_\Lambda$. Thanks to the decomposition theorem, all elements of $\mathcal{B}_\Lambda$ can be seen as generalized cluster monomials. As another application, we obtain a new proof for the sign-coherence of g-vectors. Caldero-Chapoton algebras lead to several general conjectures on cluster algebras.

Contents

1. Introduction

1.1. Let $\mathcal{A}_Q$ be the Fomin-Zelevinsky cluster algebra [FZ1, FZ2] associated to a finite 2-acyclic quiver Q. By definition $\mathcal{A}_Q$ is generated by an inductively defined set of rational functions, called cluster variables. The cluster variables are contained in the set $\mathcal{M}_Q$ of cluster monomials, which are by definition certain monomials in the cluster variables.

Now let W be a non-degenerate potential for Q, and let $\Lambda = \mathcal{P}(Q,W)$ be the associated Jacobian algebra introduced by Derksen, Weyman and Zelevinsky [DWZ1, DWZ2]. The category of decorated representations of Λ is denoted by $\mathrm{decrep}(\Lambda)$. To any $\mathcal{M} \in \mathrm{decrep}(\Lambda)$ one can associate a Laurent polynomial $C_\Lambda(\mathcal{M})$, the Caldero-Chapoton function of $\mathcal{M}$. It

2010 *Mathematics Subject Classification.* Primary 13F60; Secondary 16G10, 16G20.

follows from [DWZ1, DWZ2] that the cluster monomials form a subset of the set $\mathcal{C}_\Lambda$ of Caldero-Chapoton functions.

1.2. The generic basis conjecture. One of the main problems in cluster algebra theory is to find a basis of $\mathcal{A}_Q$ with favourable properties. As an important requirement, this basis should contain the set $\mathcal{M}_Q$ of cluster monomials in a natural way.

The concept of strongly reduced irreducible components of varieties of decorated representations of a Jacobian algebra Λ was introduced in [GLS]. To each strongly reduced component Z one can associate a generic Caldero-Chapoton function $C_\Lambda(Z)$. It was conjectured in [GLS] that the set $\mathcal{B}_\Lambda$ of generic Caldero-Chapoton functions forms a $\mathbb{C}$-basis of $\mathcal{A}_Q$. Using a non-degenerate potential defined by Labardini [L], Plamondon [P2] found a counterexample and then conjectured that $\mathcal{B}_\Lambda$ is a basis of the upper cluster algebra $\mathcal{A}_Q^{\rm up}$. This conjecture should also be wrong in general. We replace it by yet another conjecture.

We study the *Caldero-Chapoton algebra*

$$\mathcal{A}_\Lambda := \langle C_\Lambda(\mathcal{M}) \mid \mathcal{M} \in \operatorname{decrep}(\Lambda)\rangle_{\rm alg}$$

generated by all Caldero-Chapoton functions. We do not restrict ourselves to Jacobian algebras, but work with algebras Λ defined as arbitrary quotients of completed path algebras. In particular, we generalize the notation of a Caldero-Chapoton function to this general setup. One easily checks that the functions $C_\Lambda(\mathcal{M})$ do not only generate $\mathcal{A}_\Lambda$ as an algebra but also as a vector space over the ground field $\mathbb{C}$.

Conjecture 1.1. *$\mathcal{B}_\Lambda$ is a $\mathbb{C}$-basis of $\mathcal{A}_\Lambda$.*

We show that the set $\mathcal{B}_\Lambda$ of generic Caldero-Chapoton functions is linearly independent provided the kernel of the skew-symmetric incidence matrix B_Q of Q does not contain any non-zero element in $\mathbb{Q}^n_{\geq 0}$. This generalizes [P2, Proposition 3.19].

For $\Lambda = \mathcal{P}(Q, W)$ a Jacobian algebra associated to a quiver Q with non-degenerate potential W we have

$$\mathcal{A}_Q \subseteq \mathcal{A}_\Lambda \subseteq \mathcal{A}_Q^{\rm up}$$

where $\mathcal{A}_Q$ is the cluster algebra and $\mathcal{A}_Q^{\rm up}$ is the upper cluster algebra associated to Q. (We refer to [BFZ, DWZ1, FZ1] for missing definitions.) For this special case, we have a list of conjectures, which hopefully will lead to a better understanding of the rather mysterious relation between $\mathcal{A}_Q$ and $\mathcal{A}_Q^{\rm up}$.

1.3. Parametrization of strongly reduced components. Plamondon [P2, Theorem 1.2] parametrized the strongly reduced components for finite-dimensional basic algebras. We generalize this to arbitrary basic algebras. Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a basic algebra, where the quiver Q has n vertices. Let $\operatorname{decIrr}(\Lambda)$ be the set of irreducible components of all varieties $\operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ of decorated representations of Λ, where $(\mathbf{d}, \mathbf{v})$ runs through $\mathbb{N}^n \times \mathbb{N}^n$. By $\operatorname{decIrr}^{\rm s.r.}(\Lambda)$ we denote the subset of strongly reduced components. (The definition is in Section 5.) Recall that $\operatorname{decIrr}^{\rm s.r.}(\Lambda)$ parametrizes the elements in $\mathcal{B}_\Lambda$.

Let

$$G_\Lambda^{\rm s.r.}\colon \operatorname{decIrr}^{\rm s.r.}(\Lambda) \to \mathbb{Z}^n$$

be the map sending $Z \in \operatorname{decIrr}(\Lambda)$ to the generic g-vector $g_\Lambda(Z)$ of Z. (The definition of a g-vector is in Section 3.) Using Plamondon's result for finite-dimensional algebras, and a long-path truncation argument, we get the following parametrization of strongly reduced components for arbitrary Λ.

Theorem 1.2. *For a basic algebra Λ the following hold:*

(i) *The map*
$$G_\Lambda^{\text{s.r.}}\colon \operatorname{decIrr}^{\text{s.r.}}(\Lambda) \to \mathbb{Z}^n$$
is injective.

(ii) *The following are equivalent:*
  (a) $G_\Lambda^{\text{s.r.}}$ *is surjective.*
  (b) Λ *is finite-dimensional.*

1.4. A decomposition theorem for strongly reduced components. The notion of a direct sum of irreducible components of representation varieties was introduced in [CBS]. The Zariski closure $Z := \overline{Z_1 \oplus \cdots \oplus Z_t}$ of a direct sum of irreducible components $Z_1, \ldots, Z_t$ of varieties of representations of Λ is always irreducible, but in general Z is not an irreducible component. It was shown in [CBS] that Z is an irreducible component provided the dimension of the first extension group between the components is generically zero. The following decomposition theorem is an analogue for strongly reduced components. Instead of extension groups, we work with a generalization $E_\Lambda(-,?)$ of the Derksen-Weyman-Zelevinsky E-invariant [DWZ2]. (We define $E_\Lambda(-,?)$ in Section 3.)

Theorem 1.3. *For $Z_1, \ldots, Z_t \in \operatorname{decIrr}(\Lambda)$ the following are equivalent:*

(i) $\overline{Z_1 \oplus \cdots \oplus Z_t}$ *is a strongly reduced irreducible component.*
(ii) *Each Z_i is strongly reduced and $E_\Lambda(Z_i, Z_j) = 0$ for all $i \neq j$.*

Based on Theorem 1.3, we show that all elements of $\mathcal{B}_\Lambda$ can be seen as CC-cluster monomials. (The CC-cluster monomials generalize Fomin and Zelevinsky's notion of cluster monomials.)

1.5. Sign-coherence of g-vectors. A subset U of $\mathbb{Z}^n$ is called *sign-coherent* if for each $1 \leq i \leq n$ we have either $a_i \geq 0$ for all $(a_1, \ldots, a_n) \in U$, or we have $a_i \leq 0$ for all $(a_1, \ldots, a_n) \in U$.

The following theorem generalizes [P2, Theorem 3.7(1)].

Theorem 1.4. *Let Λ be a basic algebra, and let $Z_1, \ldots, Z_t \in \operatorname{decIrr}^{\text{s.r.}}(\Lambda)$ be strongly reduced components. Assume that*
$$\overline{Z_1 \oplus \cdots \oplus Z_t}$$
is a strongly reduced component. Then $\{g_\Lambda(Z_1), \ldots, g_\Lambda(Z_t)\}$ is sign-coherent.

1.6. The paper is organized as follows. In Section 2 we recall definitions and basic properties of basic algebras and their (decorated) representations. We also introduce truncations of basic algebras, which play a crucial role in some of our proofs. In Section 3 we introduce and study g-vectors and E-invariants of decorated representations. Caldero-Chapoton functions and Caldero-Chapoton algebras are defined in Section 4. Our main results Theorem 1.2 and 1.3 are proved in Section 5. In Section 6 we introduce component graphs, component clusters and CC-clusters, and we show that the cardinality of loop-complete subgraphs of a component graph is bounded by the number of simple modules. We also present several general conjectures on the structure of component graphs and on a generalization of the Fomin-Zelevinsky Laurent phenomenon. Section 7 explains the relation between Caldero-Chapoton algebras and cluster algebras, and it contains several conjectures on the relation between cluster algebras and upper cluster algebras. Section 8 contains the proof of Theorem 1.4. Finally, in Section 9 we discuss several examples of Caldero-Chapoton algebras.

1.7. Notation. We denote the composition of maps $f\colon M \to N$ and $g\colon N \to L$ by $gf = g \circ f\colon M \to L$. We write $|U|$ for the cardinality of a set U.

A finite-dimensional module M is *basic* provided it is a direct sum of pairwise non-isomorphic indecomposable modules. For a module M and some $m \geq 1$ let M^m be the direct sum of m copies of M.

For a finite-dimensional basic algebra Λ let τ_Λ be its Auslander-Reiten translation. For an introduction to Auslander-Reiten theory we refer to the books [ARS] and [ASS].

For $n \geq 1$ and a set S, depending on the situation, we identify S^n with the set of $(n\times 1)$- or $(1\times n)$-matrices with entries in S. By $\mathbb{N}$ we denote the natural numbers, including zero. For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ let $|\mathbf{d}| := d_1 + \cdots + d_n$. For $n \in \mathbb{N}$ let $M_n(\mathbb{Z})$ be the set of $(n \times n)$-matrices with integer entries.

For a ring R let $R[x_1^{\pm}, \ldots, x_n^{\pm}]$ be the algebra of Laurent polynomials over R in n independent variables $x_1, \ldots, x_n$. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ set $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$.

2. Basic algebras and decorated representations

2.1. Basic algebras and quiver representations. Throughout, let $\mathbb{C}$ be the field of complex numbers. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a finite set of *vertices*, Q_1 is a finite set of *arrows*, and $s, t\colon Q_1 \to Q_0$ are maps. For each arrow $a \in Q_1$ we call $s(a)$ and $t(a)$ the starting and terminal vertex of a, respectively. If not mentioned otherwise, we always assume that $Q_0 = \{1, \ldots, n\}$. Let $B_Q = (b_{ij}) \in M_n(\mathbb{Z})$, where

$$b_{ij} := |\{a \in Q_1 \mid s(a) = j,\, t(a) = i\}| - |\{a \in Q_1 \mid s(a) = i,\, t(a) = j\}|.$$

A *path* in Q is a tuple $p = (a_m, \ldots, a_1)$ of arrows $a_i \in Q_1$ such that $s(a_{i+1}) = t(a_i)$ for all $1 \leq i \leq m-1$. Then $\mathrm{length}(p) := m$ is the *length* of p. Additionally, for each vertex $i \in Q_0$ there is a path e_i of length 0. We often just write $a_m \cdots a_1$ instead of $(a_m, \ldots, a_1)$.

A path $p = (a_m, \ldots, a_1)$ of length $m \geq 1$ is a *cycle* in Q, or more precisely an m-*cycle* in Q, if $s(a_1) = t(a_m)$. The quiver Q is *acyclic* if there are no cycles in Q, and for $s \geq 1$ the quiver Q is called s-*acyclic* if there are no m-cycles for $1 \leq m \leq s$.

A *representation* of a quiver $Q = (Q_0, Q_1, s, t)$ is a tuple $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$, where each M_i is a finite-dimensional $\mathbb{C}$-vector space, and $M_a\colon M_{s(a)} \to M_{t(a)}$ is a $\mathbb{C}$-linear map for each arrow $a \in Q_1$. We call $\underline{\dim}(M) := (\dim(M_1), \ldots, \dim(M_n))$ the *dimension vector* of M. Let $\dim(M) := \dim(M_1) + \cdots + \dim(M_n)$ be the *dimension* of M. For a path $p = (a_m, \ldots, a_1)$ in Q let $M_p := M_{a_m} \circ \cdots \circ M_{a_1}$. The representation M is called *nilpotent* provided there exists some $N > 0$ such that $M_p = 0$ for all paths p in Q with $\mathrm{length}(p) > N$.

For $i \in Q_0$ let $S_i := (M_i, M_a)_{i,a}$ be the representation of Q with $M_i = \mathbb{C}$, $M_j = 0$ for all $j \neq i$, and $M_a = 0$ for all $a \in Q_1$. For a nilpotent representation M the ith entry $\dim(M_i)$ of its dimension vector $\underline{\dim}(M)$ equals the Jordan-Hölder multiplicity $[M : S_i]$ of S_i in M.

For $m \in \mathbb{N}$ let $\mathbb{C}Q[m]$ be a $\mathbb{C}$-vector space with a $\mathbb{C}$-basis labeled by the paths of length m in Q. Note that $\mathbb{C}Q[m]$ is finite-dimensional. We do not distinguish between a path p of length m and the corresponding basis vector in $\mathbb{C}Q[m]$.

The *completed path algebra* of a quiver Q is denoted by $\mathbb{C}\langle\langle Q\rangle\rangle$. As a $\mathbb{C}$-vector space we have

$$\mathbb{C}\langle\langle Q\rangle\rangle = \prod_{m\geq 0} \mathbb{C}Q[m].$$

We write the elements in $\mathbb{C}\langle\langle Q\rangle\rangle$ as infinite sums $\sum_{m\geq 0} a_m$ with $a_m \in \mathbb{C}Q[m]$. The product in $\mathbb{C}\langle\langle Q\rangle\rangle$ is then defined as

$$(\sum_{i\geq 0} a_i)(\sum_{j\geq 0} b_j) := \sum_{k\geq 0}\sum_{i+j=k} a_i b_j.$$

A *potential* of Q is an element $W = \sum_{m\geq 1} w_m$ of $\mathbb{C}\langle\langle Q\rangle\rangle$, where each w_m is a $\mathbb{C}$-linear combination of m-cycles in Q. By definition, $W = 0$ is also a potential. The definition of a *non-degenerate* potential can be found in [DWZ1, Section 7].

The category $\mathrm{mod}(\mathbb{C}\langle\langle Q\rangle\rangle)$ of finite-dimensional left $\mathbb{C}\langle\langle Q\rangle\rangle$-modules can be identified with the category $\mathrm{nil}(Q)$ of nilpotent representations of Q.

By $\mathfrak{m}$ we denote the *arrow ideal* in $\mathbb{C}\langle\langle Q\rangle\rangle$, which is generated by the arrows of Q. An ideal I of $\mathbb{C}\langle\langle Q\rangle\rangle$ is *admissible* if $I \subseteq \mathfrak{m}^2$. We call an algebra Λ *basic* if $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ for some quiver Q and some admissible ideal I of $\mathbb{C}\langle\langle Q\rangle\rangle$.

A *representation* of a basic algebra $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ is a nilpotent representation of Q, which is annihilated by the ideal I. We identify the category $\mathrm{rep}(\Lambda)$ of representations of Λ with the category $\mathrm{mod}(\Lambda)$ of finite-dimensional left Λ-modules. Up to isomorphism the simple representations of Λ are the 1-dimensional representations $S_1, \ldots, S_n$.

The category of (possibly infinite dimensional) Λ-modules is denoted by $\mathrm{Mod}(\Lambda)$, we consider $\mathrm{rep}(\Lambda)$ as a subcategory of $\mathrm{Mod}(\Lambda)$.

2.2. Decorated representations of quivers. Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a basic algebra. A *decorated representation* of Λ is a pair $\mathcal{M} = (M, V)$, where M is a representation of Λ and $V = (V_1, \ldots, V_n)$ is a tuple of finite-dimensional $\mathbb{C}$-vector spaces. Let $\underline{\dim}(V) := (\dim(V_1), \ldots, \dim(V_n))$ and $\dim(V) := \dim(V_1) + \cdots + \dim(V_n)$. We call $\underline{\dim}(\mathcal{M}) := (\underline{\dim}(M), \underline{\dim}(V))$ the *dimension vector* of $\mathcal{M}$.

One defines morphisms and direct sums of decorated representations in the obvious way. Let $\mathrm{decrep}(\Lambda)$ be the category of decorated representations of Λ.

Let $\mathcal{M} = (M, V) \in \mathrm{decrep}(\Lambda)$. We write $M = 0$ if all M_i are zero, and $V = 0$ if all V_i are zero. Furthermore, $\mathcal{M} = 0$ if $M = 0$ and $V = 0$.

For $1 \leq i \leq n$ set $\mathcal{S}_i := (S_i, 0)$, and let $\mathcal{S}_i^- := (0, V)$, where $V_i = \mathbb{C}$ and $V_j = 0$ for all $j \neq i$. The representations $\mathcal{S}_i^-$ are the *negative simple* decorated representations of Λ.

2.3. Varieties of representations. For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ let $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ be the affine variety of representations of Λ with dimension vector $\mathbf{d}$. By definition the closed points of $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ are the representations $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ of Λ with $M_i = \mathbb{C}^{d_i}$ for all $i \in Q_0$. One can regard $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ as a Zariski closed subset of the affine space

$$\mathrm{rep}_{\mathbf{d}}(Q) := \prod_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}).$$

For $\mathbf{d} = (d_1, \ldots, d_n)$ let $G_{\mathbf{d}} := \prod_{i=1}^n \mathrm{GL}(\mathbb{C}^{d_i})$. The group $G_{\mathbf{d}}$ acts on $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ by conjugation. More precisely, for $g = (g_1, \ldots, g_n) \in G_{\mathbf{d}}$ and $M \in \mathrm{rep}_{\mathbf{d}}(\Lambda)$ let

$$g.M := (M_i, g_{t(a)}^{-1} M_a g_{s(a)})_{i\in Q_0, a\in Q_1}.$$

For $M \in \mathrm{rep}_{\mathbf{d}}(\Lambda)$ let $\mathcal{O}(M)$ be the $G_{\mathbf{d}}$-orbit of M. The $G_{\mathbf{d}}$-orbits are in bijection with the isomorphism classes of representations of Λ with dimension vector $\mathbf{d}$.

For $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$ let $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ be the affine variety of decorated representations $\mathcal{M} = (M, V)$ with $M \in \mathrm{rep}_{\mathbf{d}}(\Lambda)$ and $V = \mathbb{C}^{\mathbf{v}} := (\mathbb{C}^{v_1}, \ldots, \mathbb{C}^{v_n})$, where $\mathbf{v} = (v_1, \ldots, v_n)$.

For $\mathcal{M} = (M, V) \in \operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ define $g.\mathcal{M} := (g.M, V)$. This defines a $G_{\mathbf{d}}$-action on $\operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$. The $G_{\mathbf{d}}$-orbit of $\mathcal{M}$ is denoted by $\mathcal{O}(\mathcal{M})$. We have

$$\dim \mathcal{O}(\mathcal{M}) = \dim \mathcal{O}(M) = \dim G_{\mathbf{d}} - \dim \operatorname{End}_\Lambda(M), \tag{1}$$

see for example [G].

2.4. Quiver Grassmannians. Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a basic algebra. For a representation $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ of Λ and $\mathbf{e} \in \mathbb{N}^n$ let $\operatorname{Gr}_{\mathbf{e}}(M)$ be the *quiver Grassmannian* of subrepresentations U of M with $\underline{\dim}(U) = \mathbf{e}$. (By definition a *subrepresentation* of M is a tuple $U = (U_i)_{i\in Q_0}$ of subspaces $U_i \subseteq M_i$ such that $M_a(U_{s(a)}) \subseteq U_{t(a)}$ for all $a \in Q_1$.) So $\operatorname{Gr}_{\mathbf{e}}(M)$ is a projective variety, which can be seen as a closed subvariety of the product of the classical Grassmannians $\operatorname{Gr}_{e_i}(M_i)$ of e_i-dimensional subspaces of M_i, where $\mathbf{e} = (e_1, \dots, e_n)$. Let $\chi(\operatorname{Gr}_{\mathbf{e}}(M))$ be the Euler-Poincaré characteristic of $\operatorname{Gr}_{\mathbf{e}}(M)$.

2.5. Truncations of basic algebras. For a basic algebra $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ and some $p \geq 1$ let $\Lambda_p := \Lambda/J_p$, where J_p is the ideal of Λ generated by all (residue classes) of paths of length p in Q. We call Λ_p the *p-truncation* of Λ. We get canonical surjective algebra homomorphisms

$$\mathbb{C}\langle\langle Q\rangle\rangle \xrightarrow{\pi} \Lambda \xrightarrow{\pi_p} \Lambda_p$$

with $\operatorname{Ker}(\pi) = I$ and $I_p := \operatorname{Ker}(\pi_p \circ \pi) = I + \mathfrak{m}^p$, where $\mathfrak{m}^p$ is the pth power of the arrow ideal $\mathfrak{m}$. Thus we can write $\Lambda_p = \mathbb{C}\langle\langle Q\rangle\rangle/I_p$. As a vector space, Λ_p is isomorphic to

$$V_p/(V_p \cap (I + \mathfrak{m}^p))$$

where

$$V_p := \prod_{0\leq m\leq p-1} \mathbb{C}Q[m].$$

Clearly, Λ_p is a finite-dimensional basic algebra, and the canonical epimorphism $\pi_p\colon \Lambda \to \Lambda_p$ induces embeddings $\operatorname{rep}(\Lambda_p) \to \operatorname{rep}(\Lambda)$ and $\operatorname{decrep}(\Lambda_p) \to \operatorname{decrep}(\Lambda)$ in the obvious way.

Lemma 2.1. *Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a basic algebra. Then the following hold:*

(i) *Let $\mathcal{M} = (M, V) \in \operatorname{decrep}(\Lambda)$. If $p \geq \dim(M)$, then $\mathcal{M}$ is in the image of the embedding $\operatorname{decrep}(\Lambda_p) \to \operatorname{decrep}(\Lambda)$.*

(ii) *Let $M, N \in \operatorname{rep}(\Lambda)$. If $p \geq \dim(M), \dim(N)$, then*
$$\dim \operatorname{Hom}_{\Lambda_p}(M, N) = \dim \operatorname{Hom}_\Lambda(M, N).$$

(iii) *Let $M, N \in \operatorname{rep}(\Lambda)$. If $p \geq \dim(M) + \dim(N)$, then*
$$\dim \operatorname{Ext}^1_{\Lambda_p}(M, N) = \dim \operatorname{Ext}^1_\Lambda(M, N).$$

(iv) *Let $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$. If $p \geq |\mathbf{d}|$, then $\operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda_p) = \operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$.*

Proof. Let $a_m \cdots a_1$ be a path of length m in Q, and let M be a representation of Λ. For any non-zero vector $v_0 \in M$ set $v_i := a_i \cdots a_1 v_0$ for $1 \leq i \leq m$. Assume that each of the vectors $v_1, \dots, v_m$ is non-zero. We claim that $v_0, v_1, \dots, v_m$ are pairwise different and linearly independent. Let b be a path of maximal length such that $bv_0 \neq 0$. Such a path b exists, because M is nilpotent. By induction $v_1, \dots, v_m$ are linearly independent. Assume now that

$$v_0 = \sum_{i=1}^m \lambda_i v_i$$

for some $\lambda_i \in \mathbb{C}$. We have $v_i = a_i \cdots a_1 v_0$. Therefore we get

$$bv_0 = \sum_{i=1}^{m} \lambda_i b a_i \cdots a_1 v_0.$$

Since $ba_i \cdots a_1$ is either zero or a path of length length$(b) + i$, we have $ba_i \cdots a_1 v_0 = 0$ for all $1 \leq i \leq m$. Since $bv_0 \neq 0$, this is a contradiction. Therefore $v_0, v_1, \ldots, v_m$ are linearly independent. It follows that for any $\mathcal{M} \in \operatorname{decrep}(\Lambda)$ with $\underline{\dim}(\mathcal{M}) = (\mathbf{d}, \mathbf{v})$ and any path b with length$(b) \geq |\mathbf{d}|$ we have $b\mathcal{M} = 0$. This implies (i). Parts (ii) and (iv) are easy consequences of (i). Any extension of representations M and N of Λ is a representation of Λ of dimension $\dim(M) + \dim(N)$. This implies (iii). □

3. E-invariants and g-vectors of decorated representations

3.1. Definition of E-invariants and g-vectors. Let Q be a quiver, and let W be a potential of Q. Let $\Lambda = \mathcal{P}(Q, W)$ be the associated Jacobian algebra [DWZ1, Section 3].

For decorated representations $\mathcal{M}$ and $\mathcal{N}$ of Λ the *g-vector* $g(\mathcal{M})$ and the invariants $E^{\text{inj}}(\mathcal{M})$ and $E^{\text{inj}}(\mathcal{M}, \mathcal{N})$ were defined in [DWZ2], where $E^{\text{inj}}(\mathcal{M})$ is called the *E-invariant* of $\mathcal{M}$. We define invariants $g_\Lambda(\mathcal{M})$, $E_\Lambda(\mathcal{M})$ and $E_\Lambda(\mathcal{M}, \mathcal{N})$ of decorated representations $\mathcal{M}$ and $\mathcal{N}$ of an arbitrary basic algebra $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle / I$ as follows.

For a decorated representation $\mathcal{M} = (M, V)$ of Λ let

$$g_\Lambda(\mathcal{M}) := (g_1, \ldots, g_n)$$

with

$$g_i := g_i(\mathcal{M}) := -\dim \operatorname{Hom}_\Lambda(S_i, M) + \dim \operatorname{Ext}^1_\Lambda(S_i, M) + \dim(V_i)$$

be the *g-vector* of $\mathcal{M}$.

For decorated representations $\mathcal{M} = (M, V)$ and $\mathcal{N} = (N, W)$ of Λ let

$$E_\Lambda(\mathcal{M}, \mathcal{N}) := \dim \operatorname{Hom}_\Lambda(M, N) + \sum_{i=1}^{n} g_i(\mathcal{N}) \dim(M_i).$$

The *E-invariant* of $\mathcal{M}$ is defined as $E_\Lambda(\mathcal{M}) := E_\Lambda(\mathcal{M}, \mathcal{M})$.

Lemma 3.1. *Let $\Lambda = \mathcal{P}(Q, W)$, where W is a potential of Q. For $\mathcal{M}, \mathcal{N} \in \operatorname{decrep}(\Lambda)$ the following hold:*

(i) *$g_\Lambda(\mathcal{M}) = g(\mathcal{M})$.*
(ii) *$E_\Lambda(\mathcal{M}, \mathcal{N}) = E^{\text{inj}}(\mathcal{M}, \mathcal{N})$.*

Proof. Part (i) follows from [P1, Lemma 4.7, Proposition 4.8]. It can also be shown in a more elementary way by using the exact sequence displayed in [DWZ2, Equation (10.4)]. Part (ii) is a direct consequence of (i) and the definition of $E_\Lambda(\mathcal{M}, \mathcal{N})$ and $E^{\text{inj}}(\mathcal{M}, \mathcal{N})$. □

3.2. Homological interpretation of the E-invariant. For $1 \leq i \leq n$ let $I_i \in \operatorname{Mod}(\Lambda)$ be the injective envelope of the simple representation S_i of Λ. One easily checks that the socle $\operatorname{soc}(I_i)$ of I_i is isomorphic to S_i, and that

$$\dim \operatorname{Hom}_\Lambda(M, I_i) = \dim(M_i) \tag{2}$$

for all $M \in \operatorname{rep}(\Lambda)$. Note that in general I_i is infinite dimensional. For $M \in \operatorname{rep}(\Lambda)$ let

$$0 \to M \xrightarrow{f} I_0^\Lambda(M) \to I_1^\Lambda(M)$$

denote a minimal injective presentation of M. The modules $I_0^\Lambda(M)$ and $I_1^\Lambda(M)$ are up to isomorphism uniquely determined by M.

We will need the following theorem due to Auslander and Reiten.

Theorem 3.2 ([AR, Theorem 1.4 (b)])**.** *Let M and N be representations of a finite-dimensional basic algebra Λ. Then we have*

$$\dim \operatorname{Hom}_\Lambda(\tau_\Lambda^-(N), M) = \dim \operatorname{Hom}_\Lambda(M, N) - \dim \operatorname{Hom}_\Lambda(M, I_0^\Lambda(N)) + \dim \operatorname{Hom}_\Lambda(M, I_1^\Lambda(N)).$$

Lemma 3.3. *Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a finite-dimensional basic algebra, and let $M \in \operatorname{rep}(\Lambda)$. Let*

$$0 \to M \xrightarrow{f} I_0^\Lambda(M) \to I_1^\Lambda(M)$$

be a minimal injective presentation of M. Then for $1 \leq i \leq n$ we have

(i) $[\operatorname{soc}(I_0^\Lambda(M)) : S_i] = [\operatorname{soc}(M) : S_i] = \dim \operatorname{Hom}_\Lambda(S_i, M)$ *and*

$$I_0^\Lambda(M) \cong I_1^{\dim \operatorname{Hom}_\Lambda(S_1,M)} \oplus \cdots \oplus I_n^{\dim \operatorname{Hom}_\Lambda(S_n,M)}.$$

(ii) $[\operatorname{soc}(I_1^\Lambda(M)) : S_i] = [\operatorname{soc}(\operatorname{Coker}(f)) : S_i] = \dim \operatorname{Ext}_\Lambda^1(S_i, M)$ *and*

$$I_1^\Lambda(M) \cong I_1^{\dim \operatorname{Ext}_\Lambda^1(S_1,M)} \oplus \cdots \oplus I_n^{\dim \operatorname{Ext}_\Lambda^1(S_n,M)}.$$

Proof. Since $I_0^\Lambda(M)$ is the injective envelope of M, we have $\operatorname{soc}(M) \cong \operatorname{soc}(I_0^\Lambda(M))$. This implies (i). By the construction of injective presentations, $I_1^\Lambda(M)$ is the injective envelope of $\operatorname{Coker}(f)$. It follows that $\operatorname{soc}(\operatorname{Coker}(f)) \cong \operatorname{soc}(I_1^\Lambda(M))$. We apply the functor $\operatorname{Hom}_\Lambda(S_i, -)$ to the exact sequence

$$0 \to M \xrightarrow{f} I_0^\Lambda(M) \to \operatorname{Coker}(f) \to 0.$$

This yields an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(S_i, M) \xrightarrow{F} \operatorname{Hom}_\Lambda(S_i, I_0^\Lambda(M)) \to \operatorname{Hom}_\Lambda(S_i, \operatorname{Coker}(f)) \xrightarrow{G} \operatorname{Ext}_\Lambda^1(S_i, M) \to 0$$

Here we used that $I_0^\Lambda(M)$ is injective, which implies $\operatorname{Ext}_\Lambda^1(S_i, I_0^\Lambda(M)) = 0$. By (i) we know that F is an isomorphism. Thus G is also an isomorphism. This implies (ii). □

Combinining Lemma 2.1 and Lemma 3.3 yields the following result.

Lemma 3.4. *Let $\mathcal{M} = (M, V)$ be a decorated representation of a basic algebra Λ, and let $g_\Lambda(\mathcal{M}) = (g_1, \ldots, g_n)$ be the g-vector of $\mathcal{M}$. If $p > \dim(M)$, then*

$$g_i = -[I_0^{\Lambda_p}(M) : S_i] + [I_1^{\Lambda_p}(M) : S_i] + \dim(V_i)$$

for all $1 \leq i \leq n$.

The following result is a homological interpretation of the E-invariant in terms of Auslander-Reiten translations. This can be seen as a generalization of [DWZ2, Corollary 10.9].

Proposition 3.5. *Let $\mathcal{M} = (M, V)$ and $\mathcal{N} = (N, W)$ be decorated representations of a basic algebra Λ. If $p > \dim(M), \dim(N)$, then*

$$E_\Lambda(\mathcal{M}, \mathcal{N}) = E_{\Lambda_p}(\mathcal{M}, \mathcal{N}) = \dim \operatorname{Hom}_{\Lambda_p}(\tau_{\Lambda_p}^-(N), M) + \sum_{i=1}^n \dim(W_i) \dim(M_i).$$

In particular, we have

$$E_{\Lambda_p}(\mathcal{M},\mathcal{N}) = E_{\Lambda_q}(\mathcal{M},\mathcal{N})$$

and

$$\dim \operatorname{Hom}_{\Lambda_p}(\tau_{\Lambda_p}^-(N), M) = \dim \operatorname{Hom}_{\Lambda_q}(\tau_{\Lambda_q}^-(N), M)$$

for all $p, q > \dim(M), \dim(N)$*.*

Proof. Since $p > \dim(M), \dim(N)$ we can apply Lemma 2.1 and get

$$\begin{aligned}
\dim \operatorname{Hom}_{\Lambda_p}(M, N) &= \dim \operatorname{Hom}_{\Lambda}(M, N),\\
\dim \operatorname{Hom}_{\Lambda_p}(S_i, N) &= \dim \operatorname{Hom}_{\Lambda}(S_i, N),\\
\dim \operatorname{Ext}^1_{\Lambda_p}(S_i, N) &= \dim \operatorname{Ext}^1_{\Lambda}(S_i, N).
\end{aligned}$$

Let

$$0 \to N \to I_0^{\Lambda_p}(N) \to I_1^{\Lambda_p}(N)$$

be a minimal injective presentation of N, where we regard N now as a representation of Λ_p. It follows from Lemma 3.3 and Equation (2) that

$$\begin{aligned}
\dim \operatorname{Hom}_{\Lambda_p}(M, I_0^{\Lambda_p}(N)) &= \sum_{i=1}^n \dim \operatorname{Hom}_{\Lambda_p}(S_i, N) \dim(M_i),\\
\dim \operatorname{Hom}_{\Lambda_p}(M, I_1^{\Lambda_p}(N)) &= \sum_{i=1}^n \dim \operatorname{Ext}^1_{\Lambda_p}(S_i, N) \dim(M_i).
\end{aligned}$$

This implies

$$\begin{aligned}
E_{\Lambda}(\mathcal{M},\mathcal{N}) &= \dim \operatorname{Hom}_{\Lambda}(M, N) + \sum_{i=1}^n (-\dim \operatorname{Hom}_{\Lambda}(S_i, N) + \dim \operatorname{Ext}^1_{\Lambda}(S_i, N)) \dim(M_i)\\
&\quad + \sum_{i=1}^n \dim(W_i) \dim(M_i)\\
&= \dim \operatorname{Hom}_{\Lambda_p}(M, N) + \sum_{i=1}^n (-\dim \operatorname{Hom}_{\Lambda_p}(S_i, N) + \dim \operatorname{Ext}^1_{\Lambda_p}(S_i, N)) \dim(M_i)\\
&\quad + \sum_{i=1}^n \dim(W_i) \dim(M_i)\\
&= \dim \operatorname{Hom}_{\Lambda_p}(M, N) - \dim \operatorname{Hom}_{\Lambda_p}(M, I_0^{\Lambda_p}(N)) + \dim \operatorname{Hom}_{\Lambda_p}(M, I_1^{\Lambda_p}(N))\\
&\quad + \sum_{i=1}^n \dim(W_i) \dim(M_i).
\end{aligned}$$

The first equality follows from Lemmas 2.1, 3.3 and 3.4. The second equality says that $E_{\Lambda}(\mathcal{M},\mathcal{N}) = E_{\Lambda_p}(\mathcal{M},\mathcal{N})$. Applying Theorem 3.2 yields

$$E_{\Lambda_p}(\mathcal{M},\mathcal{N}) = \dim \operatorname{Hom}_{\Lambda_p}(\tau_{\Lambda_p}^-(N), M) + \sum_{i=1}^n \dim(W_i) \dim(M_i).$$

This finishes the proof. □

Corollary 3.6. *For decorated representations* $\mathcal{M}$ *and* $\mathcal{N}$ *of a basic algebra* Λ *we have*

$$E_{\Lambda}(\mathcal{M},\mathcal{N}) \geq 0.$$

4. Caldero-Chapoton algebras

4.1. Caldero-Chapoton functions. To any basic algebra $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ we associate a set of Laurent polynomials in n independent variables $x_1, \dots, x_n$ as follows. The *Caldero-Chapoton function* associated to a decorated representation $\mathcal{M} = (M, V)$ of Λ is defined as

$$C_\Lambda(\mathcal{M}) := \mathbf{x}^{g_\Lambda(\mathcal{M})} \sum_{\mathbf{e}\in\mathbb{N}^n} \chi(\mathrm{Gr}_{\mathbf{e}}(M))\mathbf{x}^{B_Q\mathbf{e}}.$$

Note that $C_\Lambda(\mathcal{M}) \in \mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$. Let

$$\mathcal{C}_\Lambda := \{C_\Lambda(\mathcal{M}) \mid \mathcal{M} \in \mathrm{decrep}(\Lambda)\}$$

be the set of Caldero-Chapoton functions associated to Λ. For $\mathcal{M} = (M, 0)$ we sometimes write $C_\Lambda(M)$ instead of $C_\Lambda(\mathcal{M})$.

The definition of $C_\Lambda(\mathcal{M})$ is motivated by the (different versions of) Caldero-Chapoton functions appearing in the theory of cluster algebras, see for example [Pa, Section 1]. Such functions first appeared in work of Caldero and Chapoton [CC, Section 3], where they show that the cluster variables of a cluster algebra of a Dynkin quiver are Caldero-Chapoton functions.

Lemma 4.1. *For decorated representations $\mathcal{M} = (M, V)$ and $\mathcal{N} = (N, W)$ the following hold:*

(i) $g_\Lambda(\mathcal{M} \oplus \mathcal{N}) = g_\Lambda(\mathcal{M}) + g_\Lambda(\mathcal{N})$.
(ii) $C_\Lambda(\mathcal{M}) = C_\Lambda(M, 0)C_\Lambda(0, V)$.
(iii) $C_\Lambda(\mathcal{M} \oplus \mathcal{N}) = C_\Lambda(\mathcal{M})C_\Lambda(\mathcal{N})$.

Proof. Part (i) follows directly from the definitions and from the additivity of the functors $\mathrm{Hom}_\Lambda(-, ?)$ and $\mathrm{Ext}^1_\Lambda(-, ?)$. To prove (ii), let $\mathcal{M} = (M, V)$ be a decorated representation of Λ. For the decorated representation $(0, V)$ we have

$$C_\Lambda(0, V) = \prod_{i=1}^{n} x_i^{v_i}$$

where $\underline{\dim}(V) = (v_1, \dots, v_n)$. For the decorated representation $(M, 0)$ we have

$$C_\Lambda(M, 0) := \mathbf{x}^{g_\Lambda(M,0)} \sum_{\mathbf{e}\in\mathbb{N}^n} \chi(\mathrm{Gr}_{\mathbf{e}}(M))\mathbf{x}^{B_Q\mathbf{e}}$$

where $g_i(M, 0) = -\dim \mathrm{Hom}_\Lambda(S_i, M) + \dim \mathrm{Ext}^1_\Lambda(S_i, M)$ for $1 \le i \le n$. Now one easily checks that $C_\Lambda(\mathcal{M}) = C_\Lambda(M, 0)C_\Lambda(0, V)$. Thus (ii) holds. Now (iii) follows from (i), (ii) and the well known formula

$$\chi(\mathrm{Gr}_{\mathbf{e}}(M \oplus N)) = \sum_{(\mathbf{e}',\mathbf{e}'')} \chi(\mathrm{Gr}_{\mathbf{e}'}(M))\chi(\mathrm{Gr}_{\mathbf{e}''}(N))$$

where the sum runs over all pairs $(\mathbf{e}', \mathbf{e}'') \in \mathbb{N}^n \times \mathbb{N}^n$ such that $\mathbf{e}' + \mathbf{e}'' = \mathbf{e}$. □

4.2. Definition of a Caldero-Chapoton algebra. In the previous section, we associated to a basic algebra Λ the set

$$\mathcal{C}_\Lambda = \{C_\Lambda(\mathcal{M}) \mid \mathcal{M} \in \mathrm{decrep}(\Lambda)\}$$

of Caldero-Chapoton functions. Clearly, $\mathcal{C}_\Lambda$ is a subset of the integer Laurent polynomial ring $\mathbb{Z}[x_1^\pm, \dots, x_n^\pm]$ generated by the variables $x_1, \dots, x_n$. By definition the *Caldero-Chapoton algebra* $\mathcal{A}_\Lambda$ associated to Λ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ generated by $\mathcal{C}_\Lambda$. The following is a direct consequence of Lemma 4.1(iii).

Lemma 4.2. *The set $\mathcal{C}_\Lambda$ generates $\mathcal{A}_\Lambda$ as a $\mathbb{C}$-vector space.*

4.3. Linear independence of Caldero-Chapoton functions. Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ be a basic algebra. Except in some trivial cases, the set $\mathcal{C}_\Lambda$ of Caldero-Chapoton functions associated to decorated representations of Λ is linearly dependent. Often the Caldero-Chapoton functions satisfy beautiful relations, which should be studies more intensively. On the other hand, by Lemma 4.2, there are $\mathbb{C}$-bases of $\mathcal{A}_\Lambda$ consisting only of Caldero-Chapoton functions. Our aim is to provide a candidate $\mathcal{B}_\Lambda$ for such a basis. Before constructing $\mathcal{B}_\Lambda$ in Section 5, we prove the following criterion for linear independence of certain sets of Caldero-Chapoton functions.

Let

$$\mathbb{Q}^n_{\geq 0} := \{(a_1, \ldots, a_n) \in \mathbb{Q}^n \mid a_i \geq 0 \text{ for all } i\},$$
$$\mathbb{Q}^n_{> 0} := \{(a_1, \ldots, a_n) \in \mathbb{Q}^n \mid a_i > 0 \text{ for all } i\}.$$

Proposition 4.3. *Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ be a basic algebra. Let $\mathcal{M}_j$, $j \in J$ be decorated representations of Λ. Assume the following:*

(i) $\operatorname{Ker}(B_Q) \cap \mathbb{Q}^n_{\geq 0} = 0$.
(ii) *The g-vectors $g_\Lambda(\mathcal{M}_j)$, $j \in J$ are pairwise different.*

Then the Caldero-Chapoton functions $C_\Lambda(\mathcal{M}_j)$, $j \in J$ are pairwise different and linearly independent in $\mathcal{A}_\Lambda$.

Proof. We treat B_Q as a linear map $\mathbb{Q}^n \to \mathbb{Q}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ define $\mathbf{a} \leq \mathbf{b}$ if there exists some $\mathbf{e} \in \mathbb{Q}^n_{\geq 0}$ such that

$$\mathbf{a} = \mathbf{b} + B_Q\mathbf{e}.$$

We claim that this defines a partial order on $\mathbb{Z}^n$. Clearly, $\mathbf{a} \leq \mathbf{a}$, so $\leq$ is reflexive. Furthermore, assume that $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a}$. Thus $\mathbf{a} = \mathbf{b} + B_Q\mathbf{e}_1$ and $\mathbf{b} = \mathbf{a} + B_Q\mathbf{e}_2$ for some $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{Q}^n_{\geq 0}$. It follows that $\mathbf{a} = \mathbf{a} + B_Q(\mathbf{e}_1 + \mathbf{e}_2)$. Thus $\mathbf{e}_1 + \mathbf{e}_2 \in \operatorname{Ker}(B_Q)$. Our assumption (i) yields $\mathbf{e}_1 = \mathbf{e}_2 = 0$. Thus $\mathbf{a} = \mathbf{b}$. This shows that $\leq$ is antisymmetric. Finally, assume that $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$. Thus we have $\mathbf{a} = \mathbf{b} + B_Q\mathbf{e}_1$ and $\mathbf{b} = \mathbf{c} + B_Q\mathbf{e}_2$ for some $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{Q}^n_{\geq 0}$. It follows that $\mathbf{a} = \mathbf{c} + B_Q(\mathbf{e}_1 + \mathbf{e}_2)$. In other words, we have $\mathbf{a} \leq \mathbf{c}$. Thus $\leq$ is transitive.

The partial order $\leq$ on $\mathbb{Z}^n$ induces obviously a partial order on the set of Laurent monomials in the variables $x_1, \ldots, x_n$. Namely, set $\mathbf{x}^\mathbf{a} \leq \mathbf{x}^\mathbf{b}$ if $\mathbf{a} \leq \mathbf{b}$. Let $\deg(\mathbf{x}^\mathbf{a}) := \mathbf{a}$ be the *degree* of $\mathbf{x}^\mathbf{a}$.

Among the Laurent monomials $\mathbf{x}^{g_\Lambda(\mathcal{M}) + B_Q\mathbf{e}}$ occuring in the expression

$$C_\Lambda(\mathcal{M}) = \mathbf{x}^{g_\Lambda(\mathcal{M})} \sum_{\mathbf{e} \in \mathbb{N}^n} \chi(\operatorname{Gr}_\mathbf{e}(M))\mathbf{x}^{B_Q\mathbf{e}} = \sum_{\mathbf{e} \in \mathbb{N}^n} \chi(\operatorname{Gr}_\mathbf{e}(M))\mathbf{x}^{g_\Lambda(\mathcal{M}) + B_Q\mathbf{e}}$$

the monomial $\mathbf{x}^{g_\Lambda(\mathcal{M})}$ is the unique monomial of maximal degree.

For $\mathbf{e} = 0$ the Grassmannian $\operatorname{Gr}_\mathbf{e}(M)$ is just a point, and $B_Q\mathbf{e} = 0$. Thus, if $\mathbf{e} = 0$, we have $\chi(\operatorname{Gr}_\mathbf{e}(M))\mathbf{x}^{B_Q\mathbf{e}} = 1$. This shows that the Laurent monomial $\mathbf{x}^{g_\Lambda(\mathcal{M})}$ really occurs as a non-trivial summand of $C_\Lambda(\mathcal{M})$. In particular, we have $C_\Lambda(\mathcal{M}) \neq C_\Lambda(\mathcal{N})$ if $g_\Lambda(\mathcal{M}) \neq g_\Lambda(\mathcal{N})$.

Now let $\mathcal{M}_1, \ldots, \mathcal{M}_t$ be decorated representations of Λ with pairwise different g-vectors. Assume that

$$\lambda_1 C_\Lambda(\mathcal{M}_1) + \cdots + \lambda_t C_\Lambda(\mathcal{M}_t) = 0$$

for some $\lambda_j \in \mathbb{C}$. Without loss of generality we assume that $\lambda_j \neq 0$ for all j. There is a (not necessarily unique) index s such that $\mathbf{x}^{g_\Lambda(\mathcal{M}_s)}$ is maximal in the set $\{\mathbf{x}^{g_\Lambda(\mathcal{M}_j)} \mid 1 \leq j \leq t\}$. It follow that the Laurent monomial $\mathbf{x}^{g_\Lambda(\mathcal{M}_s)}$ does not occur as a summand of any of the Laurent polynomials $C_\Lambda(\mathcal{M}_j)$ with $j \neq s$. (Here we use that the g-vectors of the decorated representations $\mathcal{M}_j$ are pairwise different.) This implies $\lambda_s = 0$, a contradiction. Thus $C_\Lambda(\mathcal{M}_1), \dots, C_\Lambda(\mathcal{M}_t)$ are linearly independent. □

For the example, where Λ is the path algebra of an affine quiver of type $\mathbb{A}_2$, the main argument used in the proof of Proposition 4.3 can already be found in [C, Section 6.1].

Note that condition (d) in the following lemma coincides with condition (i) in Proposition 4.3.

Lemma 4.4. *For the conditions*

(a) $\operatorname{rank}(B_Q) = n$.
(b) *Each row of B_Q has at least one non-zero entry, and there are $n - \operatorname{rank}(B_Q)$ rows of B_Q, which are non-negative linear combinations of the remaining $\operatorname{rank}(B_Q)$ rows of B_Q.*
(c) $\operatorname{Im}(B_Q) \cap \mathbb{Q}^n_{>0} \neq \varnothing$.
(d) $\operatorname{Ker}(B_Q) \cap \mathbb{Q}^n_{\geq 0} = 0$.

the implications

$$\text{(a)} \implies \text{(b)} \implies \text{(c)} \implies \text{(d)}$$

hold.

Proof. The implication (a) $\implies$ (b) is trivial. Next, assume (b) holds. Let $m := \operatorname{rank}(B_Q)$. We denote the jth row of B_Q by r_j. By assumption there are pairwise different indices $i_1, \dots, i_m \in \{1, \dots, n\}$ such that for each $1 \leq k \leq n$ with $k \notin \{i_1, \dots, i_m\}$ we have

$$r_k = \lambda_1^{(k)} r_{i_1} + \cdots + \lambda_m^{(k)} r_{i_m}$$

for some non-negative rational numbers $\lambda_j^{(k)}$. Since r_k is non-zero, at least one of the $\lambda_j^{(k)}$ is posititve. Clearly, there is some $\mathbf{e} \in \mathbb{Q}^n$ such that $r_{i_j} \cdot \mathbf{e} = 1$ for all $1 \leq j \leq m$. (The $(k \times n)$-matrix with rows $r_{i_1}, \dots, r_{i_m}$ has rank m. Thus, we can see it as a surjective homomorphism $\mathbb{Q}^n \to \mathbb{Q}^m$.) Now observe that the kth entry of $B_Q\mathbf{e}$ is $\lambda_1^{(k)} + \cdots + \lambda_m^{(k)}$ for all $1 \leq k \leq n$ with $k \notin \{i_1, \dots, i_m\}$ and that this entry is positive. It follows that $\operatorname{Im}(B_Q) \cap \mathbb{Q}^n_{>0} \neq \varnothing$.

Finally, to show (c) $\implies$ (d) let $\mathbf{b} \in \operatorname{Im}(B_Q) \cap \mathbb{Q}^n_{>0}$. Thus there is some $\mathbf{a} \in \mathbb{Q}^n$ such that $B_Q\mathbf{a} = \mathbf{b}$. Since B_Q is skew-symmetric, we get $-\mathbf{a}B_Q = \mathbf{b}$. Now let $\mathbf{e} \in \operatorname{Ker}(B_Q) \cap \mathbb{Q}^n_{\geq 0}$. We get $B_Q\mathbf{e} = 0$, and therefore $-\mathbf{a}B_Q\mathbf{e} = \mathbf{b} \cdot \mathbf{e} = 0$. Since $\mathbf{b}$ has only positive entries and $\mathbf{e}$ has only non-negative entries, we get $\mathbf{e} = 0$. This finishes the proof. □

If we replace condition (i) by condition (a), Proposition 4.3 was first proved by Fu and Keller [FK, Corollary 4.4]. Essentially the same argument was later also used by Plamondon [P1]. That the Fu-Keller argument can be applied under condition (b) was observed by Geiß and Labardini. To any triangulation T of a punctured Riemann surface with non-empty boundary, one can associate a 2-acyclic quiver Q_T. It is shown in [GL] that there is always a triangulation T such that the matrix B_{Q_T} satisfies condition (b).

5. Strongly reduced components of representation varieties

5.1. Decomposition theorems for irreducible components. Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ be a basic algebra, and let $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$. By $\mathrm{Irr}_{\mathbf{d}}(\Lambda)$ and $\mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we denote the set of irreducible components of $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ and $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$, respectively. For $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we write $\underline{\dim}(Z) := (\mathbf{d}, \mathbf{v})$. Let

$$\mathrm{Irr}(\Lambda) = \bigcup_{\mathbf{d} \in \mathbb{N}^n} \mathrm{Irr}_{\mathbf{d}}(\Lambda) \quad \text{and} \quad \mathrm{decIrr}(\Lambda) = \bigcup_{(\mathbf{d},\mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n} \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda).$$

Note that any irreducible component $Z \in \mathrm{decIrr}(\Lambda)$ can be seen as an irreducible component in $\mathrm{Irr}(\Lambda_{\mathrm{dec}})$, where $\Lambda_{\mathrm{dec}} := \Lambda \times \mathbb{C} \times \cdots \times \mathbb{C}$ is defined as the product of Λ with n copies of $\mathbb{C}$. In fact, we can identify $\mathrm{decrep}(\Lambda)$ and $\mathrm{rep}(\Lambda_{\mathrm{dec}})$. Thus statements on varieties of representations can be carried over to varieties of decorated representations.

By definition we have

$$\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) = \{(M, \mathbb{C}^{\mathbf{v}}) \mid M \in \mathrm{rep}_{\mathbf{d}}(\Lambda)\}.$$

We have an isomorphism

$$\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) \to \mathrm{rep}_{\mathbf{d}}(\Lambda)$$

of affine varieties mapping $(M, \mathbb{C}^{\mathbf{v}})$ to M. Thus the irreducible components of $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ can be interpreted as irreducible components of $\mathrm{rep}_{\mathbf{d}}(\Lambda)$. For $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ let πZ be the corresponding component in $\mathrm{Irr}_{\mathbf{d}}(\Lambda)$. Recall that the group $G_{\mathbf{d}}$ acts on $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ by

$$g.(M, \mathbb{C}^{\mathbf{v}}) := (g.M, \mathbb{C}^{\mathbf{v}}),$$

and that the $G_{\mathbf{d}}$-orbit of a decorated representation $\mathcal{M}$ is denoted by $\mathcal{O}(\mathcal{M})$.

For $Z, Z_1, Z_2 \in \mathrm{decIrr}(\Lambda)$ define

$$\begin{aligned}
c_\Lambda(Z) &:= \min\{\dim(Z) - \dim \mathcal{O}(\mathcal{M}) \mid \mathcal{M} \in Z\},\\
e_\Lambda(Z) &:= \min\{\dim \mathrm{Ext}^1_\Lambda(M, M) \mid \mathcal{M} = (M, V) \in Z\},\\
E_\Lambda(Z) &:= \min\{E_\Lambda(\mathcal{M}) \mid \mathcal{M} \in Z\},\\
\mathrm{end}_\Lambda(Z) &:= \min\{\dim \mathrm{End}_\Lambda(M) \mid \mathcal{M} = (M, V) \in Z\},\\
\hom_\Lambda(Z_1, Z_2) &:= \min\{\dim \mathrm{Hom}_\Lambda(M_1, M_2) \mid \mathcal{M}_i = (M_i, V_i) \in Z_i,\ i = 1, 2\},\\
\mathrm{ext}^1_\Lambda(Z_1, Z_2) &:= \min\{\dim \mathrm{Ext}^1_\Lambda(M_1, M_2) \mid \mathcal{M}_i = (M_i, V_i) \in Z_i,\ i = 1, 2\},\\
E_\Lambda(Z_1, Z_2) &:= \min\{E_\Lambda(\mathcal{M}_1, \mathcal{M}_2) \mid \mathcal{M}_i \in Z_i,\ i = 1, 2\}.
\end{aligned}$$

It is easy to construct examples of components $Z \in \mathrm{decIrr}(\Lambda)$ such that $\mathrm{end}_\Lambda(Z) \neq \hom_\Lambda(Z, Z)$, $e_\Lambda(Z) \neq \mathrm{ext}^1_\Lambda(Z, Z)$ and $E_\Lambda(Z) \neq E_\Lambda(Z, Z)$. Note that for $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we have

$$c_\Lambda(Z) = \dim(Z) - \dim(G_{\mathbf{d}}) + \mathrm{end}_\Lambda(Z).$$

This follows from Equation (1).

By [CBS, Lemma 4.3] the functions $\dim \mathrm{Hom}_\Lambda(-, ?)$ and $\dim \mathrm{Ext}^1_\Lambda(-, ?)$ are upper semicontinuous. Using this one easily shows the following lemma.

Lemma 5.1. *For $Z, Z_1, Z_2 \in \mathrm{decIrr}(\Lambda)$ the following hold:*

(i) *The sets*

$$\{\mathcal{M} \in Z \mid \dim(Z) - \dim \mathcal{O}(\mathcal{M}) = c_\Lambda(Z)\},$$
$$\{\mathcal{M} = (M,V) \in Z \mid \dim \operatorname{Ext}^1_\Lambda(M,M) = e_\Lambda(Z)\},$$
$$\{\mathcal{M} \in Z \mid E_\Lambda(\mathcal{M}) = E_\Lambda(Z)\},$$
$$\{\mathcal{M} = (M,V) \in Z \mid \dim \operatorname{End}_\Lambda(M) = \operatorname{end}_\Lambda(Z)\}$$

are open in Z.

(ii) *The sets*

$$\{((M_1,V_1),(M_2,V_2)) \in Z_1 \times Z_2 \mid \dim \operatorname{Hom}_\Lambda(M_1,M_2) = \hom_\Lambda(Z_1,Z_2)\},$$
$$\{((M_1,V_1),(M_2,V_2)) \in Z_1 \times Z_2 \mid \dim \operatorname{Ext}^1_\Lambda(M_1,M_2) = \operatorname{ext}^1_\Lambda(Z_1,Z_2)\},$$
$$\{(\mathcal{M}_1,\mathcal{M}_2) \in Z_1 \times Z_2 \mid E_\Lambda(\mathcal{M}_1,\mathcal{M}_2) = E_\Lambda(Z_1,Z_2)\}$$

are open in $Z_1 \times Z_2$.

For $Z \in \operatorname{decIrr}(\Lambda)$ there is a dense open subset U of Z such that $g_\Lambda(\mathcal{M}) = g_\Lambda(\mathcal{N})$ for all $\mathcal{M},\mathcal{N} \in U$. This follows again by upper semicontinuity. For $\mathcal{M} \in U$ let

$$g_\Lambda(Z) := g_\Lambda(\mathcal{M})$$

be the *generic g-vector* of Z.

Lemma 5.2. *For* $Z, Z_1, Z_2 \in \operatorname{decIrr}(\Lambda)$ *we have*

$$c_\Lambda(Z) \le e_\Lambda(Z) \le E_\Lambda(Z) \quad \textit{and} \quad \operatorname{ext}^1_\Lambda(Z_1,Z_2) \le E_\Lambda(Z_1,Z_2).$$

Proof. Let $\mathbf{d} = \underline{\dim}(\pi Z)$ and $\mathbf{d}_i = \underline{\dim}(\pi Z_i)$. Choose some $p \ge 2|\mathbf{d}|, |\mathbf{d}_1| + |\mathbf{d}_2|$. By Lemma 2.1 we can regard all the representations in Z, Z_1 and Z_2 as representations of Λ_p. Thus we can interpret Z, Z_1 and Z_2 as irreducible components in $\operatorname{decIrr}(\Lambda_p)$. Now Proposition 3.5 allows us to assume without loss of generality that $\Lambda = \Lambda_p$. Voigt's Lemma [G, Proposition 1.1] implies that $c_\Lambda(Z) \le e_\Lambda(Z)$. The Auslander-Reiten formula $\operatorname{Ext}^1_\Lambda(M,N) \cong \mathrm{D}\underline{\operatorname{Hom}}_\Lambda(\tau_\Lambda^-(N),M)$ yields

$$\dim \operatorname{Ext}^1_\Lambda(M,N) \le \dim \operatorname{Hom}_\Lambda(\tau_\Lambda^-(N),M).$$

This implies $e_\Lambda(Z) \le E_\Lambda(Z)$ and $\operatorname{ext}^1_\Lambda(Z_1,Z_2) \le E_\Lambda(Z_1,Z_2)$. (Here we used again Proposition 3.5.) □

Following [GLS] we call an irreducible component $Z \in \operatorname{decIrr}(\Lambda)$ *strongly reduced* provided

$$c_\Lambda(Z) = e_\Lambda(Z) = E_\Lambda(Z).$$

For example, if Λ is finite-dimensional, one can easily check that for any injective Λ-module $I \in \operatorname{rep}(\Lambda)$ the closure of the orbit $\mathcal{O}(I,0)$ is a strongly reduced irreducible component. Similarly, it follows directly from the definitions that for all decorated representations of the form $\mathcal{M} = (0,V)$, the closure of $\mathcal{O}(\mathcal{M})$ is a strongly reduced component. (In this case, $\mathcal{O}(\mathcal{M})$ is just a point, and it is equal to its closure.)

Let $\operatorname{decIrr}^{\text{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda)$ be the set of all strongly reduced components of $\operatorname{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$, and let

$$\operatorname{decIrr}^{\text{s.r.}}(\Lambda) := \bigcup_{(\mathbf{d},\mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n} \operatorname{decIrr}^{\text{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda).$$

An irreducible component Z in $\operatorname{Irr}(\Lambda)$ or $\operatorname{decIrr}(\Lambda)$ is called *indecomposable* provided there exists a dense open subset U of Z, which contains only indecomposable representations or decorated representations, respectively. In particular, if $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ is indecomposable, then either $\mathbf{d} = 0$ or $\mathbf{v} = 0$.

Given irreducible components Z_i of $\mathrm{decrep}_{\mathbf{d}_i,\mathbf{v}_i}(\Lambda)$ for $1 \le i \le t$, let $(\mathbf{d},\mathbf{v}) := (\mathbf{d}_1,\mathbf{v}_1) + \cdots + (\mathbf{d}_t,\mathbf{v}_t)$ and let

$$Z_1 \oplus \cdots \oplus Z_t$$

be the points of $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$, which are isomorphic to $M_1 \oplus \cdots \oplus M_t$ with $M_i \in Z_i$ for $1 \le i \le t$. The Zariski closure of $Z_1 \oplus \cdots \oplus Z_t$ in $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ is denoted by

$$\overline{Z_1 \oplus \cdots \oplus Z_t}.$$

It is quite easy to show that $\overline{Z_1 \oplus \cdots \oplus Z_t}$ is an irreducible closed subset of $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$, but in general it is not an irreducible component.

Theorem 5.3 ([CBS]). *For $Z_1,\ldots,Z_t \in \mathrm{decIrr}(\Lambda)$ the following are equivalent:*

(i) *$\overline{Z_1 \oplus \cdots \oplus Z_t}$ is an irreducible component.*
(ii) *$\mathrm{ext}^1_\Lambda(Z_i,Z_j) = 0$ for all $i \ne j$.*

Furthermore, the following hold:

(iii) *Each irreducible component $Z \in \mathrm{decIrr}(\Lambda)$ can be written as $Z = \overline{Z_1 \oplus \cdots \oplus Z_t}$ with $Z_1,\ldots,Z_t$ indecomposable irreducible components in $\mathrm{decIrr}(\Lambda)$. Suppose that*
$$\overline{Z_1 \oplus \cdots \oplus Z_t} = \overline{Z_1' \oplus \cdots \oplus Z_s'}$$
is an irreducible component with Z_i and Z_i' indecomposable irreducible components in $\mathrm{decIrr}(\Lambda)$ for all i. Then $s = t$ and there is a bijection $\sigma\colon \{1,\ldots,t\} \to \{1,\ldots,s\}$ such that $Z_i = Z'_{\sigma(i)}$ for all i.

The next lemma is an easy exercise.

Lemma 5.4. *For $1 \le i \le n$ and any decorated representation $\mathcal{M} = (M,V)$ of Λ we have*
$$E_\Lambda(\mathcal{M},\mathcal{S}_i^-) = \dim(M_i) \quad \text{and} \quad E_\Lambda(\mathcal{S}_i^-,\mathcal{M}) = 0.$$

Corollary 5.5. *For any $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we have $d_iv_i = 0$ for all $1 \le i \le n$.*

Lemma 5.6. *Let $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$, and assume that $p > |\mathbf{d}|$. Then the following are equivalent:*

(i) *$Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$.*
(ii) *$Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda_p)$.*

Proof. Since $p > |\mathbf{d}|$, we can apply Lemma 2.1 and Proposition 3.5 and get $c_{\Lambda_p}(Z) = c_\Lambda(Z)$ and $E_{\Lambda_p}(Z) = E_\Lambda(Z)$. This yields the result. □

The additivity of the functor $\mathrm{Hom}_\Lambda(-,?)$ and upper semicontinuity imply the following lemma.

Lemma 5.7. *Let $Z,Z_1,Z_2 \in \mathrm{decIrr}(\Lambda)$. Suppose that $Z = \overline{Z_1 \oplus Z_2}$. Then the following hold:*

(i) $\mathrm{end}_\Lambda(Z) = \mathrm{end}_\Lambda(Z_1) + \mathrm{end}_\Lambda(Z_2) + \mathrm{hom}_\Lambda(Z_1,Z_2) + \mathrm{hom}_\Lambda(Z_2,Z_1)$.
(ii) $E_\Lambda(Z) = E_\Lambda(Z_1) + E_\Lambda(Z_2) + E_\Lambda(Z_1,Z_2) + E_\Lambda(Z_2,Z_1)$.

For $\mathbf{a} = (a_1,\ldots,a_n)$ and $\mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{Z}^n$ set $\mathbf{a}\cdot\mathbf{b} := a_1b_1 + \cdots + a_nb_n$. The following lemma is obvious.

Lemma 5.8. *Let* $\mathbf{d}, \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{N}^n$ *with* $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$*. Then*

$$\dim(G_{\mathbf{d}}) - \dim(G_{\mathbf{d}_1}) - \dim(G_{\mathbf{d}_2}) = 2(\mathbf{d}_1 \cdot \mathbf{d}_2).$$

Lemma 5.9. *Let* $Z, Z_1, Z_2 \in \operatorname{decIrr}(\Lambda)$ *with* $Z = \overline{Z_1 \oplus Z_2}$*. We have*

$$\dim(Z) = \dim(Z_1) + \dim(Z_2) + 2(\underline{\dim}(\pi Z_1) \cdot \underline{\dim}(\pi Z_2)) - \hom_\Lambda(Z_1, Z_2) - \hom_\Lambda(Z_2, Z_1).$$

Proof. For $i = 1, 2$ let $(\mathbf{d}_i, \mathbf{v}_i) := \underline{\dim}(Z_i)$, and let $(\mathbf{d}, \mathbf{v}) := \underline{\dim}(Z)$. We have $\underline{\dim}(Z) = \underline{\dim}(Z_1) + \underline{\dim}(Z_2)$ and $\underline{\dim}(\pi Z_i) = \mathbf{d}_i$. The map

$$f \colon G_{\mathbf{d}} \times Z_1 \times Z_2 \to Z$$

defined by

$$(g, (M_1, \mathbb{C}^{\mathbf{v}_1}), (M_2, \mathbb{C}^{\mathbf{v}_2})) \mapsto (g.(M_1 \oplus M_2), \mathbb{C}^{\mathbf{v}})$$

is a morphism of affine varieties. For $(\mathcal{M}_1, \mathcal{M}_2) \in Z_1 \times Z_2$ define

$$f_{\mathcal{M}_1,\mathcal{M}_2} \colon G_{\mathbf{d}} \times \mathcal{O}(\mathcal{M}_1) \times \mathcal{O}(\mathcal{M}_2) \to \mathcal{O}(\mathcal{M}_1 \oplus \mathcal{M}_2)$$

by $(g, \mathcal{N}_1, \mathcal{N}_2) \mapsto f(g, \mathcal{N}_1, \mathcal{N}_2)$. The fibres of $f_{\mathcal{M}_1,\mathcal{M}_2}$ are of dimension

$$d_{\mathcal{M}_1,\mathcal{M}_2} := \dim(G_{\mathbf{d}}) + \dim \mathcal{O}(\mathcal{M}_1) + \dim \mathcal{O}(\mathcal{M}_2) - \dim \mathcal{O}(\mathcal{M}_1 \oplus \mathcal{M}_2).$$

Using Equation (1), an easy calculation yields

$$d_{\mathcal{M}_1,\mathcal{M}_2} = \dim(G_{\mathbf{d}_1}) + \dim(G_{\mathbf{d}_2}) + \dim \operatorname{Hom}_\Lambda(M_1, M_2) + \dim \operatorname{Hom}_\Lambda(M_2, M_1).$$

Let $\mathcal{M}$ be in the image of f. We want to compute the dimension of the fibre $f^{-1}(\mathcal{M})$. Let

$$\mathcal{T} := \{\mathcal{O}(\mathcal{N}_1) \times \mathcal{O}(\mathcal{N}_2) \subseteq Z_1 \times Z_2 \mid \mathcal{N}_1 \oplus \mathcal{N}_2 \cong \mathcal{M}\}.$$

It follows from the Krull-Remak-Schmidt Theorem that $\mathcal{T}$ is a finite set. Thus the fibre $f^{-1}(\mathcal{M})$ is the disjoint union of the fibres $f^{-1}_{\mathcal{N}_1,\mathcal{N}_2}(\mathcal{M})$, where $\mathcal{O}(\mathcal{N}_1) \times \mathcal{O}(\mathcal{N}_2)$ runs through $\mathcal{T}$. So we get

$$\dim(f^{-1}(\mathcal{M})) = \max\{d_{\mathcal{N}_1,\mathcal{N}_2} \mid \mathcal{O}(\mathcal{N}_1) \times \mathcal{O}(\mathcal{N}_2) \in \mathcal{T}\}.$$

Thus by upper semicontinuity there is a dense open subset $V \subseteq Z$ such that all fibres $f^{-1}(\mathcal{M})$ with $\mathcal{M} \in V$ have dimension

$$d_{Z_1,Z_2} := \dim(G_{\mathbf{d}_1}) + \dim(G_{\mathbf{d}_2}) + \hom_\Lambda(Z_1, Z_2) + \hom_\Lambda(Z_2, Z_1).$$

By Chevalley's Theorem we have

$$\dim(Z) + d_{Z_1,Z_2} = \dim(G_{\mathbf{d}}) + \dim(Z_1) + \dim(Z_2).$$

Using Lemma 5.8 we get

$$\dim(Z) = \dim(Z_1) + \dim(Z_2) - 2(\mathbf{d}_1 \cdot \mathbf{d}_2) - \hom_\Lambda(Z_1, Z_2) - \hom_\Lambda(Z_2, Z_1).$$

This finishes the proof. □

Lemma 5.10. *For* $Z, Z_1, Z_2 \in \operatorname{decIrr}(\Lambda)$ *with* $Z = \overline{Z_1 \oplus Z_2}$ *we have*

$$c_\Lambda(Z) = c_\Lambda(Z_1) + c_\Lambda(Z_2).$$

Proof. For $i = 1, 2$ let $(\mathbf{d}_i, \mathbf{v}_i) := \underline{\dim}(Z_i)$, and let $(\mathbf{d}, \mathbf{v}) := \underline{\dim}(Z)$. We get

$$\begin{aligned} c_\Lambda(Z) &= \dim(Z) - \dim(G_{\mathbf{d}}) + \operatorname{end}_\Lambda(Z) \\ &= \dim(Z_1) + \dim(Z_2) - \dim(G_{\mathbf{d}_1}) - \dim(G_{\mathbf{d}_2}) + \operatorname{end}_\Lambda(Z_1) + \operatorname{end}_\Lambda(Z_2) \\ &= c_\Lambda(Z_1) + c_\Lambda(Z_2). \end{aligned}$$

The first equality follows directly from the definition of $c_\Lambda(Z)$. The second equality uses Lemma 5.7(i) and Lemma 5.9. □

The following result is a version of Theorem 5.3 for strongly reduced components.

Theorem 5.11. *For $Z_1, \ldots, Z_t \in \mathrm{decIrr}(\Lambda)$ the following are equivalent:*

(i) $\overline{Z_1 \oplus \cdots \oplus Z_t}$ *is a strongly reduced irreducible component.*
(ii) *Each Z_i is strongly reduced and $E_\Lambda(Z_i, Z_j) = 0$ for all $i \neq j$.*

Proof. Without loss of generality assume that $t = 2$. The general case follows by induction. Let $Z_1 \in \mathrm{decIrr}_{\mathbf{d}_1,\mathbf{v}_1}(\Lambda)$ and $Z_2 \in \mathrm{decIrr}_{\mathbf{d}_2,\mathbf{v}_2}(\Lambda)$.

Assume that $Z := \overline{Z_1 \oplus Z_2}$ is a strongly reduced component. Thus we have $c_\Lambda(Z) = E_\Lambda(Z)$. Applying Lemma 5.10 and Lemma 5.7(ii) this implies

$$c_\Lambda(Z_1) + c_\Lambda(Z_2) = E_\Lambda(Z_1) + E_\Lambda(Z_2) + E_\Lambda(Z_1, Z_2) + E_\Lambda(Z_2, Z_1).$$

Since $c_\Lambda(Z_i) \leq E_\Lambda(Z_i)$ we get $E_\Lambda(Z_1, Z_2) = E_\Lambda(Z_2, Z_1) = 0$ and $c_\Lambda(Z_i) = E_\Lambda(Z_i)$. Thus (i) implies (ii).

To show the converse, assume that Z_1 and Z_2 are strongly reduced with $E_\Lambda(Z_1, Z_2) = E_\Lambda(Z_2, Z_1) = 0$. We claim that

$$c_\Lambda(Z) = c_\Lambda(Z_1) + c_\Lambda(Z_2) = E_\Lambda(Z_1) + E_\Lambda(Z_2) = E_\Lambda(Z).$$

For the first equality we use Lemma 5.10, the second equality is just our assumption that Z_1 and Z_2 are strongly reduced. Finally, the third equality follows from Lemma 5.7 together with our assumption that $E_\Lambda(Z_1, Z_2)$ and $E_\Lambda(Z_2, Z_1)$ are both zero. Thus Z is strongly reduced. □

Note that Theorems 5.3 and 5.11 imply that each $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ is of the form $Z = \overline{Z_1 \oplus \cdots \oplus Z_t}$ with $Z_i \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ and Z_i indecomposable for all i.

The next lemma follows directly from upper semicontinuity and Lemma 4.1(i).

Lemma 5.12. *For $Z, Z_1, Z_2 \in \mathrm{decIrr}(\Lambda)$ with $Z = \overline{Z_1 \oplus Z_2}$ we have*

$$g_\Lambda(Z) = g_\Lambda(Z_1) + g_\Lambda(Z_2).$$

Lemma 5.13. *For $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we have*

$$\mathbf{d} \cdot g_\Lambda(Z) = \dim(Z) - \dim(G_\mathbf{d}).$$

Proof. It follows from the definitions that

$$E_\Lambda(Z) = \mathrm{end}_\Lambda(Z) + \mathbf{d} \cdot g_\Lambda(Z),$$

and we have

$$c_\Lambda(Z) = \dim(Z) - \dim(G_\mathbf{d}) + \mathrm{end}_\Lambda(Z).$$

Now the claim follows, since $c_\Lambda(Z) = E_\Lambda(Z)$. □

Corollary 5.14. *Let $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda)$ with $\mathbf{d} \neq 0$. If $E_\Lambda(Z) = 0$, then*

$$\mathbf{d} \cdot g_\Lambda(Z) = -\mathrm{end}_\Lambda(Z) < 0.$$

5.2. Parametrization of strongly reduced components. Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle / I$ be a finite-dimensional basic algebra. Plamondon [P2] constructs a map

$$P_\Lambda \colon \mathrm{decIrr}(\Lambda) \to \mathbb{Z}^n,$$

which can be defined as follows: Let $Z \in \mathrm{decIrr}(\Lambda)$. Then there exist injective Λ-modules $I_0^\Lambda(Z)$ and $I_1^\Lambda(Z)$, which are uniquely determined up to isomorphism, and a dense open subset $U \subseteq \pi Z$ such that for each representation $M \in U$ we have $I_0^\Lambda(M) = I_0^\Lambda(Z)$ and $I_1^\Lambda(M) = I_1^\Lambda(Z)$. For $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}_{\mathbf{d},\mathbf{v}}(\Lambda)$ define

$$P_\Lambda(Z) := -\underline{\dim}(\mathrm{soc}(I_0^\Lambda(Z))) + \underline{\dim}(\mathrm{soc}(I_1^\Lambda(Z))) + \mathbf{v}.$$

Let

$$P_\Lambda^{\mathrm{s.r.}}\colon \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda) \to \mathbb{Z}^n$$

be the restriction of P_Λ to $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$.

For a representation M let $\mathrm{add}(M)$ be the category of all finite direct sums of direct summands of M. Plamondon [P2] obtains the following striking result.

Theorem 5.15 (Plamondon)**.** *For any finite-dimensional basic algebra Λ the following hold:*

(i)
$$P_\Lambda^{\mathrm{s.r.}}\colon \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda) \to \mathbb{Z}^n$$
is bijective.

(ii) *For every $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ we have*
$$\mathrm{add}(I_0^\Lambda(Z)) \cap \mathrm{add}(I_1^\Lambda(Z)) = 0.$$

Note that Plamondon works with irreducible components, and not with decorated irreducible components. But his results translate easily from one concept to the other.

We now generalize Theorem 5.15(i) to arbitrary basic algebras Λ. It turns out that $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ is in general no longer parametrized by $\mathbb{Z}^n$ but by a subset of $\mathbb{Z}^n$. Our proof is based on Plamondon's result and uses additionally truncations of basic algebras.

For a basic algebra Λ let

$$G_\Lambda\colon \mathrm{decIrr}(\Lambda) \to \mathbb{Z}^n$$

be the map, which sends $Z \in \mathrm{decIrr}(\Lambda)$ to the generic g-vector $g_\Lambda(Z)$ of Z. For finite-dimensional Λ, it follows immediately from Lemma 3.3 that $G_\Lambda = P_\Lambda$. Let

$$G_\Lambda^{\mathrm{s.r.}}\colon \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda) \to \mathbb{Z}^n$$

be the restriction of G_Λ to $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$.

For a basic algebra Λ let

$$\mathrm{decIrr}_{<p}(\Lambda)$$

be the set of irreducible components $Z \in \mathrm{decIrr}(\Lambda)$ such that $(\mathbf{d},\mathbf{v}) := \underline{\dim}(Z)$ satisfies $|\mathbf{d}| < p$. Define

$$\mathrm{decIrr}^{\mathrm{s.r.}}_{<p}(\Lambda) := \mathrm{decIrr}_{<p}(\Lambda) \cap \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda).$$

Lemma 5.16. *For a basic algebra Λ the following hold:*

(i) *For all $p \le q$ we have*
$$\mathrm{decIrr}^{\mathrm{s.r.}}_{<p}(\Lambda_p) \subseteq \mathrm{decIrr}^{\mathrm{s.r.}}_{<q}(\Lambda_q) \subseteq \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda).$$

(ii) *We have*
$$\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda) = \bigcup_{p>0} \mathrm{decIrr}^{\mathrm{s.r.}}_{<p}(\Lambda_p).$$

Proof. Let $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$, and let $p > |\mathbf{d}|$. Thus we have $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda_p)$ and $Z \in \mathrm{decIrr}_{<p}(\Lambda_p)$. Furthermore, we have $c_{\Lambda_p}(Z) = c_\Lambda(Z)$ and $E_{\Lambda_p}(Z) = E_\Lambda(Z)$. Thus $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ if and only if $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda_p)$. This yields the result. □

Theorem 5.17. *For a basic algebra* Λ *the following hold:*

(i) *The map*
$$G_\Lambda^{\text{s.r.}}\colon \operatorname{decIrr}^{\text{s.r.}}(\Lambda) \to \mathbb{Z}^n$$
is injective.

(ii) *The following are equivalent:*
   (a) $G_\Lambda^{\text{s.r.}}$ *is surjective.*
   (b) Λ *is finite-dimensional.*

Proof. Since Λ_p is finite-dimensional for all p, we know from Plamondon's Theorem 5.15(i) that
$$G_{\Lambda_p}^{\text{s.r.}}\colon \operatorname{decIrr}^{\text{s.r.}}(\Lambda_p) \to \mathbb{Z}^n$$
is bijective. Now Lemma 5.16 yields that the map
$$G_\Lambda^{\text{s.r.}}\colon \operatorname{decIrr}^{\text{s.r.}}(\Lambda) \to \mathbb{Z}^n$$
sends $Z \in \operatorname{decIrr}_{<p}^{\text{s.r.}}(\Lambda_p)$ to $G_{\Lambda_p}^{\text{s.r.}}(Z)$, and that $G_\Lambda^{\text{s.r.}}$ is injective. This proves (i). Theorem 5.15(i) says that (b) implies (a). To show the converse, assume that Λ is infinite dimensional. Thus there exists some $1 \le i \le n$ such that the injective envelope I_i of the simple Λ-module S_i is infinite dimensional. Let $I_{i,p}$ be the injective envelope of the simple Λ_p-module S_i. Using that I_i is infinite dimensional, one can easily show that $\dim(I_{i,p}) \ge p$.

Assume that $-e_i$ is in the image of $G_\Lambda^{\text{s.r.}}$. (Here e_i denotes the ith standard basis vector of $\mathbb{Z}^n$.) In other words, there is some $Z \in \operatorname{decIrr}^{\text{s.r.}}(\Lambda)$ such that $G_\Lambda^{\text{s.r.}}(Z) = -e_i$. By Lemma 5.16(ii) we know that $Z \in \operatorname{decIrr}_{<p}^{\text{s.r.}}(\Lambda_p)$ for some $p \ge 1$. Since $g_\Lambda(Z) = -e_i$, we have $I_0^{\Lambda_p}(Z) = I_{i,p}$ and $I_1^{\Lambda_p}(Z) = 0$. This implies that Z is the closure of the orbit of the decorated representation $(I_{i,p}, 0)$. But $\dim(I_{i,p}) \ge p$ and the dimension of all representations in Z is strictly smaller than p, a contradiction. □

The proof of Theorem 5.17(ii) yields the following result.

Corollary 5.18. *For a basic algebra* Λ *and* $1 \le i \le n$ *the following are equivalent:*

(i) $-e_i \in \operatorname{Im}(G_\Lambda^{\text{s.r.}})$.
(ii) I_i *is finite-dimensional.*

Let
$$\mathcal{G}_\Lambda := \operatorname{Im}(G_\Lambda^{\text{s.r.}}) = \{g_\Lambda(Z) \mid Z \in \operatorname{decIrr}^{\text{s.r.}}(\Lambda)\}$$
be the set of generic g-vectors of the strongly reduced irreducible components.

6. Component graphs and CC-clusters

6.1. The graph of strongly reduced components. Let Λ be a basic algebra. In [CBS] the *component graph* $\Gamma(\operatorname{Irr}(\Lambda))$ of Λ is defined as follows: The vertices of $\Gamma(\operatorname{Irr}(\Lambda))$ are the indecomposable irreducible components in $\operatorname{Irr}(\Lambda)$. There is an edge between (possibly equal) vertices Z_1 and Z_2 if $\operatorname{ext}_\Lambda^1(Z_1, Z_2) = \operatorname{ext}_\Lambda^1(Z_2, Z_1) = 0$.

We want to define an analogue of $\Gamma(\operatorname{Irr}(\Lambda))$ for strongly reduced components. The graph $\Gamma(\operatorname{decIrr}^{\text{s.r.}}(\Lambda))$ of strongly reduced components has as vertices the indecomposable components in $\operatorname{decIrr}^{\text{s.r.}}(\Lambda)$, and there is an edge between (possibly equal) vertices Z_1 and Z_2 if $E_\Lambda(Z_1, Z_2) = E_\Lambda(Z_2, Z_1) = 0$.

6.2. Component clusters. Let Γ be a graph, and let Γ_0 be the set of vertices of Γ. We allow only single edges, and we allow loops, i.e. edges from a vertex to itself. For a subset $U \subseteq \Gamma_0$ let Γ_U be the full subgraph, whose set of vertices is U. The subgraph Γ_U is *complete* if for each $i, j \in J$ with $i \neq j$ there is an edge between i and j. A complete subgraph Γ_U is *maximal* if for each complete subgraph $\Gamma_{U'}$ with $U \subseteq U'$ we have $U = U'$. We call a subgraph Γ_U *loop-complete* if Γ_U is complete and there is a loop at each vertex in U.

The set of vertices of a maximal complete subgraph of $\Gamma := \Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ is called a *component cluster* of Λ. A component cluster $\mathcal{U}$ of Λ is *E-rigid* provided $E_\Lambda(Z) = 0$ for all $Z \in \mathcal{U}$. (Recall that there is a loop at a vertex Z of Γ if and only if $E_\Lambda(Z, Z) = 0$. If $E_\Lambda(Z) = 0$, then $E_\Lambda(Z, Z) = 0$, but the converse does not hold.)

Proposition 6.1. *For each loop-complete subgraph Γ_U of $\Gamma := \Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ we have $|U| \leq n$.*

Proof. Assume that $Z_1, \ldots, Z_{n+1}$ are pairwise different vertices of a loop-complete subgraph Γ_J of $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$. For $1 \leq i \leq n+1$ let $g_\Lambda(Z_i)$ be the generic g-vector of Z_i. Since Γ_J is loop-complete we know by Theorem 5.11 that

$$Z_{\mathbf{a}} := \overline{Z_1^{a_1} \oplus \cdots \oplus Z_{n+1}^{a_{n+1}}}$$

is again a strongly reduced component for each $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{N}^{n+1}$. By the additivity of g-vectors we get

$$g_\Lambda(Z_{\mathbf{a}}) = a_1 g_\Lambda(Z_1) + \cdots + a_{n+1} g_\Lambda(Z_{n+1}).$$

Furthermore, we know from Theorem 5.3 that $Z_{\mathbf{a}} = Z_{\mathbf{b}}$ if and only if $\mathbf{a} = \mathbf{b}$. Now one can essentially copy the proof of [GS, Theorem 1.1] to show that there are $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{n+1}$ with $g_\Lambda(Z_{\mathbf{a}}) = g_\Lambda(Z_{\mathbf{b}})$ but $\mathbf{a} \neq \mathbf{b}$. By Theorem 5.17 different strongly reduced components have different g-vectors. Thus we have a contradiction. □

Corollary 6.2. *Let Λ be a finite-dimensional basic algebra. Let M be a representation of Λ with $\mathrm{Hom}_\Lambda(\tau_\Lambda^-(M), M) = 0$. Then M has at most n isomorphism classes of indecomposable direct summands.*

The following conjecture might be a bit too optimistic. But it is true for $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle$ the path algebra of an acyclic quiver Q, see [DW, Corollary 21] and Section 9.1.

Conjecture 6.3. *For any basic algebra Λ the following hold:*

(i) *The component clusters of Λ have cardinality at most n.*
(ii) *The E-rigid component clusters of Λ are exactly the component clusters of cardinality n.*

6.3. *E*-rigid representations. After most of this work was done, we learned that Iyama and Reiten [IR] obtained some beautiful results on socalled *τ-rigid modules* over finite-dimensional algebras, which fit perfectly into the framework of Caldero-Chapoton algebras.

Adapting their terminology to decorated representations of basic algebras, a decorated representation $\mathcal{M}$ of a basic algebra Λ is called *E-rigid* provided $E_\Lambda(\mathcal{M}) = 0$. The following theorem is just a reformulation of Iyama and Reiten's results on τ-rigid modules. Part (i) follows also directly from the more general statement in Proposition 6.1.

For $\mathcal{M} \in \mathrm{decrep}(\Lambda)$ let $\Sigma(\mathcal{M})$ be the number of isomorphism classes of indecomposable direct summands of $\mathcal{M}$.

Theorem 6.4 ([IR]). *Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ be a finite-dimensional basic algebra. For $\mathcal{M} \in \mathrm{decrep}(\Lambda)$ the following hold:*

(i) *If $\mathcal{M}$ is E-rigid, then $\Sigma(\mathcal{M}) \leq n$.*
(ii) *For each E-rigid $\mathcal{M} \in \mathrm{decrep}(\Lambda)$ there exists some $\mathcal{N} \in \mathrm{decrep}(\Lambda)$ such that $\mathcal{M} \oplus \mathcal{N}$ is E-rigid and $\Sigma(\mathcal{M} \oplus \mathcal{N}) = n$.*
(iii) *For each E-rigid $\mathcal{M} \in \mathrm{decrep}(\Lambda)$ with $\Sigma(\mathcal{M}) = n-1$ there are exactly two non-isomorphic indecomposable decorated representations $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{decrep}(\Lambda)$ such that $\mathcal{M} \oplus \mathcal{N}_i$ is E-rigid and $\Sigma(\mathcal{M} \oplus \mathcal{N}_i) = n$ for $i = 1, 2$.*

It is easy to find examples of infinite dimensional basic algebras Λ such that Theorem 6.4(iii) does not hold, see Section 9.3.1.

A basic algebra Λ is *representation-finite* if there are only finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}(\Lambda)$. One easily checks that Λ is finite-dimensional in this case.

Corollary 6.5. *Assume that Λ is a representation-finite basic algebra. Then the following hold:*

(i) *Each component cluster of Λ is E-rigid.*
(ii) *Each component cluster of Λ has cardinality n.*
(iii) *There is bijection between the set of isomorphism classes of E-rigid representation of Λ to the set $\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ of strongly reduced components. Namely, one maps an E-rigid representation $\mathcal{M}$ to the closure of the orbit $\mathcal{O}(\mathcal{M})$.*

Proof. Since Λ is representation-finite, every irreducible component $Z \in \mathrm{decIrr}(\Lambda)$ is a union of finitely many orbits, and exactly one of these orbits has do be dense in Z. Thus we have $c_\Lambda(Z) = 0$. This implies (i) and (iii). Now (ii) follows directly from Theorem 6.4(ii). □

6.4. Generic Caldero-Chapoton functions. For each $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$ let

$$C_{\mathbf{d},\mathbf{v}} \colon \mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) \to \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$$

be the function defined by $\mathcal{M} \mapsto C_\Lambda(\mathcal{M})$. The map $C_{\mathbf{d},\mathbf{v}}$ is a constructible function. In particular, the image of $C_{\mathbf{d},\mathbf{v}}$ is finite. Thus for an irreducible component $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ there exists a dense open subset $U \subseteq Z$ such that $C_{\mathbf{d},\mathbf{v}}$ is constant on U. Define

$$C_\Lambda(Z) := C_\Lambda(\mathcal{M})$$

with $\mathcal{M}$ any representation in U. The element $C_\Lambda(Z)$ depends only on Z and not on the choice of U.

Define

$$\mathcal{B}_\Lambda := \{C_\Lambda(Z) \mid Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)\}.$$

We refer to the elements of $\mathcal{B}_\Lambda$ as *generic Caldero-Chapoton functions*.

Proposition 6.6. *Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ be a basic algebra. If $\mathrm{Ker}(B_Q) \cap \mathbb{Q}^n_{\geq 0} = 0$, then $\mathcal{B}_\Lambda$ is linearly independent in $\mathcal{A}_\Lambda$.*

Proof. For each $Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda)$ there is some $\mathcal{M} \in Z$ such that $g_\Lambda(\mathcal{M}) = g_\Lambda(Z)$ and $C_\Lambda(\mathcal{M}) = C_\Lambda(Z)$. By Theorem 5.17(i) the generic g-vectors of the strongly reduced components of decorated representations of Λ are pairwise different. Now Proposition 4.3 yields the result. □

If $\mathcal{B}_\Lambda$ is a basis of $\mathcal{A}_\Lambda$, then we call $\mathcal{B}_\Lambda$ the *generic basis* of $\mathcal{A}_\Lambda$.

6.5. CC-clusters. For a component cluster $\mathcal{U}$ of a basic algebra Λ let

$$\mathcal{C}_\mathcal{U} := \{C_\Lambda(Z) \mid Z \in \mathcal{U}\} \quad \text{and} \quad \mathcal{M}_\mathcal{U} := \{\prod_{Z\in\mathcal{U}} C_\Lambda(Z)^{a_Z} \mid a_Z \in I_Z\}$$

where

$$I_Z := \begin{cases} \mathbb{N} & \text{if } E_\Lambda(Z,Z) = 0, \\ \{0,1\} & \text{otherwise.} \end{cases}$$

(In each of the products above we assume that $a_Z = 0$ for all but finitely many $Z \in \mathcal{U}$.) The set $\mathcal{C}_\mathcal{U}$ is called a *CC-cluster* of Λ, and the elements in $\mathcal{M}_\mathcal{U}$ are *CC-cluster monomials*. (The letters CC just indicate that we deal with sets of Caldero-Chapoton functions.) A CC-cluster $\mathcal{C}_\mathcal{U}$ is *E-rigid* provided $E_\Lambda(Z) = 0$ for all $Z \in \mathcal{U}$.

Note that

$$\mathcal{C}_\mathcal{U} \subseteq \mathcal{M}_\mathcal{U} \subseteq \mathcal{A}_\Lambda.$$

The following result is a direct consequence of the definition of $\mathcal{B}_\Lambda$ and Theorem 5.11.

Proposition 6.7. *Let Λ be a basic algebra. Then*

$$\mathcal{B}_\Lambda = \bigcup_\mathcal{U} \mathcal{M}_\mathcal{U}$$

where the union is over all component clusters $\mathcal{U}$ of Λ.

6.6. A change of perspective. The CC-clusters are a generalization of the clusters of a cluster algebra defined by Fomin and Zelevinsky. In general, the Fomin-Zelevinsky cluster monomials form just a small subset of the set of CC-cluster monomials. Recall that the Fomin-Zelevinsky cluster monomials are obtained by the inductive procedure of cluster mutation [FZ1, FZ2], and the relation between neighbouring clusters is described by the exchange relations. One can see the exchange relations as part of the definition of a cluster algebra. On the other hand, the definition of a Caldero-Chapoton algebra does not involve any mutations of CC-clusters. The CC-clusters are given by collections of irreducible components, and they do not have to be constructed inductively. One can find a meaningful notion of *neighbouring* CC-clusters, and it remains quite a challenge to actually determine the exchange relations.

6.7. Conjectures. In this section let Λ be any basic algebra. The following conjectures are again quite optimistic in this generality.

Conjecture 6.8. *$\mathcal{B}_\Lambda$ is a $\mathbb{C}$-basis of $\mathcal{A}_\Lambda$.*

Conjecture 6.8 is true for every $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle$ with Q an acyclic quiver and also for numerous other examples, see [GLS].

Conjecture 6.9. *We have*

$$\mathcal{A}_\Lambda = \langle C_\Lambda(Z) \mid Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda),\ E_\Lambda(Z) = 0\rangle_{\mathrm{alg}}.$$

If true, the following conjecture would be a vast generalization of the Laurent phenomenon [FZ1].

Conjecture 6.10. *For any E-rigid component cluster $\{Z_1, \ldots, Z_n\}$ of Λ, we have*

$$\mathcal{A}_\Lambda \subseteq \mathbb{C}[C_\Lambda(Z_1)^\pm, \ldots, C_\Lambda(Z_n)^\pm].$$

In this case, we say that $\mathcal{A}_\Lambda$ has the Laurent phenomenon property.

Conjecture 6.11. *For any E-rigid component cluster* $\{Z_1, \ldots, Z_n\}$ *of* Λ*, the generic g-vectors* $g_\Lambda(Z_1), \ldots, g_\Lambda(Z_n)$ *form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$*.*

7. Caldero-Chapoton algebras and cluster algebras

7.1. Calderon-Chapoton algebras of Jacobian algebras. Suppose that Q is a 2-acyclic quiver with a non-degenerate potential W, and let $\Lambda := \mathcal{P}(Q, W)$ be the associated Jacobian algebra. Let $\mathcal{A}_Q$ and $\mathcal{A}_Q^{\mathrm{up}}$ be the cluster algebra and upper cluster algebra of Q, respectively. Set $\mathcal{A}_{Q,W} := \mathcal{A}_\Lambda$, $\mathcal{B}_{Q,W} := \mathcal{B}_\Lambda$ and $\mathcal{G}_{Q,W} := \mathcal{G}_\Lambda$. Let

$$\mathcal{M}_{Q,W} := \{C_\Lambda(Z) \mid Z \in \mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda),\ E_\Lambda(Z) = 0\}.$$

The first part of the following proposition is a consequence of [DWZ2, Lemma 5.2], compare also the calculation at the end of [GLS, Section 6.3]. The rest follows from [DWZ2, Corollary 7.2].

Proposition 7.1. *We have*

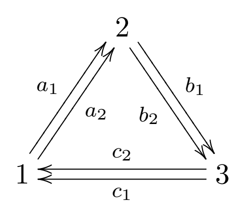
$$\mathcal{A}_Q \subseteq \mathcal{A}_{Q,W} \subseteq \mathcal{A}_Q^{\mathrm{up}}.$$

The set $\mathcal{M}_Q$ *of cluster monomials of* $\mathcal{A}_Q$ *is contained in* $\mathcal{B}_{Q,W}$*. More precisely, we have*

$$\mathcal{M}_Q \subseteq \mathcal{M}_{Q,W} \subseteq \mathcal{B}_{Q,W}.$$

In general, the sets $\mathcal{M}_Q$, $\mathcal{M}_{Q,W}$ and $\mathcal{B}_{Q,W}$ are pairwise different.

7.2. Example. Let Q be the quiver



and define

$$\begin{aligned} W_1 &:= c_1b_1a_1 + c_2b_2a_2, \\ W_2 &:= c_1b_1a_1 + c_2b_2a_2 - c_2b_1a_2c_1b_2a_1. \end{aligned}$$

It is not difficult to check that $\mathcal{P}(Q, W_1)$ is infinite dimensional and $\mathcal{P}(Q, W_2)$ is finite-dimensional. By [BFZ, Proposition 1.26] the algebras $\mathcal{A}_Q$ and $\mathcal{A}_Q^{\mathrm{up}}$ do not coincide. The potentials W_1 and W_2 are both non-degenerate, see [DWZ1, Example 8.6] and [L, Example 8.2], respectively. Furthermore, by [P2, Example 4.3] the set $\mathcal{B}_{Q,W_2}$ of generic functions is not contained in $\mathcal{A}_Q$. In particular, $\mathcal{A}_Q$ and $\mathcal{A}_{Q,W_2}$ do not coincide. We conjecture that $\mathcal{A}_Q = \mathcal{A}_{Q,W_1}$ and $\mathcal{A}_Q^{\mathrm{up}} = \mathcal{A}_{Q,W_2}$.

7.3. Conjectures. Presently there are only few examples of cluster algebras $\mathcal{A}_Q$, which do not coincide with the upper cluster algebra $\mathcal{A}_Q^{\mathrm{up}}$. So one would like to collect more evidence for the following conjectures.

Conjecture 7.2. *There exist non-degenerate potentials* W_1 *and* W_2 *of* Q *such that* $\mathcal{A}_Q = \mathcal{A}_{Q,W_1}$ *and* $\mathcal{A}_Q^{\mathrm{up}} = \mathcal{A}_{Q,W_2}$*.*

Conjecture 7.3. *For a non-degenerate potential* W *of* Q *the following are equivalent:*

(i) $\mathcal{A}_{Q,W} = \mathcal{A}_Q^{\mathrm{up}}$.
(ii) $\mathcal{P}(Q, W)$ *is finite-dimensional.*

Combining Conjectures 7.2 and 7.3 yields the following conjecture.

Conjecture 7.4. *The following are equivalent:*

(i) $\mathcal{A}_Q = \mathcal{A}_Q^{\mathrm{up}}$.
(ii) $\mathcal{P}(Q, W)$ *is finite-dimensional for all non-degenerate potentials* W.

Conjecture 7.5. *For non-degenerate potentials* W_1 *and* W_2 *of* Q *the following are equivalent:*

(i) $\mathcal{A}_{Q,W_1} \subseteq \mathcal{A}_{Q,W_2}$.
(ii) $\mathcal{B}_{Q,W_1} \subseteq \mathcal{B}_{Q,W_2}$.
(iii) $\mathcal{G}_{Q,W_1} \subseteq \mathcal{G}_{Q,W_2}$.

8. Sign-coherence of generic g-vectors

The following result implies Theorem 1.4. The special case, where $\Lambda = \mathcal{P}(Q, W)$ is a Jacobian algebra with non-degenerate potential W and $\mathcal{U}$ is an E-rigid component cluster, is proved in [P2, Theorem 3.7(1)].

Theorem 8.1. *Let* Λ *be a basic algebra, and let* $\mathcal{U}$ *be a component cluster of* Λ. *Then the set* $\{g_\Lambda(Z) \mid Z \in \mathcal{U}\}$ *is sign-coherent.*

Proof. Assume that $\{g_\Lambda(Z) \mid Z \in \mathcal{U}\}$ is not sign-coherent. Thus there are $Z_1, Z_2 \in \mathcal{U}$ such that the set $\{g_\Lambda(Z_1), g_\Lambda(Z_2)\}$ is not sign-coherent. Since $\mathcal{U}$ is a component cluster, we know from Theorem 5.11 that $Z := \overline{Z_1 \oplus Z_2}$ is a strongly reduced component. By Lemma 5.12 we have $g_\Lambda(Z) = g_\Lambda(Z_1) + g_\Lambda(Z_2)$. By Lemma 5.16(ii) there is some p such that $Z, Z_1, Z_2 \in \mathrm{decIrr}^{\mathrm{s.r.}}_{<p}(\Lambda_p)$. We also know that $g_{\Lambda_p}(Z) = g_\Lambda(Z)$ and $g_{\Lambda_p}(Z_i) = g_\Lambda(Z_i)$ for $i = 1, 2$, and that

$$I_0^{\Lambda_p}(Z) = I_0^{\Lambda_p}(Z_1) + I_0^{\Lambda_p}(Z_2) \quad \text{and} \quad I_1^{\Lambda_p}(Z) = I_1^{\Lambda_p}(Z_1) + I_1^{\Lambda_p}(Z_2).$$

For $i = 1, 2$ let $(\mathbf{d}_i, \mathbf{v}_i) := \underline{\dim}(Z_i)$.

We first assume that $\mathbf{v}_1 = \mathbf{v}_2 = 0$. Since $\{g_\Lambda(Z_1), g_\Lambda(Z_2)\}$ is not sign-coherent, we get from Lemma 3.4 that

$$\mathrm{add}(I_0^{\Lambda_p}(Z_1)) \cap \mathrm{add}(I_1^{\Lambda_p}(Z_2)) \neq 0 \quad \text{or} \quad \mathrm{add}(I_1^{\Lambda_p}(Z_1)) \cap \mathrm{add}(I_0^{\Lambda_p}(Z_2)) \neq 0,$$

a contradiction to Theorem 5.15(ii).

Next, assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both non-zero. The components Z_1 and Z_2 are indecomposable. It follows that Z_1 and Z_2 are just the orbits of some negative simple representations. But then $\{g_\Lambda(Z_1), g_\Lambda(Z_2)\}$ has to be sign-coherent, a contradiction.

Finally, let $\mathbf{v}_1 = 0$ and $\mathbf{v}_2 \neq 0$. Thus we get $Z_2 = \mathcal{O}(\mathcal{S}_i^-)$ for some $1 \leq i \leq n$. This implies $g_\Lambda(Z_2) = e_i$. Since $\{g_\Lambda(Z_1), g_\Lambda(Z_2)\}$ is not sign-coherent, the ith entry of $g_\Lambda(Z_1)$ has to be negative. It follows that the socle of each representation in Z_1 has S_i as a composition factor. In particular, the ith entry d_i of $\mathbf{d}_1$ is non-zero. But we also have $E_\Lambda(Z_1, Z_2) = 0$. Now Lemma 5.4 implies that $d_i = 0$, a contradiction. □

9. Examples

9.1. Strongly reduced components for hereditary algebras.

9.1.1. Assume that $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle$ with Q an acyclic quiver. Thus Λ is equal to the ordinary path algebra $\mathbb{C}Q$. Clearly, for each $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$ the variety $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ is an affine space. In particular, it has just one irreducible component, namely $Z_{\mathbf{d},\mathbf{v}} := \mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$.

Lemma 9.1. *The following hold:*

(i) *For irreducible components* $Z_{\mathbf{d}_1,0}, Z_{\mathbf{d}_2,0} \in \mathrm{decIrr}(\Lambda)$ *we have*

$$\mathrm{ext}^1_\Lambda(Z_{\mathbf{d}_1,0}, Z_{\mathbf{d}_2,0}) = E_\Lambda(Z_{\mathbf{d}_1,0}, Z_{\mathbf{d}_2,0}).$$

(ii) $Z_{\mathbf{d},\mathbf{v}}$ *is strongly reduced if and only if* $d_i v_i = 0$ *for all* $1 \leq i \leq n$.

Proof. Since Λ is a finite-dimensional hereditary algebra, we have

$$\dim \mathrm{Ext}^1_\Lambda(M, N) = \dim \mathrm{Hom}_\Lambda(\tau^-_\Lambda(N), M)$$

for all $M, N \in \mathrm{rep}(\Lambda)$. Now Proposition 3.5 implies (i). In particular, for $Z = Z_{\mathbf{d},0}$ we have $e_\Lambda(Z) = E_\Lambda(Z)$. Since $Z = \mathrm{decrep}_{\mathbf{d},0}(\Lambda)$ is an affine space, Voigt's Lemma implies that $c_\Lambda(Z) = e_\Lambda(Z)$. Thus Z is strongly reduced. The components Z_{0,e_i} are obviously also strongly reduced. Now Lemma 5.4 yields (ii). □

The following result is a direct consequence of Lemma 9.1 and Schofield's [Scho] ground breaking work on general representations of acyclic quivers. For all unexplained terminology we refer to [Scho].

Proposition 9.2. *Let* $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle$ *with* Q *an acyclic quiver. Then the indecomposable strongly reduced components are the components* $Z_{\mathbf{d},0}$*, where* $\mathbf{d}$ *is a Schur root, and the components* $Z_{0,e_1}, \ldots, Z_{0,e_n}$*, where* e_i *is the* i*th standard basis vector of* $\mathbb{Z}^n$*.*

For a finite-dimensional path algebra $\Lambda = \mathbb{C}Q$ one can use Schofield's algorithm [Scho] (see also [DW] for a more efficient version of the algorithm) to determine the canonical decomposition of a dimension vector, and one can also use it to decide if $\mathrm{ext}^1_\Lambda(Z_1, Z_2)$ is zero or not. So at least in principle, the graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ can be computed. However, even in this case there are numerous interesting open questions on the structure of the graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$, see [Sche].

9.2. Strongly reduced components for 1-vertex algebras.

Proposition 9.3. *Let* $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle/I$ *be a basic algebra with* $n = 1$*. Then the following hold:*

(i) *If* Λ *is finite-dimensional, then the indecomposable strongly reduced components in* $\mathrm{decIrr}(\Lambda)$ *are* $\mathcal{O}(\mathcal{S}_1^-)$ *and the closure of* $\mathcal{O}(\mathcal{I}_1)$*, where* $\mathcal{I}_1 := (I_1, 0)$ *and* I_1 *is the injective envelope of the simple* Λ*-module* S_1*.*
(ii) *If* Λ *is infinite-dimensional, then the only indecomposable strongly reduced component in* $\mathrm{decIrr}(\Lambda)$ *is* $\mathcal{O}(\mathcal{S}_1^-)$*.*

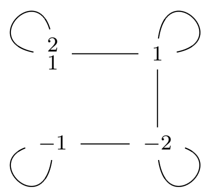
Proof. Assume that Λ is finite-dimensional. Then Theorem 5.15(i) implies that $\mathcal{G}_\Lambda = \mathbb{Z}$. For $m \geq 0$, we know that the orbit closures of $(\mathcal{S}_1^-)^m$ and $\mathcal{I}_1^m$ are E-rigid strongly reduced components with generic g-vectors me_1 and $-me_1$, respectively. This implies (i). Part (ii) follows from the proof of Theorem 5.17(ii). □

9.3. Strongly reduced components for some representation-finite algebras. In the following examples, for each E-rigid indecomposable strongly reduced component, we just display the indecomposable decorated representation whose orbit closure is the component. We describe representations by displaying their socle series and their composition factors. For $1 \leq i \leq n$ we write i and $-i$ instead of $\mathcal{S}_i$ and $\mathcal{S}_i^-$, respectively. For a decorated representation of the form $\mathcal{M} = (M, 0)$ we just display M.

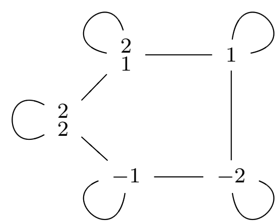
9.3.1. Let Q be the quiver

$$1 \xleftarrow{\;b\;} 2 \circlearrowright a$$

and let $\Lambda := \mathbb{C}\langle\langle Q \rangle\rangle / I$, where I is generated by ba. Then $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ looks as follows:



For $p = 2$, the component graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda_p))$ looks as follows:

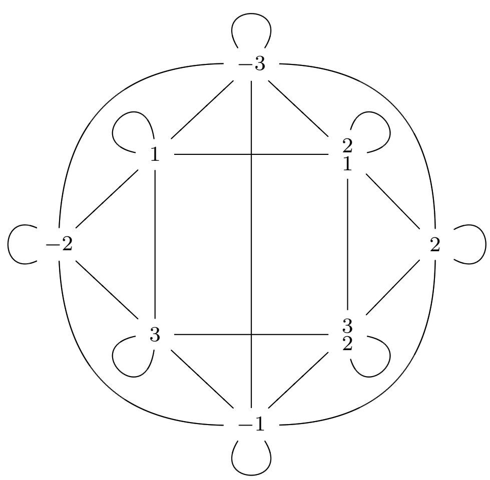


To repair the somewhat non-symmetric graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ one could insert a vertex for the infinite-dimensional indecomposable injective Λ-module I_2. Such aspects will be dealt with elsewhere.

9.3.2. Let Q be the quiver

$$1 \xleftarrow{\;b\;} 2 \xleftarrow{\;a\;} 3$$

and let $\Lambda := \mathbb{C}\langle\langle Q \rangle\rangle / I$, where I is generated by ba. Then $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ looks as follows:



Note that for $M = {3 \atop 2}$ and $N = 1$ and we have $\mathrm{Ext}^1_\Lambda(M, N) = 0$ but $E_\Lambda((M, 0), (N, 0)) = \mathrm{Hom}_\Lambda(\tau^-(N), M) \neq 0$.

9.4. Examples of Caldero-Chapoton algebras.

9.4.1. Let Q be the quiver

$$1 \circlearrowleft$$

and let $\Lambda := \mathbb{C}\langle\langle Q \rangle\rangle$. We have $B_Q = (0)$. Up to isomorphism, for each $d \geq 1$ there is a unique indecomposable representation M_d of Λ with $\dim(M_d) = d$. One easily checks that

$$C_\Lambda(M_d) = d + 1.$$

This implies $\mathcal{A}_\Lambda = \mathbb{C}$.

For $p \geq 1$ the indecomposable representations of the p-truncation Λ_p are $M_1, \ldots, M_p$, and we get

$$C_{\Lambda_p}(M_d) = \begin{cases} (d+1)x_1^{-1} & \text{if } d = p, \\ (d+1) & \text{otherwise.} \end{cases}$$

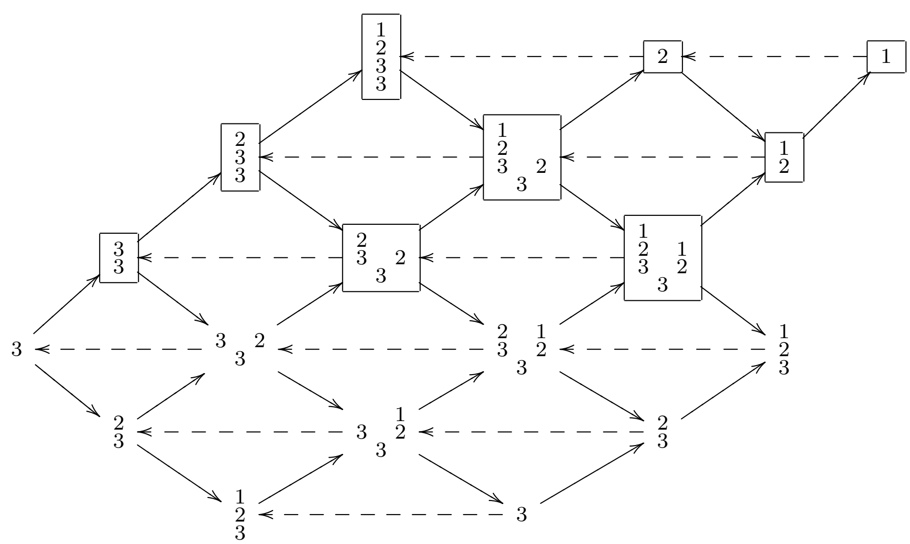
This implies $\mathcal{A}_{\Lambda_p} = \mathbb{C}[x_1^{-1}]$.

9.4.2. In this section, let Q be the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \circlearrowleft c$$

and let $\Lambda := \mathbb{C}\langle\langle Q \rangle\rangle / I$, where I is the ideal generated by c^2. We want to study the Caldero-Chapoton algebra $\mathcal{A}_\Lambda$ generated by the Caldero-Chapoton functions associated to the decorated representations of Λ.

The basic algebra Λ is a representation-finite string algebra, and its Auslander-Reiten quiver looks as follows: (Recall that there is an arrow from M to N if and only if there is an irreducible homomorphism from M to N, and for all non-injective M we draw a dashed arrow from M to its Auslander-Reiten translate $\tau_\Lambda(M)$.)



In this quiver the two south-west and the two south-east edges are identified. The framed representations are the indecomposable E-rigid decorated representations of Λ. We have

$$I_1 = {\scriptstyle 1}, \qquad I_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \qquad I_3 = \begin{smallmatrix} 1 & & \\ 2 & & 1 \\ 3 & & 2 \\ & 3 & \end{smallmatrix}.$$

We now describe explicitly the Caldero-Chapoton functions associated to the 12 indecomposable E-rigid decorated representations of Λ. By definition $C_\Lambda(\mathcal{S}_i^-) = x_i$, for $i = 1, 2, 3$. The remaining 9 functions are

$$C_\Lambda(1) = \frac{1+x_2}{x_1}, \qquad C_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) = \frac{x_1+x_3+x_2x_3}{x_1x_2},$$

$$C_\Lambda\left(\begin{smallmatrix}1\\2\\3\\3\end{smallmatrix}\right) = \frac{x_1x_2^2+x_1x_2+x_1+x_3+x_2x_3}{x_1x_2x_3}, \qquad C_\Lambda(2) = \frac{x_1+x_3}{x_2},$$

$$C_\Lambda\left(\begin{smallmatrix}3\\3\end{smallmatrix}\right) = \frac{x_2^2+x_2+1}{x_3}, \qquad C_\Lambda\left(\begin{smallmatrix}2\\3\\3\end{smallmatrix}\right) = \frac{x_1x_2^2+x_1x_2+x_1+x_3}{x_2x_3},$$

$$C_\Lambda\left(\begin{smallmatrix}1&&\\2&&\\3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1^2x_2^2+x_1^2x_2+x_1x_2x_3+2x_1x_3+x_1^2+x_1x_2x_3+x_2x_3^2+x_3^2}{x_1x_2^2x_3},$$

$$C_\Lambda\left(\begin{smallmatrix}1&&\\2&&1\\3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1^2x_2^2+x_1^2x_2+x_1^2+x_1x_2x_3+2x_1x_3+x_3^2+x_1x_2^2x_3+2x_1x_2x_3+2x_2x_3^2+x_2^2x_3^2}{x_1^2x_2^2x_3},$$

$$C_\Lambda\left(\begin{smallmatrix}2&&\\3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1^2x_2^2+x_1^2x_2+x_1^2+x_1x_2x_3+2x_1x_3+x_3^2}{x_2^2x_3}.$$

The Caldero-Chapoton functions associated to the 6 indecomposable non-E-rigid representations of Λ are

$$C_\Lambda(3) = x_2+1,$$

$$C_\Lambda\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right) = \frac{x_1x_2+x_1+x_3}{x_2},$$

$$C_\Lambda\left(\begin{smallmatrix}3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1x_2^2+x_1x_2+x_1+x_3+x_2x_3}{x_2x_3},$$

$$C_\Lambda\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) = \frac{x_1x_2+x_1+x_3+x_2x_3}{x_1x_2},$$

$$C_\Lambda\left(\begin{smallmatrix}&&1\\3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1x_2^2+x_1x_2+x_1+x_2x_3+x_2^2x_3+x_2x_3+x_3}{x_1x_2x_3},$$

$$C_\Lambda\left(\begin{smallmatrix}2&&1\\3&&2\\&3&\end{smallmatrix}\right) = \frac{x_1^2x_2^2+x_1^2x_2+x_1^2+x_1x_2x_3+2x_1x_3+x_3^2+x_1x_2^2x_3+x_1x_2x_3+x_2x_3^2}{x_1x_2^2x_3}.$$

Claim 9.4. *$\mathcal{B}_\Lambda$ generates $\mathcal{A}_\Lambda$ as a $\mathbb{C}$-vector space.*

Proof. It is enough to express the Caldero-Chapoton functions of the 6 indecomposable non-E-rigid representations in terms of the generic Caldero-Chapoton functions in $\mathcal{B}_\Lambda$. An easy calculation yields

$$C_\Lambda(3) = x_2+1,$$

$$C_\Lambda\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right) = x_1 + C_\Lambda(2),$$

$$C_\Lambda\left(\begin{smallmatrix}3&&2\\&3&\end{smallmatrix}\right) = C_\Lambda\left(\begin{smallmatrix}2\\3\\3\end{smallmatrix}\right)+1$$

$$C_\Lambda\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) = C_\Lambda\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)+1,$$

$$C_\Lambda\left(\begin{smallmatrix}&&1\\3&&2\\&3&\end{smallmatrix}\right) = C_\Lambda\left(\begin{smallmatrix}1\\2\\3\\3\end{smallmatrix}\right)+C_\Lambda(1),$$

$$C_\Lambda\left(\begin{smallmatrix}2&&1\\3&&2\\&3&\end{smallmatrix}\right) = C_\Lambda\left(\begin{smallmatrix}1&&\\2&&\\3&&2\\&3&\end{smallmatrix}\right)+1.$$

All Caldero-Chapoton functions appearing on the right hand side of the above equations are elements in $\mathcal{B}_\Lambda$. (Note that $x_i = C_\Lambda(\mathcal{S}_i^-)$ and $1 = C_\Lambda(0)$ are both in $\mathcal{B}_\Lambda$.) This finishes the proof. □

To prove the linear independence of $\mathcal{B}_\Lambda$ is left as an exercise.

Since Λ is representation-finite, each strongly reduced component contains an E-rigid decorated representation. Each vertex of $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ has a loop. Let $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))^\circ$ be the graph obtained by deleting these loops. We display $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))^\circ$ in Figure 1. Each component cluster is E-rigid and contains exactly three irreducible components.

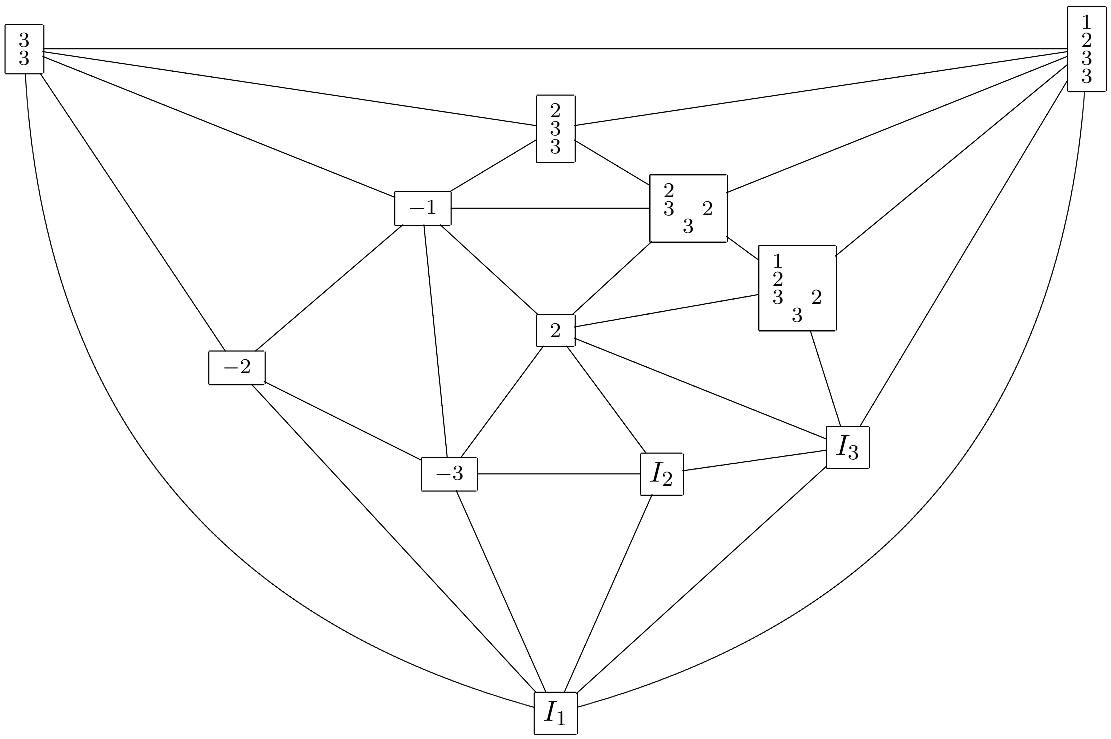


FIGURE 1. The graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))^\circ$ of indecomposable strongly reduced components

Every CC-cluster of $\mathcal{A}_\Lambda$ is a free generating set for the field $\mathbb{C}(x_1, x_2, x_3)$ of rational functions in the variables x_1, x_2, x_3. In particular, every element of $\mathcal{A}_\Lambda$ is a rational function when expressed in terms of any CC-cluster. In fact, even the following holds:

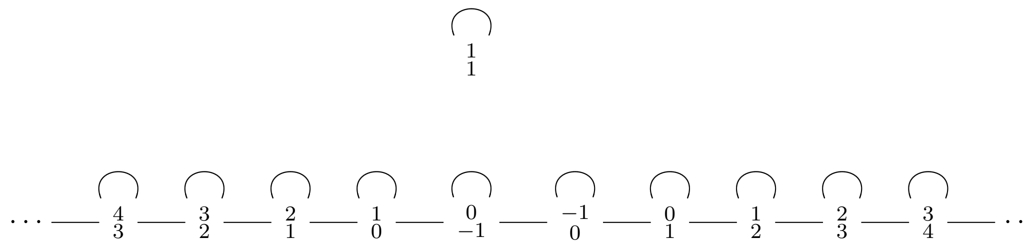
Claim 9.5. $\mathcal{A}_\Lambda$ *has the Laurent phenomenon property.*

The proof will be presented elsewhere.

9.4.3. Let Q be the 2-Kronecker quiver

$$\begin{matrix} 1 \\ \downdownarrows \\ 2 \end{matrix}$$

and let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle$. The following picture describes the quiver $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$. (For indecomposable strongly reduced components of the form $Z_{\mathbf{d},0}$ or Z_{0,e_i} we just display the vectors $\mathbf{d}$ or $-e_i$, respectively.)



Thus there is exactly one component cluster $\{Z\}$ of cardinality one, and there are infinitely many component clusters of cardinality two. One can easily check that $E_\Lambda(Z,Z) = 0$, hence the loop at Z, but $E_\Lambda(Z) \neq 0$. Thus $\{Z\}$ is not E-rigid. The other component clusters are E-rigid. The CC-cluster monomials are

$$C_\Lambda\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)^a C_\Lambda\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)^b, \quad C_\Lambda\left(\begin{smallmatrix}i+1\\i\end{smallmatrix}\right)^a C_\Lambda\left(\begin{smallmatrix}i\\i-1\end{smallmatrix}\right)^b, \quad C_\Lambda\left(\begin{smallmatrix}i-1\\i\end{smallmatrix}\right)^a C_\Lambda\left(\begin{smallmatrix}i\\i+1\end{smallmatrix}\right)^b \quad \text{and} \quad C_\Lambda\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)^a$$

where $a, b, i \geq 0$.

The set $\mathcal{B}_\Lambda$ of generic Caldero-Chapoton functions is just the set of CC-cluster monomials. Recall from [BFZ] that for any acyclic quiver Q we have $\mathcal{A}_Q = \mathcal{A}_Q^{\mathrm{up}}$. In this case, $\mathcal{B}_\Lambda$ is a $\mathbb{C}$-basis of $\mathcal{A}_Q$, see [GLS].

For acyclic quivers Q of wild representation type and $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle$, the component graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ will have vertices without loops. For example, let Q be the 3-Kronecker quiver

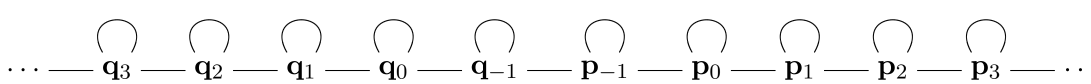
and let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle$. Let

$$\phi = \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix}$$

be the Coxeter matrix of Λ. For $k \geq 0$ define

$$\begin{aligned}
\mathbf{p}_{2k} &:= \phi^k \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \\
\mathbf{p}_{2k+1} &:= \phi^k \left(\begin{smallmatrix}1\\3\end{smallmatrix}\right), \\
\mathbf{q}_{2k} &:= \phi^{-k} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \\
\mathbf{q}_{2k+1} &:= \phi^{-k} \left(\begin{smallmatrix}3\\1\end{smallmatrix}\right).
\end{aligned}$$

Set $\mathbf{p}_{-1} := -e_1$ and $\mathbf{q}_{-1} := -e_2$. One connected component of the component graph $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$ looks as follows:



These are precisely the E-rigid vertices of $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$.

The set of Schur roots of Q consists of *real* and *imaginary* Schur roots. The above picture shows the real Schur roots (and the vectors $-e_1$ and $-e_2$). The set R_{im}^+ of imaginary Schur roots consists of all dimension vectors $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$ with $d_2 \neq 0$ such that

$$(3 - \sqrt{5})/2 \leq d_1/d_2 \leq (3 + \sqrt{5})/2,$$

see [DW, Section 3] and [K, Section 6]. There is no edge between $Z_{\mathbf{d},0}$ and any other vertex of $\Gamma(\mathrm{decIrr}^{\mathrm{s.r.}}(\Lambda))$. In particular, there is no loop at $Z_{\mathbf{d},0}$.

The CC-cluster monomials are

$$C_\Lambda(\mathbf{q}_{-1})^a C_\Lambda(\mathbf{p}_{-1})^b, \quad C_\Lambda(\mathbf{p}_{i-1})^a C_\Lambda(\mathbf{p}_i)^b, \quad C_\Lambda(\mathbf{q}_i)^a C_\Lambda(\mathbf{q}_{i-1})^b \quad \text{and} \quad C_\Lambda(\mathbf{d})$$

where $a, b, i \geq 0$ and $\mathbf{d} \in R^+_{\mathrm{im}}$. Again it follows from [GLS] that these CC-cluster monomials form a $\mathbb{C}$-basis of $\mathcal{A}_Q$. It remains a challenge to compute the exchange relations between all *neighbouring* CC-clusters. For the E-rigid CC-clusters, the exchange relations are known from the Fomin-Zelevinsky exchange relations arising from mutations of clusters. But for $\mathbf{d}, \mathbf{d}_1, \mathbf{d}_2 \in R^+_{\mathrm{im}}$ and $i \geq -1$ it remains an open problem to express the products

$$C_\Lambda(\mathbf{d})C_\Lambda(\mathbf{p}_i), \quad C_\Lambda(\mathbf{d})C_\Lambda(\mathbf{q}_i) \quad \textit{and} \quad C_\Lambda(\mathbf{d}_1)C_\Lambda(\mathbf{d}_2)$$

as linear combinations of elements from the basis $\mathcal{B}_\Lambda$.

Acknowledgements. We thank Christof Geiß and Pierre-Guy Plamondon for helpful discussions.

References

[AR] M. Auslander, I. Reiten, *Modules determined by their composition factors*, Illinois J. Math. 29 (1985), no. 2, 280–301.

[ARS] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Corrected reprint of the 1995 original. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997. xiv+425 pp.

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras*, Vol. 1. Techniques of representation theory. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006. x+458 pp.

[BFZ] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), no. 1, 1–52.

[CC] P. Caldero, F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. 81 (2006), no. 3, 595–616.

[C] G. Cerulli Irelli, *Cluster Algebras of Type* $A_2^{(1)}$, Algebras and Representation Theory (to appear), 40pp., arXiv:0904.2543

[CBS] W. Crawley-Boevey, J. Schröer, *Irreducible components of varieties of modules*, J. Reine Angew. Math. 553 (2002), 201–220.

[DW] H. Derksen, J. Weyman, *On the canonical decomposition of quiver representations*, Compositio Math. 133 (2002), no. 3, 245–265.

[DWZ1] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations*, Selecta Math. (N.S.) 14 (2008), no. 1, 59–119.

[DWZ2] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations II: applications to cluster algebras*, J. Amer. Math. Soc. 23 (2010), no. 3, 749–790.

[FZ1] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[FZ2] S. Fomin, A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Invent. Math. 154 (2003), no. 1, 63–121.

[FK] C. Fu, B. Keller, *On cluster algebras with coefficients and 2-Calabi-Yau categories*, Trans. Amer. Math. Soc. 362 (2010), no. 2, 859–895.

[G] P. Gabriel, *Finite representation type is open*, Proceedings of the International Conference on Representations of Algebras (Carleton Univ., Ottawa, Ont., 1974), Paper No. 10, 23pp. Carleton Math. Lecture Notes, No. 9, Carleton Univ., Ottawa, Ont., 1974.

[GL] C. Geiß, D. Labardini-Fragoso, Private communication (2012).

[GLS] C. Geiß, B. Leclerc, J. Schröer, *Generic bases for cluster algebras and the Chamber ansatz*, J. Amer. Math. Soc. 25 (2012), no. 1, 21–76.

[GS] C. Geiß, J. Schröer, *Extension-orthogonal components of preprojective varieties*, Trans. Amer. Math. Soc. 357 (2005), no. 5, 1953–1962.

[IR] O. Iyama, I. Reiten, *τ-tilting modules*, Talk in Trondheim on 28.03.2012.

[K] V. Kac, *Infinite root systems, representations of graphs and invariant theory. II*, J. Algebra 78 (1982), no. 1, 141–162.

[L] D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces, Part II: Arc representations*, Preprint(2009), 52pp., arXiv:0909.4100v2

[Pa] Y. Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 6, 2221–2248.

[P1] P.-G. Plamondon, *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, Compos. Math. 147 (2011), 1921–1954.

[P2] P.-G. Plamondon, *Generic bases for cluster algebras from the cluster category*, Int. Math. Res. Notices (to appear), 35 pp., arXiv:1111.4431

[Sche] S. Scherotzke, *Generalized clusters for acyclic quivers*, Preprint (2012).

[Scho] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) 65 (1992), no. 1, 46–64.

Giovanni Cerulli Irelli
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany

E-mail address: cerulli@math.uni-bonn.de

Daniel Labardini-Fragoso
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany

E-mail address: labardini@math.uni-bonn.de

Jan Schröer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany

E-mail address: schroer@math.uni-bonn.de